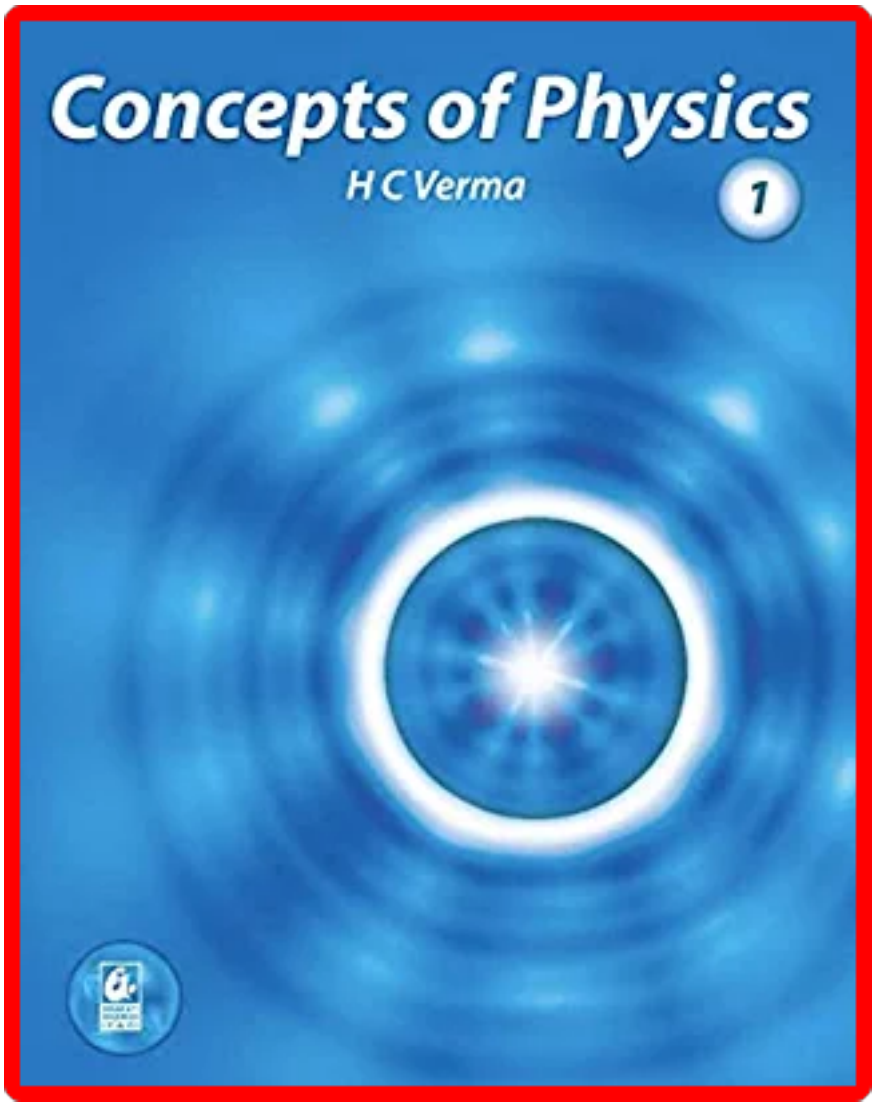
Concepts of Physics
H C Verma
1

# CONCEPTS OF PHYSICS

[VOLUME 1]

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PUBLISHERS & DISTRIBUTORS

*Dedicated to*
*Indian Philosophy & Way of Life*
*of which*
*my parents were*
*an integral part*

# FOREWORD

A few years ago I had an occasion to go through the book *Calculus* by L V Terasov. It unravels intricacies of the subject through a dialogue between Teacher and Student. I thoroughly enjoyed reading it. For me this seemed to be one of the few books which teach a difficult subject through inquisition, and using programmed concept for learning. After that book, Dr Harish Chandra Verma's book on physics, *CONCEPTS OF PHYSICS* is another such attempt, even though it is not directly in the dialogue form. I have thoroughly appreciated it. It is clear that Dr Verma has spent considerable time in formulating the structure of the book, besides its contents. I think he has been successful in this attempt. Dr Verma's book has been divided into two parts because of the size of the total manuscript. There have been several books on this subject, each one having its own flavour. However, the present book is a totally different attempt to teach physics, and I am sure it will be extremely useful to the undergraduate students. The exposition of each concept is extremely lucid. In carefully formatted chapters, besides problems and short questions, a number of objective questions have also been included. This book can certainly be extremely useful not only as a textbook, but also for preparation of various competitive examinations.

Those who have followed Dr Verma's scientific work always enjoyed the outstanding contributions he has made in various research areas. He was an outstanding student of Physics Department of IIT Kanpur during his academic career. An extremely methodical, sincere person as a student, he has devoted himself to the task of educating young minds and inculcating scientific temper amongst them. The present venture in the form of these two volumes is another attempt in that direction. I am sure that young minds who would like to *learn physics in an appropriate manner* will find these volumes extremely useful.

I must heartily congratulate Dr Harish Chandra Verma for the magnificent job he has done.

**Y R Waghmare**
Professor of Physics
IIT Kanpur.

# PREFACE

**Why a new book ?**

Excellent books exist on physics at an introductory college level so why a new one ? Why so many books exist at the same level, in the first place, and why each of them is highly appreciated ? It is because each of these books has the privilege of having an author or authors who have *experienced* physics and have their own method of communicating with the students. During my years as a physics teacher, I have developed a somewhat different methodology of presenting physics to the students. *Concepts of Physics* is a translation of this methodology into a textbook.

**Prerequisites**

The book presents a calculus-based physics course which makes free use of algebra, trigonometry and co-ordinate geometry. The level of the latter three topics is quite simple and high school mathematics is sufficient. Calculus is generally done at the introductory college level and I have assumed that the student is enrolled in a concurrent first calculus course. The relevant portions of calculus have been discussed in Chapter 2 so that the student may start using it from the beginning.

Almost no knowledge of physics is a prerequisite. I have attempted to start each topic from the zero level. A receptive mind is all that is needed to use this book.

**Basic philosophy of the book**

The motto underlying the book is *physics is enjoyable.*

Being a description of the nature around us, physics is our best friend from the day of our existence. I have extensively used this aspect of physics to introduce the physical principles starting with common day occurrences and examples. The subject then appears to be friendly and enjoyable. I have taken care that numerical values of different quantities used in problems correspond to real situations to further strengthen this approach.

**Teaching and training**

The basic aim of physics teaching has been to let the student know and understand the principles and equations of physics and their applications in real life.

However, to be able to use these principles and equations correctly in a given physical situation, one needs further training. A large number of *questions and solved and unsolved problems* are given for this purpose. Each question or problem has a specific purpose. It may be there to bring out a subtle point which might have passed unnoticed while doing the text portion. It may be a further elaboration of a concept developed in the text. It may be there to make the student react when several concepts introduced in different chapters combine and show up as a physical situation and so on. Such tools have been used to develop a culture: *analyse the situation, make a strategy to invoke correct principles and work it out.*

**Conventions**

I have tried to use symbols, names, etc., which are popular nowadays. SI units have been consistently used throughout the book. SI prefixes such as *micro, milli, mega,* etc., are used whenever they make the presentation more readable. Thus, $20\ \mu F$ is preferred over $20 \times 10^{-6}$ F. Co-ordinate sign convention is used in geometrical optics. Special emphasis has been given to dimensions of physical quantities. Numerical values of physical quantities have been mentioned with the units even in equations to maintain dimensional consistency.

I have tried my best to keep errors out of this book. I shall be grateful to the readers who point out any errors and/or make other constructive suggestions.

**H C Verma**

# ACKNOWLEDGEMENTS

The work on this book started in 1984. Since then, a large number of teachers, students and physics lovers have made valuable suggestions which I have incorporated in this work. It is not possible for me to acknowledge all of them individually. I take this opportunity to express my gratitude to them. However, to Dr S B Mathur, who took great pains in going through the entire manuscript and made valuable comments, I am specially indebted. I am also beholden to my colleagues Dr A Yadav, Dr Deb Mukherjee, Mr M M R Akhtar, Dr Arjun Prasad, Dr S K Sinha and others who gave me valuable advice and were good enough to find time for fruitful discussions. To Dr T K Dutta of B E College, Sibpur I am grateful for having taken time to go through portions of the book and making valuable comments.

I thank my student Mr Shailendra Kumar who helped me in checking the answers. I am grateful to Dr B C Rai, Mr Sunil Khijwania & Mr Tejaswi Khijwania for helping me in the preparation of rough sketches for the book.

Finally, I thank the members of my family for their support and encouragement.

**H C Verma**

# TO THE STUDENTS

Here is a brief discussion on the organisation of the book which will help you in using the book most effectively. The book contains 47 chapters divided in two volumes. Though I strongly believe in the underlying unity of physics, a broad division may be made in the book as follows:

Chapters 1–14: Mechanics

15–17: Waves including wave optics

18–22: Optics

23–28: Heat and thermodynamics

29–40: Electric and magnetic phenomena

41–47: Modern physics

Each chapter contains a description of the physical principles related to that chapter. It is well supported by mathematical derivations of equations, descriptions of laboratory experiments, historical background, etc. There are "in-text" solved examples. These examples explain the equation just derived or the concept just discussed. These will help you in fixing the ideas firmly in your mind. Your teachers may use these in-text examples in the classroom to encourage students to participate in discussions.

After the theory section, there is a section on *Worked Out Examples*. These numerical examples correspond to various thinking levels and often use several concepts introduced in that chapter or even in previous chapters. You should read the statement of a problem and try to solve it yourself. In case of difficulty, look at the solution given in the book. Even if you solve the problem successfully, you should look into the solution to compare it with your method of solution. You might have thought of a better method, but knowing more than one method is always beneficial.

Then comes the part which tests your understanding as well as develops it further. *Questions for Short Answer* generally touch very minute points of your understanding. It is not necessary that you answer these questions in a single sitting. They have great potential to initiate very fruitful dicussions. So, freely discuss these questions with your friends and see if they agree with your answer. Answers to these questions are not given for the simple reason that the answers could have cut down the span of such discussions and that would have sharply reduced the utility of these questions.

There are two sections on multiple-choice questions, namely OBJECTIVE I and OBJECTIVE II. There are four options following each of these questions. Only one option is correct for OBJECTIVE I questions. Any number of options, zero to four, may be correct for OBJECTIVE II questions. Answers to all these questions are provided.

Finally, a set of numerical problems are given for your practice. Answers to these problems are also provided. The problems are generally arranged according to the sequence of the concepts developed in the chapter but they are not grouped under section-headings. I don't want to bias your ideas beforehand by telling you that this problem belongs to that section and hence use that particular equation. You should yourself look into the problem and decide which equations or which methods should be used to solve it. Many of the problems use several concepts developed in different sections of the chapter. Many of them even use the concepts from the previous chapters. Hence, you have to plan out the strategy after understanding the problem.

Remember, no problem is difficult. Once you understand the theory, each problem will become easy. So, don't jump to exercise problems before you have gone through the theory, the worked-out problems and the objectives. Once you feel confident in theory, do the exercise problems. The exercise problems are so arranged that they gradually require more thinking.

I hope you will enjoy *Concepts of Physics*.

**H C Verma**

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## CHAPTER 1

# INTRODUCTION TO PHYSICS

### 1.1 WHAT IS PHYSICS ?

The nature around us is colourful and diverse. It contains phenomena of large varieties. The winds, the sands, the waters, the planets, the rainbow, heating of objects on rubbing, the function of a human body, the energy coming from the sun and the nucleus …… there are a large number of objects and events taking place around us.

Physics is the study of nature and its laws. We expect that all these different events in nature take place according to some basic laws and *revealing these laws of nature from the observed events* is physics. For example, the orbiting of the moon around the earth, falling of an apple from a tree and tides in a sea on a full moon night can all be explained if we know the Newton's law of gravitation and Newton's laws of motion. Physics is concerned with the basic rules which are applicable to all domains of life. Understanding of physics, therefore, leads to applications in many fields including bio and medical sciences.

The great physicist Dr R. P. Feynman has given a wonderful description of what is "understanding the nature". Suppose we do not know the rules of chess but are allowed to watch the moves of the players. If we watch the game for a long time, we may make out some of the rules. With the knowledge of these rules we may try to understand why a player played a particular move. However, this may be a very difficult task. Even if we know all the rules of chess, it is not so simple to understand all the complications of a game in a given situation and predict the correct move. Knowing the basic rules is, however, the minimum requirement if any progress is to be made.

One may guess at a wrong rule by partially watching the game. The experienced player may make use of a rule for the first time and the observer of the game may get surprised. Because of the new move some of the rules guessed at may prove to be wrong and the observer will frame new rules.

Physics goes the same way. The nature around us is like a big chess game played by Nature. The events in the nature are like the moves of the great game. We are allowed to watch the events of nature and guess at the basic rules according to which the events take place. We may come across new events which do not follow the rules guessed earlier and we may have to declare the old rules inapplicable or wrong and discover new rules.

Since physics is the study of nature, it is real. No one has been given the authority to frame the rules of physics. We only *discover* the rules that are operating in nature. Aryabhat, Newton, Einstein or Feynman are great physicists because from the observations available at that time, they could guess and frame the laws of physics which explained these observations in a convincing way. But there can be a new phenomenon any day and if the rules discovered by the great scientists are not able to explain this phenomenon, no one will hesitate to change these rules.

### 1.2 PHYSICS AND MATHEMATICS

The description of nature becomes easy if we have the freedom to use mathematics. To say that the gravitational force between two masses is proportional to the product of the masses and is inversely proportional to the square of the distance apart, is more difficult than to write

$$F \propto \frac{m_1 m_2}{r^2} . \qquad \ldots \quad (1.1)$$

Further, the techniques of mathematics such as algebra, trigonometry and calculus can be used to make predictions from the basic equations. Thus, if we know the basic rule (1.1) about the force between two particles, we can use the technique of integral calculus to find what will be the force exerted by a uniform rod on a particle placed on its perpendicular bisector.

Thus, mathematics is the language of physics. Without knowledge of mathematics it would be much more difficult to discover, understand and explain the

laws of nature. The importance of mathematics in today's world cannot be disputed. However, mathematics itself is not physics. We use a language to express our ideas. But the idea that we want to express has the main attention. If we are poor at grammar and vocabulary, it would be difficult for us to communicate our feelings but while doing so our basic interest is in the feeling that we want to express. It is nice to board a deluxe coach to go from Delhi to Agra, but the sweet memories of the deluxe coach and the video film shown on way are next to the prime goal of reaching Agra. "To understand nature" is physics, and mathematics is the deluxe coach to take us there comfortably. This relationship of physics and mathematics must be clearly understood and kept in mind while doing a physics course.

## 1.3 UNITS

Physics describes the laws of nature. This description is quantitative and involves measurement and comparison of physical quantities. To measure a physical quantity we need some standard unit of that quantity. An elephant is heavier than a goat but exactly how many times ? This question can be easily answered if we have chosen a standard mass calling it a *unit mass.* If the elephant is 200 times the unit mass and the goat is 20 times we know that the elephant is 10 times heavier than the goat. If I have the knowledge of the unit length and some one says that Gandhi Maidan is 5 times the unit length from here, I will have the idea whether I should walk down to Gandhi Maidan or I should ride a rickshaw or I should go by a bus. Thus, the physical quantities are quantitatively expressed in terms of a unit of that quantity. The measurement of the quantity is mentioned in two parts, the first part gives how many times of the standard unit and the second part gives the name of the unit. Thus, suppose I have to study for 2 hours. The numeric part 2 says that it is 2 *times* of the unit of time and the second part *hour* says that the unit chosen here is an hour.

### Who Decides the Units ?

How is a standard unit chosen for a physical quantity ? The first thing is that it should have international acceptance. Otherwise, everyone will choose his or her own unit for the quantity and it will be difficult to communicate freely among the persons distributed over the world. A body named *Conférence Générale des Poids et Mesures* or CGPM also known as *General Conference on Weight and Measures* in English has been given the authority to decide the units by international agreement. It holds its meetings and any changes in standard units are communicated through the publications of the Conference.

### Fundamental and Derived Quantities

There are a large number of physical quantities which are measured and every quantity needs a definition of unit. However, not all the quantities are independent of each other. As a simple example, if a unit of length is defined, a unit of area is automatically obtained. If we make a square with its length equal to its breadth equal to the unit length, its area can be called the unit area. All areas can then be compared to this standard unit of area. Similarly, if a unit of length and a unit of time interval are defined, a unit of speed is automatically obtained. If a particle covers a unit length in unit time interval, we say that it has a unit speed. We can define a set of *fundamental quantities* as follows :

(a) the fundamental quantities should be independent of each other, and

(b) all other quantities may be expressed in terms of the fundamental quantities.

It turns out that the number of fundamental quantities is only seven. All the rest may be derived from these quantities by multiplication and division. Many different choices can be made for the fundamental quantities. For example, one can take speed and time as fundamental quantities. Length is then a derived quantity. If something travels at unit speed, the distance it covers in unit time interval will be called a unit distance. One may also take length and time interval as the fundamental quantities and then speed will be a derived quantity. Several systems are in use over the world and in each system the fundamental quantities are selected in a particular way. The units defined for the fundamental quantities are called *fundamental units* and those obtained for the derived quantities are called the *derived units*.

Fundamental quantities are also called base quantities.

### SI Units

In 1971 CGPM held its meeting and decided a system of units which is known as the *International System of Units*. It is abbreviated as SI from the French name *Le Systéme International d'Unités.* This system is widely used throughout the world.

Table (1.1) gives the fundamental quantities and their units in SI.

**Table 1.1 : *Fundamental or Base Quantities***

| *Quantity* | *Name of the Unit* | *Symbol* |
|---|---|---|
| Length | **metre** | m |
| Mass | **kilogram** | kg |
| Time | **second** | s |
| Electric Current | **ampere** | A |
| Thermodynamic Temperature | **kelvin** | K |
| Amount of Substance | **mole** | mol |
| Luminous Intensity | **candela** | cd |

Besides the seven fundamental units two supplementary units are defined. They are for plane angle and solid angle. The unit for plane angle is *radian* with the symbol *rad* and the unit for the solid angle is *steradian* with the symbol *sr*.

**SI Prefixes**

The magnitudes of physical quantities vary over a wide range. We talk of separation between two protons inside a nucleus which is about $10^{-15}$ m and the distance of a quasar from the earth which is about $10^{26}$ m. The mass of an electron is $9{\cdot}1 \times 10^{-31}$ kg and that of our galaxy is about $2{\cdot}2 \times 10^{41}$ kg. The CGPM recommended standard prefixes for certain powers of 10. Table (1.2) shows these prefixes.

**Table 1.2 : *SI prefixes***

| *Power of* 10 | *Prefix* | *Symbol* |
|---|---|---|
| 18 | **exa** | E |
| 15 | **peta** | P |
| 12 | **tera** | T |
| 9 | **giga** | G |
| 6 | **mega** | M |
| 3 | **kilo** | k |
| 2 | **hecto** | h |
| 1 | **deka** | da |
| – 1 | **deci** | d |
| – 2 | **centi** | c |
| – 3 | **milli** | m |
| – 6 | **micro** | μ |
| – 9 | **nano** | n |
| – 12 | **pico** | p |
| – 15 | **femto** | f |
| – 18 | **atto** | a |

## 1.4 DEFINITIONS OF BASE UNITS

Any standard unit should have the following two properties :

(a) *Invariability* : The standard unit must be invariable. Thus, defining distance between the tip of the middle finger and the elbow as a unit of length is not invariable.

(b) *Availability* : The standard unit should be easily made available for comparing with other quantities.

CGPM decided in its 2018 meeting that all the SI base quantities will be defined in terms of certain universal constants and these constants will be assigned fixed numerical values by definition. In this case both the criteria of invariability and availability are automatically satisfied. The new definitions became operative since 20th May 2019. We give below the definitions of the these quantities. The fixed values given to the universal constants will appear in the definitions only. The definitions carry certain physical quantities and concepts that are beyond the scope of this book but you need not worry about it.

**Second**

1 second is the time that makes the unperturbed ground state hyperfine transition frequency $\Delta\nu_{Cs}$ to be 9192631770 when expressed in the unit Hz which is equal to $s^{-1}$.

**Metre**

1 metre is the length that makes the speed of light in vacuum to be 299792458 when expressed in the unit $m{\cdot}s^{-1}$, where the second is defined in terms of the caesium frequency $\Delta\nu_{Cs}$.

**Kilogram**

1 kilogram is the mass that makes the Planck's constant $h$ to be $6.62607015 \times 10^{-34}$ when expressed in the unit J·s which is equal to $kg{\cdot}m^2\ s^{-1}$, where the metre and the second are defined in terms of $c$ and $\Delta\nu_{Cs}$.

**Ampere**

1 ampere is the current which makes the elementary charge $e$ to be $1.602176634 \times 10^{-19}$ when expressed in the unit C which is equal to A·s, where the second is defined in terms of $\Delta\nu_{Cs}$.

**Kelvin**

1 kelvin is the temperature that makes the Boltzmann constant to be $1.380649 \times 10^{-23}$ when expressed in the unit $J{\cdot}K^{-1}$ which is equal to $kg{\cdot}m^2{\cdot}s^{-2}{\cdot}K^{-1}$, where kilogram, metre and second are defined in terms of $h$, $c$ and $\Delta\nu_{Cs}$.

**Mole**

1 mole of a substance is defined to contain exactly $6.02214076 \times 10^{23}$ elementary entities. This number is the fixed numerical value of the Avogadro constant $N_A$ when expressed in the unit $mol^{-1}$ and is called Avogadro number.

**Candela**

The candela is the SI unit of luminous intensity. 1 candela is the luminous intensity that makes the luminous efficacy of monochromatic radiation of frequency $540 \times 10^{12}$ Hz, $K_{cd}$ to be 683 when expressed in the unit $lm \cdot W^{-1}$ which is equal to $cd \cdot sr \cdot kg^{-1} m^2 s^3$, where kilogram, metre and second are defined in terms of $h$, $c$ and $\Delta\nu_{Cs}$.

## 1.5 DIMENSION

All the physical quantities of interest can be derived from the base quantities. When a quantity is expressed in terms of the base quantities, it is written as a product of different powers of the base quantities. The exponent of a base quantity that enters into the expression, is called the *dimension of the quantity in that base.* To make it clear, consider the physical quantity force. As we shall learn later, force is equal to mass times acceleration. Acceleration is change in velocity divided by time interval. Velocity is length divided by time interval. Thus,

$$\begin{aligned} force &= mass \times acceleration \\ &= mass \times \frac{velocity}{time} \\ &= mass \times \frac{length/time}{time} \\ &= mass \times length \times (time)^{-2}. \qquad \ldots (1.2) \end{aligned}$$

Thus, the dimensions of force are 1 in mass, 1 in length and –2 in time. The dimensions in all other base quantities are zero. Note that in this type of calculation the magnitudes are not considered. It is equality of the type of quantity that enters. Thus, change in velocity, initial velocity, average velocity, final velocity all are equivalent in this discussion, each one is length/time.

For convenience the base quantities are represented by one letter symbols. Generally, mass is denoted by M, length by L, time by T and electric current by I. The thermodynamic temperature, the amount of substance and the luminous intensity are denoted by the symbols of their units K, mol and cd respectively. The physical quantity that is expressed in terms of the base quantities is enclosed in square brackets to remind that the equation is among the dimensions and not among the magnitudes. Thus equation (1.2) may be written as $[\text{force}] = MLT^{-2}$.

Such an expression for a physical quantity in terms of the base quantities is called the *dimensional formula*. Thus, the dimensional formula of force is $MLT^{-2}$. The two versions given below are equivalent and are used interchangeably.

(a) The dimensional formula of force is $MLT^{-2}$.

(b) The dimensions of force are 1 in mass, 1 in length and –2 in time.

***Example 1.1***

*Calculate the dimensional formula of energy from the equation* $E = \frac{1}{2} mv^2$.

***Solution*** **:** Dimensionally, $E = mass \times (velocity)^2$, since $\frac{1}{2}$ is a number and has no dimension.

$$\therefore \quad [E] = M \times \left(\frac{L}{T}\right)^2 = ML^2 T^{-2}.$$

## 1.6 USES OF DIMENSION

### A. Homogeneity of Dimensions in an Equation

An equation contains several terms which are separated from each other by the symbols of equality, plus or minus. The dimensions of all the terms in an equation must be identical. This is another way of saying that one can add or subtract similar physical quantities. Thus, a velocity cannot be added to a force or an electric current cannot be subtracted from the thermodynamic temperature. This simple principle is called the *principle of homogeneity of dimensions* in an equation and is an extremely useful method to check whether an equation may be correct or not. If the dimensions of all the terms are not same, the equation must be wrong. Let us check the equation

$$x = ut + \frac{1}{2} at^2$$

for the dimensional homogeneity. Here $x$ is the distance travelled by a particle in time $t$ which starts at a speed $u$ and has an acceleration $a$ along the direction of motion.

$$[x] = L$$

$$[ut] = velocity \times time = \frac{length}{time} \times time = L$$

$$\left[\frac{1}{2} at^2\right] = [at^2] = acceleration \times (time)^2$$

$$= \frac{velocity}{time} \times (time)^2 = \frac{length/time}{time} \times (time)^2 = L$$

Thus, the equation is correct as far as the dimensions are concerned.

**Limitation of the Method**

Note that the dimension of $\frac{1}{2}at^2$ is same as that of $at^2$. Pure numbers are dimensionless. Dimension does not depend on the magnitude. Due to this reason the equation $x = ut + at^2$ is also dimensionally correct. Thus, a dimensionally correct equation need not be actually correct but a dimensionally wrong equation must be wrong.

***Example 1.2***

*Test dimensionally if the formula* $t = 2\pi\sqrt{\frac{m}{F/x}}$ *may be correct, where t is time period, m is mass, F is force and x is distance.*

***Solution*** **:** The dimension of force is $MLT^{-2}$. Thus, the dimension of the right-hand side is

$$\sqrt{\frac{M}{MLT^{-2}/L}} = \sqrt{\frac{1}{T^{-2}}} = T.$$

The left-hand side is time period and hence the dimension is T. The dimensions of both sides are equal and hence the formula may be correct.

## B. Conversion of Units

When we choose to work with a different set of units for the base quantities, the units of all the derived quantities must be changed. Dimensions can be useful in finding the conversion factor for the unit of a derived physical quantity from one system to other. Consider an example. When SI units are used, the unit of pressure is 1 pascal. Suppose we choose 1 cm as the unit of length, 1 g as the unit of mass and 1 s as the unit of time (this system is still in wide use and is called CGS system). The unit of pressure will be different in this system. Let us call it for the time-being 1 CGS pressure. Now, how many CGS pressure is equal to 1 pascal ?

Let us first write the dimensional formula of pressure.

We have $P = \frac{F}{A}.$

Thus, $[P] = \frac{[F]}{[A]} = \frac{MLT^{-2}}{L^2} = ML^{-1}T^{-2}$

so, $1 \text{ pascal} = (1 \text{ kg})(1 \text{ m})^{-1}(1 \text{ s})^{-2}$

and $1 \text{ CGS pressure} = (1 \text{ g})(1 \text{ cm})^{-1}(1 \text{ s})^{-2}$

Thus, $\frac{1 \text{ pascal}}{1 \text{ CGS pressure}} = \left(\frac{1 \text{ kg}}{1 \text{ g}}\right)\left(\frac{1 \text{ m}}{1 \text{ cm}}\right)^{-1}\left(\frac{1 \text{ s}}{1 \text{ s}}\right)^{-2}$

$= (10^3)(10^2)^{-1} = 10$

or, $1 \text{ pascal} = 10 \text{ CGS pressure}.$

Thus, knowing the conversion factors for the base quantities, one can work out the conversion factor for any derived quantity if the dimensional formula of the derived quantity is known.

## C. Deducing Relation among the Physical Quantities

Sometimes dimensions can be used to deduce a relation between the physical quantities. If one knows the quantities on which a particular physical quantity depends and if one guesses that this dependence is of product type, method of dimension may be helpful in the derivation of the relation. Taking an example, suppose we have to derive the expression for the time period of a simple pendulum. The simple pendulum has a bob, attached to a string, which oscillates under the action of the force of gravity. Thus, the time period may depend on the length of the string, the mass of the bob and the acceleration due to gravity. We assume that the dependence of time period on these quantities is of product type, that is,

$$t = kl^a m^b g^c \qquad \ldots (1.3)$$

where $k$ is a dimensionless constant and $a$, $b$ and $c$ are exponents which we want to evaluate. Taking the dimensions of both sides,

$$T = L^a M^b (LT^{-2})^c = L^{a+c} M^b T^{-2c}.$$

Since the dimensions on both sides must be identical, we have

$$a + c = 0$$
$$b = 0$$
and $$-2c = 1$$

giving $a = \frac{1}{2}$, $b = 0$ and $c = -\frac{1}{2}$.

Putting these values in equation (1.3)

$$t = k\sqrt{\frac{l}{g}}. \qquad \ldots (1.4)$$

Thus, by dimensional analysis we can deduce that the time period of a simple pendulum is independent of its mass, is proportional to the square root of the length of the pendulum and is inversely proportional to the square root of the acceleration due to gravity at the place of observation.

**Limitations of the Dimensional Method**

Although dimensional analysis is very useful in deducing certain relations, it cannot lead us too far. First of all we have to know the quantities on which a particular physical quantity depends. Even then the method works only if the dependence is of the product type. For example, the distance travelled by a uniformly accelerated particle depends on the initial velocity $u$, the acceleration $a$ and the time $t$. But the method of dimensions cannot lead us to the correct expression for $x$ because the expression is not of

product type. It is equal to the sum of two terms as

$$x = ut + \frac{1}{2}at^2.$$

Secondly, the numerical constants having no dimensions cannot be deduced by the method of dimensions. In the example of time period of a simple pendulum, an unknown constant $k$ remains in equation (1.4). One has to know from somewhere else that this constant is $2\pi$.

Thirdly, the method works only if there are as many equations available as there are unknowns. In mechanical quantities, only three base quantities length, mass and time enter. So, dimensions of these three may be equated in the guessed relation giving at most three equations in the exponents. If a particular quantity (in mechanics) depends on more than three quantities we shall have more unknowns and less equations. The exponents cannot be determined uniquely in such a case. Similar constraints are present for electrical or other nonmechanical quantities.

### 1.7 ORDER OF MAGNITUDE

In physics, we come across quantities which vary over a wide range. We talk of the size of a mountain and the size of the tip of a pin. We talk of the mass of our galaxy and the mass of a hydrogen atom. We talk of the age of the universe and the time taken by an electron to complete a circle around the proton in a hydrogen atom. It becomes quite difficult to get a feel of largeness or smallness of such quantities. To express such widely varying numbers, one uses the *powers of ten* method.

In this method, each number is expressed as $a \times 10^b$ where $1 \le a < 10$ and $b$ is a positive or negative integer. Thus the diameter of the sun is expressed as $1{\cdot}39 \times 10^9$ m and the diameter of a hydrogen atom as $1{\cdot}06 \times 10^{-10}$ m. To get an approximate idea of the number, one may round the number $a$ to 1 if it is less than or equal to 5 and to 10 if it is greater than 5. The number can then be expressed approximately as $10^b$. We then get the *order of magnitude* of that number. Thus, the diameter of the sun is *of the order of* $10^9$ m and that of a hydrogen atom is *of the order of* $10^{-10}$ m. More precisely, the exponent of 10 in such a representation is called the order of magnitude of that quantity. Thus, the diameter of the sun is 19 *orders of magnitude larger* than the diameter of a hydrogen atom. This is because the order of magnitude of $10^9$ is 9 and of $10^{-10}$ is $-10$. The difference is $9 - (-10) = 19$.

To quickly get an approximate value of a quantity in a given physical situation, one can make an *order of magnitude calculation.* In this all numbers are approximated to $10^b$ form and the calculation is made.

Let us estimate the number of persons that may sit in a circular field of radius 800 m. The area of the field is

$$A = \pi r^2 = 3{\cdot}14 \times (800 \text{ m})^2 \approx 10^6 \text{ m}^2.$$

The average area one person occupies in sitting $\approx 50 \text{ cm} \times 50 \text{ cm} = 0{\cdot}25 \text{ m}^2 = 2{\cdot}5 \times 10^{-1} \text{ m}^2 \approx 10^{-1} \text{ m}^2$. The number of persons who can sit in the field is

$$N \approx \frac{10^6 \text{ m}^2}{10^{-1} \text{ m}^2} = 10^7.$$

Thus of the order of $10^7$ persons may sit in the field.

### 1.8 THE STRUCTURE OF WORLD

Man has always been interested to find how the world is structured. Long long ago scientists suggested that the world is made up of certain indivisible small particles. The number of particles in the world is large but the varieties of particles are not many. Old Indian philosopher Kanadi derives his name from this proposition (In Sanskrit or Hindi *Kana* means a small particle). After extensive experimental work people arrived at the conclusion that the world is made up of just three types of ultimate particles, the proton, the neutron and the electron. All objects which we have around us, are aggregation of atoms and molecules. The molecules are composed of atoms and the atoms have at their heart a nucleus containing protons and neutrons. Electrons move around this nucleus in special arrangements. It is the number of protons, neutrons and electrons in an atom that decides all the properties and behaviour of a material. Large number of atoms combine to form an object of moderate or large size. However, the laws that we generally deduce for these macroscopic objects are not always applicable to atoms, molecules, nuclei or the elementary particles. These laws known as *classical physics* deal with large size objects only. When we say a particle in classical physics we mean an object which is small as compared to other moderate or large size objects and for which the classical physics is valid. It may still contain millions and millions of atoms in it. Thus, a particle of dust dealt in classical physics may contain about $10^{18}$ atoms.

Twentieth century experiments have revealed another aspect of the construction of world. There are perhaps no ultimate indivisible particles. Hundreds of elementary particles have been discovered and there are free transformations from one such particle to the other. Nature is seen to be a well-connected entity.

## *Worked Out Examples*

**1.** *Find the dimensional formulae of the following quantities :*
(a) *the universal constant of gravitation G,*
(b) *the surface tension S,*
(c) *the thermal conductivity k and*
(d) *the coefficient of viscosity* $\eta$.
*Some equations involving these quantities are*

$$F = \frac{Gm_1 m_2}{r^2}, \qquad S = \frac{\rho g r h}{2},$$

$$Q = k\frac{A(\theta_2 - \theta_1)t}{d} \quad \text{and} \quad F = -\eta A \frac{v_2 - v_1}{x_2 - x_1}$$

*where the symbols have their usual meanings.*

***Solution :*** (a) $F = G\dfrac{m_1 m_2}{r^2}$

or, $G = \dfrac{Fr^2}{m_1 m_2}$

or, $[G] = \dfrac{[F]L^2}{M^2} = \dfrac{MLT^{-2}.L^2}{M^2} = M^{-1}L^3T^{-2}.$

(b) $S = \dfrac{\rho g r h}{2}$

or, $[S] = [\rho]\,[g]L^2 = \dfrac{M}{L^3}\cdot\dfrac{L}{T^2}\cdot L^2 = MT^{-2}.$

(c) $Q = k\dfrac{A(\theta_2 - \theta_1)t}{d}$

or, $k = \dfrac{Qd}{A(\theta_2 - \theta_1)t}.$

Here, $Q$ is the heat energy having dimension $ML^2T^{-2}$, $\theta_2 - \theta_1$ is temperature, $A$ is area, $d$ is thickness and $t$ is time. Thus,

$$[k] = \frac{ML^2T^{-2}L}{L^2KT} = MLT^{-3}K^{-1}.$$

(d) $F = -\eta A\dfrac{v_2 - v_1}{x_2 - x_1}$

or, $MLT^{-2} = [\eta]L^2\dfrac{L/T}{L} = [\eta]\dfrac{L^2}{T}$

or, $[\eta] = ML^{-1}T^{-1}.$

**2.** *Find the dimensional formulae of*
(a) *the charge Q,*
(b) *the potential V,*
(c) *the capacitance C, and*
(d) *the resistance R.*
*Some of the equations containing these quantities are*

$$Q = It,\ U = VIt,\ Q = CV \text{ and } V = RI;$$

*where I denotes the electric current, t is time and U is energy.*

***Solution :*** (a) $Q = It$. Hence, $[Q] = IT$.

(b) $U = VIt$

or, $ML^2T^{-2} = [V]IT$ or, $[V] = ML^2I^{-1}T^{-3}.$

(c) $Q = CV$

or, $IT = [C]ML^2I^{-1}T^{-3}$ or, $[C] = M^{-1}L^{-2}I^2T^4.$

(d) $V = RI$

or, $R = \dfrac{V}{I}$ or, $[R] = \dfrac{ML^2I^{-1}T^{-3}}{I} = ML^2I^{-2}T^{-3}.$

**3.** *The SI and CGS units of energy are joule and erg respectively. How many ergs are equal to one joule ?*

***Solution :*** Dimensionally, Energy $= mass \times (velocity)^2$

$$= mass \times \left(\frac{length}{time}\right)^2 = ML^2T^{-2}.$$

Thus, 1 joule $= (1\text{ kg})(1\text{ m})^2(1\text{ s})^{-2}$

and $1\text{ erg} = (1\text{ g})(1\text{ cm})^2(1\text{ s})^{-2}$

$$\frac{1\text{ joule}}{1\text{ erg}} = \left(\frac{1\text{ kg}}{1\text{ g}}\right)\left(\frac{1\text{ m}}{1\text{ cm}}\right)^2\left(\frac{1\text{ s}}{1\text{ s}}\right)^{-2}$$

$$= \left(\frac{1000\text{ g}}{1\text{ g}}\right)\left(\frac{100\text{ cm}}{1\text{ cm}}\right)^2 = 1000 \times 10000 = 10^7.$$

So, $1\text{ joule} = 10^7\text{ erg}.$

**4.** *Young's modulus of steel is* $19 \times 10^{10}\ \text{N/m}^2$. *Express it in dyne/cm*$^2$. *Here dyne is the CGS unit of force.*

***Solution :*** The unit of Young's modulus is $\text{N/m}^2$.

This suggests that it has dimensions of $\dfrac{Force}{(distance)^2}$.

Thus, $[Y] = \dfrac{[F]}{L^2} = \dfrac{MLT^{-2}}{L^2} = ML^{-1}T^{-2}.$

$\text{N/m}^2$ is in SI units.

So, $1\text{ N/m}^2 = (1\text{ kg})(1\text{ m})^{-1}(1\text{ s})^{-2}$

and $1\text{ dyne/cm}^2 = (1\text{ g})(1\text{ cm})^{-1}(1\text{ s})^{-2}$

so, $$\frac{1\text{ N/m}^2}{1\text{ dyne/cm}^2} = \left(\frac{1\text{ kg}}{1\text{ g}}\right)\left(\frac{1\text{ m}}{1\text{ cm}}\right)^{-1}\left(\frac{1\text{ s}}{1\text{ s}}\right)^{-2}$$

$$= 1000 \times \frac{1}{100} \times 1 = 10$$

or, $1\text{ N/m}^2 = 10\text{ dyne/cm}^2$

or, $19 \times 10^{10}\text{ N/m}^2 = 19 \times 10^{11}\text{ dyne/cm}^2.$

**5.** *If velocity, time and force were chosen as basic quantities, find the dimensions of mass.*

***Solution :*** Dimensionally, $Force = mass \times acceleration$

$$= mass \times \frac{velocity}{time}$$

or, $$mass = \frac{force \times time}{velocity}$$

or, $$[mass] = FTV^{-1}.$$

**6.** *Test dimensionally if the equation $v^2 = u^2 + 2ax$ may be correct.*

***Solution*** **:** There are three terms in this equation $v^2$, $u^2$ and $2ax$. The equation may be correct if the dimensions of these three terms are equal.

$$[v^2] = \left(\frac{L}{T}\right)^2 = L^2 T^{-2};$$

$$[u^2] = \left(\frac{L}{T}\right)^2 = L^2 T^{-2};$$

and $$[2ax] = [a]\,[x] = \left(\frac{L}{T^2}\right) L = L^2 T^{-2}.$$

Thus, the equation may be correct.

**7.** *The distance covered by a particle in time t is given by $x = a + bt + ct^2 + dt^3$; find the dimensions of a, b, c and d.*

***Solution*** **:** The equation contains five terms. All of them should have the same dimensions. Since [$x$] = *length*, each of the remaining four must have the dimension of length.

Thus, $[a] = length = L$

$[bt] = L$, or, $[b] = LT^{-1}$

$[ct^2] = L$, or, $[c] = LT^{-2}$

and $[dt^3] = L$, or, $[d] = LT^{-3}$.

**8.** *If the centripetal force is of the form $m^a v^b r^c$, find the values of a, b and c.*

***Solution*** **:** Dimensionally,

$$Force = (Mass)^a \times (velocity)^b \times (length)^c$$

or, $MLT^{-2} = M^a (L^b T^{-b}) L^c = M^a L^{b+c} T^{-b}$

Equating the exponents of similar quantities,

$a = 1,\ b + c = 1,\ -b = -2$

or, $a = 1,\ b = 2,\ c = -1$ or, $F = \frac{mv^2}{r}$.

**9.** *When a solid sphere moves through a liquid, the liquid opposes the motion with a force F. The magnitude of F depends on the coefficient of viscosity $\eta$ of the liquid, the radius r of the sphere and the speed v of the sphere. Assuming that F is proportional to different powers of these quantities, guess a formula for F using the method of dimensions.*

***Solution*** **:** Suppose the formula is $F = k\,\eta^a r^b v^c$.

Then, $$MLT^{-2} = [ML^{-1}T^{-1}]^a L^b \left(\frac{L}{T}\right)^c$$

$$= M^a L^{-a+b+c} T^{-a-c}.$$

Equating the exponents of M, L and T from both sides,

$$a = 1$$

$$-a + b + c = 1$$

$$-a - c = -2$$

Solving these, $a$ = 1, $b$ = 1, and $c$ = 1.

Thus, the formula for $F$ is $F = k\eta rv$.

**10.** *The heat produced in a wire carrying an electric current depends on the current, the resistance and the time. Assuming that the dependence is of the product of powers type, guess an equation between these quantities using dimensional analysis. The dimensional formula of resistance is $ML^2 I^{-2} T^{-3}$ and heat is a form of energy.*

***Solution*** **:** Let the heat produced be $H$, the current through the wire be $I$, the resistance be $R$ and the time be $t$. Since heat is a form of energy, its dimensional formula is $ML^2 T^{-2}$.

Let us assume that the required equation is

$$H = kI^a R^b t^c,$$

where $k$ is a dimensionless constant.

Writing dimensions of both sides,

$$ML^2 T^{-2} = I^a (ML^2 I^{-2} T^{-3})^b T^c$$

$$= M^b L^{2b} T^{-3b+c} I^{a-2b}$$

Equating the exponents,

$$b = 1$$

$$2b = 2$$

$$-3b + c = -2$$

$$a - 2b = 0$$

Solving these, we get, $a$ = 2, $b$ = 1 and $c$ = 1.

Thus, the required equation is $H = kI^2 Rt$.

□

## QUESTIONS FOR SHORT ANSWER

**1.** The metre is defined as the distance travelled by light in $\frac{1}{299{,}792{,}458}$ second. Why didn't people choose some easier number such as $\frac{1}{300{,}000{,}000}$ second ? Why not 1 second ?

**2.** What are the dimensions of :
(a) volume of a cube of edge $a$,
(b) volume of a sphere of radius $a$,
(c) the ratio of the volume of a cube of edge $a$ to the volume of a sphere of radius $a$ ?

**3.** Suppose you are told that the linear size of everything in the universe has been doubled overnight. Can you test this statement by measuring sizes with a metre stick ? Can you test it by using the fact that the speed of light is a universal constant and has not changed ? What will happen if all the clocks in the universe also start running at half the speed ?

**4.** If all the terms in an equation have same units, is it necessary that they have same dimensions ? If all the terms in an equation have same dimensions, is it necessary that they have same units ?

**5.** If two quantities have same dimensions, do they represent same physical content ?

**6.** It is desirable that the standards of units be easily available, invariable, indestructible and easily reproducible. If we use foot of a person as a standard unit of length, which of the above features are present and which are not ?

**7.** Suggest a way to measure :
(a) the thickness of a sheet of paper,
(b) the distance between the sun and the moon.

## OBJECTIVE I

**1.** Which of the following sets cannot enter into the list of fundamental quantities in any system of units ?
(a) length, mass and velocity,
(b) length, time and velocity,
(c) mass, time and velocity,
(d) length, time and mass.

**2.** A physical quantity is measured and the result is expressed as $nu$ where $u$ is the unit used and $n$ is the numerical value. If the result is expressed in various units then
(a) $n \propto$ size of $u$ (b) $n \propto u^2$
(c) $n \propto \sqrt{u}$ (d) $n \propto \frac{1}{u}$.

**3.** Suppose a quantity $x$ can be dimensionally represented in terms of M, L and T, that is, $[x] = M^a L^b T^c$. The quantity mass
(a) can always be dimensionally represented in terms of L, T and $x$,
(b) can never be dimensoinally represented in terms of L, T and $x$,
(c) may be represented in terms of L, T and $x$ if $a = 0$,
(d) may be represented in terms of L, T and $x$ if $a \neq 0$.

**4.** A dimensionless quantity
(a) never has a unit, (b) always has a unit,
(c) may have a unit, (d) does not exist.

**5.** A unitless quantity
(a) never has a nonzero dimension,
(b) always has a nonzero dimension,
(c) may have a nonzero dimension,
(d) does not exist.

**6.** $\int \frac{dx}{\sqrt{2ax - x^2}} = a^n \sin^{-1}\left[\frac{x}{a} - 1\right]$.
The value of $n$ is
(a) 0 (b) –1
(c) 1 (d) none of these.
You may use dimensional analysis to solve the problem.

## OBJECTIVE II

**1.** The dimensions $ML^{-1}T^{-2}$ may correspond to
(a) work done by a force
(b) linear momentum
(c) pressure
(d) energy per unit volume.

**2.** Choose the correct statement(s) :
(a) A dimensionally correct equation may be correct.
(b) A dimensionally correct equation may be incorrect.
(c) A dimensionally incorrect equation may be correct.
(d) A dimensionally incorrect equation may be incorrect.

**3.** Choose the correct statement(s) :
(a) All quantities may be represented dimensionally in terms of the base quantities.
(b) A base quantity cannot be represented dimensionally in terms of the rest of the base quantities.
(c) The dimension of a base quantity in other base quantities is always zero.
(d) The dimension of a derived quantity is never zero in any base quantity.

## EXERCISES

**1.** Find the dimensions of
(a) linear momentum,
(b) frequency and
(c) pressure.

**2.** Find the dimensions of
(a) angular speed $\omega$, (b) angular acceleration $\alpha$,
(c) torque $\Gamma$ and (d) moment of interia $I$.
Some of the equations involving these quantities are

$\omega = \frac{\theta_2 - \theta_1}{t_2 - t_1}$, $\alpha = \frac{\omega_2 - \omega_1}{t_2 - t_1}$, $\Gamma = F.r$ and $I = mr^2$.
The symbols have standard meanings.

**3.** Find the dimensions of
(a) electric field $E$, (b) magnetic field $B$ and
(c) magnetic permeability $\mu_0$.
The relevant equations are
$F = qE$, $F = qvB$, and $B = \frac{\mu_0 I}{2\pi a}$;
where $F$ is force, $q$ is charge, $v$ is speed, $I$ is current, and $a$ is distance.

**4.** Find the dimensions of
(a) electric dipole moment $p$ and
(b) magnetic dipole moment $M$.
The defining equations are $p = q.d$ and $M = IA$;
where $d$ is distance, $A$ is area, $q$ is charge and $I$ is current.

**5.** Find the dimensions of Planck's constant $h$ from the equation $E = h\nu$ where $E$ is the energy and $\nu$ is the frequency.

**6.** Find the dimensions of
(a) the specific heat capacity $c$,
(b) the coefficient of linear expansion $\alpha$ and
(c) the gas constant $R$.
Some of the equations involving these quantities are
$Q = mc(T_2 - T_1)$, $l_t = l_0[1 + \alpha(T_2 - T_1)]$ and $PV = nRT$.

**7.** Taking force, length and time to be the fundamental quantities find the dimensions of
(a) density, (b) pressure,
(c) momentum and (d) energy.

**8.** Suppose the acceleration due to gravity at a place is 10 m/s$^2$. Find its value in cm/(minute)$^2$.

**9.** The average speed of a snail is 0·020 miles/hour and that of a leopard is 70 miles/hour. Convert these speeds in SI units.

**10.** The height of mercury column in a barometer in a Calcutta laboratory was recorded to be 75 cm. Calculate this pressure in SI and CGS units using the following data : Specific gravity of mercury = 13·6, Density of water = $10^3$ kg/m$^3$, $g$ = 9·8 m/s$^2$ at Calcutta. Pressure = $h\rho g$ in usual symbols.

**11.** Express the power of a 100 watt bulb in CGS unit.

**12.** The normal duration of I.Sc. Physics practical period in Indian colleges is 100 minutes. Express this period in microcenturies. 1 microcentury = $10^{-6} \times 100$ years. How many microcenturies did you sleep yesterday ?

**13.** The surface tension of water is 72 dyne/cm. Convert it in SI unit.

**14.** The kinetic energy $K$ of a rotating body depends on its moment of inertia $I$ and its angular speed $\omega$. Assuming the relation to be $K = kI^a\omega^b$ where $k$ is a dimensionless constant, find $a$ and $b$. Moment of inertia of a sphere about its diameter is $\frac{2}{5}Mr^2$.

**15.** Theory of relativity reveals that mass can be converted into energy. The energy $E$ so obtained is proportional to certain powers of mass $m$ and the speed $c$ of light. Guess a relation among the quantities using the method of dimensions.

**16.** Let $I$ = current through a conductor, $R$ = its resistance and $V$ = potential difference across its ends. According to Ohm's law, product of two of these quantities equals the third. Obtain Ohm's law from dimensional analysis. Dimensional formulae for $R$ and $V$ are $ML^2I^{-2}T^{-3}$ and $ML^2T^{-3}I^{-1}$ respectively.

**17.** The frequency of vibration of a string depends on the length $L$ between the nodes, the tension $F$ in the string and its mass per unit length $m$. Guess the expression for its frequency from dimensional analysis.

**18.** Test if the following equations are dimensionally correct :
(a) $h = \frac{2S\cos\theta}{\rho rg}$, (b) $v = \sqrt{\frac{P}{\rho}}$,
(c) $V = \frac{\pi P r^4 t}{8\eta l}$, (d) $\nu = \frac{1}{2\pi}\sqrt{\frac{mgl}{I}}$;
where $h$ = height, $S$ = surface tension, $\rho$ = density, $P$ = pressure, $V$ = volume, $\eta$ = coefficient of viscosity, $\nu$ = frequency and $I$ = moment of inertia.

**19.** Let $x$ and $a$ stand for distance. Is $\int \frac{dx}{\sqrt{a^2 - x^2}} = \frac{1}{a}\sin^{-1}\frac{a}{x}$ dimensionally correct ?

□

## ANSWERS

### OBJECTIVE I

1. (b) 2. (d) 3. (d) 4. (c) 5. (a) 6. (a)

### OBJECTIVE II

1. (c), (d) 2. (a), (b), (d) 3. (a), (b), (c)

### EXERCISES

1. (a) $MLT^{-1}$ (b) $T^{-1}$ (c) $ML^{-1}T^{-2}$
2. (a) $T^{-1}$ (b) $T^{-2}$ (c) $ML^2T^{-2}$ (d) $ML^2$
3. (a) $MLT^{-3}I^{-1}$ (b) $MT^{-2}I^{-1}$ (c) $MLT^{-2}I^{-2}$
4. (a) LTI (b) $L^2I$
5. $ML^2T^{-1}$
6. (a) $L^2T^{-2}K^{-1}$ (b) $K^{-1}$ (c) $ML^2T^{-2}K^{-1}(mol)^{-1}$

7. (a) $FL^{-4}T^2$ (b) $FL^{-2}$ (c) FT (d) FL
8. $36 \times 10^5$ cm/(minute)$^2$
9. 0·0089 m/s, 31 m/s
10. $10 \times 10^4$ N/m$^2$, $10 \times 10^5$ dyne/cm$^2$
11. $10^9$ erg/s
12. 1·9 microcenturies
13. 0·072 N/m
14. $a = 1,\ b = 2$
15. $E = kmc^2$
16. $V = IR$
17. $\frac{k}{L}\sqrt{\frac{F}{m}}$
18. all are dimensionally correct
19. no

□

## CHAPTER 2

# PHYSICS AND MATHEMATICS

Mathematics is the language of physics. It becomes easier to describe, understand and apply the physical principles, if one has a good knowledge of mathematics. In the present course we shall constantly be using the techniques of algebra, trigonometry and geometry as well as vector algebra, differential calculus and integral calculus. In this chapter we shall discuss the latter three topics. Errors in measurement and the concept of significant digits are also introduced.

### 2.1 VECTORS AND SCALARS

Certain physical quantities are completely described by a numerical value alone (with units specified) and are added according to the ordinary rules of algebra. As an example the mass of a system is described by saying that it is 5 kg. If two bodies one having a mass of 5 kg and the other having a mass of 2 kg are added together to make a composite system, the total mass of the system becomes 5 kg + 2 kg = 7 kg. Such quantities are called *scalars*.

The complete description of certain physical quantities requires a numerical value (with units specified) as well as a direction in space. Velocity of a particle is an example of this kind. The magnitude of velocity is represented by a number such as 5 m/s and tells us how fast a particle is moving. But the description of velocity becomes complete only when the direction of velocity is also specified. We can represent this velocity by drawing a line parallel to the velocity and putting an arrow showing the direction of velocity. We can decide beforehand a particular length to represent 1 m/s and the length of the line representing a velocity of 5 m/s may be taken as 5 times this unit length. Figure (2.1) shows representations of several velocities in this scheme. The front end (carrying the arrow) is called the head and the rear end is called the tail.

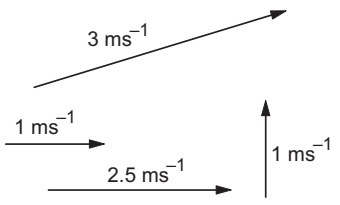



Figure 2.1

Further, if a particle is given two velocities simultaneously its resultant velocity is different from the two velocities and is obtained by using a special rule. Suppose a small ball is moving inside a long tube at a speed 3 m/s and the tube itself is moving in the room at a speed 4 m/s along a direction perpendicular to its length. In which direction and how fast is the ball moving as seen from the room ?

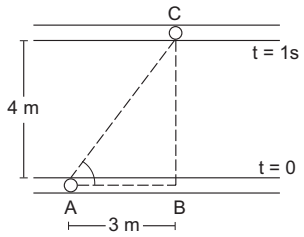



Figure 2.2

Figure (2.2) shows the positions of the tube and the ball at $t = 0$ and $t = 1$ s. Simple geometry shows that the ball has moved 5 m in a direction $\theta = 53°$ from the tube. So the resultant velocity of the ball is 5 m/s along this direction. The general rule for finding the resultant of two velocities may be stated as follows.

Draw a line $AB$ representing the first velocity with $B$ as the head. Draw another line $BC$ representing the second velocity with its tail $B$ coinciding with the head of the first line. The line $AC$ with $A$ as the tail and $C$ as the head represents the resultant velocity. Figure (2.3) shows the construction.

The resultant is also called the sum of the two velocities. We have added the two velocities $AB$ and $BC$ and have obtained the sum $AC$. This rule of addition is called the "triangle rule of addition".

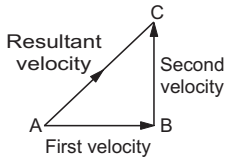



Figure 2.3

The physical quantities which have magnitude and direction and which can be added according to the triangle rule, are called *vector quantities*. Other examples of vector quantities are force, linear momentum, electric field, magnetic field etc.

The vectors are denoted by putting an arrow over the symbols representing them. Thus, we write $\overrightarrow{AB}$, $\overrightarrow{BC}$ etc. Sometimes a vector is represented by a single letter such as $\vec{v}$, $\vec{F}$ etc. Quite often in printed books the vectors are represented by bold face letters like **AB**, **BC**, **v**, **f** etc.

If a physical quantity has magnitude as well as direction but does not add up according to the triangle rule, it will not be called a vector quantity. Electric current in a wire has both magnitude and direction but there is no meaning of triangle rule there. Thus, electric current is not a vector quantity.

## 2.2 EQUALITY OF VECTORS

Two vectors (representing two values of the same physical quantity) are called equal if their magnitudes and directions are same. Thus, a parallel translation of a vector does not bring about any change in it.

## 2.3 ADDITION OF VECTORS

The triangle rule of vector addition is already described above. If $\vec{a}$ and $\vec{b}$ are the two vectors to be added, a diagram is drawn in which the tail of $\vec{b}$ coincides with the head of $\vec{a}$. The vector joining the tail of $\vec{a}$ with the head of $\vec{b}$ is the vector sum of $\vec{a}$ and $\vec{b}$. Figure (2.4a) shows the construction. The same rule

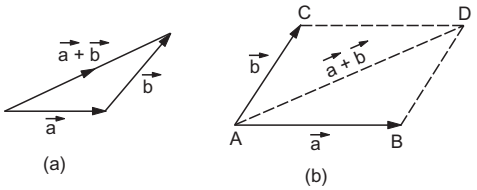



Figure 2.4

may be stated in a slightly different way. We draw the vectors $\vec{a}$ and $\vec{b}$ with both the tails coinciding (figure 2.4b). Taking these two as the adjacent sides we complete the parallelogram. The diagonal through the common tails gives the sum of the two vectors. Thus, in figure, (2.4b) $\overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AD}$.

Suppose the magnitude of $\vec{a} = a$ and that of $\vec{b} = b$. What is the magnitude of $\vec{a} + \vec{b}$ and what is its direction ? Suppose the angle between $\vec{a}$ and $\vec{b}$ is θ. It is easy to see from figure (2.5) that

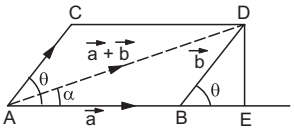



Figure 2.5

$$AD^2 = (AB + BE)^2 + (DE)^2$$
$$= (a + b\cos\theta)^2 + (b\sin\theta)^2$$
$$= a^2 + 2ab\cos\theta + b^2.$$

Thus, the magnitude of $\vec{a} + \vec{b}$ is

$$\sqrt{a^2 + b^2 + 2ab\cos\theta}. \qquad \ldots (2.1)$$

Its angle with $\vec{a}$ is α where

$$\tan\alpha = \frac{DE}{AE} = \frac{b\sin\theta}{a + b\cos\theta}. \qquad \ldots (2.2)$$

***Example 2.1***

*Two vectors having equal magnitudes A make an angle θ with each other. Find the magnitude and direction of the resultant.*

***Solution*** : The magnitude of the resultant will be

$$B = \sqrt{A^2 + A^2 + 2AA\cos\theta}$$
$$= \sqrt{2A^2(1 + \cos\theta)} = \sqrt{4A^2\cos^2\frac{\theta}{2}}$$
$$= 2A\cos\frac{\theta}{2}.$$

The resultant will make an angle α with the first vector where

$$\tan\alpha = \frac{A\sin\theta}{A + A\cos\theta} = \frac{2A\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2A\cos^2\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

or, $\alpha = \frac{\theta}{2}$

Thus, the resultant of two equal vectors bisects the angle between them.

## 2.4 MULTIPLICATION OF A VECTOR BY A NUMBER

Suppose $\vec{a}$ is a vector of magnitude $a$ and $k$ is a number. We define the vector $\vec{b} = k\vec{a}$ as a vector of magnitude $|ka|$. If $k$ is positive the direction of the vector $\vec{b} = k\vec{a}$ is same as that of $\vec{a}$. If $k$ is negative, the direction of $\vec{b}$ is opposite to $\vec{a}$. In particular, multiplication by (–1) just inverts the direction of the vector. The vectors $\vec{a}$ and $-\vec{a}$ have equal magnitudes but opposite directions.

If $\vec{a}$ is a vector of magnitude $a$ and $\vec{u}$ is a vector of unit magnitude in the direction of $\vec{a}$, we can write $\vec{a} = a\vec{u}$.

## 2.5 SUBTRACTION OF VECTORS

Let $\vec{a}$ and $\vec{b}$ be two vectors. We define $\vec{a} - \vec{b}$ as the sum of the vector $\vec{a}$ and the vector $(-\vec{b})$. To subtract $\vec{b}$ from $\vec{a}$, invert the direction of $\vec{b}$ and add to $\vec{a}$. Figure (2.6) shows the process.

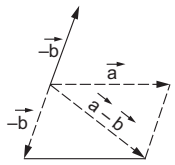



Figure 2.6

***Example 2.2***

*Two vectors of equal magnitude* 5 unit *have an angle* 60° *between them. Find the magnitude of* (a) *the sum of the vectors and* (b) *the difference of the vectors.*

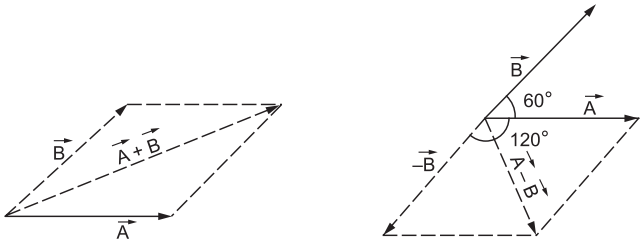



Figure 2.7

***Solution*** **:** Figure (2.7) shows the construction of the sum $\vec{A} + \vec{B}$ and the difference $\vec{A} - \vec{B}$.

(a) $\vec{A} + \vec{B}$ is the sum of $\vec{A}$ and $\vec{B}$. Both have a magnitude of 5 unit and the angle between them is 60°. Thus, the magnitude of the sum is

$$|\vec{A} + \vec{B}| = \sqrt{5^2 + 5^2 + 2 \times 5 \times 5 \cos 60°}$$
$$= 2 \times 5 \cos 30° = 5\sqrt{3} \text{ unit.}$$

(b) $\vec{A} - \vec{B}$ is the sum of $\vec{A}$ and $(-\vec{B})$. As shown in the figure, the angle between $\vec{A}$ and $(-\vec{B})$ is 120°. The magnitudes of both $\vec{A}$ and $(-\vec{B})$ is 5 unit. So,

$$|\vec{A} - \vec{B}| = \sqrt{5^2 + 5^2 + 2 \times 5 \times 5 \cos 120°}$$
$$= 2 \times 5 \cos 60° = 5 \text{ unit.}$$

## 2.6 RESOLUTION OF VECTORS

Figure (2.8) shows a vector $\vec{a} = \overrightarrow{OA}$ in the *X*-*Y* plane drawn from the origin *O*. The vector makes an angle $\alpha$ with the *X*-axis and $\beta$ with the *Y*-axis. Draw perpendiculars *AB* and *AC* from *A* to the *X* and *Y* axes respectively. The length *OB* is called the projection of $\overrightarrow{OA}$ on *X*-axis. Similarly *OC* is the projection of $\overrightarrow{OA}$ on *Y*-axis. According to the rules of vector addition

$$\vec{a} = \overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{OC}.$$

Thus, we have resolved the vector $\vec{a}$ into two parts, one along *OX* and the other along *OY*. The magnitude of the part along *OX* is $OB = a\cos\alpha$ and the magnitude of the part along *OY* is $OC = a\cos\beta$. If $\vec{i}$ and $\vec{j}$ denote vectors of unit magnitude along *OX* and *OY* respectively, we get

$$\overrightarrow{OB} = a\cos\alpha\,\vec{i} \text{ and } \overrightarrow{OC} = a\cos\beta\,\vec{j}$$

so that $\vec{a} = a\cos\alpha\,\vec{i} + a\cos\beta\,\vec{j}$.

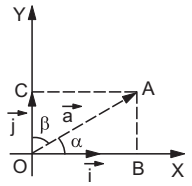



Figure 2.8

If the vector $\vec{a}$ is not in the *X*-*Y* plane, it may have nonzero projections along *X,Y,Z* axes and we can resolve it into three parts i.e., along the *X, Y* and *Z* axes. If $\alpha$, $\beta$, $\gamma$ be the angles made by the vector $\vec{a}$ with the three axes respectively, we get

$$\vec{a} = a\cos\alpha\,\vec{i} + a\cos\beta\,\vec{j} + a\cos\gamma\,\vec{k} \qquad \ldots (2.3)$$

where $\vec{i}$, $\vec{j}$ and $\vec{k}$ are the unit vectors along *X, Y* and *Z* axes respectively. The magnitude $(a\cos\alpha)$ is called the component of $\vec{a}$ along *X*-axis, $(a\cos\beta)$ is called the component along *Y*-axis and $(a\cos\gamma)$ is called the component along *Z*-axis. In general, the component of a vector $\vec{a}$ along a direction making an angle $\theta$ with it

is $a\cos\theta$ (figure 2.9) which is the projection of $\vec{a}$ along the given direction.

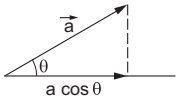



Figure 2.9

Equation (2.3) shows that any vector can be expressed as a linear combination of the three unit vectors $\vec{i}$, $\vec{j}$ and $\vec{k}$.

***Example 2.3***

*A force of* 10·5 N *acts on a particle along a direction making an angle of* 37° *with the vertical. Find the component of the force in the vertical direction.*

***Solution*** **:** The component of the force in the vertical direction will be

$$F_{\perp} = F\cos\theta = (10{\cdot}5\text{ N})\,(\cos 37^\circ)$$

$$= (10{\cdot}5\text{ N})\frac{4}{5} = 8{\cdot}40\text{ N}.$$

We can easily add two or more vectors if we know their components along the rectangular coordinate axes. Let us have

$$\vec{a} = a_x\,\vec{i} + a_y\,\vec{j} + a_z\,\vec{k}$$

$$\vec{b} = b_x\,\vec{i} + b_y\,\vec{j} + b_z\,\vec{k}$$

and $$\vec{c} = c_x\,\vec{i} + c_y\,\vec{j} + c_z\,\vec{k}$$

then

$$\vec{a} + \vec{b} + \vec{c} = (a_x + b_x + c_x)\,\vec{i} + (a_y + b_y + c_y)\,\vec{j} + (a_z + b_z + c_z)\,\vec{k}.$$

If all the vectors are in the $X$-$Y$ plane then all the $z$ components are zero and the resultant is simply

$$\vec{a} + \vec{b} + \vec{c} = (a_x + b_x + c_x)\,\vec{i} + (a_y + b_y + c_y)\,\vec{j}.$$

This is the sum of two mutually perpendicular vectors of magnitude $(a_x + b_x + c_x)$ and $(a_y + b_y + c_y)$. The resultant can easily be found to have a magnitude

$$\sqrt{(a_x + b_x + c_x)^2 + (a_y + b_y + c_y)^2}$$

making an angle $\alpha$ with the $X$-axis where

$$\tan\alpha = \frac{a_y + b_y + c_y}{a_x + b_x + c_x}.$$

## 2.7 DOT PRODUCT OR SCALAR PRODUCT OF TWO VECTORS

The dot product (also called scalar product) of two vectors $\vec{a}$ and $\vec{b}$ is defined as

$$\vec{a}\cdot\vec{b} = ab\cos\theta \qquad \ldots\ (2.4)$$

where $a$ and $b$ are the magnitudes of $\vec{a}$ and $\vec{b}$ respectively and $\theta$ is the angle between them. The dot product between two mutually perpendicular vectors is zero as $\cos 90^\circ = 0$.

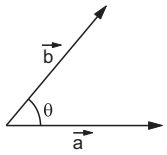



Figure 2.10

The dot product is commutative and distributive.

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$$

$$\vec{a}\cdot(\vec{b} + \vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}.$$

***Example 2.4***

*The work done by a force* $\vec{F}$ *during a displacement* $\vec{r}$ *is given by* $\vec{F}\cdot\vec{r}$. *Suppose a force of* 12 N *acts on a particle in vertically upward direction and the particle is displaced through* 2·0 m *in vertically downward direction. Find the work done by the force during this displacement.*

***Solution*** **:** The angle between the force $\vec{F}$ and the displacement $\vec{r}$ is 180°. Thus, the work done is

$$W = \vec{F}\cdot\vec{r}$$

$$= Fr\cos\theta$$

$$= (12\text{ N})(2{\cdot}0\text{ m})(\cos 180^\circ)$$

$$= -24\text{ N–m} = -24\text{ J}.$$

**Dot Product of Two Vectors in terms of the Components along the Coordinate Axes**

Consider two vectors $\vec{a}$ and $\vec{b}$ represented in terms of the unit vectors $\vec{i}$, $\vec{j}$, $\vec{k}$ along the coordinate axes as

$$\vec{a} = a_x\,\vec{i} + a_y\,\vec{j} + a_z\,\vec{k}$$

and $$\vec{b} = b_x\,\vec{i} + b_y\,\vec{j} + b_z\,\vec{k}.$$

Then

$$\vec{a}\cdot\vec{b} = (a_x\,\vec{i} + a_y\,\vec{j} + a_z\,\vec{k})\cdot(b_x\,\vec{i} + b_y\,\vec{j} + b_z\,\vec{k})$$

$$= a_x b_x\,\vec{i}\cdot\vec{i} + a_x b_y\,\vec{i}\cdot\vec{j} + a_x b_z\,\vec{i}\cdot\vec{k}$$

$$+ a_y b_x\,\vec{j}\cdot\vec{i} + a_y b_y\,\vec{j}\cdot\vec{j} + a_y b_z\,\vec{j}\cdot\vec{k}$$

$$+ a_z b_x\,\vec{k}\cdot\vec{i} + a_z b_y\,\vec{k}\cdot\vec{j} + a_z b_z\,\vec{k}\cdot\vec{k} \qquad \ldots\ (i)$$

Since, $\vec{i}$, $\vec{j}$ and $\vec{k}$ are mutually orthogonal,

we have $\vec{i}\cdot\vec{j}=\vec{i}\cdot\vec{k}=\vec{j}\cdot\vec{i}=\vec{j}\cdot\vec{k}=\vec{k}\cdot\vec{i}=\vec{k}\cdot\vec{j}=0.$

Also, $\vec{i}\cdot\vec{i}=1\times1\cos0=1.$

Similarly, $\vec{j}\cdot\vec{j}=\vec{k}\cdot\vec{k}=1.$

Using these relations in equation (i) we get

$$\vec{a}\cdot\vec{b}=a_x b_x+a_y b_y+a_z b_z.$$

## 2.8 CROSS PRODUCT OR VECTOR PRODUCT OF TWO VECTORS

The cross product or vector product of two vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a}\times\vec{b}$ is itself a vector. The magnitude of this vector is

$$|\vec{a}\times\vec{b}|=ab\sin\theta \qquad \ldots\ (2.5)$$

where $a$ and $b$ are the magnitudes of $\vec{a}$ and $\vec{b}$ respectively and θ is the smaller angle between the two. When two vectors are drawn with both the tails coinciding, two angles are formed between them (figure 2.11). One of the angles is smaller than 180°

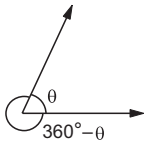



Figure 2.11

and the other is greater than 180° unless both are equal to 180°. The angle θ used in equation (2.5) is the smaller one. If both the angles are equal to 180°, sin θ = sin 180° = 0 and hence $|\vec{a}\times\vec{b}|=0$. Similarly if θ = 0, sin θ = 0 and $|\vec{a}\times\vec{b}|=0$. The cross product of two parallel vectors is zero.

The direction of $\vec{a}\times\vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$. Thus, it is perpendicular to the plane formed by $\vec{a}$ and $\vec{b}$. To determine the direction of arrow on this perpendicular several rules are in use. In order to avoid confusion we here describe just one rule.

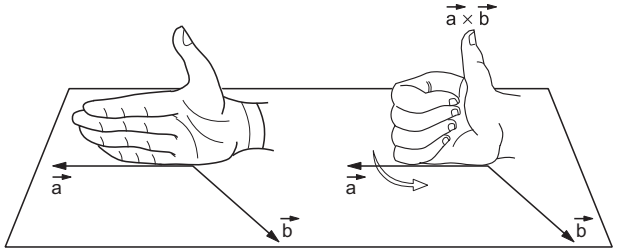



Figure 2.12

Draw the two vectors $\vec{a}$ and $\vec{b}$ with both the tails coinciding (figure 2.12). Now place your stretched right palm perpendicular to the plane of $\vec{a}$ and $\vec{b}$ in such a way that the fingers are along the vector $\vec{a}$ and when the fingers are closed they go towards $\vec{b}$. The direction of the thumb gives the direction of arrow to be put on the vector $\vec{a}\times\vec{b}$.

This is known as the *right hand thumb rule*. The left handers should be more careful in using this rule as it must be practiced with right hand only.

Note that this rule makes the cross product noncommutative. In fact

$$\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}.$$

The cross product follows the distributive law

$$\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}.$$

It does not follow the associative law

$$\vec{a}\times(\vec{b}\times\vec{c})\neq(\vec{a}\times\vec{b})\times\vec{c}.$$

When we choose a coordinate system any two perpendicular lines may be chosen as $X$ and $Y$ axes. However, once $X$ and $Y$ axes are chosen, there are two possible choices of $Z$-axis. The $Z$-axis must be perpendicular to the $X$-$Y$ plane. But the positive direction of $Z$-axis may be defined in two ways. We choose the positive direction of $Z$-axis in such a way that

$$\vec{i}\times\vec{j}=\vec{k}.$$

Such a coordinate system is called a *right handed system*. In such a system

$$\vec{j}\times\vec{k}=\vec{i} \quad \text{and} \quad \vec{k}\times\vec{i}=\vec{j}.$$

Of course $\vec{i}\times\vec{i}=\vec{j}\times\vec{j}=\vec{k}\times\vec{k}=0.$

***Example 2.5***

*The vector* $\vec{A}$ *has a magnitude of* 5 unit, $\vec{B}$ *has a magnitude of* 6 unit *and the cross product of* $\vec{A}$ *and* $\vec{B}$ *has a magnitude of* 15 unit. *Find the angle between* $\vec{A}$ *and* $\vec{B}$.

***Solution*** : If the angle between $\vec{A}$ and $\vec{B}$ is θ, the cross product will have a magnitude

$$|\vec{A}\times\vec{B}|=AB\sin\theta$$

or, $$15=5\times6\sin\theta$$

or, $$\sin\theta=\frac{1}{2}.$$

Thus, $$\theta=30^\circ \text{ or, } 150^\circ.$$

**Cross Product of Two Vectors in terms of the Components along the Coordinate Axes**

Let $$\vec{a}=a_x\vec{i}+a_y\vec{j}+a_z\vec{k}$$

and $$\vec{b}=b_x\vec{i}+b_y\vec{j}+b_z\vec{k}.$$

Then $\vec{a} \times \vec{b} = (a_x \vec{i} + a_y \vec{j} + a_z \vec{k}) \times (b_x \vec{i} + b_y \vec{j} + b_z \vec{k})$

$= a_x b_x \vec{i} \times \vec{i} + a_x b_y \vec{i} \times \vec{j} + a_x b_z \vec{i} \times \vec{k}$

$+ a_y b_x \vec{j} \times \vec{i} + a_y b_y \vec{j} \times \vec{j} + a_y b_z \vec{j} \times \vec{k}$

$+ a_z b_x \vec{k} \times \vec{i} + a_z b_y \vec{k} \times \vec{j} + a_z b_z \vec{k} \times \vec{k}$

$= a_x b_y \vec{k} + a_x b_z(-\vec{j}) + a_y b_x(-\vec{k}) + a_y b_z(\vec{i})$

$+ a_z b_x(\vec{j}) + a_z b_y(-\vec{i})$

$= (a_y b_z - a_z b_y)\vec{i} + (a_z b_x - a_x b_z)\vec{j}$

$+ (a_x b_y - a_y b_x)\vec{k}.$

**Zero Vector**

If we add two vectors $\vec{A}$ and $\vec{B}$, we get a vector. Suppose the vectors $\vec{A}$ and $\vec{B}$ have equal magnitudes but opposite directions. What is the vector $\vec{A} + \vec{B}$ ? The magnitude of this vector will be zero. For mathematical consistency it is convenient to have a vector of zero magnitude although it has little significance in physics. This vector is called zero vector. The direction of a zero vector is indeterminate. We can write this vector as $\vec{0}$. The concept of zero vector is also helpful when we consider vector product of parallel vectors. If $\vec{A} \parallel \vec{B}$, the vector $\vec{A} \times \vec{B}$ is zero vector. For any vector $\vec{A}$,

$$\vec{A} + \vec{0} = \vec{A}$$

$$\vec{A} \times \vec{0} = \vec{0}$$

and for any number $\lambda$,

$$\lambda \vec{0} = \vec{0}.$$

## 2.9 DIFFERENTIAL CALCULUS : $\frac{dy}{dx}$ AS RATE MEASURER

Consider two quantities $y$ and $x$ interrelated in such a way that for each value of $x$ there is one and only one value of $y$. Figure (2.13) represents the graph

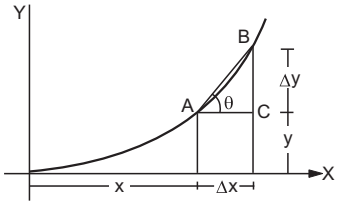



Figure 2.13

of $y$ versus $x$. The value of $y$ at a particular $x$ is obtained by the height of the ordinate at that $x$. Let $x$ be changed by a small amount $\Delta x$, and the corresponding change in $y$ be $\Delta y$. We can define the "rate of change" of $y$ with respect to $x$ in the following way. When $x$ changes by $\Delta x$, $y$ changes by $\Delta y$ so that the rate of change seems to be equal to $\frac{\Delta y}{\Delta x}$. If $A$ be the point $(x, y)$ and $B$ be the point $(x + \Delta x, y + \Delta y)$, the rate $\frac{\Delta y}{\Delta x}$ equals the slope of the line $AB$. We have

$$\frac{\Delta y}{\Delta x} = \frac{BC}{AC} = \tan\theta.$$

However, this cannot be the precise definition of the rate. Because the rate also varies between the points $A$ and $B$. The curve is steeper at $B$ than at $A$. Thus, to know the rate of change of $y$ at a particular value of $x$, say at $A$, we have to take $\Delta x$ very small. However small we take $\Delta x$, as long as it is not zero the rate may vary within that small part of the curve. However, if we go on drawing the point $B$ closer to $A$ and everytime calculate $\frac{\Delta y}{\Delta x} = \tan\theta$, we shall see that as $\Delta x$ is made smaller and smaller the slope $\tan\theta$ of the line $AB$ approaches the slope of the tangent at $A$. This slope of the tangent at $A$ thus gives the rate of change of $y$ with respect to $x$ at $A$. This rate is denoted by $\frac{dy}{dx}$. Thus,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

For small changes $\Delta x$ we can approximately write

$$\Delta y = \frac{dy}{dx} \Delta x.$$

Note that if the function $y$ increases with an increase in $x$ at a point, $\frac{dy}{dx}$ is positive there, because both $\Delta y$ and $\Delta x$ are positive. If the function $y$ decreases with an increase in $x$, $\Delta y$ is negative when $\Delta x$ is positive. Then $\frac{\Delta y}{\Delta x}$ and hence $\frac{dy}{dx}$ is negative.

***Example 2.6***

*From the curve given in figure (2.14) find* $\frac{dy}{dx}$ *at* $x$ = 2, 6 *and* 10.

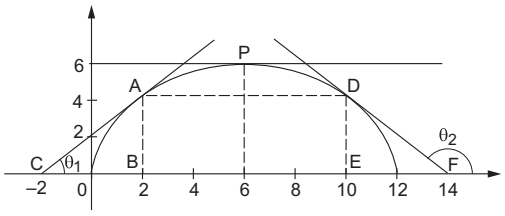



Figure 2.14

***Solution*** **:** The tangent to the curve at $x$ = 2 is $AC$. Its slope is $\tan\theta_1 = \frac{AB}{BC} = \frac{5}{4}$.

Thus, $\frac{dy}{dx} = \frac{5}{4}$ at $x = 2$.

The tangent to the curve at $x = 6$ is parallel to the $X$-axis.

Thus, $\frac{dy}{dx} = \tan\theta = 0$ at $x = 6$.

The tangent to the curve at $x = 10$ is $DF$. Its slope is

$$\tan\theta_2 = \frac{DE}{EF} = -\frac{5}{4}.$$

Thus, $\frac{dy}{dx} = -\frac{5}{4}$ at $x = 10$.

If we are given the graph of $y$ versus $x$, we can find $\frac{dy}{dx}$ at any point of the curve by drawing the tangent at that point and finding its slope. Even if the graph is not drawn and the algebraic relation between $y$ and $x$ is given in the form of an equation, we can find $\frac{dy}{dx}$ algebraically. Let us take an example.

The area $A$ of a square of length $L$ is $A = L^2$.

If we change $L$ to $L + \Delta L$, the area will change from $A$ to $A + \Delta A$ (figure 2.15).

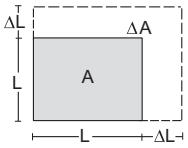



Figure 2.15

$$A + \Delta A = (L + \Delta L)^2$$

$$= L^2 + 2L\,\Delta L + (\Delta L)^2$$

or, $$\Delta A = 2L(\Delta L) + (\Delta L)^2$$

or, $$\frac{\Delta A}{\Delta L} = 2L + \Delta L.$$

Now if $\Delta L$ is made smaller and smaller, $2L + \Delta L$ will approach $2L$.

Thus, $$\frac{dA}{dL} = \lim_{\Delta L \to 0} \frac{\Delta A}{\Delta L} = 2L.$$

Table (2.1) gives the formulae for $\frac{dy}{dx}$ for some of the important functions. $\frac{dy}{dx}$ is called the differential coefficient or derivative of $y$ with respect to $x$.

**Table 2.1 : $\frac{dy}{dx}$ *for some common functions***

| $y$ | $\frac{dy}{dx}$ | $y$ | $\frac{dy}{dx}$ |
|---|---|---|---|
| $x^n$ | $nx^{n-1}$ | $\sec x$ | $\sec x \tan x$ |
| $\sin x$ | $\cos x$ | $\operatorname{cosec} x$ | $-\operatorname{cosec} x \cot x$ |
| $\cos x$ | $-\sin x$ | $\ln x$ | $\frac{1}{x}$ |
| $\tan x$ | $\sec^2 x$ | $e^x$ | $e^x$ |
| $\cot x$ | $-\operatorname{cosec}^2 x$ | | |

Besides, there are certain rules for finding the derivatives of composite functions.

(a) $\frac{d}{dx}(cy) = c\frac{dy}{dx}$ ($c$ is a constant)

(b) $\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$

(c) $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

(d) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

(e) $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

With these rules and table 2.1 derivatives of almost all the functions of practical interest may be evaluated.

***Example 2.7***

*Find* $\frac{dy}{dx}$ *if* $y = e^x \sin x$.

***Solution*** : $y = e^x \sin x$.

So $$\frac{dy}{dx} = \frac{d}{dx}(e^x \sin x) = e^x \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(e^x)$$

$$= e^x \cos x + e^x \sin x = e^x(\cos x + \sin x).$$

## 2.10 MAXIMA AND MINIMA

Suppose a quantity $y$ depends on another quantity $x$ in a manner shown in figure (2.16). It becomes maximum at $x_1$ and minimum at $x_2$.

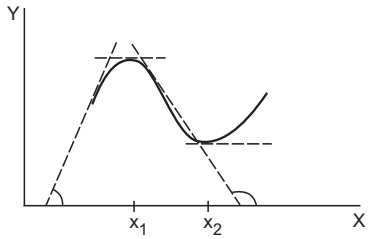



Figure 2.16

At these points the tangent to the curve is parallel to the $X$-axis and hence its slope is $\tan\theta = 0$. But the slope of the curve $y$-$x$ equals the rate of change $\frac{dy}{dx}$. Thus, at a maximum or a minimum,

$$\frac{dy}{dx} = 0.$$

Just before the maximum the slope is positive, at the maximum it is zero and just after the maximum it is negative. Thus, $\frac{dy}{dx}$ decreases at a maximum and hence the rate of change of $\frac{dy}{dx}$ is negative at a maximum i.e.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) < 0 \text{ at a maximum.}$$

The quantity $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is the rate of change of the slope. It is written as $\frac{d^2y}{dx^2}$. Thus, the condition of a maximum is

$$\left.\begin{aligned}\frac{dy}{dx} &= 0\\ \frac{d^2y}{dx^2} &< 0\end{aligned}\right\} \text{— maximum.} \qquad \ldots\ (2.6)$$

Similarly, at a minimum the slope changes from negative to positive. The slope increases at such a point and hence $\frac{d}{dx}\left(\frac{dy}{dx}\right) > 0$. The condition of a minimum is

$$\left.\begin{aligned}\frac{dy}{dx} &= 0\\ \frac{d^2y}{dx^2} &> 0\end{aligned}\right\} \text{— minimum.} \qquad \ldots\ (2.7)$$

Quite often it is known from the physical situation whether the quantity is a maximum or a minimum. The test on $\frac{d^2y}{dx^2}$ may then be omitted.

***Example 2.8***

*The height reached in time t by a particle thrown upward with a speed u is given by*

$$h = ut - \frac{1}{2}gt^2$$

*where $g = 9{\cdot}8$ m/s$^2$ is a constant. Find the time taken in reaching the maximum height.*

***Solution*** : The height $h$ is a function of time. Thus, $h$ will be maximum when $\frac{dh}{dt} = 0$. We have,

$$h = ut - \frac{1}{2}gt^2$$

or,
$$\frac{dh}{dt} = \frac{d}{dt}(ut) - \frac{d}{dt}\left(\frac{1}{2}gt^2\right)$$

$$= u\frac{dt}{dt} - \frac{1}{2}g\frac{d}{dt}(t^2)$$

$$= u - \frac{1}{2}g(2t) = u - gt.$$

For maximum $h$,

$$\frac{dh}{dt} = 0$$

or, $\quad u - gt = 0 \quad$ or, $\quad t = \frac{u}{g}$.

## 2.11 INTEGRAL CALCULUS

Let $PQ$ be a curve representing the relation between two quantities $x$ and $y$ (figure 2.17). The point $P$ corresponds to $x = a$ and $Q$ corresponds to $x = b$. Draw perpendiculars from $P$ and $Q$ on the $X$-axis so as to cut it at $A$ and $B$ respectively. We are interested in finding the area $PABQ$. Let us denote the value of $y$ at $x$ by the symbol $y = f(x)$.

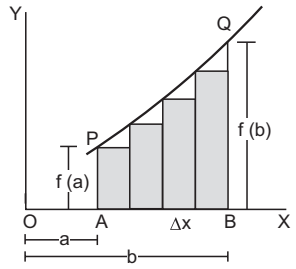



Figure 2.17

Let us divide the length $AB$ in $N$ equal elements each of length $\Delta x = \frac{b-a}{N}$. From the ends of each small length we draw lines parallel to the $Y$-axis. From the points where these lines cut the given curve, we draw short lines parallel to the $X$-axis. This constructs the rectangular bars shown shaded in the figure. The sum of the areas of these $N$ rectangular bars is

$$I' = f(a)\,\Delta x + f(a + \Delta x)\,\Delta x + f(a + 2\Delta x)\,\Delta x + \ldots$$
$$\ldots + f[a + (N-1)\,\Delta x]\,\Delta x\,.$$

This may be written as

$$I' = \sum_{i=1}^{N} f(x_i)\Delta x \qquad \ldots\ (2.8)$$

where $x_i$ takes the values $a, a + \Delta x, a + 2\Delta x, \ldots, b - \Delta x$.

This area differs slightly from the area $PABQ$. This difference is the sum of the small triangles formed just under the curve. Now the important point is the following. As we increase the number of intervals $N$, the vertices of the bars touch the curve $PQ$ at more points and the total area of the small triangles decreases. As $N$ tends to infinity ($\Delta x$ tends to zero

because $\Delta x = \frac{b-a}{N}$) the vertices of the bars touch the curve at infinite number of points and the total area of the triangles tends to zero. In such a limit the sum (2.8) becomes the area $I$ of $PABQ$. Thus, we may write,

$$I = \lim_{\Delta x \to 0} \sum_{i=1}^{N} f(x_i)\Delta x.$$

The limit is taken as $\Delta x$ tends to zero or as $N$ tends to infinity. In mathematics this quantity is denoted as

$$I = \int_a^b f(x)\, dx$$

and is read as the integral of $f(x)$ with respect to $x$ within the limits $x = a$ to $x = b$. Here $a$ is called the lower limit and $b$ the upper limit of integration. The integral is the sum of a large number of terms of the type $f(x)\,\Delta x$ with $x$ continuously varying from $a$ to $b$ and the number of terms tending to infinity.

Let us use the above method to find the area of a trapezium. Let us suppose the line $PQ$ is represented by the equation $y = x$.

The points $A$ and $B$ on the $X$-axis represent $x = a$ and $x = b$. We have to find the area of the trapezium $PABQ$.

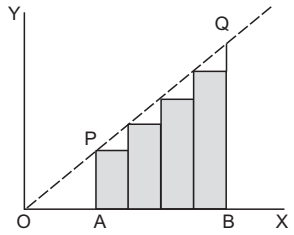



Figure 2.18

Let us divide the length $AB$ in $N$ equal intervals. The length of each interval is $\Delta x = \frac{b-a}{N}$. The height of the first shaded bar is $y = x = a$, of the second bar is $y = x = a + \Delta x$, that of the third bar is $y = x = a + 2\,\Delta x$ etc. The height of the $N$th bar is $y = x = a + (N-1)\Delta x$. The width of each bar is $\Delta x$, so that the total area of all the bars is

$$I' = a\Delta x + (a + \Delta x)\,\Delta x + (a + 2\Delta x)\,\Delta x + \ldots + [a + (N-1)\Delta x]\Delta x$$

$$= \big[a + (a + \Delta x) + (a + 2\Delta x) + \ldots + \{a + (N-1)\Delta x\}\big]\Delta x \quad \ldots (2.9)$$

This sum can be written as

$$I' = \sum_{i=1}^{N} x_i\, \Delta x$$

where $\Delta x = \frac{b-a}{N}$ and $x_i = a,\ a + \Delta x, \ldots b - \Delta x$.

As $\Delta x \to 0$ the total area of the bars becomes the area of the shaded part $PABQ$.

Thus, the required area is

$$I = \lim_{\Delta x \to 0} \sum_{i=1}^{N} x_i\, \Delta x = \int_a^b x dx. \quad \ldots (i)$$

Now the terms making the series in the square bracket in equation (2.9) are in arithmetic progression so that this series may be summed up using the formula $S = \frac{n}{2}(a + l)$. Equation (2.9) thus becomes

$$I' = \frac{N}{2}[a + \{a + (N-1)\Delta x\}]\Delta x$$

$$= \frac{N\Delta x}{2}[2a + N\Delta x - \Delta x]$$

$$= \frac{b-a}{2}[2a + b - a - \Delta x]$$

$$= \frac{b-a}{2}[a + b - \Delta x].$$

Thus, the area $PABQ$ is

$$I = \lim_{\Delta x \to 0}\left[\frac{b-a}{2}\right][a + b - \Delta x]$$

$$= \frac{b-a}{2}(a + b)$$

$$= \frac{1}{2}(b^{\,2} - a^{\,2}). \quad \ldots (ii)$$

Thus, from (i) and (ii)

$$\int_a^b x\, dx = \frac{1}{2}(b^{\,2} - a^{\,2}).$$

In mathematics, special methods have been developed to find the integration of various functions $f(x)$. A very useful method is as follows. Suppose we wish to find

$$\int_a^b f(x)\, dx = \lim_{\Delta x \to 0} \sum_{i=1}^{N} f(x_i)\, \Delta x$$

where $\Delta x = \frac{b-a}{N}$; $x_i = a,\ a + \Delta x, \ldots b - \Delta x$.

Now look for a function $F(x)$ such that the derivative of $F(x)$ is $f(x)$ that is, $\frac{dF(x)}{dx} = f(x)$. If you can find such a function $F(x)$, then

$$\int_a^b f(x)\, dx = F(b) - F(a);$$

$F(b) - F(a)$ is also written as $[F(x)]_a^b$.

$F(x)$ is called the indefinite integration or the antiderivative of $f(x)$. We also write $\int f(x)\,dx = F(x)$. This may be treated as another way of writing $\frac{dF(x)}{dx} = f(x)$ .

For example, $\frac{d}{dx}\left(\frac{1}{2}x^2\right) = \frac{1}{2}\frac{d}{dx}(x^2) = \frac{1}{2}\cdot 2x = x.$

Thus, $$\int_a^b x\,dx = \left[\frac{1}{2}x^2\right]_a^b$$
$$= \left(\frac{1}{2}b^2\right) - \left(\frac{1}{2}a^2\right)$$
$$= \frac{1}{2}(b^2 - a^2)$$

as deduced above.

Table (2.2) lists some important integration formulae. Many of them are essentially same as those given in table (2.1).

**Table 2.2 : *Integration Formulae***

| $f(x)$ | $F(x) = \int f(x)\,dx$ | $f(x)$ | $F(x) = \int f(x)\,dx$ |
|---|---|---|---|
| $\sin x$ | $-\cos x$ | $x^n (n \neq -1)$ | $\frac{x^{n+1}}{n+1}$ |
| $\cos x$ | $\sin x$ | $\frac{1}{x}$ | $\ln x$ |
| $\sec^2 x$ | $\tan x$ | $\frac{1}{x^2 + a^2}$ | $\frac{1}{a}\tan^{-1}\frac{x}{a}$ |
| $\text{cosec}^2 x$ | $-\cot x$ | $\frac{1}{\sqrt{a^2 - x^2}}$ | $\sin^{-1}\frac{x}{a}$ |
| $\sec x \tan x$ | $\sec x$ | | |
| $\text{cosec}\, x \cot x$ | $-\text{cosec}\, x$ | | |

Some useful rules for integration are as follows:

(a) $\int c\,f(x)\,dx = c\int f(x)\,dx$ where $c$ is a constant

(b) Let $\int f(x)\,dx = F(x)$

then $\int f(cx)\,dx = \frac{1}{c}F(cx).$

(c) $\int [\,f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx.$

***Example 2.9***

*Evaluate* $\int_3^6 (2x^2 + 3x + 5)\,dx.$

***Solution*** : $\int (2x^2 + 3x + 5)\,dx$

$$= \int 2x^2 dx + \int 3x\,dx + \int 5\,dx$$
$$= 2\int x^2 dx + 3\int x\,dx + 5\int x^0 dx$$
$$= 2\frac{x^3}{3} + 3\frac{x^2}{2} + 5\frac{x^1}{1}$$
$$= \frac{2}{3}x^3 + \frac{3}{2}x^2 + 5x.$$

Thus, $\int_3^6 (2x^2 + 3x + 5)\,dx = \left[\frac{2}{3}x^3 + \frac{3}{2}x^2 + 5x\right]_3^6$

$$= \frac{2}{3}(216 - 27) + \frac{3}{2}(36 - 9) + 5(6 - 3)$$
$$= 126 + 40{\cdot}5 + 15 = 181{\cdot}5.$$

## 2.12 SIGNIFICANT DIGITS

When a measurement is made, a numerical value is read generally from some calibrated scale. To measure the length of a body we can place a metre scale in contact with the body. One end of the body may be made to coincide with the zero of the metre scale and the reading just in front of the other end is noted from the scale. When an electric current is measured with an ammeter the reading of the pointer on the graduation of the ammeter is noted. The value noted down includes all the digits that can be directly read from the scale and one doubtful digit at the end. The doubtful digit corresponds to the eye estimation within the smallest subdivision of the scale. This smallest subdivision is known as the *least count* of the instrument. In a metre scale, the major graduations are at an interval of one centimetre and ten subdivisions are made between two consecutive major graduations. Thus, the smallest subdivision measures a millimetre. If one end of the object coincides with the zero of the metre scale, the other end may fall between 10·4 cm and 10·5 cm mark of the scale (figure 2.19). We can estimate the distance between the 10·4 cm mark and the edge of the body as follows.

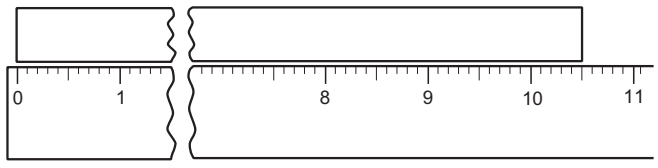



Figure 2.19

We mentally divide the 1 mm division in 10 equal parts and guess on which part is the edge falling. We may note down the reading as 10·46 cm. The digits 1, 0 and 4 are certain but 6 is doubtful. All these digits are called *significant digits*. We say that the length is measured up to four significant digits. The rightmost or the doubtful digit is called the *least significant digit* and the leftmost digit is called the *most significant digit.*

There may be some confusion if there are zeroes at the right end of the number. For example, if a measurement is quoted as 600 mm and we know nothing about the least count of the scale we cannot be sure whether the last zeros are significant or not. If the scale had marking only at each metre then the edge must be between the marks 0 m and 1 m and the digit 6 is obtained only through the eye estimation. Thus, 6 is the doubtful digit and the zeros after that are insignificant. But if the scale had markings at centimetres, the number read is 60 and these two digits are significant, the last zero is insignificant. If the scale used had markings at millimetres, all the three digits 6, 0, 0 are significant. To avoid confusion one may report only the significant digits and the magnitude may be correctly described by proper powers of 10. For example, if only 6 is significant in 600 mm we may write it as $6 \times 10^2$ mm. If 6 and the first zero are significant we may write it as $6{\cdot}0 \times 10^2$ mm and if all the three digits are significant we may write it as $6{\cdot}00 \times 10^2$ mm.

If the integer part is zero, any number of continuous zeros just after the decimal part is insignificant. Thus, the number of significant digits in 0·0023 is two and in 1·0023 is five.

## 2.13 SIGNIFICANT DIGITS IN CALCULATIONS

When two or more numbers are added, subtracted, multiplied or divided, how to decide about the number of significant digits in the answer ? For example, suppose the mass of a body $A$ is measured to be 12·0 kg and of another body $B$ to be 7·0 kg. What is the ratio of the mass of $A$ to the mass of $B$ ? Arithmetic will give this ratio as

$$\frac{12{\cdot}0}{7{\cdot}0} = 1{\cdot}714285\ldots$$

However, all the digits of this answer cannot be significant. The zero of 12·0 is a doubtful digit and the zero of 7·0 is also doubtful. The quotient cannot have so many reliable digits. The rules for deciding the number of significant digits in an arithmetic calculation are listed below.

1. In a multiplication or division of two or more quantities, the number of significant digits in the answer is equal to the number of significant digits in the quantity which has the minimum number of significant digits. Thus, $\frac{12{\cdot}0}{7{\cdot}0}$ will have two significant digits only.

The insignificant digits are dropped from the result if they appear after the decimal point. They are replaced by zeros if they appear to the left of the decimal point. The least significant digit is rounded according to the rules given below.

If the digit next to the one rounded is more than 5, the digit to be rounded is increased by 1. If the digit next to the one rounded is less than 5, the digit to be rounded is left unchanged. If the digit next to the one rounded is 5, then the digit to be rounded is increased by 1 if it is odd and is left unchanged if it is even.

2. For addition or subtraction write the numbers one below the other with all the decimal points in one line. Now locate the first column from left that has a doubtful digit. All digits right to this column are dropped from all the numbers and rounding is done to this column. The addition or subtraction is now performed to get the answer.

***Example 2.10***

*Round off the following numbers to three significant digits (a)* 15462*, (b)* 14·745*, (c)* 14·750 *and (d)* 14·650 $\times 10^{12}$.

***Solution*** : (a) The third significant digit is 4. This digit is to be rounded. The digit next to it is 6 which is greater than 5. The third digit should, therefore, be increased by 1. The digits to be dropped should be replaced by zeros because they appear to the left of the decimal. Thus, 15462 becomes 15500 on rounding to three significant digits.

(b) The third significant digit in 14·745 is 7. The number next to it is less than 5. So 14·745 becomes 14·7 on rounding to three significant digits.

(c) 14·750 will become 14·8 because the digit to be rounded is odd and the digit next to it is 5.

(d) $14{\cdot}650 \times 10^{12}$ will become $14{\cdot}6 \times 10^{12}$ because the digit to be rounded is even and the digit next to it is 5.

***Example 2.11***

*Evaluate* $\frac{25{\cdot}2 \times 1374}{33{\cdot}3}$. *All the digits in this expression are significant.*

***Solution*** : We have $\frac{25{\cdot}2 \times 1374}{33{\cdot}3} = 1039{\cdot}7838....$

Out of the three numbers given in the expression 25·2 and 33·3 have 3 significant digits and 1374 has four. The answer should have three significant digits. Rounding 1039·7838... to three significant digits, it becomes 1040. Thus, we write

$$\frac{25{\cdot}2 \times 1374}{33{\cdot}3} = 1040.$$

***Example 2.12***

*Evaluate* 24·36 + 0·0623 + 256·2.

***Solution :***

24·36
0·0623
256·2

Now the first column where a doubtful digit occurs is the one just next to the decimal point (256·2). All digits right to this column must be dropped after proper rounding. The table is rewritten and added below

24·4
0·1
256·2
280·7

The sum is 280·7.

## 2.14 ERRORS IN MEASUREMENT

While doing an experiment several errors can enter into the results. Errors may be due to faulty equipment, carelessness of the experimenter or random causes. The first two types of errors can be removed after detecting their cause but the random errors still remain. No specific cause can be assigned to such errors.

When an experiment is repeated many times, the random errors are sometimes positive and sometimes negative. Thus, the average of a large number of the results of repeated experiments is close to the true value. However, there is still some uncertainty about the truth of this average. The uncertainty is estimated by calculating the standard deviation described below.

Let $x_1, x_2, x_3, \ldots, x_N$ are the results of an experiment repeated $N$ times. The standard deviation $\sigma$ is defined as

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2}$$

where $\bar{x} = \frac{1}{N}\sum_i x_i$ is the average of all the values of $x$. The best value of $x$ derived from these experiments is $\bar{x}$ and the uncertainty is of the order of $\pm\sigma$. In fact $\bar{x} \pm 1{\cdot}96\,\sigma$ is quite often taken as the interval in which the true value should lie. It can be shown that there is a 95% chance that the true value lies within $\bar{x} \pm 1{\cdot}96\,\sigma$.

If one wishes to be more sure, one can use the interval $\bar{x} \pm 3\,\sigma$ as the interval which will contain the true value. The chances that the true value will be within $\bar{x} \pm 3\,\sigma$ is more that 99%.

All this is true if the number of observations $N$ is large. In practice if $N$ is greater than 8, the results are reasonably correct.

***Example 2.13***

*The focal length of a concave mirror obtained by a student in repeated experiments are given below. Find the average focal length with uncertainty in $\pm\sigma$ limit.*

| No. of observation | focal length in cm |
|---|---|
| 1 | 25·4 |
| 2 | 25·2 |
| 3 | 25·6 |
| 4 | 25·1 |
| 5 | 25·3 |
| 6 | 25·2 |
| 7 | 25·5 |
| 8 | 25·4 |
| 9 | 25·3 |
| 10 | 25·7 |

***Solution*** : The average focal length $\bar{f} = \frac{1}{10}\sum_{i=1}^{10} f_i$

$= 25{\cdot}37 \approx 25{\cdot}4.$

The calculation of $\sigma$ is shown in the table below:

| $i$ | $f_i$ cm | $f_i - \bar{f}$ cm | $(f_i - \bar{f})^2$ cm$^2$ | $\Sigma (f_i - \bar{f})^2$ cm$^2$ |
|---|---|---|---|---|
| 1 | 25·4 | 0·0 | 0·00 | |
| 2 | 25·2 | – 0·2 | 0·04 | |
| 3 | 25·6 | 0·2 | 0·04 | |
| 4 | 25·1 | – 0·3 | 0·09 | |
| 5 | 25·3 | – 0·1 | 0·01 | 0·33 |
| 6 | 25·2 | – 0·2 | 0·04 | |
| 7 | 25·5 | 0·1 | 0·01 | |
| 8 | 25·4 | 0·0 | 0·00 | |
| 9 | 25·3 | – 0·1 | 0·01 | |
| 10 | 25·7 | 0·3 | 0·09 | |

$$\sigma = \sqrt{\frac{1}{10}\sum_i (f_i - \bar{f})^2} = \sqrt{0{\cdot}033\ \text{cm}^2} = 0{\cdot}18\ \text{cm}$$

$\cong 0{\cdot}2$ cm.

Thus, the focal length is likely to be within (25·4 ± 0·2 cm) and we write

$$f = (25{\cdot}4 \pm 0{\cdot}2)\ \text{cm}.$$

## *Worked Out Examples*

**1.** *A vector has component along the X-axis equal to* 25 unit *and along the Y-axis equal to* 60 unit. *Find the magnitude and direction of the vector.*

***Solution*** **:** The given vector is the resultant of two perpendicular vectors, one along the $X$-axis of magnitude 25 unit and the other along the $Y$-axis of magnitude 60 units. The resultant has a magnitude $A$ given by

$$A = \sqrt{(25)^2 + (60)^2 + 2 \times 25 \times 60 \cos 90^\circ}$$

$$= \sqrt{(25)^2 + (60)^2} = 65.$$

The angle $\alpha$ between this vector and the $X$-axis is given by

$$\tan\alpha = \frac{60}{25}.$$

**2.** *Find the resultant of the three vectors shown in figure (2-W1).*

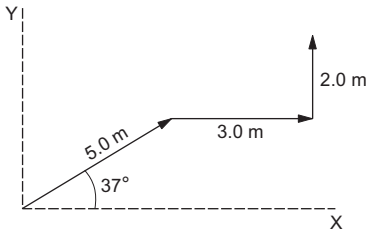



Figure 2-W1

***Solution*** **:** Take the axes as shown in the figure.

The $x$-component of the 5·0 m vector = 5·0 m cos37°

= 4·0 m,

the $x$-component of the 3·0 m vector = 3·0 m

and the $x$-component of the 2·0 m vector = 2·0 m cos90°

= 0.

Hence, the $x$-component of the resultant

= 4·0 m + 3·0 m + 0 = 7·0 m.

The $y$-component of the 5·0 m vector = 5·0 m sin37°

= 3·0 m,

the $y$-component of the 3·0 m vector = 0

and the $y$-component of the 2·0 m vector = 2·0 m.

Hence, the $y$-component of the resultant

= 3·0 m + 0 + 2·0 m = 5·0 m.

The magnitude of the resultant vector

$$= \sqrt{(7{\cdot}0 \text{ m})^2 + (5{\cdot}0 \text{ m})^2}$$

$$= 8{\cdot}6 \text{ m}.$$

If the angle made by the resultant with the $X$-axis is $\theta$, then

$$\tan\theta = \frac{y\text{-component}}{x\text{-component}} = \frac{5{\cdot}0}{7{\cdot}0} \quad \text{or,} \quad \theta = 35{\cdot}5^\circ.$$

**3.** *The sum of the three vectors shown in figure (2-W2) is zero. Find the magnitudes of the vectors* $\vec{OB}$ *and* $\vec{OC}$.

***Solution*** **:** Take the axes as shown in the figure

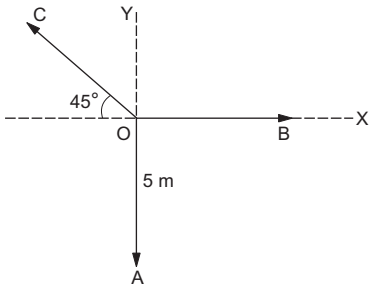



Figure 2-W2

The $x$-component of $\vec{OA} = (OA)\cos 90^\circ = 0$.

The $x$-component of $\vec{OB} = (OB)\cos 0^\circ = OB$.

The $x$-component of $\vec{OC} = (OC)\cos 135^\circ = -\frac{1}{\sqrt{2}} OC$.

Hence, the $x$-component of the resultant

$$= OB - \frac{1}{\sqrt{2}} OC. \qquad \ldots \text{(i)}$$

It is given that the resultant is zero and hence its $x$-component is also zero. From (i),

$$OB = \frac{1}{\sqrt{2}} OC. \qquad \ldots \text{(ii)}$$

The $y$-component of $\vec{OA} = OA \cos 180^\circ = -OA$.

The $y$-component of $\vec{OB} = OB \cos 90^\circ = 0$.

The $y$-component of $\vec{OC} = OC \cos 45^\circ = \frac{1}{\sqrt{2}} OC$.

Hence, the $y$-component of the resultant

$$= \frac{1}{\sqrt{2}} OC - OA \qquad \ldots \text{(iii)}$$

As the resultant is zero, so is its $y$-component. From (iii),

$$\frac{1}{\sqrt{2}} OC = OA, \quad \text{or,} \quad OC = \sqrt{2}\, OA = 5\sqrt{2} \text{ m}.$$

From (ii), $OB = \frac{1}{\sqrt{2}} OC = 5$ m.

**4.** *The magnitudes of vectors* $\vec{OA}$, $\vec{OB}$ *and* $\vec{OC}$ *in figure (2-W3) are equal. Find the direction of* $\vec{OA} + \vec{OB} - \vec{OC}$.

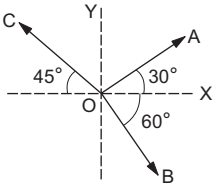



Figure 2-W3

***Solution*** : Let $OA = OB = OC = F$.

$x$-component of $\vec{OA} = F\cos30° = F\frac{\sqrt{3}}{2}$.

$x$-component of $\vec{OB} = F\cos60° = \frac{F}{2}$.

$x$-component of $\vec{OC} = F\cos135° = -\frac{F}{\sqrt{2}}$.

$x$-component of $\vec{OA} + \vec{OB} - \vec{OC}$

$$= \left(\frac{F\sqrt{3}}{2}\right) + \left(\frac{F}{2}\right) - \left(-\frac{F}{\sqrt{2}}\right)$$

$$= \frac{F}{2}(\sqrt{3} + 1 + \sqrt{2}).$$

$y$-component of $\vec{OA} = F\cos60° = \frac{F}{2}$.

$y$-component of $\vec{OB} = F\cos150° = -\frac{F\sqrt{3}}{2}$.

$y$-component of $\vec{OC} = F\cos45° = \frac{F}{\sqrt{2}}$.

$y$-component of $\vec{OA} + \vec{OB} - \vec{OC}$

$$= \left(\frac{F}{2}\right) + \left(-\frac{F\sqrt{3}}{2}\right) - \left(\frac{F}{\sqrt{2}}\right)$$

$$= \frac{F}{2}(1 - \sqrt{3} - \sqrt{2}).$$

Angle of $\vec{OA} + \vec{OB} - \vec{OC}$ with the $X$-axis

$$= \tan^{-1}\frac{\frac{F}{2}(1 - \sqrt{3} - \sqrt{2})}{\frac{F}{2}(1 + \sqrt{3} + \sqrt{2})} = \tan^{-1}\frac{(1 - \sqrt{3} - \sqrt{2})}{(1 + \sqrt{3} + \sqrt{2})}.$$

**5**. *Find the resultant of the three vectors $\vec{OA}$, $\vec{OB}$ and $\vec{OC}$ shown in figure (2-W4). Radius of the circle is R.*

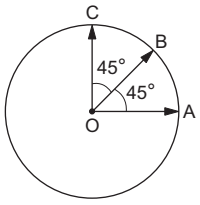



Figure 2-W4

***Solution*** : $OA = OC$.

$\vec{OA} + \vec{OC}$ is along $\vec{OB}$ (bisector) and its magnitude is

$$2R\cos45° = R\sqrt{2}.$$

$(\vec{OA} + \vec{OC}) + \vec{OB}$ is along $\vec{OB}$ and its magnitude is

$$R\sqrt{2} + R = R(1 + \sqrt{2}).$$

**6**. *The resultant of vectors $\vec{OA}$ and $\vec{OB}$ is perpendicular to $\vec{OA}$ (figure 2-W5). Find the angle AOB.*

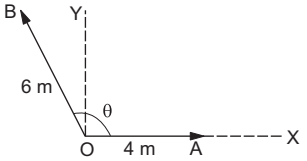



Figure 2-W5

***Solution*** : Take the dotted lines as $X$, $Y$ axes.

$x$-component of $\vec{OA} = 4$ m, $x$-component of $\vec{OB} = 6\text{ m}\cos\theta$.

$x$-component of the resultant $= (4 + 6\cos\theta)$ m.

But it is given that the resultant is along $Y$-axis. Thus, the $x$-component of the resultant = 0

$4 + 6\cos\theta = 0$ or, $\cos\theta = -2/3$.

**7**. *Write the unit vector in the direction of $\vec{A} = 5\vec{i} + \vec{j} - 2\vec{k}$.*

***Solution*** : $|\vec{A}| = \sqrt{5^2 + 1^2 + (-2)^2} = \sqrt{30}$.

The required unit vector is $\frac{\vec{A}}{|\vec{A}|}$

$$= \frac{5}{\sqrt{30}}\vec{i} + \frac{1}{\sqrt{30}}\vec{j} - \frac{2}{\sqrt{30}}\vec{k}.$$

**8**. *If $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$ show that $\vec{a} \perp \vec{b}$.*

***Solution*** : We have $|\vec{a} + \vec{b}|^2 = (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b})$

$$= \vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b}$$

$$= a^2 + b^2 + 2\vec{a}\cdot\vec{b}.$$

Similarly,

$$|\vec{a} - \vec{b}|^2 = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b})$$

$$= a^2 + b^2 - 2\vec{a}\cdot\vec{b}.$$

If $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$,

$a^2 + b^2 + 2\vec{a}\cdot\vec{b} = a^2 + b^2 - 2\vec{a}\cdot\vec{b}$

or, $\vec{a}\cdot\vec{b} = 0$

or, $\vec{a} \perp \vec{b}$.

**9**. *If $\vec{a} = 2\vec{i} + 3\vec{j} + 4\vec{k}$ and $\vec{b} = 4\vec{i} + 3\vec{j} + 2\vec{k}$, find the angle between $\vec{a}$ and $\vec{b}$.*

***Solution*** : We have $\vec{a}\cdot\vec{b} = ab\cos\theta$

or, $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{ab}$

where $\theta$ is the angle between $\vec{a}$ and $\vec{b}$.

Now $\vec{a}\cdot\vec{b} = a_x b_x + a_y b_y + a_z b_z$

$$= 2\times4 + 3\times3 + 4\times2 = 25.$$

Also $a = \sqrt{a_x^2 + a_y^2 + a_z^2}$

$$= \sqrt{4 + 9 + 16} = \sqrt{29}$$

and $b = \sqrt{b_x^2 + b_y^2 + b_z^2} = \sqrt{16 + 9 + 4} = \sqrt{29}$.

Thus, $\cos\theta = \frac{25}{29}$

or, $\theta = \cos^{-1}\left(\frac{25}{29}\right)$.

**10.** *If* $\vec{A} = 2\vec{i} - 3\vec{j} + 7\vec{k}$, $\vec{B} = \vec{i} + 2\vec{k}$ *and* $\vec{C} = \vec{j} - \vec{k}$ *find* $\vec{A} \cdot (\vec{B} \times \vec{C})$.

***Solution :*** $\vec{B} \times \vec{C} = (\vec{i} + 2\vec{k}) \times (\vec{j} - \vec{k})$

$= \vec{i} \times (\vec{j} - \vec{k}) + 2\vec{k} \times (\vec{j} - \vec{k})$

$= \vec{i} \times \vec{j} - \vec{i} \times \vec{k} + 2\vec{k} \times \vec{j} - 2\vec{k} \times \vec{k}$

$= \vec{k} + \vec{j} - 2\vec{i} - 0 = -2\vec{i} + \vec{j} + \vec{k}$.

$\vec{A} \cdot (\vec{B} \times \vec{C}) = (2\vec{i} - 3\vec{j} + 7\vec{k}) \cdot (-2\vec{i} + \vec{j} + \vec{k})$

$= (2)(-2) + (-3)(1) + (7)(1)$

$= 0$.

**11.** *The volume of a sphere is given by*

$$V = \frac{4}{3}\pi R^3$$

*where R is the radius of the sphere. (a) Find the rate of change of volume with respect to R. (b) Find the change in volume of the sphere as the radius is increased from* 20·0 cm *to* 20·1 cm. *Assume that the rate does not appreciably change between* $R$ = 20·0 cm *to* $R$ = 20·1 cm.

***Solution :*** (a) $V = \frac{4}{3}\pi R^3$

or, $\frac{dV}{dR} = \frac{4}{3}\pi \frac{d}{dR}(R)^3 = \frac{4}{3}\pi \cdot 3R^2 = 4\pi R^2$.

(b) At $R$ = 20 cm, the rate of change of volume with the radius is

$\frac{dV}{dR} = 4\pi R^2 = 4\pi (400 \text{ cm}^2)$

$= 1600\pi \text{ cm}^2$.

The change in volume as the radius changes from 20·0 cm to 20·1 cm is

$\Delta V = \frac{dV}{dR}\Delta R$

$= (1600\pi \text{ cm}^2)(0{\cdot}1 \text{ cm})$

$= 160\pi \text{ cm}^3$.

**12.** *Find the derivative of the following functions with respect to x. (a)* $y = x^2 \sin x$, *(b)* $y = \frac{\sin x}{x}$ *and (c)* $y = \sin(x^2)$.

***Solution :***

(a) $y = x^2 \sin x$

$\frac{dy}{dx} = x^2 \frac{d}{dx}(\sin x) + (\sin x)\frac{d}{dx}(x^2)$

$= x^2 \cos x + (\sin x)(2x)$

$= x(2\sin x + x\cos x)$.

(b) $y = \frac{\sin x}{x}$

$\frac{dy}{dx} = \frac{x\frac{d}{dx}(\sin x) - \sin x\left(\frac{dx}{dx}\right)}{x^2}$

$= \frac{x\cos x - \sin x}{x^2}$.

(c) $\frac{dy}{dx} = \frac{d}{dx^2}(\sin x^2) \cdot \frac{d(x^2)}{dx}$

$= \cos x^2 (2x)$

$= 2x \cos x^2$.

**13.** *Find the maximum or minimum values of the function* $y = x + \frac{1}{x}$ *for* $x > 0$.

***Solution :*** $y = x + \frac{1}{x}$

$\frac{dy}{dx} = \frac{d}{dx}(x) + \frac{d}{dx}(x^{-1})$

$= 1 + (-x^{-2})$

$= 1 - \frac{1}{x^2}$.

For $y$ to be maximum or minimum,

$\frac{dy}{dx} = 0$

or, $1 - \frac{1}{x^2} = 0$

Thus, $x = 1$ or $-1$.

For $x > 0$ the only possible maximum or minimum is at $x = 1$. At $x = 1$, $y = x + \frac{1}{x} = 2$.

Near $x = 0$, $y = x + \frac{1}{x}$ is very large because of the term $\frac{1}{x}$. For very large $x$, again $y$ is very large because of the term $x$. Thus $x$ = 1 must correspond to a minimum. Thus, $y$ has only a minimum for $x > 0$. This minimum occurs at $x$ = 1 and the minimum value of $y$ is $y$ = 2.

**14.** *Figure (2-W6) shows the curve* $y = x^2$. *Find the area of the shaded part between* $x$ = 0 *and* $x$ = 6.

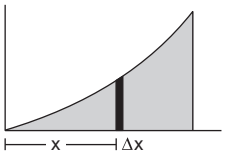



Figure 2-W6

***Solution :*** The area can be divided into strips by drawing ordinates between $x = 0$ and $x = 6$ at a regular interval of $dx$. Consider the strip between the ordinates at $x$ and $x + dx$. The height of this strip is $y = x^2$. The area of this strip is $dA = y\,dx = x^2 dx$.

The total area of the shaded part is obtained by summing up these strip-areas with $x$ varying from 0 to 6. Thus

$$A = \int_0^6 x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^6 = \frac{216 - 0}{3} = 72.$$

**15.** *Evaluate* $\int_0^t A \sin \omega t\,dt$ *where* $A$ *and* $\omega$ *are constants.*

***Solution :***

$$\int_0^t A \sin \omega t\,dt$$

$$= A\left[\frac{-\cos \omega t}{\omega}\right]_0^t = \frac{A}{\omega}(1 - \cos \omega t).$$

**16.** *The velocity $v$ and displacement $x$ of a particle executing simple harmonic motion are related as*

$$v\frac{dv}{dx} = -\omega^2 x.$$

*At* $x = 0$, $v = v_0$. *Find the velocity $v$ when the displacement becomes $x$.*

***Solution :*** We have

$$v\frac{dv}{dx} = -\omega^2 x$$

or, $$v\,dv = -\omega^2 x\,dx$$

or, $$\int_{v_0}^{v} v\,dv = \int_0^x -\omega^2 x\,dx \qquad \ldots \text{(i)}$$

When summation is made on $-\omega^2 x\,dx$ the quantity to be varied is $x$. When summation is made on $v\,dv$ the quantity to be varied is $v$. As $x$ varies from 0 to $x$ the velocity varies from $v_0$ to $v$. Therefore, on the left the limits of integration are from $v_0$ to $v$ and on the right they are from 0 to $x$. Simiplifying (i),

$$\left[\frac{1}{2}v^2\right]_{v_0}^{v} = -\omega^2\left[\frac{x^2}{2}\right]_0^x$$

or, $$\frac{1}{2}(v^2 - v_0^2) = -\omega^2\frac{x^2}{2}$$

or, $$v^2 = v_0^2 - \omega^2 x^2$$

or, $$v = \sqrt{v_0^2 - \omega^2 x^2}.$$

**17.** *The charge flown through a circuit in the time interval between $t$ and $t + dt$ is given by $dq = e^{-t/\tau}dt$, where $\tau$ is a constant. Find the total charge flown through the circuit between $t = 0$ to $t = \tau$.*

***Solution :*** The total charge flown is the sum of all the $dq$'s for $t$ varying from $t = 0$ to $t = \tau$. Thus, the total charge flown is

$$Q = \int_0^\tau e^{-t/\tau}\,dt$$

$$= \left[\frac{e^{-t/\tau}}{-1/\tau}\right]_0^\tau = \tau\left(1 - \frac{1}{e}\right).$$

**18.** *Evaluate* $(21{\cdot}6002 + 234 + 2732{\cdot}10) \times 13$.

***Solution :***

| | | |
|---|---|---|
| 21.6002 | | 22 |
| 234 | $\Rightarrow$ | 234 |
| 2732·10 | | 2732 |
| | | 2988 |

The three numbers are arranged with their decimal points aligned (shown on the left part above). The column just left to the decimals has 4 as the doubtful digit. Thus, all the numbers are rounded to this column. The rounded numbers are shown on the right part above. The required expression is $2988 \times 13 = 38844$. As 13 has only two significant digits the product should be rounded off after two significant digits. Thus the result is 39000.

□

## QUESTIONS FOR SHORT ANSWER

**1.** Is a vector necessarily changed if it is rotated through an angle ?

**2.** Is it possible to add two vectors of unequal magnitudes and get zero ? Is it possible to add three vectors of equal magnitudes and get zero ?

**3.** Does the phrase "direction of zero vector" have physical significance ? Discuss in terms of velocity, force etc.

**4.** Can you add three unit vectors to get a unit vector ? Does your answer change if two unit vectors are along the coordinate axes ?

**5.** Can we have physical quantities having magnitude and direction which are not vectors ?

**6.** Which of the following two statements is more appropriate ?

(a) Two forces are added using triangle rule because force is a vector quantity.
(b) Force is a vector quantity because two forces are added using triangle rule.

**7.** Can you add two vectors representing physical quantities having different dimensions ? Can you multiply two vectors representing physical quantities having different dimensions ?

**8.** Can a vector have zero component along a line and still have nonzero magnitude ?

**9.** Let $\varepsilon_1$ and $\varepsilon_2$ be the angles made by $\vec{A}$ and $-\vec{A}$ with the positive $X$-axis. Show that $\tan\varepsilon_1 = \tan\varepsilon_2$. Thus, giving $\tan\varepsilon$ does not uniquely determine the direction of $\vec{A}$.

**10.** Is the vector sum of the unit vectors $\vec{i}$ and $\vec{j}$ a unit vector ? If no, can you multiply this sum by a scalar number to get a unit vector ?

**11.** Let $\vec{A} = 3\vec{i} + 4\vec{j}$. Write four vector $\vec{B}$ such that $\vec{A} \neq \vec{B}$ but $A = B$.

**12.** Can you have $\vec{A} \times \vec{B} = \vec{A} \cdot \vec{B}$ with $A \neq 0$ and $B \neq 0$ ? What if one of the two vectors is zero ?

**13.** If $\vec{A} \times \vec{B} = 0$, can you say that (a) $\vec{A} = \vec{B}$, (b) $\vec{A} \neq \vec{B}$ ?

**14.** Let $\vec{A} = 5\vec{i} - 4\vec{j}$ and $\vec{B} = -7{\cdot}5\vec{i} + 6\vec{j}$. Do we have $\vec{B} = k\vec{A}$ ? Can we say $\frac{\vec{B}}{\vec{A}} = k$ ?

## OBJECTIVE I

**1.** A vector is not changed if
(a) it is rotated through an arbitrary angle
(b) it is multiplied by an arbitrary scalar
(c) it is cross multiplied by a unit vector
(d) it is slid parallel to itself.

**2.** Which of the sets given below may represent the magnitudes of three vectors adding to zero ?
(a) 2, 4, 8 (b) 4, 8, 16 (c) 1, 2, 1 (d) 0·5, 1, 2.

**3.** The resultant of $\vec{A}$ and $\vec{B}$ makes an angle $\alpha$ with $\vec{A}$ and $\beta$ with $\vec{B}$,
(a) $\alpha < \beta$ (b) $\alpha < \beta$ if $A < B$
(c) $\alpha < \beta$ if $A > B$ (d) $\alpha < \beta$ if $A = B$.

**4.** The component of a vector is
(a) always less than its magnitude
(b) always greater than its magnitude
(c) always equal to its magnitude
(d) none of these.

**5.** A vector $\vec{A}$ points vertically upward and $\vec{B}$ points towards north. The vector product $\vec{A} \times \vec{B}$ is
(a) along west (b) along east
(c) zero (d) vertically downward.

**6.** The radius of a circle is stated as 2·12 cm. Its area should be written as
(a) 14 $cm^2$ (b) 14·1 $cm^2$ (c) 14·11 $cm^2$ (d) 14·1124 $cm^2$.

## OBJECTIVE II

**1.** A situation may be described by using different sets of coordinate axes having different orientations. Which of the following do not depend on the orientation of the axes ?
(a) the value of a scalar (b) component of a vector
(c) a vector (d) the magnitude of a vector.

**2.** Let $\vec{C} = \vec{A} + \vec{B}$.
(a) $|\vec{C}|$ is always greater than $|\vec{A}|$
(b) It is possible to have $|\vec{C}| < |\vec{A}|$ and $|\vec{C}| < |\vec{B}|$
(c) $C$ is always equal to $A + B$
(d) $C$ is never equal to $A + B$.

**3.** Let the angle between two nonzero vectors $\vec{A}$ and $\vec{B}$ be 120° and its resultant be $\vec{C}$.
(a) $C$ must be equal to $|A - B|$
(b) $C$ must be less than $|A - B|$
(c) $C$ must be greater than $|A - B|$
(d) $C$ may be equal to $|A - B|$.

**4.** The $x$-component of the resultant of several vectors
(a) is equal to the sum of the $x$-components of the vectors
(b) may be smaller than the sum of the magnitudes of the vectors
(c) may be greater than the sum of the magnitudes of the vectors
(d) may be equal to the sum of the magnitudes of the vectors.

**5.** The magnitude of the vector product of two vectors $|\vec{A}|$ and $|\vec{B}|$ may be
(a) greater than $AB$ (b) equal to $AB$
(c) less than $AB$ (d) equal to zero.

## EXERCISES

**1.** A vector $\vec{A}$ makes an angle of 20° and $\vec{B}$ makes an angle of 110° with the *X*-axis. The magnitudes of these vectors are 3 m and 4 m respectively. Find the resultant.

**2.** Let $\vec{A}$ and $\vec{B}$ be the two vectors of magnitude 10 unit each. If they are inclined to the *X*-axis at angles 30° and 60° respectively, find the resultant.

**3.** Add vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ each having magnitude of 100 unit and inclined to the *X*-axis at angles 45°, 135° and 315° respectively.

**4.** Let $\vec{a} = 4\,\vec{i} + 3\,\vec{j}$ and $\vec{b} = 3\,\vec{i} + 4\,\vec{j}$. (a) Find the magnitudes of (a) $\vec{a}$, (b) $\vec{b}$, (c) $\vec{a} + \vec{b}$ and (d) $\vec{a} - \vec{b}$.

**5.** Refer to figure (2-E1). Find (a) the magnitude, (b) $x$ and $y$ components and (c) the angle with the *X*-axis of the resultant of $\vec{OA}$, $\vec{BC}$ and $\vec{DE}$.

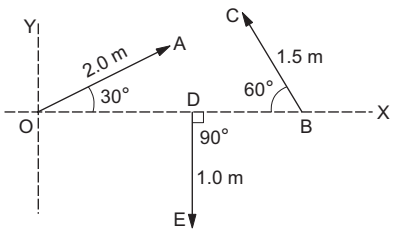



Figure 2-E1

**6.** Two vectors have magnitudes 3 unit and 4 unit respectively. What should be the angle between them if the magnitude of the resultant is (a) 1 unit, (b) 5 unit and (c) 7 unit.

**7.** A spy report about a suspected car reads as follows. "The car moved 2·00 km towards east, made a perpendicular left turn, ran for 500 m, made a perpendicular right turn, ran for 4·00 km and stopped". Find the displacement of the car.

**8.** A carrom board (4 ft × 4 ft square) has the queen at the centre. The queen, hit by the striker moves to the front edge, rebounds and goes in the hole behind the striking line. Find the magnitude of displacement of the queen (a) from the centre to the front edge, (b) from the front edge to the hole and (c) from the centre to the hole.

**9.** A mosquito net over a 7 ft × 4 ft bed is 3 ft high. The net has a hole at one corner of the bed through which a mosquito enters the net. It flies and sits at the diagonally opposite upper corner of the net. (a) Find the magnitude of the displacement of the mosquito. (b) Taking the hole as the origin, the length of the bed as the *X*-axis, its width as the *Y*-axis, and vertically up as the *Z*-axis, write the components of the displacement vector.

**10.** Suppose $\vec{a}$ is a vector of magnitude 4·5 unit due north. What is the vector (a) $3\,\vec{a}$, (b) $-4\,\vec{a}$?

**11.** Two vectors have magnitudes 2 m and 3 m. The angle between them is 60°. Find (a) the scalar product of the two vectors, (b) the magnitude of their vector product.

**12.** Let $A_1A_2A_3A_4A_5A_6A_1$ be a regular hexagon. Write the $x$-components of the vectors represented by the six sides taken in order. Use the fact that the resultant of these six vectors is zero, to prove that
$\cos 0 + \cos\pi/3 + \cos 2\pi/3 + \cos 3\pi/3 + \cos 4\pi/3 + \cos 5\pi/3 = 0$.
Use the known cosine values to verify the result.

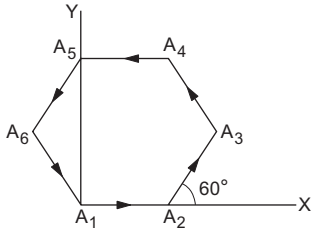



Figure 2-E2

**13.** Let $\vec{a} = 2\,\vec{i} + 3\,\vec{j} + 4\,\vec{k}$ and $\vec{b} = 3\,\vec{i} + 4\,\vec{j} + 5\,\vec{k}$. Find the angle between them.

**14.** Prove that $\vec{A} \cdot (\vec{A} \times \vec{B}) = 0$.

**15.** If $\vec{A} = 2\,\vec{i} + 3\,\vec{j} + 4\,\vec{k}$ and $\vec{B} = 4\,\vec{i} + 3\,\vec{j} + 2\,\vec{k}$, find $\vec{A} \times \vec{B}$.

**16.** If $\vec{A}$, $\vec{B}$, $\vec{C}$ are mutually perpendicular, show that $\vec{C} \times (\vec{A} \times \vec{B}) = 0$. Is the converse true ?

**17.** A particle moves on a given straight line with a constant speed $v$. At a certain time it is at a point $P$ on its straight line path. $O$ is a fixed point. Show that $\vec{OP} \times \vec{v}$ is independent of the position $P$.

**18.** The force on a charged particle due to electric and magnetic fields is given by $\vec{F} = q\,\vec{E} + q\,\vec{v} \times \vec{B}$. Suppose $\vec{E}$ is along the *X*-axis and $\vec{B}$ along the *Y*-axis. In what direction and with what minimum speed $v$ should a positively charged particle be sent so that the net force on it is zero ?

**19.** Give an example for which $\vec{A} \cdot \vec{B} = \vec{C} \cdot \vec{B}$ but $\vec{A} \neq \vec{C}$.

**20.** Draw a graph from the following data. Draw tangents at $x$ = 2, 4, 6 and 8. Find the slopes of these tangents. Verify that the curve drawn is $y = 2x^2$ and the slope of tangent is $\tan\theta = \frac{dy}{dx} = 4x$.

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | 2 | 8 | 18 | 32 | 50 | 72 | 98 | 128 | 162 | 200 |

**21.** A curve is represented by $y = \sin x$. If $x$ is changed from $\frac{\pi}{3}$ to $\frac{\pi}{3} + \frac{\pi}{100}$, find approximately the change in $y$.

**22.** The electric current in a charging *R*–*C* circuit is given by $i = i_0 e^{-t/RC}$ where $i_0$, $R$ and $C$ are constant parameters of the circuit and $t$ is time. Find the rate of change of current at (a) $t = 0$, (b) $t = RC$, (c) $t = 10\,RC$.

**23.** The electric current in a discharging R–C circuit is given by $i = i_0 e^{-t/RC}$ where $i_0$, $R$ and $C$ are constant parameters and $t$ is time. Let $i_0 = 2{\cdot}00$ A, $R = 6{\cdot}00 \times 10^5\ \Omega$

and $C = 0{\cdot}500\ \mu F$. (a) Find the current at $t = 0{\cdot}3$ s. (b) Find the rate of change of current at $t = 0{\cdot}3$ s. (c) Find approximately the current at $t = 0{\cdot}31$ s.

**24.** Find the area bounded under the curve $y = 3x^2 + 6x + 7$ and the $X$-axis with the ordinates at $x = 5$ and $x = 10$.

**25.** Find the area enclosed by the curve $y = \sin x$ and the $X$-axis between $x = 0$ and $x = \pi$.

**26.** Find the area bounded by the curve $y = e^{-x}$, the $X$-axis and the $Y$-axis.

**27.** A rod of length $L$ is placed along the $X$-axis between $x = 0$ and $x = L$. The linear density (mass/length) $\rho$ of the rod varies with the distance $x$ from the origin as $\rho = a + bx$. (a) Find the SI units of $a$ and $b$. (b) Find the mass of the rod in terms of $a$, $b$ and $L$.

**28.** The momentum $p$ of a particle changes with time $t$ according to the relation $\frac{dp}{dt} = (10\ \text{N}) + (2\ \text{N/s})t$. If the momentum is zero at $t = 0$, what will the momentum be at $t = 10$ s ?

**29.** The changes in a function $y$ and the independent variable $x$ are related as $\frac{dy}{dx} = x^2$. Find $y$ as a function of $x$.

**30.** Write the number of significant digits in (a) 1001, (b) 100·1, (c) 100·10, (d) 0·001001.

**31.** A metre scale is graduated at every millimetre. How many significant digits will be there in a length measurement with this scale ?

**32.** Round the following numbers to 2 significant digits. (a) 3472, (b) 84·16, (c) 2·55 and (d) 28·5.

**33.** The length and the radius of a cylinder measured with a slide callipers are found to be 4·54 cm and 1·75 cm respectively. Calculate the volume of the cylinder.

**34.** The thickness of a glass plate is measured to be 2·17 mm, 2·17 mm and 2·18 mm at three different places. Find the average thickness of the plate from this data.

**35.** The length of the string of a simple pendulum is measured with a metre scale to be 90·0 cm. The radius of the bob plus the length of the hook is calculated to be 2·13 cm using measurements with a slide callipers. What is the effective length of the pendulum ? (The effective length is defined as the distance between the point of suspension and the centre of the bob.)

□

## ANSWERS

### OBJECTIVE I

1. (d) 2. (c) 3. (c) 4. (d) 5. (a) 6. (b)

### OBJECTIVE II

1. (a), (c), (d) 2. (b) 3. (c) 4. (a), (b), (d)
5. (b), (c), (d)

### EXERCISES

1. 5 m at 73° with $X$-axis
2. $20 \cos 15°$ unit at 45° with $X$-axis
3. 100 unit at 45° with $X$-axis
4. (a) 5 (b) 5 (c) $7\sqrt{2}$ (d) $\sqrt{2}$
5. (a) 1·6 m (b) 0·98 m and 1·3 m respectively
   (c) $\tan^{-1}(1{\cdot}32)$
6. (a) 180° (b) 90° (c) 0
7. 6·02 km, $\tan^{-1}\frac{1}{12}$.
8. (a) $\frac{2}{3}\sqrt{10}$ ft (b) $\frac{4}{3}\sqrt{10}$ ft (c) $2\sqrt{2}$ ft
9. (a) $\sqrt{74}$ ft (b) 7 ft, 4 ft, 3 ft
10. (a) 13·5 unit due north (b) 18 unit due south
11. (a) $3\ \text{m}^2$ (b) $3\sqrt{3}\ \text{m}^2$
13. $\cos^{-1}(38/\sqrt{1450})$
15. $-6\vec{i} + 12\vec{j} - 6\vec{k}$
16. no
18. along $Z$-axis with speed $E/B$
21. 0·0157
22. (a) $\frac{-i_0}{RC}$ (b) $\frac{-i_0}{RCe}$ (c) $\frac{-i_0}{RCe^{10}}$
23. (a) $\frac{2{\cdot}00}{e}$ A (b) $\frac{-20}{3e}$ A/s (c) $\frac{5{\cdot}8}{3e}$ A
24. 1135
25. 2
26. 1
27. (a) kg/m, $\text{kg/m}^2$ (b) $aL + bL^2/2$
28. 200 kg–m/s
29. $y = \frac{x^3}{3} + C$
30. (a) 4 (b) 4 (c) 5 (d) 4
31. 1, 2, 3 or 4
32. (a) 3500 (b) 84 (c) 2·6 (d) 28
33. $43{\cdot}7\ \text{cm}^3$
34. 2·17 mm
35. 92·1 cm

□

## CHAPTER 3

# REST AND MOTION : KINEMATICS

### 3.1 REST AND MOTION

When do we say that a body is at rest and when do we say that it is in motion ? You may say that if a body does not change its position as time passes it is at rest. If a body changes its position with time, it is said to be moving. But when do we say that it is not changing its position ? A book placed on the table remains on the table and we say that the book is at rest. However, if we station ourselves on the moon (the Appollo missions have made it possible), the whole earth is found to be changing its position and so the room, the table and the book are all continuously changing their positions. The book is at rest if it is viewed from the room, it is moving if it is viewed from the moon.

Motion is a combined property of the object under study and the observer. There is no meaning of rest or motion without the viewer. Nothing is in absolute rest or in absolute motion. The moon is moving with respect to the book and the book moves with respect to the moon. Take another example. A robber enters a train moving at great speed with respect to the ground, brings out his pistol and says "Don't move, stand still". The passengers stand still. The passengers are at rest with respect to the robber but are moving with respect to the rail track.

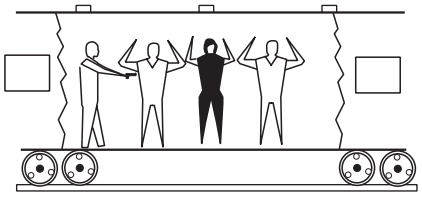


Figure 3.1

To locate the position of a particle we need a *frame of reference.* A convenient way to fix up the frame of reference is to choose three mutually perpendicular axes and name them *X-Y-Z* axes. The coordinates, (*x, y, z*) of the particle then specify the position of the particle with respect to that frame. Add a clock into the frame of reference to measure the time. If all the three coordinates *x, y* and *z* of the particle remain unchanged as time passes, we say that the particle is at rest with respect to this frame. If any one or more coordinates change with time, we say that the body is moving with respect to this frame.

There is no rule or restriction on the choice of a frame. We can choose a frame of reference according to our convenience to describe the situation under study. Thus, when we are in a train it is convenient to choose a frame attached to our compartment. The coordinates of a suitcase placed on the upper berth do not change with time (unless the train gives a jerk) and we say that the suitcase is at rest in the train-frame. The different stations, electric poles, trees etc. change their coordinates and we say that they are moving in the train-frame. Thus, we say that "Bombay is coming" and "Pune has already passed".

In the following sections we shall assume that the frame of reference is already chosen and we are describing the motion of the objects in the chosen frame. Sometimes the choice of the frame is clear from the context and we do not mention it. Thus, when one says the car is travelling and the rickshaw is not, it is clear that all positions are measured from a frame attached to the road.

### 3.2 DISTANCE AND DISPLACEMENT

Suppose a particle is at $A$ at time $t_1$ and at $B$ at time $t_2$ with respect to a given frame (figure 3.2).

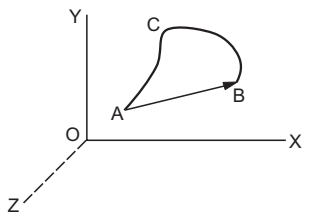



Figure 3.2

During the time interval $t_1$ to $t_2$ the particle moves along the path *ACB*. The length of the path *ACB* is called the *distance* travelled during the time interval $t_1$ to $t_2$. If we connect the initial position *A* with the final position *B* by a straight line, we get the *displacement* of the particle. The magnitude of the displacement is the length of the straight line joining the initial and the final position. The direction is from the initial to the final position. The displacement has both the magnitude as well as the direction. Further the displacements add according to the triangle rule of vector addition. Suppose a particle kept on a table is displaced on the table and at the same time the table is also displaced in the room. The net displacement of the particle in the room is obtained by the vector sum of the two displacements. Thus, displacement is a vector quantity. In contrast the distance covered has only a magnitude and is thus, a scalar quantity.

***Example 3.1***

*An old person moves on a semi-circular track of radius* 40·0 m *during a morning walk. If he starts at one end of the track and reaches at the other end, find the distance covered and the displacement of the person.*

***Solution*** : The distance covered by the person equals the length of the track. It is equal to $\pi R = \pi \times 40{\cdot}0$ m = 126 m.

The displacement is equal to the diameter of the semi-circular track joining the two ends. It is $2R = 2 \times 40{\cdot}0$ m = 80 m. The direction of this displacement is from the initial point to the final point.

## 3.3 AVERAGE SPEED AND INSTANTANEOUS SPEED

The average speed of a particle in a time interval is defined as the distance travelled by the particle divided by the time interval. If the particle travels a distance $s$ in time $t_1$ to $t_2$, the *average speed* is defined as

$$v_{av} = \frac{s}{t_2 - t_1}. \qquad \ldots (3.1)$$

The average speed gives the overall "rapidity" with which the particle moves in this interval. In a one-day cricket match, the average run rate is quoted as the total runs divided by the total number of overs used to make these runs. Some of the overs may be expensive and some may be economical. Similarly, the average speed gives the total effect in the given interval. The rapidity or slowness may vary from instant to instant. When an athelete starts running, he or she runs slowly and gradually

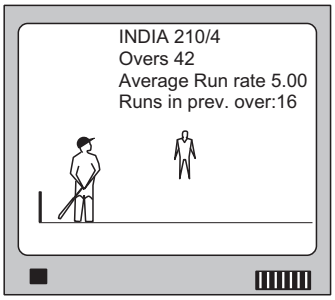



Figure 3.3

increases the rate. We define the *instantaneous speed* at a time $t$ as follows.

Let $\Delta s$ be the distance travelled in the time interval $t$ to $t + \Delta t$. The average speed in this time interval is

$$v_{av} = \frac{\Delta s}{\Delta t}.$$

Now make $\Delta t$ vanishingly small and look for the value of $\frac{\Delta s}{\Delta t}$. Remember $\Delta s$ is the distance travelled in the chosen time interval $\Delta t$. As $\Delta t$ approaches 0, the distance $\Delta s$ also approaches zero but the ratio $\frac{\Delta s}{\Delta t}$ has a finite limit.

The instantaneous speed at a time $t$ is defined as

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt} \qquad \ldots (3.2)$$

where $s$ is the distance travelled in time $t$. The average speed is defined for a time interval and the instantaneous speed is defined at a particular instant. Instantaneous speed is also called "speed".

***Example 3.2***

*The distance travelled by a particle in time t is given by* $s = (2{\cdot}5 \text{ m/s}^2)\, t^2$. *Find (a) the average speed of the particle during the time* 0 *to* 5·0 s, *and (b) the instantaneous speed at* $t$ = 5·0 s.

***Solution*** **:** (a) The distance travelled during time 0 to 5·0 s is

$$s = (2{\cdot}5 \text{ m/s}^2)\,(5{\cdot}0 \text{ s})^2 = 62{\cdot}5 \text{ m}.$$

The average speed during this time is

$$v_{av} = \frac{62{\cdot}5 \text{ m}}{5 \text{ s}} = 12{\cdot}5 \text{ m/s}.$$

(b) $s = (2{\cdot}5 \text{ m/s}^2)\, t^2$

or, $\frac{ds}{dt} = (2{\cdot}5 \text{ m/s}^2)\,(2t) = (5{\cdot}0 \text{ m/s}^2)\, t.$

At $t$ = 5·0 s the speed is

$$v = \frac{ds}{dt} = (5{\cdot}0 \text{ m/s}^2)\,(5{\cdot}0 \text{ s}) = 25 \text{ m/s}.$$

If we plot the distance $s$ as a function of time (figure 3.4), the speed at a time $t$ equals the slope of

the tangent to the curve at the time $t$. The average speed in a time interval $t$ to $t + \Delta t$ equals the slope of the chord $AB$ where $A$ and $B$ are the points on the

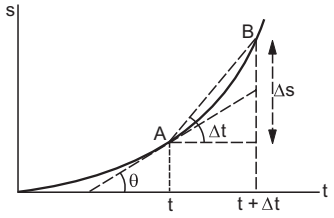



Figure 3.4

curve corresponding to the time $t$ and $t + \Delta t$. As $\Delta t$ approaches zero, the chord $AB$ becomes the tangent at $A$ and the average speed $\frac{\Delta s}{\Delta t}$ becomes the slope of the tangent which is $\frac{ds}{dt}$.

If the speed of the particle at time $t$ is $v$, the distance $ds$ travelled by it in the short time interval $t$ to $t + dt$ is $v\ dt$. Thus, $ds = vdt$. The total distance travelled by the particle in a finite time interval $t_1$ to $t_2$ can be obtained by summing over these small distances $ds$ as time changes from $t_1$ to $t_2$. Thus, the distance travelled by a particle in the time interval

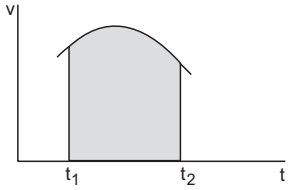



Figure 3.5

$t_1$ to $t_2$ is

$$s = \int_{t_1}^{t_2} v\ dt. \qquad \ldots\ (3.3)$$

If we plot a graph of the speed $v$ versus time $t$, the distance travelled by the particle can be obtained by finding the area under the curve. Figure (3.5) shows such a speed-time graph. To find the distance travelled in the time interval $t_1$ to $t_2$ we draw ordinates from $t = t_1$ and $t = t_2$. The area bounded by the curve $v - t$, the $X$-axis and the two ordinates at $t = t_1$ and $t = t_2$ (shown shaded in the figure) gives the total distance covered.

The dimension of speed is $LT^{-1}$ and its SI unit is metre/second abbreviated as m/s.

***Example 3.3***

*Figure (3.6) shows the speed versus time graph for a particle. Find the distance travelled by the particle during the time $t$ = 0 to $t$ = 3 s.*

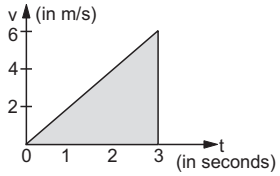



Figure 3.6

***Solution*** **:** The distance travelled by the particle in the time 0 to 3 s is equal to the area shaded in the figure. This is a right angled triangle with height = 6 m/s and the base = 3 s. The area is $\frac{1}{2}$ (base) (height) $= \frac{1}{2} \times$ (3 s) (6 m/s) = 9 m. Thus, the particle covered a distance of 9 m during the time 0 to 3 s.

## 3.4 AVERAGE VELOCITY AND INSTANTANEOUS VELOCITY

The *average velocity* of a particle in a time interval $t_1$ to $t_2$ is defined as its displacement divided by the time interval. If the particle is at a point $A$ (figure 3.7) at time $t = t_1$ and at $B$ at time $t = t_2$, the displacement in this time interval is the vector $\overrightarrow{AB}$. The average velocity in this time interval is then,

$$\vec{v}_{av} = \frac{\overrightarrow{AB}}{t_2 - t_1}.$$

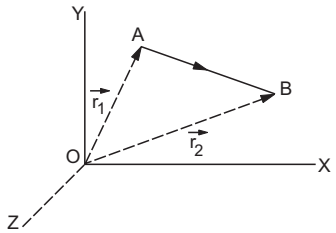



Figure 3.7

Like displacement, it is a vector quantity.

**Position vector :** If we join the origin to the position of the particle by a straight line and put an arrow towards the position of the particle, we get the *position vector* of the particle. Thus, the position vector of the particle shown in figure (3.7) at time $t = t_1$ is $\overrightarrow{OA}$ and that at $t = t_2$ is $\overrightarrow{OB}$. The displacement of the particle in the time interval $t_1$ to $t_2$ is

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{r}_2 - \vec{r}_1.$$

The average velocity of a particle in the time interval $t_1$ to $t_2$ can be written as

$$\vec{v}_{av} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1}. \quad \ldots (3.4)$$

Note that only the positions of the particle at time $t = t_1$ and $t = t_2$ are used in calculating the average velocity. The positions in between $t_1$ and $t_2$ are not needed, hence the actual path taken in going from $A$ to $B$ is not important in calculating the average velocity.

***Example 3.4***

*A table clock has its minute hand* 4·0 cm *long. Find the average velocity of the tip of the minute hand (a) between* 6·00 a.m. *to* 6·30 a.m. *and (b) between* 6·00 a.m. *to* 6·30 p.m.

***Solution*** **:** At 6·00 a.m. the tip of the minute hand is at 12 mark and at 6·30 a.m. or 6·30 p.m. it is 180° away. Thus, the straight line distance between the initial and final position of the tip is equal to the diameter of the clock.

Displacement = 2 $R$ = 2 × 4·0 cm = 8·0 cm.

The displacement is from the 12 mark to the 6 mark on the clock panel. This is also the direction of the average velocity in both cases.

(a) The time taken from 6·00 a.m. to 6·30 a.m. is 30 minutes = 1800 s. The average velocity is

$$v_{av} = \frac{\text{Displacement}}{\text{time}} = \frac{8{\cdot}0 \text{ cm}}{1800 \text{ s}} = 4{\cdot}4 \times 10^{-3} \text{ cm/s}.$$

(b) The time taken from 6·00 a.m. to 6·30 p.m. is 12 hours and 30 minutes = 45000 s. The average velocity is

$$v_{av} = \frac{\text{Displacement}}{\text{time}} = \frac{8{\cdot}0 \text{ cm}}{45000 \text{ s}} = 1{\cdot}8 \times 10^{-4} \text{ cm/s}.$$

The *instantaneous velocity* of a particle at a time $t$ is defined as follows. Let the average velocity of the particle in a short time interval $t$ to $t + \Delta t$ be $\vec{v}_{av}$. This average velocity can be written as

$$\vec{v}_{av} = \frac{\Delta \vec{r}}{\Delta t}$$

where $\Delta \vec{r}$ is the displacement in the time interval $\Delta t$. We now make $\Delta t$ vanishingly small and find the limiting value of $\frac{\Delta \vec{r}}{\Delta t}$. This value is instantaneous velocity $\vec{v}$ of the particle at time $t$.

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}. \quad \ldots (3.5)$$

For very small intervals the displacement $\Delta \vec{r}$ is along the line of motion of the particle. Thus, the length $\Delta r$ equals the distance $\Delta s$ travelled in that interval. So the magnitude of the velocity is

$$v = \left| \frac{d\vec{r}}{dt} \right| = \frac{|d\vec{r}|}{dt} = \frac{ds}{dt} \quad \ldots (3.6)$$

which is the instantaneous speed at time $t$. Instantaneous velocity is also called the "velocity".

## 3.5 AVERAGE ACCELERATION AND INSTANTANEOUS ACCELERATION

If the velocity of a particle remains constant as time passes, we say that it is moving with uniform velocity. If the velocity changes with time, it is said to be accelerated. The acceleration is the rate of change of velocity. Velocity is a vector quantity hence a change in its magnitude or direction or both will change the velocity.

Suppose the velocity of a particle at time $t_1$ is $\vec{v}_1$ and at time $t_2$ it is $\vec{v}_2$. The change produced in time interval $t_1$ to $t_2$ is $\vec{v}_2 - \vec{v}_1$. We define the *average acceleration* $\vec{a}_{av}$ as the change in velocity divided by the time interval. Thus,

$$\vec{a}_{av} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1}. \quad \ldots (3.7)$$

Again the average acceleration depends only on the velocities at time $t_1$ and $t_2$. How the velocity changed in between $t_1$ and $t_2$ is not important in defining the average acceleration.

*Instantaneous acceleration* of a particle at time $t$ is defined as

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} \quad \ldots (3.8)$$

where $\Delta \vec{v}$ is the change in velocity between the time $t$ and $t + \Delta t$. At time $t$ the velocity is $\vec{v}$ and at time $t + \Delta t$ it becomes $\vec{v} + \Delta \vec{v}$. $\frac{\Delta \vec{v}}{\Delta t}$ is the average acceleration of the particle in the interval $\Delta t$. As $\Delta t$ approaches zero, this average acceleration becomes the instantaneous acceleration. Instantaneous acceleration is also called *"acceleration"*.

The dimension of acceleration is $LT^{-2}$ and its SI unit is metre/second$^2$ abbreviated as m/s$^2$.

## 3.6 MOTION IN A STRAIGHT LINE

When a particle is constrained to move on a straight line, the description becomes fairly simple. We choose the line as the $X$-axis and a suitable time instant as $t = 0$. Generally the origin is taken at the point where the particle is situated at $t = 0$. The position of the particle at time $t$ is given by its coordinate $x$ at that time. The velocity is

$$v = \frac{dx}{dt} \quad \ldots (3.9)$$

and the acceleration is $a = \frac{dv}{dt}$ … (3.10)

$$= \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}. \quad \ldots (3.11)$$

If $\frac{dx}{dt}$ is positive, the direction of the velocity is along the positive $X$-axis and if $\frac{dx}{dt}$ is negative, the direction, is along the negative $X$-axis. Similarly if $\frac{dv}{dt}$ is positive, the acceleration is along the positive $X$-axis and if $\frac{dv}{dt}$ is negative, the acceleration is along the negative $X$-axis. The magnitude of $v$ is speed. If the velocity and the acceleration are both positive, the speed increases. If both of them are negative then also the speed increases but if they have opposite signs, the speed decreases. When the speed decreases, we say that the particle is decelerating. Deceleration is equivalent to negative acceleration. An acceleration of $2{\cdot}0$ m/s$^2$ towards east is same as a deceleration of $2{\cdot}0$ m/s$^2$ towards west.

**Motion with Constant Acceleration**

Suppose the acceleration of a particle is $a$ and remains constant. Let the velocity at time 0 be $u$ and the velocity at time $t$ be $v$. Thus,

$$\frac{dv}{dt} = a, \quad \text{or,} \quad dv = a\,dt$$

$$\text{or,} \quad \int_u^v dv = \int_0^t a\,dt.$$

As time changes from 0 to $t$ the velocity changes from $u$ to $v$. So on the left hand side the summation is made over $v$ from $u$ to $v$ whereas on the right hand side the summation is made on time from 0 to $t$. Evaluating the integrals we get,

$$[v]_u^v = a[t]_0^t$$

$$\text{or,} \quad v - u = at$$

$$\text{or,} \quad v = u + at. \quad \ldots (3.12)$$

Equation (3.12) may be written as

$$\frac{dx}{dt} = u + at$$

$$\text{or,} \quad dx = (u + at)dt$$

$$\text{or,} \quad \int_0^x dx = \int_0^t (u + at)dt.$$

At $t = 0$ the particle is at $x = 0$. As time changes from 0 to $t$ the position changes from 0 to $x$. So on the left hand side the summation is made on position from 0 to $x$ whereas on the right hand side the summation is made on time from 0 to $t$. Evaluating the integrals, the above equation becomes

$$[x]_0^x = \int_0^t u\,dt + \int_0^t at\,dt$$

$$\text{or,} \quad x = u\int_0^t dt + a\int_0^t t\,dt$$

$$= u[t]_0^t + a\left[\frac{t^2}{2}\right]_0^t$$

$$\text{or,} \quad x = ut + \frac{1}{2}at^2. \quad \ldots (3.13)$$

From equation (3.12),

$$v^2 = (u + at)^2$$

$$\text{or,} \quad = u^2 + 2uat + a^2t^2$$

$$\text{or,} \quad = u^2 + 2a\left(ut + \frac{1}{2}at^2\right)$$

$$\text{or,} \quad = u^2 + 2ax. \quad \ldots (3.14)$$

The three equations (3.12) to (3.14) are collected below in table 3.1. They are very useful in solving the problems of motion in a straight line with constant acceleration.

**Table 3.1**

$$v = u + at$$
$$x = ut + \frac{1}{2}at^2$$
$$v^2 = u^2 + 2ax$$

Remember that $x$ represents the position of the particle at time $t$ and not (in general) the distance travelled by it in time 0 to $t$. For example, if the particle starts from the origin and goes upto $x = 4$ m, then turns and is at $x = 2$ m at time $t$, the distance travelled is 6 m but the position is still given by $x = 2$ m.

The quantities $u$, $v$ and $a$ may take positive or negative values depending on whether they are directed along the positive or negative direction. Similarly $x$ may be positive or negative.

***Example 3.5***

*A particle starts with an initial velocity* $2{\cdot}5$ m/s *along the positive x direction and it accelerates uniformly at the rate* $0{\cdot}50$ m/s$^2$. *(a) Find the distance travelled by it in the first two seconds. (b) How much time does it take to reach the velocity* $7{\cdot}5$ m/s ? *(c) How much distance will it cover in reaching the velocity* $7{\cdot}5$ m/s ?

***Solution*** : (a) We have,

$$x = ut + \frac{1}{2}at^2$$

$$= (2{\cdot}5 \text{ m/s})(2 \text{ s}) + \frac{1}{2}(0{\cdot}50 \text{ m/s}^2)(2 \text{ s})^2$$

$$= 5{\cdot}0 \text{ m} + 1{\cdot}0 \text{ m} = 6{\cdot}0 \text{ m}.$$

Since the particle does not turn back it is also the distance travelled.

(b) We have,

$$v = u + at$$

or, $7{\cdot}5 \text{ m/s} = 2{\cdot}5 \text{ m/s} + (0{\cdot}50 \text{ m/s}^2)\, t$

or, $$t = \frac{7{\cdot}5 \text{ m/s} - 2{\cdot}5 \text{ m/s}}{0{\cdot}50 \text{ m/s}^2} = 10 \text{ s}$$

(c) We have,

$$v^2 = u^2 + 2ax$$

or, $(7{\cdot}5 \text{ m/s})^2 = (2{\cdot}5 \text{ m/s})^2 + 2(0{\cdot}50 \text{ m/s}^2)x$

or, $$x = \frac{(7{\cdot}5 \text{ m/s})^2 - (2{\cdot}5 \text{ m/s})^2}{2 \times 0{\cdot}50 \text{ m/s}^2} = 50 \text{ m}.$$

***Example 3.6***

*A particle having initial velocity u moves with a constant acceleration a for a time t. (a) Find the displacement of the particle in the last* 1 second. *(b) Evaluate it for* $u = 5$ m/s, $a = 2$ m/s$^2$ *and* $t = 10$ s.

***Solution*** : (a) The position at time $t$ is

$$s = ut + \frac{1}{2}at^2$$

The position at time $(t - 1 \text{ s})$ is

$$s' = u(t - 1 \text{ s}) + \frac{1}{2}a(t - 1 \text{ s})^2$$

$$= ut - u(1 \text{ s}) + \frac{1}{2}at^2 - at(1 \text{ s}) + \frac{1}{2}a(1 \text{ s})^2$$

Thus, the displacement in the last 1 s is

$$s_t = s - s'$$

$$= u(1 \text{ s}) + at(1 \text{ s}) - \frac{1}{2}a(1 \text{ s})^2$$

or, $$s_t = u(1 \text{ s}) + \frac{a}{2}(2t - 1 \text{ s})(1 \text{ s}). \quad \ldots \text{ (i)}$$

(b) Putting the given values in (i)

$$s_t = \left(5 \frac{\text{m}}{\text{s}}\right)(1 \text{ s}) + \frac{1}{2}\left(2 \frac{\text{m}}{\text{s}^2}\right)(2 \times 10 \text{ s} - 1 \text{ s})(1 \text{ s})$$

$$= 5 \text{ m} + \left(1 \frac{\text{m}}{\text{s}^2}\right)(19 \text{ s})(1 \text{ s})$$

$$= 5 \text{ m} + 19 \text{ m} = 24 \text{ m}.$$

Sometimes, we are not careful in writing the units appearing with the numerical values of physical quantities. If we forget to write the unit of second in equation (i), we get,

$$s_t = u + \frac{a}{2}(2t - 1).$$

This equation is often used to calculate the displacement in the "$t$th second". However, as you can verify, different terms in this equation have different dimensions and hence the above equation is dimensionally incorrect. Equation (i) is the correct form which was used to solve part (b).

Also note that this equation gives the displacement of the particle in the last 1 second and not necessarily the distance covered in that second.

**Freely Falling Bodies**

A common example of motion in a straight line with constant acceleration is free fall of a body near the earth's surface. If air resistance is neglected and a body is dropped near the surface of the earth, it falls along a vertical straight line. The acceleration is in the vertically downward direction and its magnitude is almost constant if the height is small as compared with the radius of the earth (6400 km). This magnitude is approximately equal to $9{\cdot}8$ m/s$^2$ or $32$ ft/s$^2$ and is denoted by the letter $g$.

If we take vertically upward as the positive $Y$-axis, acceleration is along the negative $Y$-axis and we write $a = -g$. The equation (3.12) to (3.14) may be written in this case as

$$v = u - gt$$

$$y = ut - \frac{1}{2}gt^2$$

$$v^2 = u^2 - 2gy.$$

Here $y$ is the $y$-coordinate (that is the height above the origin) at time $t$, $u$ is the velocity in $y$ direction at $t = 0$ and $v$ is the velocity in $y$ direction at time $t$. The position of the particle at $t = 0$ is $y = 0$.

Sometimes it is convenient to choose vertically downward as the positive $Y$-axis. Then $a = g$ and the equations (3.12) to (3.14) become

$$v = u + gt$$

$$y = ut + \frac{1}{2}gt^2$$

$$v^2 = u^2 + 2gy.$$

***Example 3.7***

*A ball is thrown up at a speed of* $4{\cdot}0$ m/s. *Find the maximum height reached by the ball. Take* $g = 10$ m/s$^2$.

***Solution*** : Let us take vertically upward direction as the positive $Y$-axis. We have $u = 4{\cdot}0$ m/s and $a = -10$ m/s$^2$. At the highest point the velocity becomes zero. Using the formula.

$$v^2 = u^2 + 2ay,$$

$$0 = (4{\cdot}0 \text{ m/s})^2 + 2(-10 \text{ m/s}^2)y$$

or, $$y = \frac{16 \text{ m}^2/\text{s}^2}{20 \text{ m/s}^2} = 0{\cdot}80 \text{ m}.$$

### 3.7 MOTION IN A PLANE

If a particle is free to move in a plane, its position can be located with two coordinates. We choose the plane of motion as the *X-Y* plane. We choose a suitable instant as $t = 0$ and choose the origin at the place where the particle is situated at $t = 0$. Any two convenient mutually perpendicular directions in the *X-Y* plane are chosen as the *X* and *Y*-axes.

The position of the particle at a time $t$ is completely specified by its coordinates $(x, y)$. The coordinates at time $t + \Delta t$ are $(x + \Delta x, y + \Delta y)$. Figure (3.8) shows the positions at $t$ and $t + \Delta t$ as $A$ and $B$ respectively. The displacement during the time interval $t$ to $t + \Delta t$ is

$$\Delta\vec{r} = \vec{AB} = \vec{AC} + \vec{CB}$$

$$= \Delta x\,\vec{i} + \Delta y\,\vec{j}$$

or, $$\frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta x}{\Delta t}\vec{i} + \frac{\Delta y}{\Delta t}\vec{j}.$$

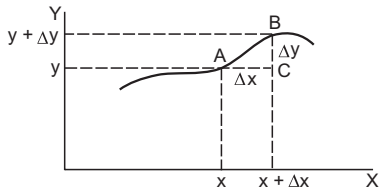



Figure 3.8

Taking limits as $\Delta t \to 0$

$$\vec{v} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j}. \quad \ldots (3.15)$$

Thus, we see that the $x$-component of the velocity is

$$v_x = \frac{dx}{dt} \quad \ldots (3.16)$$

and the $y$-component is

$$v_y = \frac{dy}{dt}. \quad \ldots (3.17)$$

Differentiating (3.15) with respect to time,

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\vec{i} + \frac{dv_y}{dt}\vec{j}$$

Thus, the acceleration has components

$$a_x = \frac{dv_x}{dt} \quad \ldots (3.18)$$

and $$a_y = \frac{dv_y}{dt}. \quad \ldots (3.19)$$

We see that the $x$-coordinate, the $x$-component of velocity $v_x$ and the $x$-component of acceleration $a_x$ are related by

$$v_x = \frac{dx}{dt} \quad \text{and} \quad a_x = \frac{dv_x}{dt}.$$

These equations are identical to equations (3.9) and (3.10). Thus, if $a_x$ is constant, integrating these equations we get

$$\left.\begin{aligned} v_x &= u_x + a_x t \\ x &= u_x t + \frac{1}{2} a_x t^2 \\ v_x^2 &= u_x^2 + 2a_x x \end{aligned}\right| \quad \ldots (3.20)$$

where $u_x$ is the $x$-component of the velocity at $t = 0$. Similarly we have

$$v_y = \frac{dy}{dt} \quad \text{and} \quad a_y = \frac{dv_y}{dt}$$

and if $a_y$ is constant,

$$\left.\begin{aligned} v_y &= u_y + a_y t \\ y &= u_y t + \frac{1}{2} a_y t^2 \\ v_y^2 &= u_y^2 + 2a_y y \end{aligned}\right| \quad \ldots (3.21)$$

The general scheme for the discussion of motion in a plane is therefore simple. The $x$-coordinate, the $x$-component of velocity and the $x$-component of acceleration are related by equations of straight line motion along the *X*-axis. Similarly the $y$-coordinate, the $y$-component of velocity and the $y$-component of acceleration are related by the equations of straight line motion along the *Y*-axis. The problem of motion in a plane is thus, broken up into two independent problems of straight line motion, one along the *X*-axis and the other along the *Y*-axis.

***Example 3.8***

*A particle moves in the X-Y plane with a constant acceleration of* $1{\cdot}5 \text{ m/s}^2$ *in the direction making an angle of* $37°$ *with the X-axis. At* $t = 0$ *the particle is at the origin and its velocity is* $8{\cdot}0$ m/s *along the X-axis. Find the velocity and the position of the particle at* $t = 4{\cdot}0$ s.

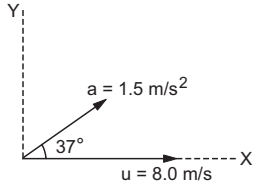



Figure 3.9

***Solution*** :
$$a_x = (1{\cdot}5 \text{ m/s}^2)(\cos 37^\circ)$$
$$= (1{\cdot}5 \text{ m/s}^2) \times \frac{4}{5} = 1{\cdot}2 \text{ m/s}^2$$
and
$$a_y = (1{\cdot}5 \text{ m/s}^2)(\sin 37^\circ)$$
$$= (1{\cdot}5 \text{ m/s}^2) \times \frac{3}{5} = 0{\cdot}90 \text{ m/s}^2.$$

The initial velocity has components
$$u_x = 8{\cdot}0 \text{ m/s}$$
and
$$u_y = 0$$

At $t = 0$, $x = 0$ and $y = 0$.

The $x$-component of the velocity at time $t = 4{\cdot}0$ s is given by
$$v_x = u_x + a_x t$$
$$= 8{\cdot}0 \text{ m/s} + (1{\cdot}2 \text{ m/s}^2)(4{\cdot}0 \text{ s})$$
$$= 8{\cdot}0 \text{ m/s} + 4{\cdot}8 \text{ m/s} = 12{\cdot}8 \text{ m/s}.$$

The $y$-component of velocity at $t = 4{\cdot}0$ s is given by
$$v_y = u_y + a_y t$$
$$= 0 + (0{\cdot}90 \text{ m/s}^2)(4{\cdot}0 \text{ s}) = 3{\cdot}6 \text{ m/s}.$$

The velocity of the particle at $t = 4{\cdot}0$ s is
$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(12{\cdot}8 \text{ m/s})^2 + (3{\cdot}6 \text{ m/s})^2}$$
$$= 13{\cdot}3 \text{ m/s}.$$

The velocity makes an angle $\theta$ with the $X$-axis where
$$\tan\theta = \frac{v_y}{v_x} = \frac{3{\cdot}6 \text{ m/s}}{12{\cdot}8 \text{ m/s}} = \frac{9}{32}.$$

The $x$-coordinate at $t = 4{\cdot}0$ s is
$$x = u_x t + \frac{1}{2} a_x t^2$$
$$= (8{\cdot}0 \text{ m/s})(4{\cdot}0 \text{ s}) + \frac{1}{2}(1{\cdot}2 \text{ m/s}^2)(4{\cdot}0 \text{ s})^2$$
$$= 32 \text{ m} + 9{\cdot}6 \text{ m} = 41{\cdot}6 \text{ m}.$$

The $y$-coordinate at $t = 4{\cdot}0$ s is
$$y = u_y t + \frac{1}{2} a_y t^2$$
$$= \frac{1}{2}(0{\cdot}90 \text{ m/s}^2)(4{\cdot}0 \text{ s})^2$$
$$= 7{\cdot}2 \text{ m}.$$

Thus, the particle is at (41·6 m, 7·2 m) at 4·0 s.

## 3.8 PROJECTILE MOTION

An important example of motion in a plane with constant acceleration is the projectile motion. When a particle is thrown obliquely near the earth's surface, it moves along a curved path. Such a particle is called a *projectile* and its motion is called *projectile motion*. We shall assume that the particle remains close to the surface of the earth and the air resistance is negligible. The acceleration of the particle is then almost constant. It is in the vertically downward direction and its magnitude is $g$ which is about $9{\cdot}8 \text{ m/s}^2$.

Let us first make ourselves familiar with certain terms used in discussing projectile motion. Figure (3.10) shows a particle projected from the point $O$ with an initial velocity $u$ at an angle $\theta$ with the horizontal. It goes through the highest point $A$ and falls at $B$ on the horizontal surface through $O$. The point $O$ is called the *point of projection*, the angle $\theta$ is called the *angle of projection* and the distance $OB$ is called the *horizontal range* or simply *range*. The total time taken by the particle in describing the path $OAB$ is called the *time of flight*.

The motion of the projectile can be discussed separately for the horizontal and vertical parts. We take the origin at the point of projection. The instant

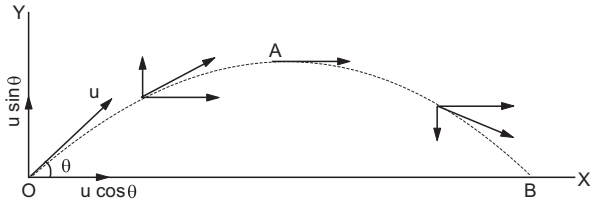



Figure 3.10

when the particle is projected is taken as $t = 0$. The plane of motion is taken as the $X$-$Y$ plane. The horizontal line $OX$ is taken as the $X$-axis and the vertical line $OY$ as the $Y$-axis. Vertically upward direction is taken as the positive direction of the $Y$-axis.

We have
$$u_x = u\cos\theta\,; \quad a_x = 0$$
$$u_y = u\sin\theta\,; \quad a_y = -g.$$

### Horizontal Motion

As $a_x = 0$, we have
$$v_x = u_x + a_x t = u_x = u\cos\theta$$
and
$$x = u_x t + \frac{1}{2} a_x t^2 = u_x t = ut\cos\theta.$$

As indicated in figure (3.10), the $x$-component of the velocity remains constant as the particle moves.

### Vertical Motion

The acceleration of the particle is $g$ in the downward direction. Thus, $a_y = -g$. The $y$-component of the initial velocity is $u_y$. Thus,
$$v_y = u_y - gt$$
and
$$y = u_y t - \frac{1}{2} g t^2.$$

Also we have,
$$v_y^2 = u_y^2 - 2gy.$$

The vertical motion is identical to the motion of a particle projected vertically upward with speed $u \sin\theta$. The horizontal motion of the particle is identical to a particle moving horizontally with uniform velocity $u \cos\theta$.

**Time of Flight**

Consider the situation shown in figure (3.10). The particle is projected from the point $O$ and reaches the same horizontal plane at the point $B$. The total time taken to reach $B$ is the time of flight.

Suppose the particle is at $B$ at a time $t$. The equation for horizontal motion gives

$$OB = x = ut \cos\theta$$

The $y$-coordinate at the point $B$ is zero. Thus, from the equation of vertical motion,

$$y = ut \sin\theta - \frac{1}{2} gt^2$$

or, $$0 = ut \sin\theta - \frac{1}{2} gt^2$$

or, $$t(u \sin\theta - \frac{1}{2} gt) = 0.$$

Thus, either $t = 0$ or, $t = \frac{2u \sin\theta}{g}$.

Now $t = 0$ corresponds to the position $O$ of the particle. The time at which it reaches $B$ is thus,

$$T = \frac{2u \sin\theta}{g}. \quad \ldots (3.22)$$

This is the time of flight.

**Range**

The distance $OB$ is the horizontal range. It is the distance travelled by the particle in time $T = \frac{2u \sin\theta}{g}$. By the equation of horizontal motion,

$$x = (u\cos\theta)t$$

or, $$OB = (u \cos\theta)\left(\frac{2u \sin\theta}{g}\right)$$

$$= \frac{2u^2 \sin\theta \cos\theta}{g}$$

$$= \frac{u^2 \sin 2\theta}{g}. \quad \ldots (3.23)$$

**Maximum Height Reached**

At the maximum height ($A$ in figure 3.10) the velocity of the particle is horizontal. The vertical component of velocity is thus, zero at the highest point. The maximum height is the $y$-coordinate of the particle when the vertical component of velocity becomes zero.

We have,

$$v_y = u_y - gt$$

$$= u \sin\theta - gt.$$

At the maximum height

$$0 = u \sin\theta - gt$$

or, $$t = \frac{u \sin\theta}{g}. \quad \ldots (3.24)$$

The maximum height is

$$H = u_y t - \frac{1}{2} gt^2$$

$$= (u \sin\theta)\left(\frac{u \sin\theta}{g}\right) - \frac{1}{2} g \left(\frac{u \sin\theta}{g}\right)^2$$

$$= \frac{u^2 \sin^2\theta}{g} - \frac{1}{2} \frac{u^2 \sin^2\theta}{g}$$

$$= \frac{u^2 \sin^2\theta}{2g}. \quad \ldots (3.25)$$

Equation (3.24) gives the time taken in reaching the maximum height. Comparison with equation (3.22) shows that it is exactly half the time of the flight. Thus, the time taken in ascending the maximum height equals the time taken in descending back to the same horizontal plane.

***Example 3.9***

*A ball is thrown from a field with a speed of* 12·0 m/s *at an angle of* 45° *with the horizontal. At what distance will it hit the field again ? Take* $g$ = 10·0 m/s$^2$.

***Solution*** : The horizontal range $= \frac{u^2 \sin 2\theta}{g}$

$$= \frac{(12 \text{ m/s})^2 \times \sin(2 \times 45°)}{10 \text{ m/s}^2}$$

$$= \frac{144 \text{ m}^2/\text{s}^2}{10{\cdot}0 \text{ m/s}^2} = 14{\cdot}4 \text{ m}.$$

Thus, the ball hits the field at 14·4 m from the point of projection.

## 3.9 CHANGE OF FRAME

So far we have discussed the motion of a particle with respect to a given frame of reference. The frame can be chosen according to the convenience of the problem. The position $\vec{r}$, the velocity $\vec{v}$ and the acceleration $\vec{a}$ of a particle depend on the frame chosen. Let us see how can we relate the position, velocity and acceleration of a particle measured in two different frames.

Consider two frames of reference $S$ and $S'$ and suppose a particle $P$ is observed from both the frames. The frames may be moving with respect to each other. Figure (3.11) shows the situation.

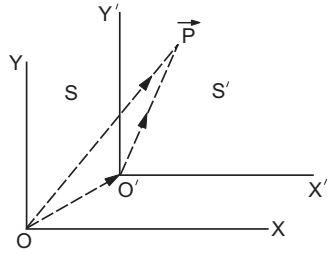



Figure 3.11

The position vector of the particle $P$ with respect to the frame $S$ is $\vec{r}_{P,S} = \vec{OP}$ . The position vector of the particle with respect to the frame $S'$ is $\vec{r}_{P,S'} = \vec{O'P}$ . The position of the frame $S'$ (the origin of frame $S'$ in fact) with respect to the frame $S$ is $OO'$.

It is clear that

$$\vec{OP} = \vec{OO'} + \vec{O'P} = \vec{O'P} + \vec{OO'}$$

or, $$\vec{r}_{P,S} = \vec{r}_{P,S'} + \vec{r}_{S',S}. \quad \ldots (3.26)$$

The position of the particle with respect to $S$ is equal to the position of the particle with respect to $S'$ plus the position of $S'$ with respect to $S$.

If we differentiate equation (3.26) with respect to time, we get

$$\frac{d}{dt}(\vec{r}_{P,S}) = \frac{d}{dt}(\vec{r}_{P,S'}) + \frac{d}{dt}(\vec{r}_{S',S})$$

or, $$\vec{v}_{P,S} = \vec{v}_{P,S'} + \vec{v}_{S',S} \quad \ldots (3.27)$$

where $\vec{v}_{P,S}$ is the velocity of the particle with respect to $S$, $\vec{v}_{P,S'}$ is the velocity of the particle with respect to $S'$ and $\vec{v}_{S',S}$ is the velocity of the frame $S'$ with respect to $S$. The velocity of the particle with respect to $S$ is equal to the velocity of the particle with respect to $S'$ plus the velocity of $S'$ with respect to $S$.

It is assumed that the meaning of time is same in both the frames. Similarly it is assumed that $\frac{d}{dt}$ has same meaning in both the frames. These assumptions are not correct if the velocity of one frame with respect to the other is so large that it is comparable to $3 \times 10^8$ m/s, or if one frame rotates with respect to the other. If the frames only translate with respect to each other with small velocity, the above assumptions are correct.

Equation (3.27) may be rewritten as

$$\vec{v}_{P,S'} = \vec{v}_{P,S} - \vec{v}_{S',S}. \quad \ldots (3.28)$$

Thus, if the velocities of two bodies (here the particle and the frame $S'$) are known with respect to a common frame (here $S$) we can find the velocity of one body with respect to the other body. The velocity of body 1 with respect to the body 2 is obtained by subtracting the velocity of body 2 from the velocity of body 1.

When we say that the muzzle velocity of a bullet is 60 m/s we mean the velocity of the bullet with respect to the gun. If the gun is mounted in a train moving with a speed of 20 m/s with respect to the ground and the bullet is fired in the direction of the train's motion, its velocity with respect to the ground will be 80 m/s. Similarly, when we say that a swimmer can swim at a speed of 5 km/h we mean the velocity of the swimmer with respect to the water. If the water itself is flowing at 3 km/h with respect to the ground and the swimmer swims in the direction of the current, he or she will move at the speed of 8 km/h with respect to the ground.

***Example 3.10***

*A swimmer can swim in still water at a rate* 4·0 km/h. *If he swims in a river flowing at* 3·0 km/h *and keeps his direction (with respect to water) perpendicular to the current, find his velocity with respect to the ground.*

***Solution :*** The velocity of the swimmer with respect to water is $\vec{v}_{S,R} = 4{\cdot}0$ km/h in the direction perpendicular to the river. The velocity of river with respect to the ground is $\vec{v}_{R,G} = 3{\cdot}0$ km/h along the length of the river. The velocity of the swimmer with respect to the ground is $\vec{v}_{S,G}$ where

$$\vec{v}_{S,G} = \vec{v}_{S,R} + \vec{v}_{R,G}.$$

Figure (3.12) shows the velocities. It is clear that,

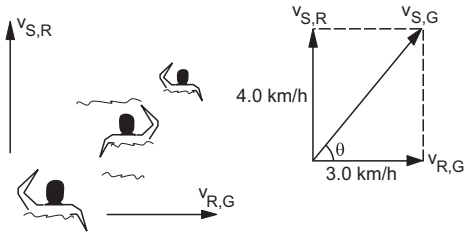



Figure 3.12

$$v_{S,G} = \sqrt{(4{\cdot}0 \text{ km/h})^2 + (3{\cdot}0 \text{ km/h})^2}$$

$$= 5{\cdot}0 \text{ km/h}$$

The angle θ made with the direction of flow is

$$\tan\theta = \frac{4{\cdot}0 \text{ km/h}}{3{\cdot}0 \text{ km/h}} = \frac{4}{3}.$$

***Example 3.11***

*A man is walking on a level road at a speed of* 3·0 km/h. *Rain drops fall vertically with a speed of* 4·0 km/h. *Find the velocity of the raindrops with respect to the man.*

***Solution*** **:** We have to find the velocity of raindrops with respect to the man. The velocity of the rain as well as the velocity of the man are given with respect to the street. We have

$$\vec{v}_{rain,\,man} = \vec{v}_{rain,\,street} - \vec{v}_{man,\,street}\,.$$

Figure (3.13) shows the velocities.

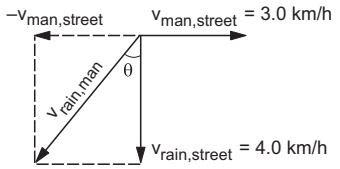



Figure 3.13

It is clear from the figure that

$$v_{rain,\,man} = \sqrt{(4{\cdot}0\text{ km/h})^2 + (3{\cdot}0\text{ km/h})^2}$$
$$= 5{\cdot}0\text{ km/h}.$$

The angle with the vertical is θ, where

$$\tan\theta = \frac{3{\cdot}0\text{ km/h}}{4{\cdot}0\text{ km/h}} = \frac{3}{4}\,.$$

Thus, the rain appears to fall at an angle $\tan^{-1}(3/4)$ with the speed 5·0 km/h as viewed by the man.

The relation between the accelerations measured from two frames can be obtained by differentiating equation (3.27) with respect to time.

We have,

$$\frac{d}{dt}(\vec{v}_{P,S}) = \frac{d}{dt}(\vec{v}_{P,S'}) + \frac{d}{dt}(\vec{v}_{S',S})$$

or, $$\vec{a}_{P,S} = \vec{a}_{P,S'} + \vec{a}_{S',S}\,. \quad \ldots\ (3.29)$$

If *S′ moves with respect to S at a uniform velocity,* $\vec{a}_{S',S} = 0$ *and so*

$$\vec{a}_{P,S} = \vec{a}_{P,S'}\,.$$

If two frames are moving with respect to each other with uniform velocity, acceleration of a body is same in both the frames.

## *Worked Out Examples*

**1.** *A man walks at a speed of* 6 km/hr *for* 1 km *and* 8 km/hr *for the next* 1 km. *What is his average speed for the walk of* 2 km ?

***Solution*** **:** Distance travelled is 2 km.

$$\text{Time taken} = \frac{1\text{ km}}{6\text{ km/hr}} + \frac{1\text{ km}}{8\text{ km/hr}}$$
$$= \left(\frac{1}{6} + \frac{1}{8}\right)\text{hr} = \frac{7}{24}\text{ hr}.$$

$$\text{Average speed} = \frac{2\text{ km} \times 24}{7\text{ hr}} = \frac{48}{7}\text{ km/hr}$$
$$\approx 7\text{ km/hr}.$$

**2.** *The I.Sc. lecture theatre of a college is* 40 ft *wide and has a door at a corner. A teacher enters at* 12·00 noon *through the door and makes* 10 *rounds along the* 40 ft *wall back and forth during the period and finally leaves the class-room at* 12·50 p.m. *through the same door. Compute his average speed and average velocity.*

***Solution*** **:** Total distance travelled in 50 minutes = 800 ft.

$$\text{Average speed} = \frac{800}{50}\text{ ft/min} = 16\text{ ft/min}.$$

At 12·00 noon he is at the door and at 12·50 pm he is again at the same door.

The displacement during the 50 min interval is zero.

Average velocity = zero.

**3.** *The position of a particle moving on X-axis is given by*

$$x = At^3 + Bt^2 + Ct + D.$$

*The numerical values of A, B, C, D are* 1, 4, –2 *and* 5 *respectively and SI units are used. Find (a) the dimensions of A, B, C and D, (b) the velocity of the particle at* $t$ = 4 s, *(c) the acceleration of the particle at* $t$ = 4 s, *(d) the average velocity during the interval* $t = 0$ *to* $t$ = 4 s, *(e) the average acceleration during the interval* $t = 0$ to $t = 4$ s.

***Solution*** : (a) Dimensions of $x$, $At^3$, $Bt^2$, $Ct$ and $D$ must be identical and in this case each is length. Thus,

$$[At^3] = \text{L}, \quad \text{or,}\ [A] = \text{LT}^{-3}$$
$$[Bt^2] = \text{L}, \quad \text{or,}\ [B] = \text{LT}^{-2}$$
$$[Ct] = \text{L}, \quad \text{or,}\ [C] = \text{LT}^{-1}$$

and $$[D] = \text{L}.$$

(b) $x = At^3 + Bt^2 + Ct + D$

or, $$v = \frac{dx}{dt} = 3At^2 + 2Bt + C.$$

Thus, at $t$ = 4 s, the velocity

$= 3(1\text{ m/s}^3)(16\text{ s}^2) + 2(4\text{ m/s}^2)(4\text{ s}) + (-2\text{ m/s})$

$= (48 + 32 - 2)\text{ m/s} = 78\text{ m/s}.$

(c) $v = 3At^2 + 2Bt + C$

or, $$a = \frac{dv}{dt} = 6At + 2B.$$

At $t$ = 4 s, $a = 6(1\text{ m/s}^3)(4\text{ s}) + 2(4\text{ m/s}^2) = 32\text{ m/s}^2$.

(d) $x = At^3 + Bt^2 + Ct + D$.

Position at $t$ = 0 is $x = D$ = 5 m.

Position at $t$ = 4 s is

$(1\text{ m/s}^3)(64\text{ s}^3) + (4\text{ m/s}^2)(16\text{ s}^2) - (2\text{ m/s})(4\text{ s}) + 5\text{ m}$

$= (64 + 64 - 8 + 5)\text{ m} = 125\text{ m}.$

Thus, the displacement during 0 to 4 s is
125 m – 5 m = 120 m.

Average velocity $= \frac{120 \text{ m}}{4 \text{ s}} = 30$ m/s.

(e) $v = 3At^2 + 2Bt + C$.

Velocity at $t = 0$ is $C = -2$ m/s.

Velocity at $t$ = 4 s is = 78 m/s.

Average acceleration $= \frac{v_2 - v_1}{t_2 - t_1} = 20 \text{ m/s}^2$.

**4.** *From the velocity-time graph of a particle given in figure (3-W1), describe the motion of the particle qualitatively in the interval* 0 *to* 4 s. *Find (a) the distance travelled during first two seconds, (b) during the time* 2 s *to* 4 s, *(c) during the time* 0 *to* 4 s, *(d) displacement during* 0 to 4 s, *(e) acceleration at* $t$ = 1/2 s *and (f) acceleration at* $t$ = 2 s.

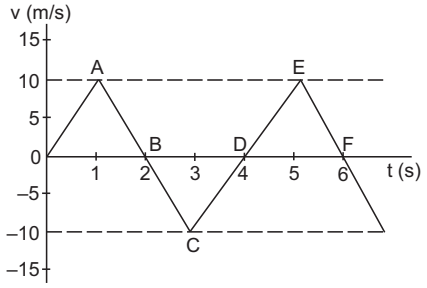



Figure 3-W1

***Solution*** **:** At $t = 0$, the particle is at rest, say at the origin. After that the velocity is positive, so that the particle moves in the positive $x$ direction. Its speed increases till 1 second when it starts decreasing. The particle continues to move further in positive $x$ direction. At $t = 2$ s, its velocity is reduced to zero, it has moved through a maximum positive $x$ distance. Then it changes its direction, velocity being negative, but increasing in magnitude. At $t$ = 3 s velocity is maximum in the negative $x$ direction and then the magnitude starts decreasing. It comes to rest at $t$ = 4 s.

(a) Distance during 0 to 2 s = Area of $OAB$

$$= \frac{1}{2} \times 2 \text{ s} \times 10 \text{ m/s} = 10 \text{ m}.$$

(b) Distance during 2 to 4 s = Area of $BCD$ = 10 m. The particle has moved in negative $x$ direction during this period.

(c) The distance travelled during 0 to 4s = 10 m + 10 m
= 20 m.

(d) displacement during 0 to 4 s = 10 m + (– 10 m) = 0.

(e) at $t$ = 1/2 s acceleration = slope of line $OA$ = 10 m/s$^2$.

(f) at $t$ = 2 s acceleration = slope of line $ABC$ = – 10 m/s$^2$.

**5.** *A particle starts from rest with a constant acceleration. At a time* $t$ *second, the speed is found to be* 100 m/s *and one second later the speed becomes* 150 m/s. *Find (a) the acceleration and (b) the distance travelled during the* $(t+1)^{th}$ *second.*

***Solution*** **:** (a) Velocity at time $t$ second is

100 m/s = $a$.($t$ second) … (1)

and velocity at time ($t$ + 1) second is

150 m/s = $a$.($t$ + 1) second. … (2)

Subtracting (1) from (2), $a$ = 50 m/s$^2$

(b) Consider the interval $t$ second to ($t$ + 1) second,

time elapsed = 1 s

initial velocity = 100 m/s

final velocity = 150 m/s.

Thus, $(150 \text{ m/s})^2 = (100 \text{ m/s})^2 + 2(50 \text{ m/s}^2)\, x$

or, $x = 125$ m.

**6.** *A boy stretches a stone against the rubber tape of a catapult or 'gulel' (a device used to detach mangoes from the tree by boys in Indian villages) through a distance of* 24 cm *before leaving it. The tape returns to its normal position accelerating the stone over the stretched length. The stone leaves the gulel with a speed* 2·2 m/s. *Assuming that the acceleration is constant while the stone was being pushed by the tape, find its magnitude.*

***Solution*** **:** Consider the accelerated 24 cm motion of the stone.

Initial velocity = 0

Final velocity = 2·2 m/s

Distance travelled = 24 cm = 0·24 m

Using $v^2 = u^2 + 2ax$,

$$a = \frac{4{\cdot}84 \text{ m}^2/\text{s}^2}{2 \times 0{\cdot}24 \text{ m}} = 10{\cdot}1 \text{ m/s}^2.$$

**7.** *A police inspector in a jeep is chasing a pickpocket on a straight road. The jeep is going at its maximum speed* $v$ *(assumed uniform). The pickpocket rides on the motorcycle of a waiting friend when the jeep is at a distance* $d$ *away, and the motorcycle starts with a constant acceleration* $a$. *Show that the pickpocket will be caught if* $v \geq \sqrt{2ad}$.

***Solution*** **:** Suppose the pickpocket is caught at a time $t$ after the motorcycle starts. The distance travelled by the motorcycle during this interval is

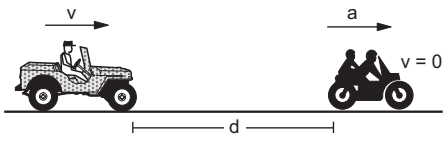



Figure 3-W2

$$s = \frac{1}{2} at^2. \quad \ldots \text{(i)}$$

During this interval the jeep travels a distance

$$s + d = vt. \quad \ldots \text{(ii)}$$

By (i) and (ii),

$$\frac{1}{2} at^2 - vt + d = 0$$

or, $$t = \frac{v \pm \sqrt{v^2 - 2ad}}{a}.$$

The pickpocket will be caught if $t$ is real and positive. This will be possible if

$$v^2 \geq 2ad \quad \text{or,} \quad v \geq \sqrt{2ad}.$$

**8.** *A car is moving at a constant speed of* 40 km/h *along a straight road which heads towards a large vertical wall and makes a sharp* 90° *turn by the side of the wall. A fly flying at a constant speed of* 100 km/h, *starts from the wall towards the car at an instant when the car is* 20 km *away, flies until it reaches the glasspane of the car and returns to the wall at the same speed. It continues to fly between the car and the wall till the car makes the* 90° *turn. (a) What is the total distance the fly has travelled during this period ? (b) How many trips has it made between the car and the wall ?*

***Solution*** **:** (a) The time taken by the car to cover 20 km before the turn is $\frac{20 \text{ km}}{40 \text{ km/h}} = \frac{1}{2}$ h. The fly moves at a constant speed of 100 km/h during this time. Hence the total distance coverd by it is $100 \frac{\text{km}}{\text{h}} \times \frac{1}{2} \text{ h} = 50 \text{ km}.$

(b) Suppose the car is at a distance $x$ away (at $A$) when the fly is at the wall (at $O$). The fly hits the glasspane at $B$, taking a time $t$. Then

$$AB = (40 \text{ km/h})t,$$

and $$OB = (100 \text{ km/h})t.$$

Thus, $$x = AB + OB$$

$$= (140 \text{ km/h})t$$

or, $$t = \frac{x}{140 \text{ km/h}}, \text{ or } OB = \frac{5}{7}x.$$

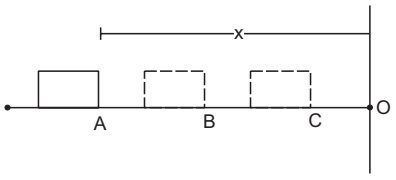



Figure 3-W3

The fly returns to the wall and during this period the car moves the distance $BC$. The time taken by the fly in this return path is

$$\left(\frac{5x/7}{100 \text{ km/h}}\right) = \frac{x}{140 \text{ km/h}}.$$

Thus, $$BC = \frac{40x}{140} = \frac{2}{7}x$$

or, $$OC = OB - BC = \frac{3}{7}x.$$

If at the beginning of the round trip (wall to the car and back) the car is at a distance $x$ away, it is $\frac{3}{7}x$ away when the next trip again starts.

Distance of the car at the beginning of the 1st trip = 20 km.

Distance of the car at the beginning of the 2nd trip

$$= \frac{3}{7} \times 20 \text{ km}.$$

Distance of the car at the beginning of the 3rd trip

$$= \left(\frac{3}{7}\right)^2 \times 20 \text{ km}.$$

Distance of the car at the beginning of the 4th trip

$$= \left(\frac{3}{7}\right)^3 \times 20 \text{ km}.$$

Distance of the car at the beginning of the $n$th trip

$$= \left(\frac{3}{7}\right)^{n-1} \times 20 \text{ km}.$$

Trips will go on till the car reaches the turn that is the distance reduces to zero. This will be the case when $n$ becomes infinity. Hence the fly makes an infinite number of trips before the car takes the turn.

**9.** *A ball is dropped from a height of* 19·6 m *above the ground. It rebounds from the ground and raises itself up to the same height. Take the starting point as the origin and vertically downward as the positive X-axis. Draw approximate plots of x versus t, v versus t and a versus t. Neglect the small interval during which the ball was in contact with the ground.*

***Solution*** **:** Since the acceleration of the ball during the contact is different from *'g'*, we have to treat the downward motion and the upward motion separately.

For the downward motion : $a = g = 9{\cdot}8 \text{ m/s}^2$,

$$x = ut + \frac{1}{2} at^2 = (4{\cdot}9 \text{ m/s}^2)t^2.$$

The ball reaches the ground when $x$ = 19·6 m. This gives $t$ = 2 s. After that it moves up, $x$ decreases and at $t$ = 4 s, $x$ becomes zero, the ball reaching the initial point.

We have at $t = 0$, $x = 0$

$t = 1$ s, $x = 4{\cdot}9$ m

$t = 2$ s, $x = 19{\cdot}6$ m

$t = 3$ s, $x = 4{\cdot}9$ m

$t = 4$ s, $x = 0$.

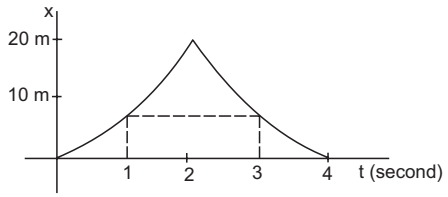



Figure 3-W4

**Velocity** : During the first two seconds,

$v = u + at = (9{\cdot}8 \text{ m/s}^2)t$

at $t = 0$ $v = 0$

at $t = 1$ s, $v = 9{\cdot}8$ m/s

at $t = 2$ s, $v = 19{\cdot}6$ m/s.

During the next two seconds the ball goes upward, velocity is negative, magnitude decreasing and at $t = 4$ s, $v = 0$. Thus,

at $t = 2$ s, $v = -19{\cdot}6$ m/s

at $t = 3$ s, $v = -9{\cdot}8$ m/s

at $t = 4$ s, $v = 0$.

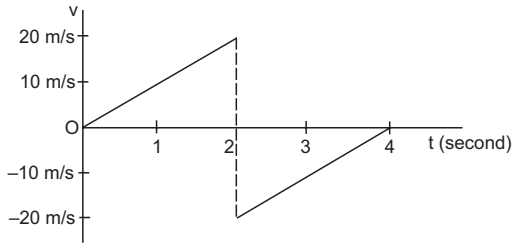



Figure 3-W5

At $t = 2$ s there is an abrupt change in velocity from 19·6 m/s to – 19·6 m/s. In fact this change in velocity takes place over a small interval during which the ball remains in contact with the ground.

**Acceleration :** The acceleration is constant 9·8 m/s$^2$ throughout the motion (except at $t$ = 2 s).

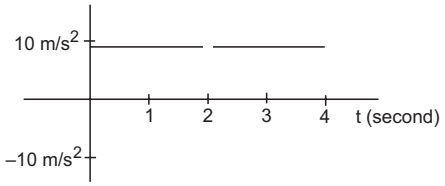



Figure 3-W6

**10.** *A stone is dropped from a balloon going up with a uniform velocity of* 5·0 m/s. *If the balloon was* 50 m *high when the stone was dropped, find its height when the stone hits the ground. Take* $g = 10$ m/s$^2$.

***Solution*** **:** At $t$ = 0, the stone was going up with a velocity of 5·0 m/s. After that it moved as a freely falling particle with downward acceleration $g$. Take vertically upward as the positive $X$-axis. If it reaches the ground at time $t$,

$$x = -50 \text{ m}, \quad u = 5 \text{ m/s}, \quad a = -10 \text{ m/s}^2.$$

We have $x = ut + \frac{1}{2}at^2$

or, $-50 \text{ m} = (5 \text{ m/s}).t + \frac{1}{2} \times (-10 \text{ m/s}^2)t^2$

or, $t = \frac{1 \pm \sqrt{41}}{2}$ s.

or, $t = -2{\cdot}7$ s or, $3{\cdot}7$ s.

Negative $t$ has no significance in this problem. The stone reaches the ground at $t$ = 3·7 s. During this time the balloon has moved uniformly up. The distance covered by it is

$$5 \text{ m/s} \times 3{\cdot}7 \text{ s} = 18{\cdot}5 \text{ m}.$$

Hence, the height of the balloon when the stone reaches the ground is 50 m + 18·5 m = 68·5 m.

**11.** *A football is kicked with a velocity of* 20 m/s *at an angle of* 45° *with the horizontal. (a) Find the time taken by the ball to strike the ground. (b) Find the maximum height it reaches. (c) How far away from the kick does it hit the ground* ? *Take* $g$ = 10 m/s$^2$.

***Solution*** **:** (a) Take the origin at the point where the ball is kicked, vertically upward as the $Y$-axis and the horizontal in the plane of motion as the $X$-axis. The initial velocity has the components

$$u_x = (20 \text{ m/s}) \cos 45° = 10\sqrt{2} \text{ m/s}$$

and $u_y = (20 \text{ m/s}) \sin 45° = 10\sqrt{2}$ m/s.

When the ball reaches the ground, $y$ = 0.

Using $y = u_y t - \frac{1}{2}gt^2$,

$$0 = (10\sqrt{2} \text{ m/s})\, t - \frac{1}{2} \times (10 \text{ m/s}^2) \times t^2$$

or, $t = 2\sqrt{2}$ s = 2·8 s.

Thus, it takes 2·8 s for the football to fall on the ground.

(b) At the highest point $v_y = 0$. Using the equation

$$v_y^2 = u_y^2 - 2gy,$$

$$0 = (10\sqrt{2} \text{ m/s})^2 - 2 \times (10 \text{ m/s}^2)\, H$$

or, $H = 10$ m.

Thus, the maximum height reached is 10 m.

(c) The horizontal distance travelled before falling to the ground is $x = u_x t$

$$= (10\sqrt{2} \text{ m/s})\, (2\sqrt{2} \text{ s}) = 40 \text{ m}.$$

**12.** *A helicopter on flood relief mission, flying horizontally with a speed u at an altitude H, has to drop a food packet for a victim standing on the ground. At what distance from the victim should the packet be dropped* ? *The victim stands in the vertical plane of the helicopter's motion.*

***Solution*** **:** The velocity of the food packet at the time of release is $u$ and is horizontal. The vertical velocity at the time of release is zero.

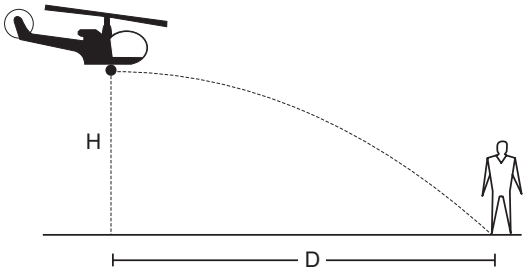



Figure 3-W7

**Vertical motion :** If $t$ be the time taken by the packet to reach the victim, we have for vertical motion,

$$H = \frac{1}{2} g t^2 \quad \text{or,} \quad t = \sqrt{\frac{2H}{g}} \; . \qquad \ldots \text{(i)}$$

**Horizontal motion :** If $D$ be the horizontal distance travelled by the packet, we have $D = ut$. Putting $t$ from (i),

$$D = u\sqrt{\frac{2H}{g}} \; .$$

The distance between the victim and the packet at the time of release is

$$\sqrt{D^2 + H^2} = \sqrt{\frac{2u^2 H}{g} + H^2} \; .$$

**13.** *A particle is projected horizontally with a speed u from the top of a plane inclined at an angle* θ *with the horizontal. How far from the point of projection will the particle strike the plane* ?

***Solution*** **:** Take $X,Y$-axes as shown in figure (3-W8). Suppose that the particle strikes the plane at a point $P$ with coordinates $(x, y)$. Consider the motion between $A$ and $P$.

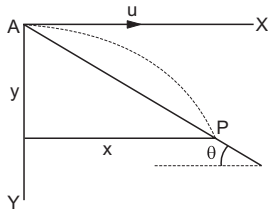



Figure 3-W8

Motion in $x$-direction :

$$\text{Initial velocity} = u$$
$$\text{Acceleration} = 0$$
$$x = ut. \qquad \ldots \text{(i)}$$

Motion in $y$-direction :

$$\text{Initial velocity} = 0$$
$$\text{Acceleration} = g$$
$$y = \frac{1}{2} g t^2. \qquad \ldots \text{(ii)}$$

Eliminating $t$ from (i) and (ii)

$$y = \frac{1}{2} g \frac{x^2}{u^2} \; .$$

Also $y = x \tan\theta$.

Thus, $\frac{gx^2}{2u^2} = x \tan\theta$ giving $x = 0$, or, $\frac{2u^2 \tan\theta}{g}$ .

Clearly the point $P$ corresponds to $x = \frac{2u^2 \tan\theta}{g}$ ,

then $y = x \tan\theta = \frac{2u^2 \tan^2\theta}{g}$ .

The distance $AP = l = \sqrt{x^2 + y^2}$

$$= \frac{2u^2}{g} \tan\theta \sqrt{1 + \tan^2\theta}$$
$$= \frac{2u^2}{g} \tan\theta \sec\theta.$$

**Alternatively :** Take the axes as shown in figure 3-W9. Consider the motion between $A$ and $P$.

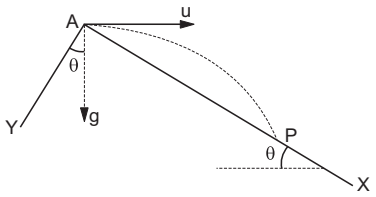



Figure 3-W9

Motion along the $X$-axis :

$$\text{Initial velocity} = u \cos\theta$$
$$\text{Acceleration} = g \sin\theta$$
$$\text{Displacement} = AP.$$

Thus, $AP = (u \cos\theta)\, t + \frac{1}{2} (g \sin\theta)\, t^2$. … (i)

Motion along the $Y$-axis :

$$\text{Initial velocity} = -u \sin\theta$$
$$\text{Acceleration} = g \cos\theta$$
$$\text{Displacement} = 0.$$

Thus, $0 = -ut \sin\theta + \frac{1}{2} g t^2 \cos\theta$

or, $t = 0, \quad \frac{2u \sin\theta}{g \cos\theta}$ .

Clearly, the point $P$ corresponds to $t = \frac{2u \sin\theta}{g \cos\theta}$.

Putting this value of $t$ in (i),

$$AP = (u \cos\theta) \left( \frac{2u \sin\theta}{g \cos\theta} \right) + \frac{g \sin\theta}{2} \left( \frac{2u \sin\theta}{g \cos\theta} \right)^2$$
$$= \frac{2u^2 \sin\theta}{g} + \frac{2u^2 \sin\theta \tan^2\theta}{g}$$
$$= \frac{2u^2}{g} \sin\theta \sec^2\theta = \frac{2u^2}{g} \tan\theta \sec\theta.$$

**14.** *A projectile is fired with a speed u at an angle* θ *with the horizontal. Find its speed when its direction of motion makes an angle* α *with the horizontal.*

***Solution*** : Let the speed be $v$ when it makes an angle α with the horizontal. As the horizontal component of velocity remains constant,

$$v\cos\alpha = u\cos\theta$$

or, $$v = u\cos\theta\sec\alpha.$$

**15.** *A bullet is fired horizontally aiming at an object which starts falling at the instant the bullet is fired. Show that the bullet will hit the object.*

***Solution*** **:** The situation is shown in figure (3-W10). The object starts falling from the point $B$. Draw a vertical line $BC$ through $B$. Suppose the bullet reaches the line $BC$ at a point $D$ and it takes a time $t$ in doing so.

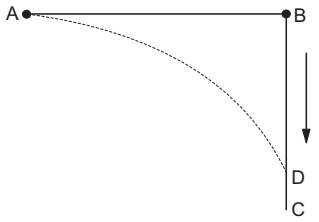



Figure 3-W10

Consider the vertical motion of the bullet. The initial vertical velocity = 0. The distance travelled vertically $= BD = \frac{1}{2}gt^2$. In time $t$ the object also travels a distance $\frac{1}{2}gt^2 = BD$. Hence at time $t$, the object will also be at the same point $D$. Thus, the bullet hits the object at point $D$.

**16.** *A man can swim in still water at a speed of* 3 km/h. *He wants to cross a river that flows at* 2 km/h *and reach the point directly opposite to his starting point. (a) In which direction should he try to swim (that is, find the angle his body makes with the river flow)* ? *(b) How much time will he take to cross the river if the river is* 500 m *wide* ?

***Solution*** **:** (a) The situation is shown in figure (3-W11). The $X$-axis is chosen along the river flow and the origin at the starting position of the man. The direction of the velocity of man with respect to ground is along the $Y$-axis (perpendicular to the river). We have to find the direction of velocity of the man with respect to water.

Let $\vec{v}_{r,g}$ = velocity of the river with respect to the ground

= 2 km/h along the $X$-axis

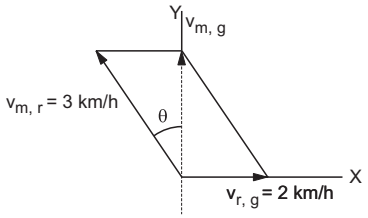



Figure 3-W11

$\vec{v}_{m,r}$ = velocity of the man with respect to the river

= 3 km/h making an angle θ with the $Y$-axis

and $\vec{v}_{m,g}$ = velocity of the man with respect to the ground along the $Y$-axis.

We have

$$\vec{v}_{m,g} = \vec{v}_{m,r} + \vec{v}_{r,g}. \quad \ldots (i)$$

Taking components along the $X$-axis

$$0 = -(3\text{ km/h})\sin\theta + 2\text{ km/h}$$

or, $$\sin\theta = \frac{2}{3}.$$

(b) Taking components in equation (i) along the $Y$-axis,

$$v_{m,g} = (3\text{ km/h})\cos\theta + 0$$

or, $$v_{m,g} = \sqrt{5}\text{ km/h}.$$

$$\text{Time} = \frac{\text{Displacement in } y \text{ direction}}{\text{Velocity in } y \text{ direction}}$$

$$= \frac{0{\cdot}5\text{ km}}{\sqrt{5}\text{ km/h}} = \frac{\sqrt{5}}{10}\text{ h}.$$

**17.** *A man can swim at a speed of* 3 km/h *in still water. He wants to cross a* 500 m *wide river flowing at* 2 km/h. *He keeps himself always at an angle of* 120° *with the river flow while swimming.*
*(a) Find the time he takes to cross the river. (b) At what point on the opposite bank will he arrive* ?

***Solution*** **:** The situation is shown in figure (3-W12).

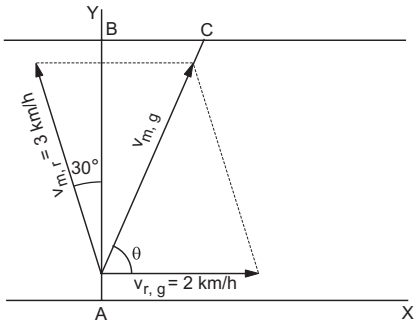



Figure 3-W12

Here $\vec{v}_{r,g}$ = velocity of the river with respect to the ground

$\vec{v}_{m,r}$ = velocity of the man with respect to the river

$\vec{v}_{m,g}$ = velocity of the man with respect to the ground.

(a) We have,

$$\vec{v}_{m,g} = \vec{v}_{m,r} + \vec{v}_{r,g} \quad \ldots \text{(i)}$$

Hence, the velocity with respect to the ground is along *AC*. Taking *y*-components in equation (i),

$$\vec{v}_{m,g} \sin\theta = 3 \text{ km/h } \cos 30° + 2 \text{ km/h } \cos 90° = \frac{3\sqrt{3}}{2} \text{ km/h}.$$

Time taken to cross the river

$$= \frac{\text{displacement along the } Y\text{-axis}}{\text{velocity along the } Y\text{-axis}}$$

$$= \frac{1/2 \text{ km}}{3\sqrt{3}/2 \text{ km/h}} = \frac{1}{3/3} \text{ h}.$$

(b) Taking *x*-components in equation (i),

$$\vec{v}_{m,g} \cos\theta = -3 \text{ km/h } \sin 30° + 2 \text{ km/h}$$

$$= \frac{1}{2} \text{ km/h}.$$

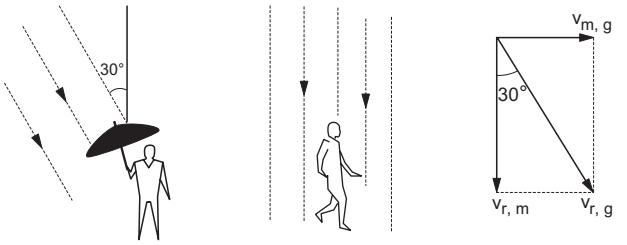
Displacement along the *X*-axis as the man crosses the river

$$= (\text{velocity along the } X\text{-axis}) \cdot (\text{time})$$

$$= \left(\frac{1 \text{ km}}{2 \text{ h}}\right) \times \left(\frac{1}{3\sqrt{3}} \text{ h}\right) = \frac{1}{6\sqrt{3}} \text{ km}.$$

**18.** *A man standing on a road has to hold his umbrella at* 30° *with the vertical to keep the rain away. He throws the umbrella and starts running at* 10 km/h. *He finds that raindrops are hitting his head vertically. Find the speed of raindrops with respect to (a) the road, (b) the moving man.*

***Solution*** **:** When the man is at rest with respect to the ground, the rain comes to him at an angle 30° with the vertical. This is the direction of the velocity of raindrops with respect to the ground. The situation when the man runs is shown in the figure (3-W13b).




(a) (b) (c)

Figure 3-W13

Here $\vec{v}_{r,g}$ = velocity of the rain with respect to the ground

$\vec{v}_{m,g}$ = velocity of the man with respect to the ground

and $\vec{v}_{r,m}$ = velocity of the rain with respect to the man.

We have, $\vec{v}_{r,g} = \vec{v}_{r,m} + \vec{v}_{m,g}$. … (i)

Taking horizontal components, equation (i) gives

$$v_{r,g} \sin 30° = v_{m,g} = 10 \text{ km/h}$$

or, $$v_{r,g} = \frac{10 \text{ km/h}}{\sin 30°} = 20 \text{ km/h},$$

Taking vertical components, equation (i) gives

$$v_{r,g} \cos 30° = v_{r,m}$$

or, $$v_{r,m} = (20 \text{ km/h}) \frac{\sqrt{3}}{2}$$

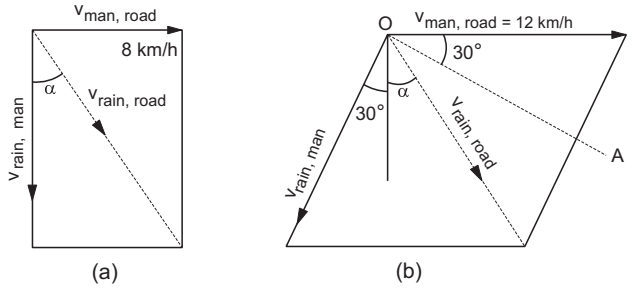
$$= 10\sqrt{3} \text{ km/h}.$$

**19.** *A man running on a horizontal road at* 8 km/h *finds the rain falling vertically. He increases his speed to* 12 km/h *and finds that the drops make angle* 30° *with the vertical. Find the speed and direction of the rain with respect to the road.*

***Solution*** **:**

We have $\vec{v}_{rain,\,road} = \vec{v}_{rain,\,man} + \vec{v}_{man,\,road}$ … (i)

The two situations given in the problem may be represented by the following figure.




(a) (b)

Figure 3-W14

$v_{rain,\,road}$ is same in magnitude and direction in both the figures.

Taking horizontal components in equation (i) for figure (3-W14a),

$$v_{rain,\,road} \sin\alpha = 8 \text{ km/h}. \quad \ldots \text{(ii)}$$

Now consider figure (3-W14b). Draw a line $OA \perp v_{rain,\,man}$ as shown.

Taking components in equation (i) along the line *OA*.

$$v_{rain,\,road} \sin(30° + \alpha) = 12 \text{ km/h } \cos 30°. \quad \ldots \text{(iii)}$$

From (ii) and (iii),

$$\frac{\sin(30° + \alpha)}{\sin\alpha} = \frac{12 \times \sqrt{3}}{8 \times 2}$$

or, $$\frac{\sin 30° \cos\alpha + \cos 30° \sin\alpha}{\sin\alpha} = \frac{3\sqrt{3}}{4}$$

or, $$\frac{1}{2} \cot\alpha + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4}$$

or, $$\cot\alpha = \frac{\sqrt{3}}{2}$$

or, $$\alpha = \cot^{-1} \frac{\sqrt{3}}{2}.$$

From (ii), $$v_{rain,\,road} = \frac{8 \text{ km/h}}{\sin\alpha} = 4\sqrt{7} \text{ km/h}.$$

**20.** *Three particles A, B and C are situated at the vertices of an equilateral triangle ABC of side d at t = 0. Each*

*of the particles moves with constant speed v. A always has its velocity along AB, B along BC and C along CA. At what time will the particles meet each other ?*

***Solution*** **:** The motion of the particles is roughly sketched in figure (3-W15). By symmetry they will meet at the

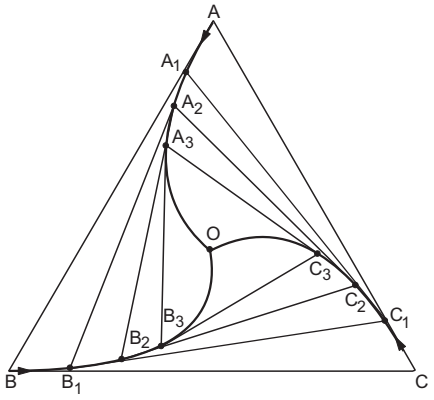



Figure 3-W15

centroid $O$ of the triangle. At any instant the particles will form an equilateral triangle $ABC$ with the same centriod $O$. Concentrate on the motion of any one particle, say $A$. At any instant its velocity makes angle 30° with $AO$.

The component of this velocity along $AO$ is $v \cos 30°$. This component is the rate of decrease of the distance $AO$. Initially,

$$AO = \frac{2}{3}\sqrt{d^2 - \left(\frac{d}{2}\right)^2} = \frac{d}{\sqrt{3}}.$$

Therefore, the time taken for $AO$ to become zero

$$= \frac{d/\sqrt{3}}{v\cos 30°} = \frac{2d}{\sqrt{3}\, v \times \sqrt{3}} = \frac{2d}{3v}.$$

**Alternative :** Velocity of $A$ is $v$ along $AB$. The velocity of $B$ is along $BC$. Its component along $BA$ is $v \cos 60° = v/2$. Thus, the separation $AB$ decreases at the rate

$$v + \frac{v}{2} = \frac{3v}{2}.$$

Since this rate is constant, the time taken in reducing the separation $AB$ from $d$ to zero is

$$t = \frac{d}{\frac{3v}{2}} = \frac{2d}{3v}.$$

□

## QUESTIONS FOR SHORT ANSWER

**1.** Galileo was punished by the Church for teaching that the sun is stationary and the earth moves around it. His opponents held the view that the earth is stationary and the sun moves around it. If the absolute motion has no meaning, are the two viewpoints not equally correct or equally wrong ?

**2.** When a particle moves with constant velocity, its average velocity, its instantaneous velocity and its speed are all equal. Comment on this statement.

**3.** A car travels at a speed of 60 km/hr due north and the other at a speed of 60 km/hr due east. Are the velocities equal ? If no, which one is greater ? If you find any of the questions irrelevant, explain.

**4.** A ball is thrown vertically upward with a speed of 20 m/s. Draw a graph showing the velocity of the ball as a function of time as it goes up and then comes back.

**5.** The velocity of a particle is towards west at an instant. Its acceleration is not towards west, not towards east, not towards north and not towards south. Give an example of this type of motion.

**6.** At which point on its path a projectile has the smallest speed ?

**7.** Two particles $A$ and $B$ start from rest and move for equal time on a straight line. The particle $A$ has an acceleration $a$ for the first half of the total time and $2a$ for the second half. The particle $B$ has an acceleration $2a$ for the first half and $a$ for the second half. Which particle has covered larger distance ?

**8.** If a particle is accelerating, it is either speeding up or speeding down. Do you agree with this statement ?

**9.** A food packet is dropped from a plane going at an altitude of 100 m. What is the path of the packet as seen from the plane ? What is the path as seen from the ground ? If someone asks “what is the actual path”, what will you answer ?

**10.** Give examples where (a) the velocity of a particle is zero but its acceleration is not zero, (b) the velocity is opposite in direction to the acceleration, (c) the velocity is perpendicular to the acceleration.

**11.** Figure (3-Q1) shows the $x$ coordinate of a particle as a function of time. Find the signs of $v_x$ and $a_x$ at $t = t_1$, $t = t_2$ and $t = t_3$.

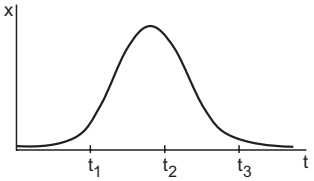



Figure 3-Q1

**12.** A player hits a baseball at some angle. The ball goes high up in space. The player runs and catches the ball before it hits the ground. Which of the two (the player or the ball) has greater displacement ?

**13.** The increase in the speed of a car is proportional to the additional petrol put into the engine. Is it possible to accelerate a car without putting more petrol or less petrol into the engine ?

**14.** Rain is falling vertically. A man running on the road keeps his umbrella tilted but a man standing on the street keeps his umbrella vertical to protect himself from the rain. But both of them keep their umbrella vertical to avoid the vertical sun-rays. Explain.

## OBJECTIVE I

**1.** A motor car is going due north at a speed of 50 km/h. It makes a 90° left turn without changing the speed. The change in the velocity of the car is about
(a) 50 km/h towards west
(b) 70 km/h towards south-west
(c) 70 km/h towards north-west
(d) zero.

**2.** Figure (3-Q2) shows the displacement-time graph of a particle moving on the $X$-axis.

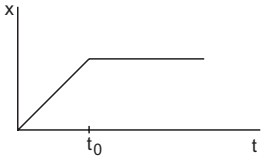



Figure 3-Q2

(a) the particle is continuously going in positive $x$ direction
(b) the particle is at rest
(c) the velocity increases up to a time $t_0$, and then becomes constant
(d) the particle moves at a constant velocity up to a time $t_0$, and then stops.

**3.** A particle has a velocity $u$ towards east at $t = 0$. Its acceleration is towards west and is constant. Let $x_A$ and $x_B$ be the magnitude of displacements in the first 10 seconds and the next 10 seconds
(a) $x_A < x_B$ (b) $x_A = x_B$ (c) $x_A > x_B$
(d) the information is insufficient to decide the relation of $x_A$ with $x_B$.

**4.** A person travelling on a straight line moves with a uniform velocity $v_1$ for some time and with uniform velocity $v_2$ for the next equal time. The average velocity $v$ is given by
(a) $v = \dfrac{v_1 + v_2}{2}$ (b) $v = \sqrt{v_1 v_2}$
(c) $\dfrac{2}{v} = \dfrac{1}{v_1} + \dfrac{1}{v_2}$ (d) $\dfrac{1}{v} = \dfrac{1}{v_1} + \dfrac{1}{v_2}$.

**5.** A person travelling on a straight line moves with a uniform velocity $v_1$ for a distance $x$ and with a uniform velocity $v_2$ for the next equal distance. The average velocity $v$ is given by
(a) $v = \dfrac{v_1 + v_2}{2}$ (b) $v = \sqrt{v_1 v_2}$
(c) $\dfrac{2}{v} = \dfrac{1}{v_1} + \dfrac{1}{v_2}$ (d) $\dfrac{1}{v} = \dfrac{1}{v_1} + \dfrac{1}{v_2}$.

**6.** A stone is released from an elevator going up with an acceleration $a$. The acceleration of the stone after the release is
(a) $a$ upward (b) $(g - a)$ upward
(c) $(g - a)$ downward (d) $g$ downward.

**7.** A person standing near the edge of the top of a building throws two balls $A$ and $B$. The ball $A$ is thrown vertically upward and $B$ is thrown vertically downward with the same speed. The ball $A$ hits the ground with a speed $v_A$ and the ball $B$ hits the ground with a speed $v_B$. We have
(a) $v_A > v_B$, (b) $v_A < v_B$ (c) $v_A = v_B$
(d) the relation between $v_A$ and $v_B$ depends on height of the building above the ground.

**8.** In a projectile motion the velocity
(a) is always perpendicular to the acceleration
(b) is never perpendicular to the acceleration
(c) is perpendicular to the acceleration for one instant only
(d) is perpendicular to the acceleration for two instants.

**9.** Two bullets are fired simultaneously, horizontally and with different speeds from the same place. Which bullet will hit the ground first ?
(a) the faster one (b) the slower one
(c) both will reach simultaneously
(d) depends on the masses.

**10.** The range of a projectile fired at an angle of 15° is 50 m. If it is fired with the same speed at an angle of 45°, its range will be
(a) 25 m (b) 37 m (c) 50 m (d) 100 m.

**11.** Two projectiles $A$ and $B$ are projected with angle of projection 15° for the projectile $A$ and 45° for the projectile $B$. If $R_A$ and $R_B$ be the horizontal range for the two projectiles, then
(a) $R_A < R_B$ (b) $R_A = R_B$ (c) $R_A > R_B$
(d) the information is insufficient to decide the relation of $R_A$ with $R_B$.

**12.** A river is flowing from west to east at a speed of 5 metres per minute. A man on the south bank of the river, capable of swimming at 10 metres per minute in still water, wants to swim across the river in the shortest time. He should swim in a direction

(a) due north (b) 30° east of north
(c) 30° north of west (d) 60° east of north.

**13.** In the arrangement shown in figure (3-Q3), the ends $P$ and $Q$ of an inextensible string move downwards with uniform speed $u$. Pulleys $A$ and $B$ are fixed. The mass $M$ moves upwards with a speed
(a) $2u\cos\theta$ (b) $u/\cos\theta$ (c) $2u/\cos\theta$ (d) $u\cos\theta$.

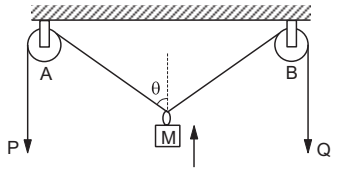


Figure 3-Q3

## OBJECTIVE II

**1.** Consider the motion of the tip of the minute hand of a clock. In one hour
(a) the displacement is zero
(b) the distance covered is zero
(c) the average speed is zero
(d) the average velocity is zero

**2.** A particle moves along the $X$-axis as
$$x = u(t - 2\text{ s}) + a(t - 2\text{ s})^2.$$
(a) the initial velocity of the particle is $u$
(b) the acceleration of the particle is $a$
(c) the acceleration of the particle is $2a$
(d) at $t$ = 2 s particle is at the origin.

**3.** Pick the correct statements :
(a) Average speed of a particle in a given time is never less than the magnitude of the average velocity.
(b) It is possible to have a situation in which $\left|\frac{d\vec{v}}{dt}\right| \neq 0$ but $\frac{d}{dt}|\vec{v}| = 0$.
(c) The average velocity of a particle is zero in a time interval. It is possible that the instantaneous velocity is never zero in the interval.
(d) The average velocity of a particle moving on a straight line is zero in a time interval. It is possible that the instantaneous velocity is never zero in the interval. (Infinite accelerations are not allowed.)

**4.** An object may have
(a) varying speed without having varying velocity
(b) varying velocity without having varying speed
(c) nonzero acceleration without having varying velocity
(d) nonzero acceleration without having varying speed.

**5.** Mark the correct statements for a particle going on a straight line :
(a) If the velocity and acceleration have opposite sign, the object is slowing down.
(b) If the position and velocity have opposite sign, the particle is moving towards the origin.
(c) If the velocity is zero at an instant, the acceleration should also be zero at that instant.
(d) If the velocity is zero for a time interval, the acceleration is zero at any instant within the time interval.

**6.** The velocity of a particle is zero at $t$ = 0.
(a) The acceleration at $t$ = 0 must be zero.
(b) The acceleration at $t$ = 0 may be zero.
(c) If the acceleration is zero from $t$ = 0 to $t$ = 10 s, the speed is also zero in this interval.
(d) If the speed is zero from $t$ = 0 to $t$ = 10 s the acceleration is also zero in this interval.

**7.** Mark the correct statements :
(a) The magnitude of the velocity of a particle is equal to its speed.
(b) The magnitude of average velocity in an interval is equal to its average speed in that interval.
(c) It is possible to have a situation in which the speed of a particle is always zero but the average speed is not zero.
(d) It is possible to have a situation in which the speed of the particle is never zero but the average speed in an interval is zero.

**8.** The velocity-time plot for a particle moving on a straight line is shown in the figure (3-Q4).

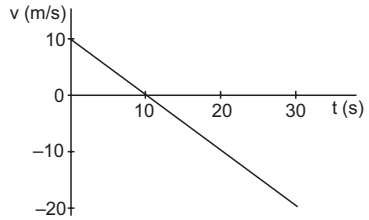


Figure 3-Q4

(a) The particle has a constant acceleration.
(b) The particle has never turned around.
(c) The particle has zero displacement.
(d) The average speed in the interval 0 to 10 s is the same as the average speed in the interval 10 s to 20 s.

**9.** Figure (3-Q5) shows the position of a particle moving on the $X$-axis as a function of time.
(a) The particle has come to rest 6 times.
(b) The maximum speed is at $t$ = 6 s.
(c) The velocity remains positive for $t$ = 0 to $t$ = 6 s.
(d) The average velocity for the total period shown is negative.

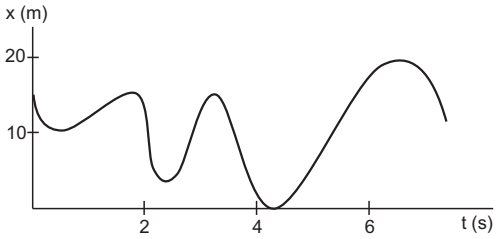



Figure 3-Q5

**10.** The accelerations of a particle as seen from two frames $S_1$ and $S_2$ have equal magnitude 4 m/s$^2$.
(a) The frames must be at rest with respect to each other.
(b) The frames may be moving with respect to each other but neither should be accelerated with respect to the other.
(c) The acceleration of $S_2$ with respect to $S_1$ may either be zero or 8 m/s$^2$.
(d) The acceleration of $S_2$ with respect to $S_1$ may be anything between zero and 8 m/s$^2$.

## EXERCISES

**1.** A man has to go 50 m due north, 40 m due east and 20 m due south to reach a field. (a) What distance he has to walk to reach the field ? (b) What is his displacement from his house to the field ?

**2.** A particle starts from the origin, goes along the $X$-axis to the point (20 m, 0) and then returns along the same line to the point (–20 m, 0). Find the distance and displacement of the particle during the trip.

**3.** It is 260 km from Patna to Ranchi by air and 320 km by road. An aeroplane takes 30 minutes to go from Patna to Ranchi whereas a delux bus takes 8 hours. (a) Find the average speed of the plane. (b) Find the average speed of the bus. (c) Find the average velocity of the plane. (d) Find the average velocity of the bus.

**4.** When a person leaves his home for sightseeing by his car, the meter reads 12352 km. When he returns home after two hours the reading is 12416 km. (a) What is the average speed of the car during this period ? (b) What is the average velocity ?

**5.** An athelete takes 2·0 s to reach his maximum speed of 18·0 km/h. What is the magnitude of his average acceleration ?

**6.** The speed of a car as a function of time is shown in figure (3-E1). Find the distance travelled by the car in 8 seconds and its acceleration.

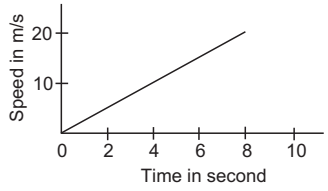



Figure 3-E1

**7.** The acceleration of a cart started at $t = 0$, varies with time as shown in figure (3-E2). Find the distance travelled in 30 seconds and draw the position-time graph.

**8.** Figure (3-E3) shows the graph of velocity versus time for a particle going along the $X$-axis. Find (a) the acceleration, (b) the distance travelled in 0 to 10 s and (c) the displacement in 0 to 10 s.

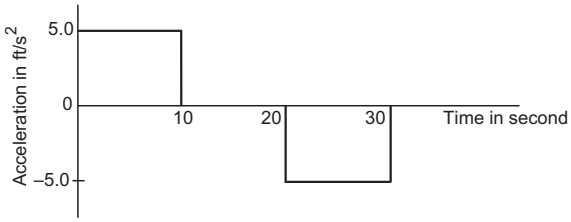



Figure 3-E2

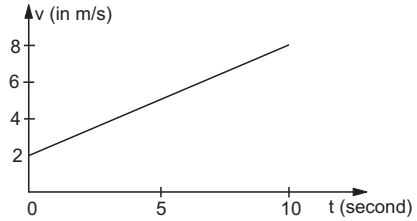



Figure 3-E3

**9.** Figure (3-E4) shows the graph of the $x$-coordinate of a particle going along the $X$-axis as a function of time. Find (a) the average velocity during 0 to 10 s, (b) instantaneous velocity at 2, 5, 8 and 12s.

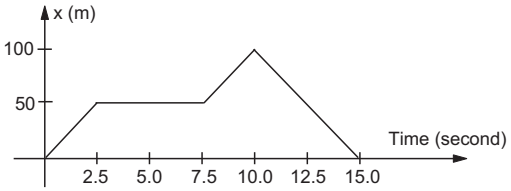



Figure 3-E4

**10.** From the velocity–time plot shown in figure (3-E5), find the distance travelled by the particle during the first 40

seconds. Also find the average velocity during this period.

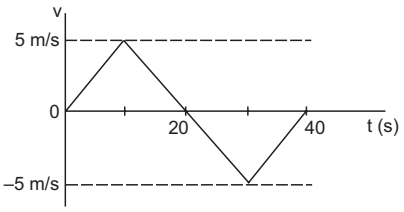



Figure 3-E5

**11.** Figure (3-E6) shows $x$-$t$ graph of a particle. Find the time $t$ such that the average velocity of the particle during the period 0 to $t$ is zero.

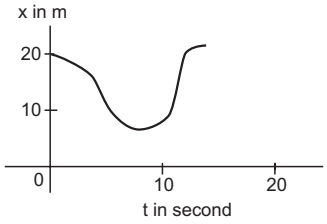



Figure 3-E6

**12.** A particle starts from a point $A$ and travels along the solid curve shown in figure (3-E7). Find approximately the position $B$ of the particle such that the average velocity between the positions $A$ and $B$ has the same direction as the instantaneous velocity at $B$.

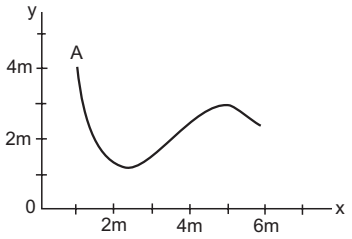



Figure 3-E7

**13.** An object having a velocity 4·0 m/s is accelerated at the rate of 1·2 m/s$^2$ for 5·0 s. Find the distance travelled during the period of acceleration.

**14.** A person travelling at 43·2 km/h applies the brake giving a deceleration of 6·0 m/s$^2$ to his scooter. How far will it travel before stopping ?

**15.** A train starts from rest and moves with a constant acceleration of 2·0 m/s$^2$ for half a minute. The brakes are then applied and the train comes to rest in one minute. Find (a) the total distance moved by the train, (b) the maximum speed attained by the train and (c) the position(s) of the train at half the maximum speed.

**16.** A bullet travelling with a velocity of 16 m/s penetrates a tree trunk and comes to rest in 0·4 m. Find the time taken during the retardation.

**17.** A bullet going with speed 350 m/s enters a concrete wall and penetrates a distance of 5·0 cm before coming to rest. Find the deceleration.

**18.** A particle starting from rest moves with constant acceleration. If it takes 5·0 s to reach the speed 18·0 km/h find (a) the average velocity during this period, and (b) the distance travelled by the particle during this period.

**19.** A driver takes 0·20 s to apply the brakes after he sees a need for it. This is called the reaction time of the driver. If he is driving a car at a speed of 54 km/h and the brakes cause a deceleration of 6·0 m/s$^2$, find the distance travelled by the car after he sees the need to put the brakes on.

**20.** Complete the following table :

| Car Model | Driver $X$<br>Reaction time 0·20 s | Driver $Y$<br>Reaction time 0·30 s |
|---|---|---|
| $A$ (deceleration on hard braking = 6·0 m/s$^2$) | Speed = 54 km/h<br>Braking distance<br>$a$ = .....<br>Total stopping distance<br>$b$ = ..... | Speed = 72 km/h<br>Braking distance<br>$c$ = .....<br>Total stopping distance<br>$d$ = ..... |
| $B$ (deceleration on hard braking = 7·5 m/s$^2$) | Speed = 54 km/h<br>Braking distance<br>$e$ = .....<br>Total stopping distance<br>$f$ = ..... | Speed = 72km/h<br>Braking distance<br>$g$ = .....<br>Total stopping distance<br>$h$ = ..... |

**21.** A police jeep is chasing a culprit going on a motorbike. The motorbike crosses a turning at a speed of 72 km/h. The jeep follows it at a speed of 90 km/h, crossing the turning ten seconds later than the bike. Assuming that they travel at constant speeds, how far from the turning will the jeep catch up with the bike ?

**22.** A car travelling at 60 km/h overtakes another car travelling at 42 km/h. Assuming each car to be 5·0 m long, find the time taken during the overtake and the total road distance used for the overtake.

**23.** A ball is projected vertically upward with a speed of 50 m/s. Find (a) the maximum height, (b) the time to reach the maximum height, (c) the speed at half the maximum height. Take $g$ = 10 m/s$^2$.

**24.** A ball is dropped from a balloon going up at a speed of 7 m/s. If the balloon was at a height 60 m at the time of dropping the ball, how long will the ball take in reaching the ground ?

**25.** A stone is thrown vertically upward with a speed of 28 m/s. (a) Find the maximum height reached by the stone. (b) Find its velocity one second before it reaches the maximum height. (c) Does the answer of part (b) change if the initial speed is more than 28 m/s such as 40 m/s or 80 m/s ?

**26.** A person sitting on the top of a tall building is dropping balls at regular intervals of one second. Find the positions of the 3rd, 4th and 5th ball when the 6th ball is being dropped.

**27.** A healthy youngman standing at a distance of 7 m from a 11·8 m high building sees a kid slipping from the top floor. With what speed (assumed uniform) should he run to catch the kid at the arms height (1·8 m) ?

**28.** An NCC parade is going at a uniform speed of 6 km/h through a place under a berry tree on which a bird is sitting at a height of 12·1 m. At a particular instant the bird drops a berry. Which cadet (give the distance from the tree at the instant) will receive the berry on his uniform ?

**29.** A ball is dropped from a height. If it takes 0·200 s to cross the last 6·00 m before hitting the ground, find the height from which it was dropped. Take $g = 10\ m/s^2$.

**30.** A ball is dropped from a height of 5 m onto a sandy floor and penetrates the sand up to 10 cm before coming to rest. Find the retardation of the ball in sand assuming it to be uniform.

**31.** An elevator is descending with uniform acceleration. To measure the acceleration, a person in the elevator drops a coin at the moment the elevator starts. The coin is 6 ft above the floor of the elevator at the time it is dropped. The person observes that the coin strikes the floor in 1 second. Calculate from these data the acceleration of the elevator.

**32.** A ball is thrown horizontally from a point 100 m above the ground with a speed of 20 m/s. Find (a) the time it takes to reach the ground, (b) the horizontal distance it travels before reaching the ground, (c) the velocity (direction and magnitude) with which it strikes the ground.

**33.** A ball is thrown at a speed of 40 m/s at an angle of 60° with the horizontal. Find (a) the maximum height reached and (b) the range of the ball. Take $g = 10\ m/s^2$.

**34.** In a soccer practice session the football is kept at the centre of the field 40 yards from the 10 ft high goalposts. A goal is attempted by kicking the football at a speed of 64 ft/s at an angle of 45° to the horizontal. Will the ball reach the goal post ?

**35.** A popular game in Indian villages is *goli* which is played with small glass balls called golis. The goli of one player is situated at a distance of 2·0 m from the goli of the second player. This second player has to project his goli by keeping the thumb of the left hand at the place of his goli, holding the goli between his two middle fingers and making the throw. If the projected goli hits the goli of the first player, the second player wins. If the height from which the goli is projected is 19·6 cm from the ground and the goli is to be projected horizontally, with what speed should it be projected so that it directly hits the stationary goli without falling on the ground earlier ?

**36.** Figure (3-E8) shows a 11·7 ft wide ditch with the approach roads at an angle of 15° with the horizontal. With what minimum speed should a motorbike be moving on the road so that it safely crosses the ditch ? Assume that the length of the bike is 5 ft, and it leaves the road when the front part runs out of the approach road.

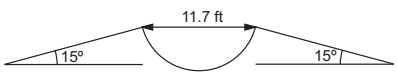



Figure 3-E8

**37.** A person standing on the top of a cliff 171 ft high has to throw a packet to his friend standing on the ground 228 ft horizontally away. If he throws the packet directly aiming at the friend with a speed of 15·0 ft/s, how short will the packet fall ?

**38.** A ball is projected from a point on the floor with a speed of 15 m/s at an angle of 60° with the horizontal. Will it hit a vertical wall 5 m away from the point of projection and perpendicular to the plane of projection without hitting the floor ? Will the answer differ if the wall is 22 m away ?

**39.** Find the average velocity of a projectile between the instants it crosses half the maximum height. It is projected with a speed $u$ at an angle $\theta$ with the horizontal.

**40.** A bomb is dropped from a plane flying horizontally with uniform speed. Show that the bomb will explode vertically below the plane. Is the statement true if the plane flies with uniform speed but not horizontally ?

**41.** A boy standing on a long railroad car throws a ball straight upwards. The car is moving on the horizontal road with an acceleration of $1\ m/s^2$ and the projection velocity in the vertical direction is 9·8 m/s. How far behind the boy will the ball fall on the car ?

**42.** A staircase contains three steps each 10 cm high and 20 cm wide (figure 3-E9). What should be the minimum horizontal velocity of a ball rolling off the uppermost plane so as to hit directly the lowest plane ?

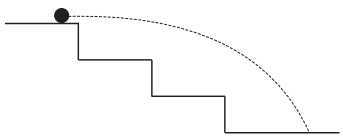


Figure 3-E9

**43.** A person is standing on a truck moving with a constant velocity of 14·7 m/s on a horizontal road. The man throws a ball in such a way that it returns to the truck after the truck has moved 58·8 m. Find the speed and the angle of projection (a) as seen from the truck, (b) as seen from the road.

**44.** The benches of a gallery in a cricket stadium are 1 m wide and 1 m high. A batsman strikes the ball at a level one metre above the ground and hits a mammoth sixer. The ball starts at 35 m/s at an angle of 53° with the horizontal. The benches are perpendicular to the plane of motion and the first bench is 110 m from the batsman. On which bench will the ball hit ?

**45.** A man is sitting on the shore of a river. He is in the line of a 1·0 m long boat and is 5·5 m away from the centre of the boat. He wishes to throw an apple into the boat. If he can throw the apple only with a speed of 10 m/s, find the minimum and maximum angles of projection for successful shot. Assume that the point of

projection and the edge of the boat are in the same horizontal level.

**46.** A river 400 m wide is flowing at a rate of 2·0 m/s. A boat is sailing at a velocity of 10 m/s with respect to the water, in a direction perpendicular to the river. (a) Find the time taken by the boat to reach the opposite bank. (b) How far from the point directly opposite to the starting point does the boat reach the opposite bank ?

**47.** A swimmer wishes to cross a 500 m wide river flowing at 5 km/h. His speed with respect to water is 3 km/h. (a) If he heads in a direction making an angle $\theta$ with the flow, find the time he takes to cross the river. (b) Find the shortest possible time to cross the river.

**48.** Consider the situation of the previous problem. The man has to reach the other shore at the point directly opposite to his starting point. If he reaches the other shore somewhere else, he has to walk down to this point. Find the minimum distance that he has to walk.

**49.** An aeroplane has to go from a point $A$ to another point $B$, 500 km away due 30° east of north. A wind is blowing due north at a speed of 20 m/s. The air-speed of the plane is 150 m/s. (a) Find the direction in which the pilot should head the plane to reach the point $B$. (b) Find the time taken by the plane to go from $A$ to $B$.

**50.** Two friends $A$ and $B$ are standing a distance $x$ apart in an open field and wind is blowing from $A$ to $B$. $A$ beats a drum and $B$ hears the sound $t_1$ time after he sees the event. $A$ and $B$ interchange their positions and the experiment is repeated. This time $B$ hears the drum $t_2$ time after he sees the event. Calculate the velocity of sound in still air $v$ and the velocity of wind $u$. Neglect the time light takes in travelling between the friends.

**51.** Suppose $A$ and $B$ in the previous problem change their positions in such a way that the line joining them becomes perpendicular to the direction of wind while maintaining the separation $x$. What will be the time lag $B$ finds between seeing and hearing the drum beating by $A$ ?

**52.** Six particles situated at the corners of a regular hexagon of side $a$ move at a constant speed $v$. Each particle maintains a direction towards the particle at the next corner. Calculate the time the particles will take to meet each other.

□

## ANSWERS

### OBJECTIVE I

1. (b) 2. (d) 3. (d) 4. (a) 5. (c) 6. (d)
7. (c) 8. (c) 9. (c) 10. (d) 11. (d) 12. (a)
13. (b)

### OBJECTIVE II

1. (a), (d) 2. (c), (d) 3. (a), (b), (c)
4. (b), (d) 5. (a), (b), (d) 6. (b), (c), (d)
7. (a) 8. (a), (d) 9. (a)
10. (d)

### EXERCISES

1. (a) 110 m (b) 50 m, $\tan^{-1} 3/4$ north to east
2. 60 m, 20 m in the negative direction
3. (a) 520 km/h (b) 40 km/h
   (c) 520 km/h Patna to Ranchi
   (d) 32·5 km/h Patna to Ranchi
4. 32 km/h (b) zero
5. 2·5 m/s$^2$
6. 80 m, 2·5 m/s$^2$
7. 1000 ft
8. (a) 0·6 m/s$^2$ (b) 50 m (c) 50 m
9. (a) 10 m/s (b) 20 m/s, zero, 20 m/s, – 20 m/s
10. 100 m, zero
11. 12 s
12. $x$ = 5 m, $y$ = 3 m
13. 35 m
14. 12 m
15. (a) 2·7 km (b) 60 m/s (c) 225 m and 2·25 km
16. 0·05 s
17. $12{\cdot}2 \times 10^5$ m/s$^2$
18. (a) 2·5 m/s (b) 12·5 m
19. 22 m
20. (a) 19 m (b) 22 m (c) 33 m (d) 39 m
    (e) 15 m (f) 18 m (g) 27 m (h) 33 m
21. 1·0 km
22. 2 s, 38 m
23. (a) 125 m (b) 5 s (c) 35 m/s
24. 4·3 s
25. (a) 40 m (b) 9·8 m/s (c) No
26. 44·1 m, 19·6 m and 4·9 m below the top
27. 4·9 m/s
28. 2·62 m
29. 48 m
30. 490 m/s$^2$
31. 20 ft/s$^2$
32. (a) 4·5 s (b) 90 m (c) 49 m/s, $\theta$ = 66° with horizontal
33. (a) 60 m (b) $80\sqrt{3}$ m
34. Yes
35. 10 m/s
36. 32 ft/s

37. 192 ft
38. Yes, Yes
39. $u\cos\theta$, horizontal in the plane of projection
41. 2 m
42. 2 m/s
43. (a) 19·6 m/s upward
    (b) 24·5 m/s at 53° with horizontal
44. Sixth
45. Minimum angle 15°, maximum angle 75° but there is an interval of 53° between 15° and 75°, which is not allowed for successful shot
46. (a) 40 s (b) 80 m
47. (a) $\frac{10 \text{ minutes}}{\sin\theta}$ (b) 10 minutes
48. 2/3 km
49. (a) $\sin^{-1}(1/15)$ east of the line $AB$ (b) 50 min
50. $\frac{x}{2}\left(\frac{1}{t_1}+\frac{1}{t_2}\right), \frac{x}{2}\left(\frac{1}{t_1}-\frac{1}{t_2}\right)$
51. $\frac{x}{\sqrt{v^2-u^2}}$
52. $2a/v$.

□

## CHAPTER 4

# THE FORCES

### 4.1 INTRODUCTION

Force is a word which we have all heard about. When you push or pull some object you exert a force on it. If you push a body you exert a force away from yourself; when you pull, you exert a force toward yourself. When you hold a heavy block in your hand you exert a large force; when you hold a light block, you exert a small force.

Can nonliving bodies exert a force ? Yes, they can. If we stand in a great storm, we feel that the wind is exerting a force on us. When we suspend a heavy block from a rope, the rope holds the block just as a man can hold it in air. When we comb our dry hair and bring the comb close to small pieces of paper, the pieces jump to the comb. The comb has attracted the paper pieces i.e. the comb has exerted force on the pieces. When a cork is dipped in water it comes to the surface; if we want to keep it inside water, we have to push it downward. We say that water exerts a force on the cork in the upward direction.

The SI unit for measuring the force is called a *newton.* Approximately, it is the force needed to hold a body of mass 102 g near the earth's surface. An accurate quantitative definition can be framed using Newton's laws of motion to be studied in the next chapter.

Force is an interaction between two objects. Force is exerted by an object $A$ on another object $B$. For any force you may ask two questions, (i) who exerted this force and (ii) on which object was this force exerted ? Thus, when a block is kept on a table, the table exerts a force on the block to hold it.

Force is a vector quantity and if more than one forces act on a particle we can find the resultant force using the laws of vector addition. Note that in all the examples quoted above, if a body $A$ exerts a force on $B$, the body $B$ also exerts a force on $A$. Thus, the table exerts a force on the block to hold it and the block exerts a force on the table to press it down. When a heavy block is suspended by a rope, the rope exerts a force on the block to hold it and the block exerts a force on the rope to make it tight and stretched. In fact these are a few examples of Newton's third law of motion which may be stated as follows.

**Newton's Third Law of Motion**

*If a body A exerts a force* $\vec{F}$ *on another body B, then B exerts a force* $-\vec{F}$ *on A, the two forces acting along the line joining the bodies.*

The two forces $\vec{F}$ and $-\vec{F}$ connected by Newton's third law are called *action-reaction pair*. Any one may be called 'action' and the other 'reaction'.

We shall discuss this law in greater detail in the next chapter.

The various types of forces in nature can be grouped in four categories :

(a) Gravitational, (b) Electromagnetic,
(c) Nuclear and (d) Weak.

### 4.2 GRAVITATIONAL FORCE

Any two bodies attract each other by virtue of their masses. The force of attraction between two point masses is $F = G\frac{m_1 m_2}{r^2}$, where $m_1$ and $m_2$ are the masses of the particles and $r$ is the distance between them. $G$ is a universal constant having the value $6{\cdot}67 \times 10^{-11}$ N-m$^2$/kg$^2$. To find the gravitational force on an extended body by another such body, we have to write the force on each particle of the 1st body by all the particles of the second body and then we have to sum up vectorially all the forces acting on the first body. For example, suppose each body contains just three particles, and let $\vec{F}_{ij}$ denote the force on the $i$ th particle of the first body due to the $j$ th particle of the second body. To find the resultant force on the first body (figure 4.1), we have to add the following 9 forces :

$$\vec{F}_{11},\ \vec{F}_{12},\ \vec{F}_{13},\ \vec{F}_{21},\ \vec{F}_{22},\ \vec{F}_{23},\ \vec{F}_{31},\ \vec{F}_{32},\ \vec{F}_{33}.$$

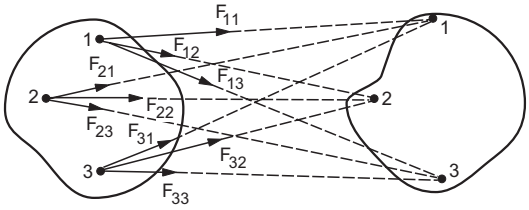



Figure 4.1

For large bodies having a large number of particles, we have to add quite a large number of forces. If the bodies are assumed continuous (a good approximation in our course), one has to go through the integration process for the infinite summation involved. However, the integration yields a particularly simple result for a special case which is of great practical importance and we quote it below. The proof of this result will be given in a later chapter.

*The gravitational force exerted by a spherically symmetric body of mass $m_1$ on another such body of mass $m_2$ kept outside the first body is $G\frac{m_1m_2}{r^2}$, where r is the distance between the centres of the two bodies. Thus, for the calculation of gravitational force between two spherically symmetric bodies, they can be treated as point masses placed at their centres.*

**Gravitational Force on Small Bodies by the Earth**

The force of attraction exerted by the earth on other objects is called *gravity*. Consider the earth to be a homogeneous sphere of radius $R$ and mass $M$. The values of $R$ and $M$ are roughly 6400 km and $6 \times 10^{24}$ kg respectively. Assuming that the earth is spherically symmetric, the force it exerts on a particle of mass $m$ kept near its surface is by the previous result, $F = G\frac{Mm}{R^2}$. The direction of this force is towards the centre of the earth which is called the *vertically downward* direction.

The same formula is valid to a good approximation even if we have a body of some other shape instead of a particle, provided the body is very small in size as compared to the earth. The quantity $G\frac{M}{R^2}$ is a constant and has the dimensions of acceleration. It is called the *acceleration due to gravity*, and is denoted by the letter $g$ (a quantity much different from $G$). Its value is approximately 9·8 m/s$^2$. For simplicity of calculations we shall often use $g$ = 10 m/s$^2$. We shall find in the next chapter that all bodies falling towards earth (remaining all the time close to the earth's surface) have this particular value of acceleration and hence the name acceleration due to gravity. Thus, the force exerted by the earth on a small body of mass $m$, kept near the earth's surface is $mg$ in the vertically downward direction.

The gravitational constant $G$ is so small that the gravitational force becomes appreciable only if at least one of the two bodies has a large mass. To have an idea of the magnitude of gravitational forces in practical life, consider two small bodies of mass 10 kg each, separated by 0·5 m. The gravitational force is

$$F = \frac{6{\cdot}7 \times 10^{-11}\ \text{N–m}^2/\text{kg}^2 \times 10^2\ \text{kg}^2}{0{\cdot}25\ \text{m}^2}$$

$$= 2{\cdot}7 \times 10^{-8}\ \text{N}$$

a force needed to hold about 3 microgram. In many of the situations we encounter, it is a good approximation to neglect all the gravitational forces other than that exerted by the earth.

### 4.3 ELECTROMAGNETIC (EM) FORCE

Over and above the gravitational force $G\frac{m_1m_2}{r^2}$, the particles may exert upon each other electromagnetic forces. If two particles having charges $q_1$ and $q_2$ are at rest with respect to the observer, the force between them has a magnitude

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r^2}$$

where $\varepsilon = 8{\cdot}85419 \times 10^{-12}\ \text{C}^2/\text{N–m}^2$ is a constant. The quantity $\frac{1}{4\pi\varepsilon_0}$ is $9{\cdot}0 \times 10^9\ \frac{\text{N–m}^2}{\text{C}^2}$.

This is called *Coulomb force* and it acts along the line joining the particles. If $q_1$ and $q_2$ are of same nature (both positive or both negative), the force is repulsive otherwise it is attractive. It is this force which is responsible for the attraction of small paper pieces when brought near a recently used comb. The electromagnetic force between moving charged paritcles is comparatively more complicated and contains terms other than the Coulomb force.

Ordinary matter is composed of electrons, protons and neutrons. Each electron has $1{\cdot}6 \times 10^{-19}$ coulomb of negative charge and each proton has an equal amount of positive charge. In atoms, the electrons are bound by the electromagnetic force acting on them due to the protons. The atoms combine to form molecules due to the electromagnetic forces. A lot of atomic and molecular phenomena result from electromagnetic forces between the subatomic particles (electrons, protons, charged mesons, etc.).

Apart from the atomic and molecular phenomena, the electromagnetic forces show up in many forms in

daily experience. Some examples having practical importance given below.

**(a) Forces between Two Surfaces in Contact**

When we put two bodies in contact with each other, the atoms at the two surfaces come close to each other. The charged constituents of the atoms of the two bodies exert great forces on each other and a measurable force results out of it. We say that the two bodies in contact exert forces on each other. When you place a book on a table, the table exerts an upward force on the book to hold it. This force comes from the electromagnetic forces acting between the atoms and molecules of the surface of the book and of the table.

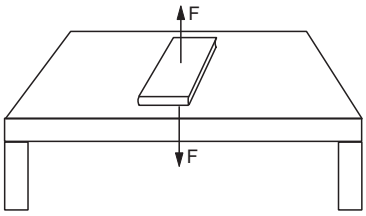



Figure 4.2

Generally, the forces between the two objects in contact are along the common normal (perpendicular) to the surfaces of contact and is that of a push or repulsion. Thus, the table pushes the book away from it (i.e., upward) and the book pushes the table downward (again away from it).

However, the forces between the two bodies in contact may have a component parallel to the surface of contact. This component is known as *friction*. We assume existence of frictionless surfaces which can exert forces only along the direction perpendicular to them. The bodies with smooth surfaces can exert only small amount of forces parallel to the surface and hence are close to frictionless surface. Thus, it is difficult to stay on a smooth metallic lamp-post, because it cannot exert enough vertical force and so it will not hold you there. The same is not true if you try to stay on the trunk of a tree which is quite rough. We shall often use the word smooth to mean frictionless.

The contact forces obey Newton's third law. Thus the book in figure (4.2) exerts a downward force $F$ on the table to press it down and the table exerts an equal upward force $F$ on the book to hold it there. When you stay on the trunk of a tree, it exerts a frictional upward force (frictional force because it is parallel to the surface of the tree) on you to hold you there, and you exert an equal frictional downward force on the tree.

**(b) Tension in a String or a Rope**

In a tug of war, two persons hold the two ends of a rope and try to pull the rope on their respective sides. The rope becomes tight and its length is slightly increased. In many situations this increase is very small and goes undetected. Such a stretched rope is said to be in a state of tension.

Similarly, if a heavy block hangs from a ceiling by a string, the string is in a state of tension. The electrons and protons of the string near the lower end exert forces on the electrons and protons of the block and the resultant of these forces is the force exerted by the string on the block. It is the resultant of these electromagnetic forces which supports the block and prevents it from falling. A string or rope under tension exerts electromagnetic forces on the bodies attached at the two ends to *pull* them.

**(c) Force due to a Spring**

When a metallic wire is coiled it becomes a spring. The straight line distance between the ends of a spring is called its length. If a spring is placed on a horizontal surface with no horizontal force on it, its length is called the *natural length*. Every spring has its own natural length. The spring can be stretched to increase its length and it can be compressed to decrease its length. When a spring is stretched, it pulls the bodies attached to its ends and when compressed, it pushes the bodies attached to its ends. If the extension or the compression is not too large, the force exerted by the spring is proportional to the change in its length. Thus, if the spring has a length $x$ and its natural length is $x_0$ the magnitude of the force exerted by it will be

$$F = k|x - x_0| = k|\Delta x|.$$

If the spring is extended, the force will be directed towards its centre and if compressed, it will be directed away from the centre. The proportionality constant $k$, which is the force per unit extension or compression, is called the *spring constant* of the spring. This force again comes into picture due to the electromagnetic forces between the atoms of the material.

The macroscopic bodies which we have to generally deal with are electrically neutral. Hence two bodies not in contact do not exert appreciable electromagnetic forces. The forces between the charged particles of the first body and those of the second body have both attractive and repulsive nature and hence they largely cancel each other. This is not the case with gravitational forces. The gravitational forces between the particles of one body and those of the other body are all attractive and hence they add to give an appreciable gravitational force in many cases. Thus, the gravitational force between the earth and a 1 kg

block kept 100 m above the earth's surface is about 9·8 N whereas the electromagnetic force between the earth and this block is almost zero even though both these bodies contain a very large number of charged particles, the electrons and the protons.

***Example 4.1***

*Suppose the exact charge neutrality does not hold in a world and the electron has a charge* 1% *less in magnitude than the proton. Calculate the Coulomb force acting between two blocks of iron each of mass* 1 kg *separated by a distance of* 1 m. *Number of protons in an iron atom* = 26 *and* 58 kg *of iron contains* $6 \times 10^{26}$ *atoms.*

***Solution :*** Each atom of iron will have a net positive charge $26 \times 0{\cdot}01 \times 1{\cdot}6 \times 10^{-19}$ C on it in the assumed world. The total positive charge on a 1 kg block would be

$$\frac{6 \times 10^{26}}{58} \times 26 \times 1{\cdot}6 \times 10^{-21} \text{ C}$$

$$= 4{\cdot}3 \times 10^{5} \text{ C}.$$

The Coulomb force between the two blocks is

$$= \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2} = \frac{9{\cdot}0 \times 10^{9} \text{ N–m}^2/\text{C}^2 \times (4{\cdot}3 \times 10^{5} \text{ C})^2}{(1 \text{ m})^2}$$

$$= 9 \times 10^{9} \times 18{\cdot}49 \times 10^{10} \text{ N}$$

$$= 1{\cdot}7 \times 10^{21} \text{ N}.$$

A tremendous force indeed !

## 4.4 NUCLEAR FORCES

Each atom contains a certain number of protons and neutrons in its nucleus. The nucleus occupies a volume of about $10^{-44}$ m$^3$ whereas the atom itself has a volume of about $10^{-23}$ m$^3$. Thus, the nucleus occupies only $1/10^{21}$ of the volume of the atom. Yet it contains about 99·98% of the mass of the atom. The atomic nucleus of a non-radioactive element is a stable particle. For example, if both the electrons are removed from a helium atom, we get the bare nucleus of helium which is called an *alpha particle.* The alpha particle is a stable object and once created it can remain intact until it is not made to interact with other objects.

An alpha particle contains two protons and two neutrons. The protons will repel each other due to the Coulomb force and will try to break the nucleus. Neutrons will be silent spectators in this electromagnetic drama (Remember, neutron is an uncharged particle). Then, why does the Coulomb force fail to break the nucleus ? Can it be the gravitational attractive force which keeps the nucleus bound ? All the protons and the neutrons will take part in this attraction, but if calculated, the gravitational attraction will turn out to be totally negligible as compared to the Coulomb repulsion.

In fact, a third kind of force, altogether different and over and above the gravitational and electromagnetic force, is operating here. These forces are called *Nuclear forces* and are exerted only if the interacting particles are protons or neutrons or both. (There are some more cases where this force operates but we shall not deal with them.) These forces are largely attractive, but are short ranged. The forces are much weaker than the Coulomb force if the separation between the particles is more than say $10^{-14}$ m. But for smaller separation ($\approx 10^{-15}$ m) the nuclear force is much stronger than the Coulomb force and being attractive it holds the nucleus stable.

Being short ranged, these forces come into picture only if the changes within the nucleus are discussed. As bare nuclei are less frequently encountered in daily life, one is generally unaware of these forces. Radioactivity, nuclear energy (fission, fusion) etc. result from nuclear forces.

## 4.5 WEAK FORCES

Yet another kind of forces is encountered when reactions involving protons, electrons and neutrons take place. A neutron can change itself into a proton and simultaneously emit an electron and a particle called *antinutrino*. This is called $\beta^-$ decay. Never think that a neutron is made up of a proton, an electron and an antineutrino. A proton can also change into neutron and simultaneously emit a positron (and a neutrino). This is called $\beta^+$ decay. The forces responsible for these changes are different from gravitational, electromagnetic or nuclear forces. Such forces are called weak forces. The range of weak forces is very small, in fact much smaller than the size of a proton or a neutron. Thus, its effect is experienced inside such particles only.

## 4.6 SCOPE OF CLASSICAL PHYSICS

The behaviour of all the bodies of linear sizes greater than $10^{-6}$ m are adequately described on the basis of relatively a small number of very simple laws of nature. These laws are the Newton's laws of motion, Newton's law of gravitation, Maxwell's electromagnetism, Laws of thermodynamics and the Lorentz force. The principles of physics based on them is called the *classical physics*. The formulation of classical physics is quite accurate for heavenly bodies like the sun, the earth, the moon etc. and is equally good for the behaviour of grains of sand and the raindrops. However, for the subatomic particles much smaller

than $10^{-6}$ m (such as atoms, nuclei etc.) these rules do not work well. The behaviour of such particles is governed by *quantum physics*. In fact, at such short dimensions the very concept of "particle" breaks down. The perception of the nature is altogether different at this scale. The validity of classical physics also depends on the velocities involved. The classical mechanics as formulated by Newton has to be considerably changed when velocities comparable to $3 \times 10^8$ m/s are involved. This is the speed of light in vacuum and is the upper limit of speed which material particle can ever reach. No matter how great and how long you apply a force, you can never get a particle going with a speed greater than $3 \times 10^8$ m/s. The mechanics of particles moving with these large velocities is known as *relativistic mechanics* and was formulated by Einstein in 1905.

Thus, classical physics is a good description of the nature if we are concerned with the particles of linear size > $10^{-6}$ m moving with velocities < $10^8$ m/s. In a major part of this book, we shall work within these restrictions and hence learn the techniques of classical physics. The size restriction automatically excludes any appreciable effects of nuclear or weak forces and we need to consider only the gravitational and electromagnetic forces. We might consider the subatomic particles here and there but shall assume the existence of gravitational and electromagnetic forces only and that classical physics is valid for these particles. The results arrived at by our analysis may only be approximately true because we shall be applying the laws which are not correct in that domain. But even that may play an important role in the understanding of nature. We shall also assume that the Newton's third law is valid for the forces which we shall be dealing with. In the final chapters we shall briefly discuss quantum physics and some of its important consequences.

## *Worked Out Examples*

**1.** *Figure (4-W1) shows two hydrogen atoms. Show on a separate diagram all the electric forces acting on different particles of the system.*

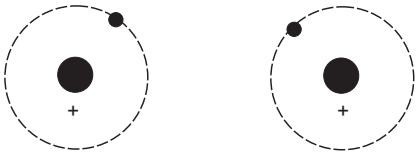



Figure 4-W1

***Solution*** **:** Each particle exerts electric forces on the remaining three particles. Thus there exist 4 × 3 = 12 forces in all. Figure (4-W2) shows them.

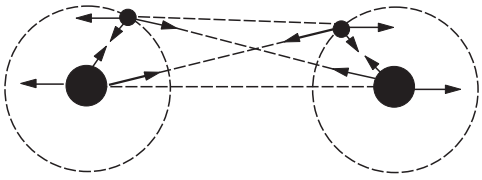


Figure 4-W2

**2.** *Figure (4-W3) shows two rods each of length l placed side by side, with their facing ends separated by a distance a. Charges + q, – q reside on the rods as shown. Calculate the electric force on the rod A due to the rod B. Discuss the cases when l>>a, a>>l.*

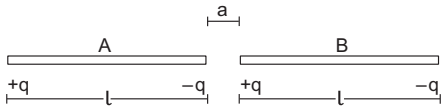



Figure 4-W3

***Solution*** **:** The force on the rod $A$ due to the charge $+q$ of the rod $B$

$$= -\frac{q^2}{4\pi\varepsilon_0 (l+a)^2} + \frac{q^2}{4\pi\varepsilon_0 a^2}$$

towards right. The force on this rod due to the charge $-q$

$$= \frac{q^2}{4\pi\varepsilon_0 (2l+a)^2} - \frac{q^2}{4\pi\varepsilon_0 (l+a)^2}$$

towards right.

The resultant force on the rod is

$$F = \frac{q^2}{4\pi\varepsilon_0}\left[\frac{1}{a^2} - \frac{2}{(l+a)^2} + \frac{1}{(2l+a)^2}\right] \text{ towards right.}$$

If $l >> a$, the last two terms in the square bracket are negligible as compared to the first term. Then,

$$F \approx \frac{q^2}{4\pi\varepsilon_0 a^2}.$$

If $a >> l$

$$F \approx \frac{q^2}{4\pi\varepsilon_0}\left[\frac{1}{a^2} - \frac{2}{a^2} + \frac{1}{a^2}\right] \approx 0.$$

Two neutral objects placed far away exert only negligible force on each other but when they are placed closer they may exert appreciable force.

**3.** *Calculate the ratio of electric to gravitational force between two electrons.*

***Solution*** **:** The electric force $= \dfrac{e^2}{4\pi\varepsilon_0 r^2}$

and the gravitational force $= \dfrac{G(m_e)^2}{r^2}$.

The ratio is $\dfrac{e^2}{4\pi\varepsilon_0 G(m_e)^2}$

$$= \frac{9\times10^9\ \frac{\text{N–m}^2}{\text{C}^2}\times(1{\cdot}6\times10^{-19}\ \text{C})^2}{6{\cdot}67\times10^{-11}\ \frac{\text{N–m}^2}{\text{kg}^2}\times(9{\cdot}1\times10^{-31}\ \text{kg})^2} = 4{\cdot}17\times10^{42}.$$

□

## QUESTIONS FOR SHORT ANSWER

**1.** A body of mass $m$ is placed on a table. The earth is pulling the body with a force $mg$. Taking this force to be the action what is the reaction ?

**2.** A boy is sitting on a chair placed on the floor of a room. Write as many action-reaction pairs of forces as you can.

**3.** A lawyer alleges in court that the police has forced his client to issue a statement of confession. What kind of force is this ?

**4.** When you hold a pen and write on your notebook, what kind of force is exerted by you on the pen ? By the pen on the notebook ? By you on the notebook ?

**5.** Is it true that the reaction of a gravitational force is always gravitational, of an electromagnetic force is always electromagnetic and so on ?

**6.** Suppose the magnitude of Nuclear force between two protons varies with the distance between them as shown in figure (4-Q1). Estimate the ratio "Nuclear force/Coulomb force" for (a) $x$ = 8 fm (b) $x$ = 4 fm, (c) $x$ = 2 fm and (d) $x$ = 1 fm (1 fm = $10^{-15}$ m).

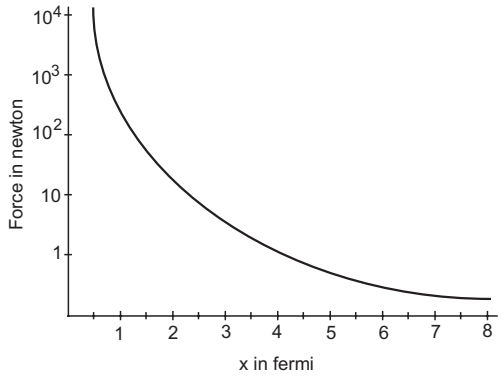



Figure 4-Q1

**7.** List all the forces acting on the block $B$ in figure (4-Q2).

**8.** List all the forces acting on (a) the pulley $A$, (b) the boy and (c) the block $C$ in figure (4-Q3).

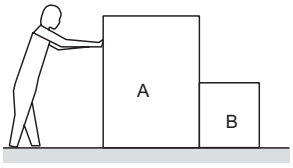



Figure 4-Q2

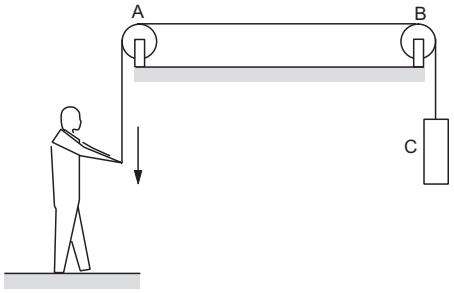



Figure 4-Q3

**9.** Figure (4-Q4) shows a boy pulling a wagon on a road. List as many forces as you can which are relevant with this figure. Find the pairs of forces connected by Newton's third law of motion.

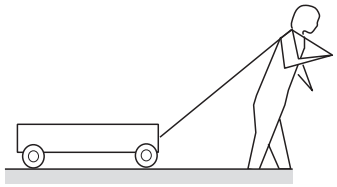


Figure 4-Q4

**10.** Figure (4-Q5) shows a cart. Complete the table shown below.

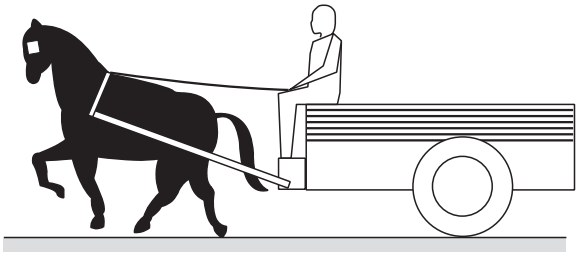


Figure 4-Q5

| Force on | Force by | Nature of the force | Direction |
|---|---|---|---|
| Cart | 1<br>2<br>3<br>: | | |
| Horse | 1<br>2<br>3<br>: | | |
| Driver | 1<br>2<br>3<br>: | | |

## OBJECTIVE I

**1.** When Neils Bohr shook hand with Werner Heisenberg, what kind of force they exerted ?
(a) Gravitational (b) Electromagnetic
(c) Nuclear (d) Weak.

**2.** Let $E$, $G$ and $N$ represent the magnitudes of electromagnetic, gravitational and nuclear forces between two electrons at a given separation. Then
(a) $N>E>G$ (b) $E>N>G$ (c) $G>N>E$ (d) $E>G>N$.

**3.** The sum of all electromagnetic forces between different particles of a system of charged particles is zero
(a) only if all the particles are positively charged
(b) only if all the particles are negatively charged
(c) only if half the particles are positively charged and half are negatively charged
(d) irrespective of the signs of the charges.

**4.** A 60 kg man pushes a 40 kg man by a force of 60 N. The 40 kg man has pushed the other man with a force of
(a) 40 N (b) 0 (c) 60 N (d) 20 N.

## OBJECTIVE II

**1.** A neutron exerts a force on a proton which is
(a) gravitational (b) electromagnetic
(c) nuclear (d) weak.

**2.** A proton exerts a force on a proton which is
(a) gravitational (b) electromagnetic
(c) nuclear (d) weak.

**3.** Mark the correct statements :
(a) The nuclear force between two protons is always greater than the electromagnetic force between them.
(b) The electromagnetic force between two protons is always greater than the gravitational force between them.
(c) The gravitational force between two protons may be greater than the nuclear force between them.
(d) Electromagnetic force between two protons may be greater than the nuclear force acting between them.

**4.** If all matter were made of electrically neutral particles such as neutrons,
(a) there would be no force of friction
(b) there would be no tension in the string
(c) it would not be possible to sit on a chair
(d) the earth could not move around the sun.

**5.** Which of the following systems may be adequately described by classical physics ?
(a) motion of a cricket ball
(b) motion of a dust particle
(c) a hydrogen atom
(d) a neutron changing to a proton.

**6.** The two ends of a spring are displaced along the length of the spring. All displacements have equal mangnitudes. In which case or cases the tension or compression in the spring will have a maximum magnitude ?
(a) the right end is displaced towards right and the left end towards left
(b) both ends are displaced towards right
(c) both ends are displaced towards left
(d) the right end is displaced towards left and the left end towards right.

**7.** Action and reaction
(a) act on two different objects
(b) have equal magnitude
(c) have opposite directions
(d) have resultant zero.

## EXERCISES

**1.** The gravitational force acting on a particle of 1 g due to a similar particle is equal to $6{\cdot}67 \times 10^{-17}$ N. Calculate the separation between the particles.

**2.** Calculate the force with which you attract the earth.

**3.** At what distance should two charges, each equal to 1 C, be placed so that the force between them equals your weight ?

**4.** Two spherical bodies, each of mass 50 kg, are placed at a separation of 20 cm. Equal charges are placed on the bodies and it is found that the force of Coulomb repulsion equals the gravitational attraction in magnitude. Find the magnitude of the charge placed on either body.

**5.** A monkey is sitting on a tree limb. The limb exerts a normal force of 48 N and a frictional force of 20 N. Find the magnitude of the total force exerted by the limb on the monkey.

**6.** A body builder exerts a force of 150 N against a bullworker and compresses it by 20 cm. Calculate the spring constant of the spring in the bullworker.

**7.** A satellite is projected vertically upwards from an earth station. At what height above the earth's surface will the force on the satellite due to the earth be reduced to half its value at the earth station ? (Radius of the earth is 6400 km.)

**8.** Two charged particles placed at a separation of 20 cm exert 20 N of Coulomb force on each other. What will be the force if the separation is increased to 25 cm ?

**9.** The force with which the earth attracts an object is called the weight of the object. Calculate the weight of the moon from the following data : The universal constant of gravitation $G = 6{\cdot}67 \times 10^{-11}$ N–m$^2$/kg$^2$, mass of the moon $= 7{\cdot}36 \times 10^{22}$ kg, mass of the earth $= 6 \times 10^{24}$ kg and the distance between the earth and the moon $= 3{\cdot}8 \times 10^5$ km.

**10.** Find the ratio of the magnitude of the electric force to the gravitational force acting between two protons.

**11.** The average separation between the proton and the electron in a hydrogen atom in ground state is $5{\cdot}3 \times 10^{-11}$ m. (a) Calculate the Coulomb force between them at this separation. (b) When the atom goes into its first excited state the average separation between the proton and the electron increases to four times its value in the ground state. What is the Coulomb force in this state ?

**12.** The geostationary orbit of the earth is at a distance of about 36000 km from the earth's surface. Find the weight of a 120-kg equipment placed in a geostationary satellite. The radius of the earth is 6400 km.

□

## ANSWERS

### OBJECTIVE I

1. (b) 2. (d) 3. (d) 4. (c)

### OBJECTIVE II

1. (a), (c) 2. (a), (b), (c) 3. (b), (c), (d)
4. (a), (b), (c) 5. (a), (b) 6. (a), (d)
7. (a), (b), (c), (d)

### EXERCISES

1. 1 m
4. $4{\cdot}3 \times 10^{-9}$ C
5. 52 N
6. 750 N/m
7. 2650 km
8. 13 N
9. $2 \times 10^{20}$ N
10. $1{\cdot}24 \times 10^{36}$
11. (a) $8{\cdot}2 \times 10^{-8}$ N,(b) $5{\cdot}1 \times 10^{-9}$ N
12. 27 N

□

## CHAPTER 5

# NEWTON'S LAWS OF MOTION

Newton's laws of motion are of central importance in classical physics. A large number of principles and results may be derived from Newton's laws. The first two laws relate to the type of motion of a system that results from a given set of forces. These laws may be interpreted in a variety of ways and it is slightly uninteresting and annoying at the outset to go into the technical details of the interpretation. The precise definitions of mass, force and acceleration should be given before we relate them. And these definitions themselves need use of Newton's laws. Thus, these laws turn out to be definitions to some extent. We shall assume that we know how to assign mass to a body, how to assign the magnitude and direction to a force and how to measure the acceleration with respect to a given frame of reference. Some discussions of these aspects were given in the previous chapters. The development here does not follow the historical track these laws have gone through, but are explained to make them simple to apply.

### 5.1 FIRST LAW OF MOTION

*If the (vector) sum of all the forces acting on a particle is zero then and only then the particle remains unaccelerated (i.e., remains at rest or moves with constant velocity).*

If the sum of all the forces on a given particle is $\vec{F}$ and its acceleration is $\vec{a}$, the above statement may also be written as

"$\vec{a} = 0$ *if and only if* $\vec{F} = 0$".

Thus, if the sum of the forces acting on a particle is known to be zero, we can be sure that the particle is unaccelerated, or if we know that a particle is unaccelerated, we can be sure that the sum of the forces acting on the particle is zero.

However, the concept of rest, motion or acceleration is meaningful only when a frame of reference is specified. Also the acceleration of the particle is, in general, different when measured from different frames. Is it possible then, that the first law is valid in all frames of reference ?

Let us consider the situation shown in figure (5.1). An elevator cabin falls down after the cable breaks. The cabin and all the bodies fixed in the cabin are accelerated with respect to the earth and the acceleration is about 9·8 m/s$^2$ in the downward direction.

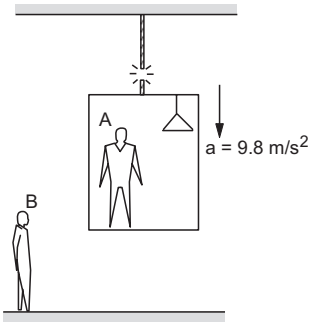



Figure 5.1

Consider the lamp in the cabin. The forces acting on the lamp are (a) the gravitational force $W$ by the earth and (b) the electromagnetic force $T$ (tension) by the rope. The direction of $W$ is downward and the directon of $T$ is upward. The sum is $(W - T)$ downward.

Measure the acceleration of the lamp from the frame of reference of the cabin. The lamp is at rest. The acceleration of the lamp is zero. The person $A$ who measured this acceleration is a learned one and uses Newton's first law to conclude that the sum of the forces acting on the particle is zero, i.e.,

$$W - T = 0 \text{ or, } W = T.$$

Instead, if we measure the acceleration from the ground, the lamp has an acceleration of 9·8 m/s$^2$. Thus, $a \neq 0$ and hence the person $B$ who measured this acceleration, concludes from Newton's first law that the sum of the forces is not zero. Thus, $W - T \neq 0$ or $W \neq T$. If $A$ measures acceleration and applies the first

law he gets $W = T$. If $B$ measures acceleration and applies the same first law, he gets $W \neq T$. Both of them cannot be correct simultaneously as $W$ and $T$ can be either equal or unequal. At least one of the two frames is a bad frame and one should not apply the first law in that frame.

There are some frames of reference in which Newton's first law is valid. Measure acceleration from such a frame and you are allowed to say that "$\vec{a} = 0$ if and only if $\vec{F} = 0$". But there are frames in which Newton's first law is not valid. You may find that even if the sum of the forces is not zero, the acceleration is still zero. Or you may find that the sum of the forces is zero, yet the particle is accelerated. So the validity of Newton's first law depends on the frame of reference from which the observer measures the state of rest, motion and acceleration of the particle.

A frame of reference in which Newton's first law is valid is called an *inertial frame of reference*. A frame in which Newton's first law is not valid is called a *noninertial frame of reference*.

Newton's first law, thus, reduces to a definition of inertial frame. Why do we call it a law then? Suppose after going through this lesson, you keep the book on your table fixed rigidly with the earth (figure 5.2).

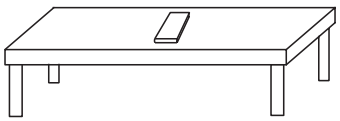


Figure 5.2

The book is at rest with respect to the earth. The acceleration of the book with respect to the earth is zero. The forces on the book are (a) the gravitational force $\vec{W}$ exerted by the earth and (b) the contact force $\vec{\mathcal{N}}$ by the table. Is the sum of $\vec{W}$ and $\vec{\mathcal{N}}$ zero? A very accurate measurement will give the answer "No". The sum of the forces is not zero although the book is at rest. The earth is not strictly an inertial frame. However, the sum is not too different from zero and we can say that the earth is an inertial frame of reference to a good approximation. Thus, for routine affairs, "$\vec{a} = 0$ if and only if $\vec{F} = 0$" is true in the earth frame of reference. This fact was identified and formulated by Newton and is known as Newton's *first law*. If we restrict that all measurements will be made from the earth frame, indeed it becomes a law. If we try to universalise this to different frames, it becomes a definition. We shall assume that unless stated otherwise, we are working from an inertial frame of reference.

***Example 5.1***

*A heavy particle of mass* 0·50 kg *is hanging from a string fixed with the roof. Find the force exerted by the string on the particle* (*Figure* 5.3). *Take* $g$ = 9·8 m/s$^2$.

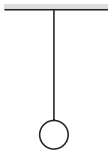


Figure 5.3

***Solution*** **:** The forces acting on the particle are

(a) pull of the earth, 0·50 kg × 9·8 m/s$^2$ = 4·9 N, vertically downward

(b) pull of the string, $T$ vertically upward.

The particle is at rest with respect to the earth (which we assume to be an inertial frame). Hence, the sum of the forces should be zero. Therefore, $T$ is 4·9 N acting vertically upward.

**Inertial Frames other than Earth**

Suppose $S$ is an inertial frame and $S'$ a frame moving uniformly with respect to $S$. Consider a particle $P$ having acceleration $\vec{a}_{P,S}$ with respect to $S$ and $\vec{a}_{P,S'}$ with respect to $S'$.

We know that,

$$\vec{a}_{P,S} = \vec{a}_{P,S'} + \vec{a}_{S',S}.$$

As $S'$ moves uniformly with respect to $S$,

$$\vec{a}_{S',S} = 0.$$

Thus, $$\vec{a}_{P,S} = \vec{a}_{P,S'} \quad \ldots \text{(i)}$$

Now $S$ is an inertial frame. So from definition, $\vec{a}_{P,S} = 0$, if and only if $\vec{F} = 0$ and hence, from (i), $\vec{a}_{P,S'} = 0$ if and only if $\vec{F} = 0$.

Thus, $S'$ is also an inertial frame. We arrive at an important result : *All frames moving uniformly with respect to an inertial frame are themselves inertial*. Thus, a train moving with uniform velocity with respect to the ground, a plane flying with uniform velocity with respect to a highway, etc., are examples of inertial frames. The sum of the forces acting on a suitcase kept on the shelf of a ship sailing smoothly and uniformly on a calm sea is zero.

## 5.2 SECOND LAW OF MOTION

*The acceleration of a particle as measured from an inertial frame is given by the (vector) sum of all the forces acting on the particle divided by its mass.*

In symbols : $\vec{a} = \vec{F}/m$ or, $\vec{F} = m\,\vec{a}$. ... (5.2)

The inertial frame is already defined by the first law of motion. A force $\vec{F}$ acting on a particle of mass $m$ produces an acceleration $\vec{F}/m$ in it with respect to an inertial frame. This is a law of nature. If the force ceases to act at some instant, the acceleration becomes zero at the same instant. In equation (5.2) $\vec{a}$ and $\vec{F}$ are measured at the same instant of time.

## 5.3 WORKING WITH NEWTON'S FIRST AND SECOND LAW

Newton's laws refer to a particle and relate the forces acting on the particle with its acceleration and its mass. Before attempting to write an equation from Newton's law, we should very clearly understand which particle we are considering. In any practical situation, we deal with extended bodies which are collection of a large number of particles. The laws as stated above may be used even if the object under consideration is an extended body, provided each part of this body has the same acceleration (in magnitude and direction). A systematic algorithm for writing equations from Newton's laws is as follows :

**Step 1 : Decide the System**

The first step is to decide the system on which the laws of motion are to be applied. The system may be a single particle, a block, a combination of two blocks one kept over the other, two blocks connected by a string, a piece of string etc. The only restriction is that all parts of the system should have identical acceleration.

Consider the situation shown in figure (5.4). The block $B$ does not slip over $A$, the disc $D$ slides over the string and all parts of the string are tight.

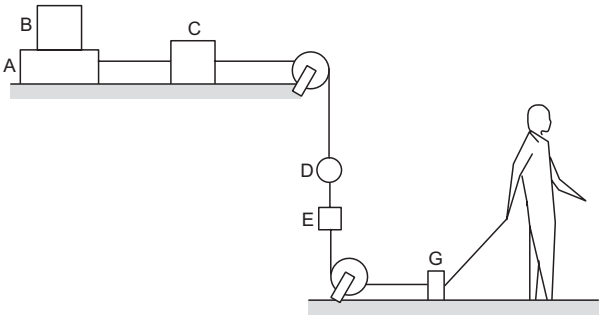



Figure 5.4

$A$ and $B$ move together. $C$ is not in contact with $A$ or $B$. But as the length of the string between $A$ and $C$ does not change, the distance moved by $C$ in any time interval is same as that by $A$. The same is true for $G$. The distance moved by $G$ in any time interval is same as that by $A$, $B$ or $C$. The direction of motion is also the same for $A$, $B$, $C$ and $G$. They have identical accelerations. We can take any of these blocks as a system or any combination of the blocks from these as a system. Some of the examples are ($A$), ($B$), ($A + B$), ($B + C$), ($A + B + C$), ($C + G$), ($A + C + G$), ($A + B + C + G$) etc. The distance covered by $E$ is also the same as the distance covered by $G$ but their directions are different. $E$ moves in a vertical line whereas $G$ in a horizontal line. ($E + G$) should not be taken as a system. At least at this stage we are unable to apply Newton's law treating $E + G$ as a single particle. As the disc $D$ slides over the string the distance covered by $D$ is not equal to that by $E$ in the same time interval. We should not treat $D + E$ as a system. *Think carefully.*

**Step 2 : Identify the Forces**

Once the system is decided, make a list of the forces acting *on* the system due to all the objects other than the system. Any force applied *by* the system should not be included in the list of the forces.

Consider the situation shown in figure (5.5). The boy stands on the floor balancing a heavy load on his head. The load presses the boy, the boy pushes the load upward the boy presses the floor downward, the floor pushes the boy upward, the earth attracts the load downward, the load attracts the earth upward, the boy attracts the earth upward and the earth attracts the boy downward. There are many forces operating in this world. Which of these forces should we include in the list of forces ?

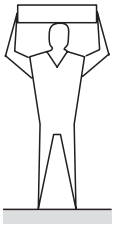


Figure 5.5

We cannot answer this question. Not because we do not know, but because we have not yet specified the system. Which is the body under consideration ? Do not try to identify forces before you have decided the system. Suppose we concentrate on the state of motion of the boy. We should then concentrate on the forces acting *on* the boy. The forces are listed in the upper half of table (5.1). Instead, if we take the load as the system and discuss the equilibrium of the load,

the list of the forces will be different. These forces appear in the lower half of table (5.1).

**Table 5.1**

| System | Force exerted by | Magnitude of the force | Direction of the force | Nature of the force |
|---|---|---|---|---|
| | Earth | $W$ | Downward | Gravitational |
| Boy | Floor | $\mathscr{N}$ | Upward | Electro-magnetic |
| | Load | $\mathscr{N}_1$ | Downward | ,, |
| | Earth | $W'$ | Downward | Gravitational |
| Load | | | | |
| | Boy | $\mathscr{N}_1$ | Upward | Electro-magnetic |

One may furnish as much information as one has about the magnitude and direction of the forces. The contact forces may have directions other than normal to the contact surface if the surfaces are rough. We shall discuss more about it under the heading of friction.

**Step 3 : Make a Free Body Diagram**

Now, represent the system by a point in a separate diagram and draw vectors representing the forces acting on the system with this point as the common origin. The forces may lie along a line, may be distributed in a plane (coplanar) or may be distributed in the space (non-planar). We shall rarely encounter situations dealing with non-planar forces. For coplanar forces the plane of diagram represents the plane of the forces acting on the system. Indicate the magnitudes and directions of the forces in this diagram. This is called a *free body diagram*. The free body diagram for the example discussed above with the boy as the system and with the load as the system are shown in figure (5.6).

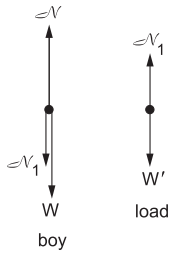



Figure 5.6

**Step 4 : Choose Axes and Write Equations**

Any three mutually perpendicular directions may be chosen as the $X$-$Y$-$Z$ axes. We give below some suggestions for choosing the axes to solve problems.

If the forces are coplanar, only two axes, say $X$ and $Y$, taken in the plane of forces are needed. Choose the $X$-axis along the direction in which the system is known to have or is likely to have the acceleration. A direction perpendicular to it may be chosen as the $Y$-axis. If the system is in equilibrium, any mutually perpendicular directions in the plane of the diagram may be chosen as the axes. Write the components of all the forces along the $X$-axis and equate their sum to the product of the mass of the system and its acceleration. This gives you one equation. Write the components of the forces along the $Y$-axis and equate the sum to zero. This gives you another equation. If the forces are collinear, this second equation is not needed.

If necessary you can go to step 1, choose another object as the system, repeat steps 2, 3 and 4 to get more equations. These are called equations of motion. Use mathematical techniques to get the unknown quantities out of these equations. This completes the algorithm.

The magnitudes of acceleration of different objects in a given situation are often related through kinematics. This should be properly foreseen and used together with the equations of motion. For example in figure (5.4) the accelerations of $C$ and $E$ have same magnitudes. Equations of motion for $C$ and for $E$ should use the same variable $a$ for acceleration.

***Example 5.2***

*A block of mass M is pulled on a smooth horizontal table by a string making an angle* $\theta$ *with the horizontal as shown in figure (5.7). If the acceleration of the block is a, find the force applied by the string and by the table on the block.*

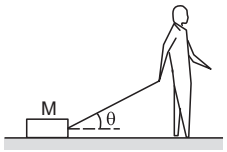



Figure 5.7

***Solution*** **:** Let us consider the block as the system.

The forces on the block are

(a) pull of the earth, $Mg$, vertically downward,

(b) contact force by the table, $\mathscr{N}$, vertically upward,

(c) pull of the string, $T$, along the string.

The free body diagram for the block is shown in figure (5.8).

The acceleration of the block is horizontal and towards the right. Take this direction as the $X$-axis and vertically upward direction as the $Y$-axis. We have,

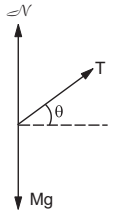



Figure 5.8

component of $Mg$ along the $X$-axis $= 0$

component of $N$ along the $X$-axis $= 0$

component of $T$ along the $X$-axis $= T\cos\theta$.

Hence the total force along the $X$-axis $= T\cos\theta$.

Using Newton's law, $T\cos\theta = Ma$. … (i)

Component of $Mg$ along the $Y$-axis $= -Mg$

component of $N$ along the $Y$-axis $= N$

component of $T$ along the $Y$-axis $= T\sin\theta$.

Total force along the $Y$-axis $= N + T\sin\theta - Mg$.

Using Newton's law, $N + T\sin\theta - Mg = 0$. … (ii)

From equation (i), $T = \dfrac{Ma}{\cos\theta}$. Putting this in equation (ii)

$N = Mg - Ma\tan\theta$.

## 5.4 NEWTON'S THIRD LAW OF MOTION

Newton's third law has already been introduced in chapter 4. "*If a body A exerts a force $\vec{F}$ on another body B, then B exerts a force $-\vec{F}$ on A.*"

Thus, the force exerted by $A$ on $B$ and that by $B$ on $A$ are equal in magnitude but opposite in direction. This law connects the forces exerted by two bodies on one another. The forces connected by the third law act on two different bodies and hence will never appear together in the list of forces at step 2 of applying Newton's first or second law.

For example, suppose a table exerts an upward force $N$ on a block placed on it. This force should be accounted if we consider the block as the system. The block pushes the table down with an equal force $N$. But this force acts on the table and should be considered only if we take the table as the system. Thus, only one of the two forces connected by the third law may appear in the equation of motion depending on the system chosen. The force exerted by the earth on a particle of mass $M$ is $Mg$ downward and therefore, by the particle on the earth is $Mg$ upward. These two forces will not cancel each other. The downward force on the particle will cause acceleration of the particle and that on the earth will cause acceleration (how large ?) of the earth.

Newton's third law of motion is not strictly correct when interaction between two bodies separated by a large distance is considered. We come across such deviations when we study electric and magnetic forces.

### Working with the Tension in a String

The idea of tension was qualitatively introduced in chapter 4. Suppose a block of mass $M$ is hanging through a string from the ceiling (figure 5.9).

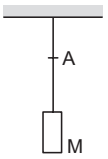



Figure 5.9

Consider a cross-section of the string at $A$. The cross-section divides the string in two parts, lower part and the upper part. The two parts are in physical contact at the cross-section at $A$. The lower part of the string will exert an electromagnetic force on the upper part and the upper part will exert an electromagnetic force on the lower part. According to the third law, these two forces will have equal magnitude. The lower part pulls down the upper part with a force $T$ and the upper part pulls up the lower part with equal force $T$. The common magnitude of the forces exerted by the two parts of the string on each other is called the tension in the string at $A$. What is the tension in the string at the lower end ? The block and the string are in contact at this end and exert electromagnetic forces on each other. The common magnitude of these forces is the tension in the string at the lower end. What is the tension in the string at the upper end ? At this end, the string and the ceiling meet. The string pulls the ceiling down and the ceiling pulls the string up. The common magnitude of these forces is the tension in the string at the upper end.

***Example 5.3***

*The mass of the part of the string below A in figure (5.9) is m. Find the tension of the string at the lower end and at A.*

***Solution*** : To get the tension at the lower end we need the force exerted by the string on the block.

Take the block as the system. The forces on it are

(a) pull of the string, $T$, upward,

(b) pull of the earth, $Mg$, downward,

The free body diagram for the block is shown in figure (5.10a). As the acceleration of the block is zero, these forces should add to zero. Hence the tension at the lower end is $T = Mg$.

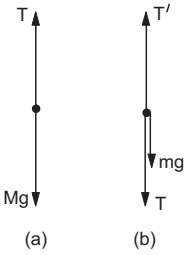



Figure 5.10

To get the tension $T$ at $A$ we need the force exerted by the upper part of the string on the lower part of the string. For this we may write the equation of motion for the lower part of the string. So take the string below $A$ as the system. The forces acting on this part are

(a) $T'$, upward, by the upper part of the string

(b) $mg$, downward, by the earth

(c) $T$, downward, by the block.

Note that in (c) we have written $T$ for the force by the block on the string. We have already used the symbol $T$ for the force by the string on the block. We have used Newton's third law here. The force exerted by the block on the string is equal in magnitude to the force exerted by the string on the block.

The free body diagram for this part is shown in figure (5.10b). As the system under consideration (the lower part of the string) is in equilibrium, Newton's first law gives

$$T' = T + mg$$

But $T = Mg$ hence, $T' = (M + m)g$.

***Example 5.4***

*The block shown in figure (5.11) has a mass M and descends with an acceleration a. The mass of the string below the point A is m. Find the tension of the string at the point A and at the lower end.*

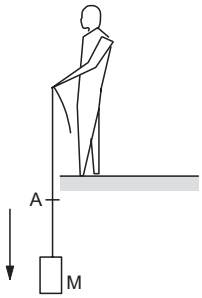



Figure 5.11

***Solution*** **:** Consider "the block + the part of the string below $A$" as the system. Let the tension at $A$ be $T$. The forces acting on this system are

(a) $(M + m)g$, downward, by the earth

(b) $T$, upward, by the upper part of the string.

The first is gravitational and the second is electromagnetic. We do not have to write the force by the string on the block. This electromagnetic force is by one part of the system on the other part. Only the forces acting on the system by the objects other than the system are to be included.

The system is descending with an acceleration $a$. Taking the downward direction as the $X$-axis, the total force along the $X$-axis is $(M + m)g - T$. Using Newton's law

$$(M + m)g - T = (M + m)a.$$

or, $$T = (M + m)(g - a). \quad \ldots \text{(i)}$$

We have omitted the free body diagram. This you can do if you can draw the free body diagram in your mind and write the equations correctly.

To get the tension $T'$ at the lower end we can put $m = 0$ in (i).

Effectively, we take the point $A$ at the lower end. Thus, we get $T' = M(g - a)$.

Suppose the string in ***Example 5.3*** or ***5.4*** is very light so that we can neglect the mass of the string. Then $T' = T$. The tension is then the same throughout the string. This result is of general nature. The tension at all the points in a string or a spring is the same provided it is assumed massless and no massive particle or body is connected in between.

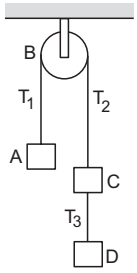



Figure 5.12

If the string in figure (5.12) is light, the tension $T_1$ of the string is same at all the points between the block $A$ and the pulley $B$. The tension $T_2$ is same at all the points between the pulley $B$ and the block $C$. The tension $T_3$ is same at all the points between the block $C$ and the block $D$. The three tensions $T_1$, $T_2$ and $T_3$ may be different from each other. If the pulley $B$ is also light, then $T_1 = T_2$.

## 5.5 PSEUDO FORCES

In this section we discuss the techniques of solving the motion of a body with respect to a noninertial frame of reference.

Consider the situation shown in figure (5.13). Suppose the frame of reference $S'$ moves with a

constant acceleration $\vec{a}_0$ with respect to an inertial frame $S$. The acceleration of a particle $P$ measured with respect to $S'$ is $\vec{a}_{P,S'} = \vec{a}$ and that with respect to $S$ is $\vec{a}_{P,S}$. The acceleration of $S'$ with respect to $S$ is $\vec{a}_{S',S} = \vec{a}_0$.

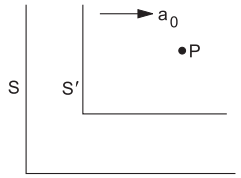



Figure 5.13

Since $S'$ is translating with respect to $S$ we have,

$$\vec{a}_{P,S'} = \vec{a}_{P,S} + \vec{a}_{S,S'} = \vec{a}_{P,S} - \vec{a}_{S',S}$$

or, $$\vec{a} = \vec{a}_{P,S} - \vec{a}_0$$

or, $$m\vec{a} = m\vec{a}_{P,S} - m\vec{a}_0$$

where $m$ is the mass of the particle $P$. Since $S$ is an inertial frame $m\vec{a}_{P,S}$ is equal to the sum of all the forces acting on $P$. Writing this sum as $\vec{F}$, we get

$$m\vec{a} = \vec{F} - m\vec{a}_0$$

or, $$\vec{a} = \frac{\vec{F} - m\vec{a}_0}{m}. \quad \ldots (5.3)$$

This equation relates the acceleration of the particle and the forces acting on it. Compare it with equation (5.2) which relates the acceleration and the force when the acceleration is measured with respect to an inertial frame. The acceleration of the frame (with respect to an inertial frame) comes into the equation of a particle. Newton's second law $\vec{a} = \vec{F}/m$ is not valid in such a noninertial frame. An extra term $-m\vec{a}_0$ has to be added to the sum of all the forces acting on the particle before writing the equation $\vec{a} = \vec{F}/m$. Note that in this extra term, $m$ is the mass of the particle under consideration and $\vec{a}_0$ is the acceleration of the working frame of reference with respect to some inertial frame.

However, we people spend most of our lifetime on the earth which is an (approximate) inertial frame. We are so familiar with the Newton's laws that we would still like to use the terminology of Newton's laws even when we use a noninertial frame. This can be done if we agree to call $(-m\vec{a}_0)$ a *force* acting on the particle. Then while preparing the list of the forces acting on the particle $P$, we include all the (real) forces acting on $P$ by all other objects and also include an imaginary force $-m\vec{a}_0$. Applying Newton's second law will then lead to equation (5.3). Such correction terms $-m a_0$ in the list of forces are called *pseudo forces*. This so-called force is to be included in the list only because we are discussing the motion from a noninertial frame and still want to use Newton's second law as "total force equals mass times acceleration". If we work from an inertial frame, the acceleration $\vec{a}_0$ of the frame is zero and no pseudo force is needed. The pseudo forces are also called *inertial forces* although their need arises because of the use of noninertial frames.

***Example 5.5***

*A pendulum is hanging from the ceiling of a car having an acceleration $a_0$ with respect to the road. Find the angle made by the string with the vertical.*

***Solution*** **:** The situation is shown in figure (5.14a). Suppose the mass of the bob is $m$ and the string makes an angle $\theta$ with the vertical. We shall work from the car frame. This frame is noninertial as it has an acceleration $\vec{a}_0$ with respect to an inertial frame (the road). Hence, if we use Newton's second law we shall have to include a pseudo force.

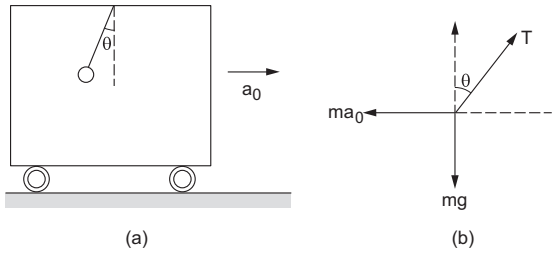



Figure 5.14

Take the bob as the system.

The forces are :

(a) $T$ along the string, by the string

(b) $mg$ downward, by the earth

(c) $ma_0$ towards left (pseudo force).

The free body diagram is shown in figure (5.14b). As the bob is at rest (remember we are discussing the motion with respect to the car) the force in (a), (b) and (c) should add to zero. Take $X$-axis along the forward horizontal direction and $Y$-axis along the upward vertical direction. The components of the forces along the $X$-axis give

$$T\sin\theta - m a_0 = 0 \quad \text{or,} \quad T\sin\theta = m a_0 \quad \ldots (i)$$

and the components along the $Y$-axis give

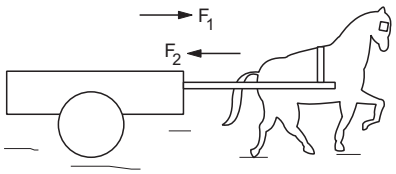
$$T\cos\theta - mg = 0 \quad \text{or,} \quad T\cos\theta = mg. \quad \ldots (ii)$$

Dividing (i) by (ii) $\tan\theta = a_0/g$.

Thus, the string makes an angle $\tan^{-1}(a_0/g)$ with the vertical.

### 5.6 THE HORSE AND THE CART

A good example which illustrates the ideas discussed in this chapter is the motion of a cart pulled by a horse. Suppose the cart is at rest when the driver whips the horse. The horse pulls the cart and the cart accelerates forward. The question posed is as follows. The horse pulls the cart by a force $F_1$ in the forward direction. From the third law of motion the cart pulls the horse by an equal force $F_2 = F_1$ in the backward direction. The sum of these forces is, therefore, zero (figure 5.15). Why should then the cart accelerate forward ?




$F_1$ : Force on the cart by the horse

$F_2$ : Force on the horse by the cart

$F_1 = F_2 = F$

Figure 5.15

Try to locate the mistake in the argument. According to our scheme, we should first decide the system. We can take the horse as the system or the cart as the system or the cart and the horse taken together as the system. Suppose you take the cart as the system. Then the forces on the cart should be listed and the forces on the horse should not enter the discussion. The force on the cart is $F_1$ in the forward direction and the acceleration of the cart is also in the forward direction. How much is this acceleration ? Take the mass of the cart to be $M_C$. Is the acceleration of the cart $a = F_1 / M_C$ in forward direction ? Think carefully. We shall return to this question.

Let us now try to understand the motion of the horse. This time we have to consider the forces on the horse. The forward force $F_1$ by the horse acts on the cart and it should not be taken into account when we discuss the motion of the horse. The force on the horse by the cart is $F_2$ in the backward direction. Why does the horse go in the forward direction when whipped ? The horse exerts a force on the cart in the forward direction and hence the cart is accelerated forward. But the cart exerts an equal force on the horse in the backward direction. Why is the horse not accelerated in backward direction ? (Imagine this situation. If the cart is accelerated forward and the horse backward, the horse will sit on the cart kicking out the driver and the passengers.) Where are we wrong ? We have not considered *all* the forces acting on the horse. The road pushes the horse by a force $P$ which has a forward component. This force acts on the horse and we must add this force when we discuss the motion of the horse. The horse accelerates forward if the forward component $f$ of the force $P$ exceeds $F_2$ (Figure 5.16). The acceleration of the horse is $(f - F_2)/M_h$. We should make sure that *all* the forces acting on the system are added. Note that the force of gravity acting on the horse has no forward component.

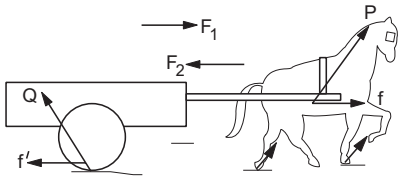



Figure 5.16

Going back to the previous paragraph the acceleration of the cart may not be $F_1 / M_C$. The road exerts a force $Q$ on the cart which may have a backward component $f'$. The total force on the cart is $F_1 - f'$. The acceleration of the cart is then $a = \dfrac{F_1 - f'}{M_C}$ in the forward direction.

The forces $f$ and $f'$ are self adjustable and they so adjust their values that $\dfrac{F_1 - f'}{M_C} = \dfrac{f - F_2}{M_h}$. The acceleration of the horse and that of the cart are equal in magnitude and direction and hence they move together.

So, once again we remind you that only the forces on the system are to be considered to discuss the motion of the system and all the forces acting on the system are to be considered. Only then apply $\vec{F} = m\vec{a}$.

### 5.7 INERTIA

A particle is accelerated (in an inertial frame) if and only if a resultant force acts on it. Loosely speaking, the particle does not change its state of rest or of uniform motion along a straight line unless it is forced to do this. This unwillingness of a particle to change its state of rest or of uniform motion along a straight line is called as *inertia.* We can understand the property of inertia in more precise terms as follows. If equal forces are applied on two particles, in general, the acceleration of the particles will be different. The property of a particle to allow a smaller acceleration is called *inertia.* It is clear that larger the mass of the particle, smaller will be the acceleration and hence larger will be the inertia.

## *Worked Out Examples*

**1.** *A body of mass m is suspended by two strings making angles* $\alpha$ *and* $\beta$ *with the horizontal. Find the tensions in the strings.*

***Solution*** **:** Take the body of mass $m$ as the system. The forces acting on the system are

(i) $mg$ downwards (by the earth),

(ii) $T_1$ along the first string (by the first string) and

(iii) $T_2$ along the second string (by the second string).

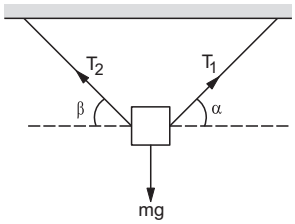



Figure 5-W1

These forces are shown in figure (5-W1). As the body is in equilibrium, these forces must add to zero. Taking horizontal components,

$$T_1 \cos\alpha - T_2 \cos\beta + mg \cos\frac{\pi}{2} = 0$$

or, $$T_1 \cos\alpha = T_2 \cos\beta. \quad \text{... (i)}$$

Taking vertical components,

$$T_1 \sin\alpha + T_2 \sin\beta - mg = 0. \quad \text{... (ii)}$$

Eliminating $T_2$ from (i) and (ii),

$$T_1 \sin\alpha + T_1 \frac{\cos\alpha}{\cos\beta} \sin\beta = mg$$

or, $$T_1 = \frac{mg}{\sin\alpha + \frac{\cos\alpha}{\cos\beta}\sin\beta} = \frac{mg\cos\beta}{\sin(\alpha+\beta)}.$$

From (i), $$T_2 = \frac{mg\cos\alpha}{\sin(\alpha+\beta)}.$$

**2.** *Two bodies of masses* $m_1$ *and* $m_2$ *are connected by a light string going over a smooth light pulley at the end of an incline. The mass* $m_1$ *lies on the incline and* $m_2$ *hangs vertically. The system is at rest. Find the angle of the incline and the force exerted by the incline on the body of mass* $m_1$ *(figure 5-W2).*

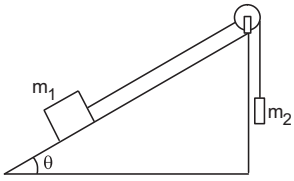



Figure 5-W2

***Solution*** : Figure (5-W3) shows the situation with the forces on $m_1$ and $m_2$ shown. Take the body of mass $m_2$ as the system. The forces acting on it are

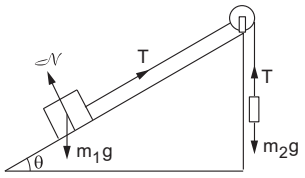



Figure 5-W3

(i) $m_2 g$ vertically downward (by the earth),

(ii) $T$ vertically upward (by the string).

As the system is at rest, these forces should add to zero.

This gives $$T = m_2 g. \quad \text{... (i)}$$

Next, consider the body of mass $m_1$ as the system. The forces acting on this system are

(i) $m_1 g$ vertically downward (by the earth),

(ii) $T$ along the string up the incline (by the string),

(iii) $\mathcal{N}$ normal to the incline (by the incline).

As the string and the pulley are all light and smooth, the tension in the string is uniform everywhere. Hence, same $T$ is used for the equations of $m_1$ and $m_2$. As the system is in equilibrium, these forces should add to zero.

Taking components parallel to the incline,

$$T = m_1 g \cos\left(\frac{\pi}{2} - \theta\right) = m_1 g \sin\theta. \quad \text{... (ii)}$$

Taking components along the normal to the incline,

$$\mathcal{N} = m_1 g \cos\theta. \quad \text{... (iii)}$$

Eliminating $T$ from (i) and (ii),

$$m_2 g = m_1 g \sin\theta$$

or, $$\sin\theta = m_2 / m_1$$

giving $$\theta = \sin^{-1}(m_2/m_1).$$

From (iii) $$\mathcal{N} = m_1 g \sqrt{1 - (m_2/m_1)^2}.$$

**3.** *A bullet moving at* 250 m/s *penetrates* 5 cm *into a tree limb before coming to rest. Assuming that the force exerted by the tree limb is uniform, find its magnitude. Mass of the bullet is* 10 g.

***Solution*** : The tree limb exerts a force on the bullet in the direction opposite to its velocity. This force causes deceleration and hence the velocity decreases from 250 m/s to zero in 5 cm. We have to find the force exerted by the tree limb on the bullet. If $a$ be the deceleration of the bullet, we have,

$$u = 250 \text{ m/s}, \; v = 0, \; x = 5 \text{ cm} = 0{\cdot}05 \text{ m}$$

giving, $a = \dfrac{(250 \text{ m/s})^2 - 0^2}{2 \times 0{\cdot}05 \text{ m}} = 625000 \text{ m/s}^2$.

The force on the bullet is $F = ma = 6250$ N.

**4.** *The force on a particle of mass* 10 g *is* $(\vec{i}\,10 + \vec{j}\,5)$ N. *If it starts from rest what would be its position at time* $t = 5$ s ?

***Solution*** : We have $F_x = 10$ N giving

$a_x = \dfrac{F_x}{m} = \dfrac{10 \text{ N}}{0{\cdot}01 \text{ kg}} = 1000 \text{ m/s}^2$.

As this is a case of constant acceleration in $x$-direction,

$$x = u_x t + \frac{1}{2} a_x t^2 = \frac{1}{2} \times 1000 \text{ m/s}^2 \times (5 \text{ s})^2$$
$$= 12500 \text{ m}$$

Similarly, $a_y = \dfrac{F_y}{m} = \dfrac{5 \text{ N}}{0{\cdot}01 \text{ kg}} = 500 \text{ m/s}^2$

and $y = 6250$ m.

Thus, the position of the particle at $t = 5$ s is,

$$\vec{r} = (\vec{i}\,12500 + \vec{j}\,6250) \text{ m}.$$

**5.** *With what acceleration 'a' should the box of figure (5-W4) descend so that the block of mass M exerts a force* $Mg/4$ *on the floor of the box* ?

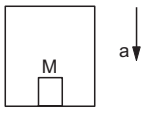



Figure 5-W4

***Solution*** : The block is at rest with respect to the box which is accelerated with respect to the ground. Hence, the acceleration of the block with respect to the ground is '$a$' downward. The forces on the block are

(i) $Mg$ downward (by the earth) and

(ii) $\mathcal{N}$ upward (by the floor).

The equation of motion of the block is, therefore

$$Mg - \mathcal{N} = Ma.$$

If $\mathcal{N} = Mg/4$, the above equation gives $a = 3g/4$. The block and hence the box should descend with an acceleration $3g/4$.

**6.** *A block 'A' of mass m is tied to a fixed point C on a horizontal table through a string passing round a massless smooth pulley B (figure 5-W5). A force F is applied by the experimenter to the pulley. Show that if the pulley is displaced by a distance x, the block will be displaced by 2x. Find the acceleration of the block and the pulley.*

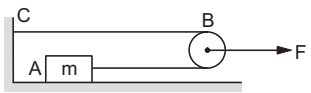



Figure 5-W5

***Solution*** : Suppose the pulley is displaced to $B'$ and the block to $A'$ (figure 5-W6). The length of the string is $CB + BA$ and is also equal to $CB + BB' + B'B + BA'$. Hence, $CB + BA' + A'A = CB + BB' + B'B + BA'$

or, $A'A = 2\,BB'$.

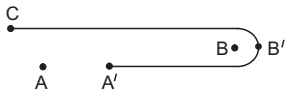



Figure 5-W6

The displacement of $A$ is, therefore, twice the displacement of $B$ in any given time interval. Diffrentiating twice, we find that the acceleration of $A$ is twice the acceleration of $B$.

To find the acceleration of the block we will need the tension in the string. That can be obtained by considering the pulley as the system.

The forces acting on the pulley are

(i) $F$ towards right by the experimenter,

(ii) $T$ towards left by the portion $BC$ of the string and

(iii) $T$ towards left by the portion $BA$ of the string.

The vertical forces, if any, add to zero as there is no vertical motion.

As the mass of the pulley is zero, the equation of motion is

$F - 2T = 0$ giving $T = F/2$.

Now consider the block as the system. The only horizontal force acting on the block is the tension $T$ towards right. The acceleration of the block is, therefore, $a = T/m = \dfrac{F}{2m}$. The acceleration of the pulley is $a/2 = \dfrac{F}{4m}$.

**7.** *A smooth ring A of mass m can slide on a fixed horizontal rod. A string tied to the ring passes over a fixed pulley B and carries a block C of mass M* $(= 2m)$ *as shown in figure (5-W7). At an instant the string between the ring and the pulley makes an angle* $\theta$ *with the rod. (a) Show that, if the ring slides with a speed v, the block descends with speed* $v \cos\theta$. *(b) With what acceleration will the ring start moving if the system is released from rest with* $\theta = 30°$ ?

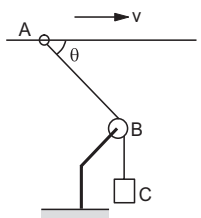



Figure 5-W7

***Solution*** **:** (a) Suppose in a small time interval $\Delta t$ the ring is displaced from $A$ to $A'$ (figure 5-W8) and the block from $C$ to $C'$. Drop a perpendicular $A'P$ from $A'$ to $AB$. For small displacement $A'B \approx PB$. Since the length of the string is constant, we have

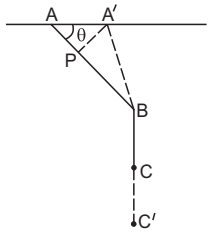



Figure 5-W8

$$AB + BC = A'B + BC'$$

or, $$AP + PB + BC = A'B + BC'$$

or, $$AP = BC' - BC = CC' \quad (\text{as } A'B \approx PB)$$

or, $$AA' \cos\theta = CC'$$

or, $$\frac{AA' \cos\theta}{\Delta t} = \frac{CC'}{\Delta t}$$

or, (velocity of the ring) cos θ = (velocity of the block).

(b) If the initial acceleration of the ring is $a$, that of the block will be $a \cos\theta$. Let $T$ be the tension in the string at this instant. Consider the block as the system. The forces acting on the block are

(i) $Mg$ downward due to the earth, and

(ii) $T$ upward due to the string.

The equation of motion of the block is

$$Mg - T = Ma\cos\theta. \qquad \ldots \text{(i)}$$

Now consider the ring as the system. The forces on the ring are

(i) $Mg$ downward due to gravity,

(ii) $\not\perp N$ upward due to the rod,

(iii) $T$ along the string due to the string.

Taking components along the rod, the equation of motion of the ring is

$$T\cos\theta = ma. \qquad \ldots \text{(ii)}$$

From (i) and (ii)

$$Mg - \frac{ma}{\cos\theta} = M\,a\cos\theta$$

or, $$a = \frac{M\,g\cos\theta}{m + M\cos^2\theta}.$$

Putting θ = 30°, $M = 2\,m$ and $g = 9{\cdot}8$ m/s$^2$; therefore

$$a = 6{\cdot}78 \text{ m/s}^2.$$

**8.** *A light rope fixed at one end of a wooden clamp on the ground passes over a tree branch and hangs on the other side (figure 5-W9). It makes an angle of* 30° *with the ground. A man weighing* (60 kg) *wants to climb up the rope. The wooden clamp can come out of the ground if an upward force greater than* 360 N *is applied to it. Find the maximum acceleration in the upward direction with which the man can climb safely. Neglect friction at the tree branch. Take* $g = 10$ m/s$^2$.

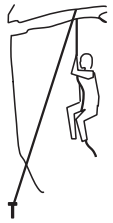


Figure 5-W9

***Solution*** : Let $T$ be the tension in the rope. The upward force on the clamp is $T\sin 30° = T/2$. The maximum tension that will not detach the clamp from the ground is, therefore, given by

$$\frac{T}{2} = 360 \text{ N}$$

or, $$T = 720 \text{ N}.$$

If the acceleration of the man in the upward direction is $a$, the equation of motion of the man is

$$T - 600 \text{ N} = (60 \text{ kg})\,a$$

The maximum acceleration of the man for safe climbing is, therefore

$$a = \frac{720 \text{ N} - 600 \text{ N}}{60 \text{ kg}} = 2 \text{ m/s}^2.$$

**9.** *Three blocks of masses* $m_1, m_2$ *and* $m_3$ *are connected as shown in the figure (5–W10). All the surfaces are frictionless and the string and the pulleys are light. Find the acceleration of* $m_1$.

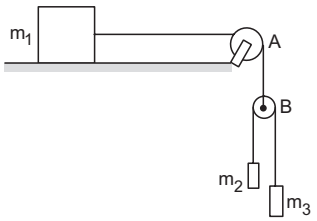



Figure 5-W10

***Solution*** : Suppose the acceleration of $m_1$ is $a_0$ towards right. That will also be the downward acceleration of the pulley $B$ because the string connecting $m_1$ and $B$ is constant in length. Also the string connecting $m_2$ and $m_3$ has a constant length. This implies that the decrease in the separation between $m_2$ and $B$ equals the increase in the separation between $m_3$ and $B$. So, the upward acceleration of $m_2$ with respect to $B$ equals the downward acceleration of $m_3$ with respect to $B$. Let this acceleration be $a$.

The acceleration of $m_2$ with respect to the ground $= a_0 - a$ (downward) and the acceleration of $m_3$ with respect to the ground $= a_0 + a$ (downward).

These accelerations will be used in Newton's laws. Let the tension be $T$ in the upper string and $T'$ in the lower string. Consider the motion of the pulley $B$.

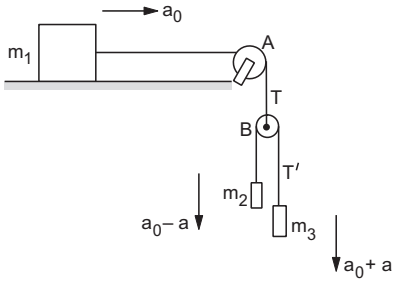



Figure 5-W11

The forces on this light pulley are
(a) $T$ upwards by the upper string and
(b) $2T'$ downwards by the lower string.
As the mass of the pulley is negligible,

$$2T' - T = 0$$

giving

$$T' = T/2. \qquad \ldots \text{(i)}$$

Motion of $m_1$ :

The acceleration is $a_0$ in the horizontal direction. The forces on $m_1$ are

(a) $T$ by the string (horizontal).

(b) $m_1 g$ by the earth (vertically downwards) and

(c) $\mathcal{N}$ by the table (vertically upwards).

In the horizontal direction, the equation is

$$T = m_1 a_0. \qquad \ldots \text{(ii)}$$

Motion of $m_2$ : acceleration is $a_0 - a$ in the downward direction. The forces on $m_2$ are

(a) $m_2 g$ downward by the earth and
(b) $T' = T/2$ upward by the string.

Thus, $$m_2 g - T/2 = m_2 (a_0 - a) \qquad \ldots \text{(iii)}$$

Motion of $m_3$ : The acceleration is $(a_0 + a)$ downward. The forces on $m_3$ are

(a) $m_3 g$ downward by the earth and
(b) $T' = T/2$ upward by the string. Thus,

$$m_3 g - T/2 = m_3 (a_0 + a). \qquad \ldots \text{(iv)}$$

We want to calculate $a_0$, so we shall eliminate $T$ and $a$ from (ii), (iii) and (iv).

Putting $T$ from (ii) in (iii) and (iv),

$$a_0 - a = \frac{m_2 g - m_1 a_0/2}{m_2} = g - \frac{m_1 a_0}{2 m_2}$$

and $$a_0 + a = \frac{m_3 g - m_1 a_0/2}{m_3} = g - \frac{m_1 a_0}{2 m_3}.$$

Adding, $$2a_0 = 2g - \frac{m_1 a_0}{2}\left(\frac{1}{m_2} + \frac{1}{m_3}\right)$$

or, $$a_0 = g - \frac{m_1 a_0}{4}\left(\frac{1}{m_2} + \frac{1}{m_3}\right)$$

or, $$a_0\left[1 + \frac{m_1}{4}\left(\frac{1}{m_2} + \frac{1}{m_3}\right)\right] = g$$

or, $$a_0 = \frac{g}{1 + \frac{m_1}{4}\left(\frac{1}{m_2} + \frac{1}{m_3}\right)}.$$

**10.** *A particle slides down a smooth inclined plane of elevation* $\theta$*, fixed in an elevator going up with an acceleration* $a_0$ *(figure 5-W12). The base of the incline has a length L. Find the time taken by the particle to reach the bottom.*

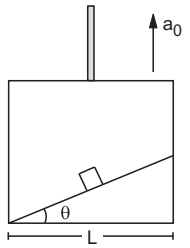



Figure 5-W12

***Solution*** : Let us work in the elevator frame. A pseudo force $ma_0$ in the downward direction is to be applied on the particle of mass $m$ together with the real forces. Thus, the forces on $m$ are (figure 5-W13)

(i) $\mathcal{N}$ normal force,
(ii) $mg$ downward (by the earth),
(iii) $ma_0$ downward (pseudo).

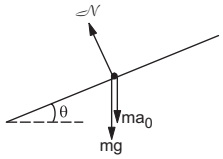



Figure 5-W13

Let $a$ be the acceleration of the particle with respect to the incline. Taking components of the forces parallel to the incline and applying Newton's law,

$$mg \sin\theta + ma_0 \sin\theta = ma$$

or, $$a = (g + a_0) \sin\theta.$$

This is the acceleration with respect to the elevator. In this frame, the distance travelled by the particle is $L/\cos\theta$. Hence,

$$\frac{L}{\cos\theta} = \frac{1}{2}(g + a_0) \sin\theta . t^2$$

or, $$t = \left[\frac{2L}{(g + a_0) \sin\theta \cos\theta}\right]^{1/2}.$$

**11.** *All the surfaces shown in figure (5-W14) are assumed to be frictionless. The block of mass m slides on the prism which in turn slides backward on the horizontal surface. Find the acceleration of the smaller block with respect to the prism.*

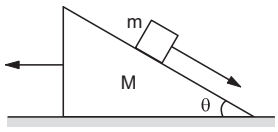



Figure 5-W14

***Solution*** : Let the acceleration of the prism be $a_0$ in the backward direction. Consider the motion of the smaller block from the frame of the prism.

The forces on the block are (figure 5-W15a)

(i) $\mathcal{N}$ normal force,

(ii) $mg$ downward (gravity),

(iii) $ma_0$ forward (pseudo).

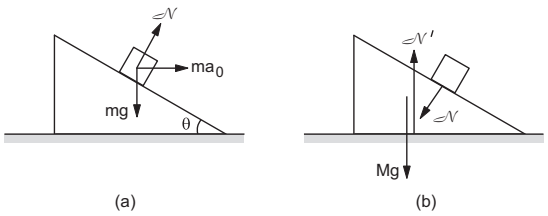



Figure 5-W15

The block slides down the plane. Components of the forces parallel to the incline give

$$ma_0 \cos\theta + mg \sin\theta = ma$$

or, $$a = a_0 \cos\theta + g \sin\theta. \quad \ldots \text{(i)}$$

Components of the force perpendicular to the incline give

$$\mathcal{N} + ma_0 \sin\theta = mg \cos\theta. \quad \ldots \text{(ii)}$$

Now consider the motion of the prism from the lab frame. No pseudo force is needed as the frame used is inertial. The forces are (figure 5-W15b)

(i) $Mg$ downward,

(ii) $\mathcal{N}$ normal to the incline (by the block),

(iii) $\mathcal{N}'$ upward (by the horizontal surface).

Horizontal components give,

$$\mathcal{N} \sin\theta = Ma_0 \quad \text{or,} \quad \mathcal{N} = Ma_0/\sin\theta. \quad \ldots \text{(iii)}$$

Putting in (ii)

$$\frac{Ma_0}{\sin\theta} + ma_0 \sin\theta = mg \cos\theta$$

or, $$a_0 = \frac{m\,g \sin\theta \cos\theta}{M + m \sin^2\theta}.$$

From (i), $$a = \frac{m\,g \sin\theta \cos^2\theta}{M + m \sin^2\theta} + g \sin\theta$$

$$= \frac{(M+m)\,g \sin\theta}{M + m \sin^2\theta}.$$

□

## QUESTIONS FOR SHORT ANSWER

**1.** The apparent weight of an object increases in an elevator while accelerating upward. A moongphaliwala sells his moongphali using a beam balance in an elevator. Will he gain more if the elevator is accelerating up ?

**2.** A boy puts a heavy box of mass $M$ on his head and jumps down from the top of a multistoried building to the ground. How much is the force exerted by the box on his head during his free fall ? Does the force greatly increase during the period he balances himself after striking the ground ?

**3.** A person drops a coin. Describe the path of the coin as seen by the person if he is in (a) a car moving at constant velocity and (b) in a freely falling elevator.

**4.** Is it possible for a particle to describe a curved path if no force acts on it ? Does your answer depend on the frame of reference chosen to view the particle ?

**5.** You are riding in a car. The driver suddenly applies the brakes and you are pushed forward. Who pushed you forward ?

**6.** It is sometimes heard that inertial frame of reference is only an ideal concept and no such inertial frame actually exists. Comment.

**7.** An object is placed far away from all the objects that can exert force on it. A frame of reference is constructed by taking the origin and axes fixed in this object. Will the frame be necessarily inertial ?

**8.** Figure (5-Q1) shows a light spring balance connected to two blocks of mass 20 kg each. The graduations in the balance measure the tension in the spring. (a) What is the reading of the balance? (b) Will the reading change if the balance is heavy, say 2·0 kg ? (c) What will happen if the spring is light but the blocks have unequal masses ?

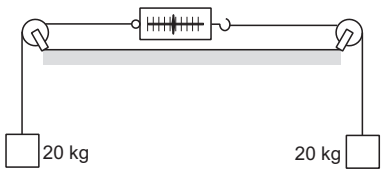



Figure 5-Q1

**9.** The acceleration of a particle is zero as measured from an inertial frame of reference. Can we conclude that no force acts on the particle ?

**10.** Suppose you are running fast in a field when you suddenly find a snake in front of you. You stop quickly. Which force is responsible for your deceleration ?

**11.** If you jump barefooted on a hard surface, your legs get injured. But they are not injured if you jump on a soft surface like sand or pillow. Explain.

**12.** According to Newton's third law each team pulls the opposite team with equal force in a tug of war. Why then one team wins and the other loses ?

**13.** A spy jumps from an airplane with his parachute. The spy accelerates downward for some time when the parachute opens. The acceleration is suddenly checked and the spy slowly falls on the ground. Explain the action of parachute in checking the acceleration.

**14.** Consider a book lying on a table. The weight of the book and the normal force by the table on the book are equal in magnitude and opposite in direction. Is this an example of Newton's third law ?

**15.** Two blocks of unequal masses are tied by a spring. The blocks are pulled stretching the spring slightly and the system is released on a frictionless horizontal platform. Are the forces due to the spring on the two blocks equal and opposite ? If yes, is it an example of Newton's third law ?

**16.** When a train starts, the head of a standing passenger seems to be pushed backward. Analyse the situation from the ground frame. Does it really go backward ? Coming back to the train frame, how do you explain the backward movement of the head on the basis of Newton's laws ?

**17.** A plumb bob is hung from the ceiling of a train compartment. If the train moves with an acceleration '$a$' along a straight horizontal track, the string supporting the bob makes an angle $\tan^{-1}(a/g)$ with the normal to the ceiling. Suppose the train moves on an inclined straight track with uniform velocity. If the angle of incline is $\tan^{-1}(a/g)$, the string again makes the same angle with the normal to the ceiling. Can a person sitting inside the compartment tell by looking at the plumb line whether the train is accelerated on a horizontal straight track or it is going on an incline ? If yes, how ? If no, suggest a method to do so.

## OBJECTIVE I

**1.** A body of weight $w_1$ is suspended from the ceiling of a room through a chain of weight $w_2$. The ceiling pulls the chain by a force

(a) $w_1$ (b) $w_2$ (c) $w_1 + w_2$ (d) $\frac{w_1 + w_2}{2}$.

**2.** When a horse pulls a cart, the force that helps the horse to move forward is the force exerted by
(a) the cart on the horse (b) the ground on the horse
(c) the ground on the cart (d) the horse on the ground.

**3.** A car accelerates on a horizontal road due to the force exerted by
(a) the engine of the car (b) the driver of the car
(c) the earth (d) the road.

**4.** A block of mass 10 kg is suspended through two light spring balances as shown in figure (5-Q2).

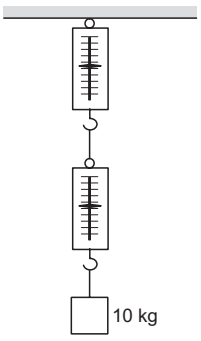



Figure 5-Q2

(a) Both the scales will read 10 kg.
(b) Both the scales will read 5 kg.
(c) The upper scale will read 10 kg and the lower zero.
(d) The readings may be anything but their sum will be 10 kg.

**5.** A block of mass $m$ is placed on a smooth inclined plane of inclination $\theta$ with the horizontal. The force exerted by the plane on the block has a magnitude
(a) $mg$ (b) $mg/\cos\theta$ (c) $mg\cos\theta$ (d) $mg\tan\theta$.

**6.** A block of mass $m$ is placed on a smooth wedge of inclination $\theta$. The whole system is accelerated horizontally so that the block does not slip on the wedge. The force exerted by the wedge on the block has a magnitude
(a) $mg$ (b) $mg/\cos\theta$ (c) $mg\cos\theta$ (d) $mg\tan\theta$.

**7.** Neglect the effect of rotation of the earth. Suppose the earth suddenly stops attracting objects placed near its surface. A person standing on the surface of the earth will
(a) fly up (b) slip along the surface
(c) fly along a tangent to the earth's surface
(d) remain standing.

**8.** Three rigid rods are joined to form an equilateral triangle $ABC$ of side 1 m. Three particles carrying charges 20 μC each are attached to the vertices of the triangle. The whole system is at rest in an inertial frame. The resultant force on the charged particle at $A$ has the magnitude
(a) zero (b) 3·6 N (c) $3{\cdot}6\sqrt{3}$ N (d) 7·2 N.

**9.** A force $F_1$ acts on a particle so as to accelerate it from rest to a velocity $v$. The force $F_1$ is then replaced by $F_2$ which decelerates it to rest.
(a) $F_1$ must be equal to $F_2$ (b) $F_1$ may be equal to $F_2$
(c) $F_1$ must be unequal to $F_2$ (d) none of these.

**10.** Two objects $A$ and $B$ are thrown upward simultaneously with the same speed. The mass of $A$ is greater than the mass of $B$. Suppose the air exerts a constant and equal force of resistance on the two bodies.
(a) The two bodies will reach the same height.
(b) $A$ will go higher than $B$.
(c) $B$ will go higher than $A$.
(d) Any of the above three may happen depending on the speed with which the objects are thrown.

**11.** A smooth wedge $A$ is fitted in a chamber hanging from a fixed ceiling near the earth's surface. A block $B$ placed at the top of the wedge takes a time $T$ to slide down the length of the wedge. If the block is placed at the top of the wedge and the cable supporting the chamber is broken at the same instant, the block will
(a) take a time longer than $T$ to slide down the wedge
(b) take a time shorter than $T$ to slide down the wedge
(c) remain at the top of the wedge
(d) jump off the wedge.

**12.** In an imaginary atmosphere, the air exerts a small force $F$ on any particle in the direction of the particle's motion. A particle of mass $m$ projected upward takes a time $t_1$ in reaching the maximum height and $t_2$ in the return journey to the original point. Then
(a) $t_1 < t_2$ (b) $t_1 > t_2$ (c) $t_1 = t_2$ (d) the relation between $t_1$ and $t_2$ depends on the mass of the particle.

**13.** A person standing on the floor of an elevator drops a coin. The coin reaches the floor of the elevator in a time $t_1$ if the elevator is stationary and in time $t_2$ if it is moving uniformly. Then
(a) $t_1 = t_2$ (b) $t_1 < t_2$ (c) $t_1 > t_2$ (d) $t_1 < t_2$ or $t_1 > t_2$ depending on whether the lift is going up or down.

**14.** A free $^{238}U$ nucleus kept in a train emits an alpha particle. When the train is stationary, a nucleus decays and a passenger measures that the separation between the alpha particle and the recoiling nucleus becomes $x$ at time $t$ after the decay. If the decay takes place while the train is moving at a uniform velocity $v$, the distance between the alpha particle and the recoiling nucleus at a time $t$ after the decay as measured by the passenger is
(a) $x + v\,t$ (b) $x - v\,t$ (c) $x$
(d) depends on the direction of the train.

## OBJECTIVE II

**1.** A reference frame attached to the earth
(a) is an inertial frame by definition
(b) cannot be an inertial frame because the earth is revolving around the sun
(c) is an inertial frame because Newton's laws are applicable in this frame
(d) cannot be an inertial frame because the earth is rotating about its axis.

**2.** A particle stays at rest as seen in a frame. We can conclude that
(a) the frame is inertial
(b) resultant force on the particle is zero
(c) the frame may be inertial but the resultant force on the particle is zero
(d) the frame may be noninertial but there is a nonzero resultant force.

**3.** A particle is found to be at rest when seen from a frame $S_1$ and moving with a constant velocity when seen from another frame $S_2$. Mark out the possible options.
(a) Both the frames are inertial.
(b) Both the frames are noninertial.
(c) $S_1$ is inertial and $S_2$ is noninertial.
(d) $S_1$ is noninertial and $S_2$ is inertial.

**4.** Figure (5-Q3) shows the displacement of a particle going along the $X$-axis as a function of time. The force acting on the particle is zero in the region
(a) $AB$ (b) $BC$ (c) $CD$ (d) $DE$.

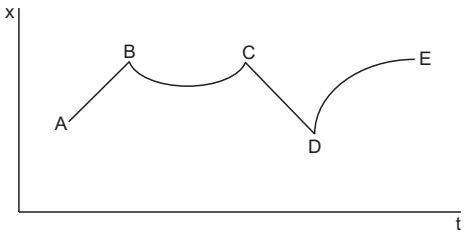



Figure 5-Q3

**5.** Figure (5-Q4) shows a heavy block kept on a frictionless surface and being pulled by two ropes of equal mass $m$. At $t = 0$, the force on the left rope is withdrawn but the force on the right end continues to act. Let $F_1$ and $F_2$ be the magnitudes of the forces by the right rope and the left rope on the block respectively.

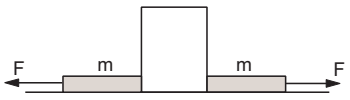



Figure 5-Q4

(a) $F_1 = F_2 = F$ for $t < 0$
(b) $F_1 = F_2 = F + mg$ for $t < 0$
(c) $F_1 = F$, $F_2 = F$ for $t > 0$
(d) $F_1 < F$, $F_2 = F$ for $t > 0$.

**6.** The force exerted by the floor of an elevator on the foot of a person standing there is more than the weight of the person if the elevator is
(a) going up and slowing down
(b) going up and speeding up
(c) going down and slowing down
(d) going down and speeding up.

**7.** If the tension in the cable supporting an elevator is equal to the weight of the elevator, the elevator may be
(a) going up with increasing speed
(b) going down with increasing speed
(c) going up with uniform speed
(d) going down with uniform speed.

**8.** A particle is observed from two frames $S_1$ and $S_2$. The frame $S_2$ moves with respect to $S_1$ with an acceleration $a$. Let $F_1$ and $F_2$ be the pseudo forces on the particle when seen from $S_1$ and $S_2$ respectively. Which of the following are not possible ?
(a) $F_1 = 0, \ F_2 \neq 0$ (b) $F_1 \neq 0, \ F_2 = 0$
(c) $F_1 \neq 0, \ F_2 \neq 0$ (d) $F_1 = 0, \ F_2 = 0$.

**9.** A person says that he measured the acceleration of a particle to be nonzero while no force was acting on the particle.
(a) He is a liar.
(b) His clock might have run slow.
(c) His meter scale might have been longer than the standard.
(d) He might have used noninertial frame.

## EXERCISES

**1.** A block of mass 2 kg placed on a long frictionless horizontal table is pulled horizontally by a constant force $F$. It is found to move 10 m in the first two seconds. Find the magnitude of $F$.

**2.** A car moving at 40 km/h is to be stopped by applying brakes in the next 4·0 m. If the car weighs 2000 kg, what average force must be applied on it ?

**3.** In a TV picture tube electrons are ejected from the cathode with negligible speed and reach a velocity of $5 \times 10^6$ m/s in travelling one centimeter. Assuming straight line motion, find the constant force exerted on the electron. The mass of the electron is $9{\cdot}1 \times 10^{-31}$ kg.

**4.** A block of mass 0·2 kg is suspended from the ceiling by a light string. A second block of mass 0·3 kg is suspended from the first block through another string. Find the tensions in the two strings. Take $g = 10$ m/s$^2$.

**5.** Two blocks of equal mass $m$ are tied to each other through a light string and placed on a smooth horizontal table. One of the blocks is pulled along the line joining them with a constant force $F$. Find the tension in the string joining the blocks.

**6.** A particle of mass 50 g moves on a straight line. The variation of speed with time is shown in figure (5-E1). Find the force acting on the particle at $t = 2$, 4 and 6 seconds.

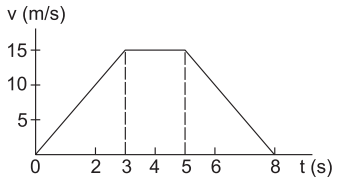



Figure 5-E1

**7.** Two blocks $A$ and $B$ of mass $m_A$ and $m_B$ respectively are kept in contact on a frictionless table. The experimenter pushes the block $A$ from behind so that the blocks accelerate. If the block $A$ exerts a force $F$ on the block $B$, what is the force exerted by the experimenter on $A$ ?

**8.** Raindrops of radius 1 mm and mass 4 mg are falling with a speed of 30 m/s on the head of a bald person. The drops splash on the head and come to rest. Assuming equivalently that the drops cover a distance equal to their radii on the head, estimate the force exerted by each drop on the head.

**9.** A particle of mass 0·3 kg is subjected to a force $F = -kx$ with $k$ = 15 N/m. What will be its initial acceleration if it is released from a point $x$ = 20 cm ?

**10.** Both the springs shown in figure (5-E2) are unstretched. If the block is displaced by a distance $x$ and released, what will be the initial acceleration?

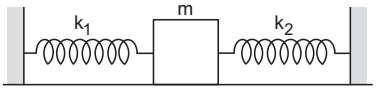



Figure 5-E2

**11.** A small block $B$ is placed on another block $A$ of mass 5 kg and length 20 cm. Initially the block B is near the right end of block $A$ (figure 5-E3). A constant horizontal force of 10 N is applied to the block $A$. All the surfaces are assumed frictionless. Find the time elapsed before the block $B$ separates from $A$.

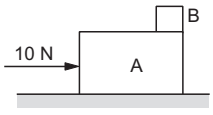



Figure 5-E3

**12.** A man has fallen into a ditch of width $d$ and two of his friends are slowly pulling him out using a light rope and two fixed pulleys as shown in figure (5-E4). Show that

the force (assumed equal for both the friends) exerted by each friend on the rope increases as the man moves up. Find the force when the man is at a depth $h$.

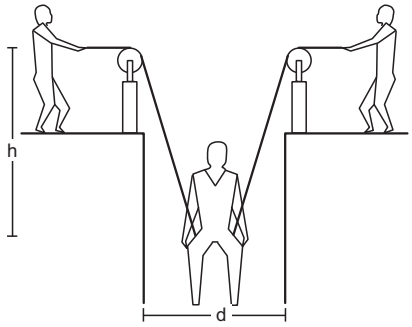



Figure 5-E4

**13.** The elevator shown in figure (5-E5) is descending with an acceleration of 2 m/s$^2$. The mass of the block $A$ is 0·5 kg. What force is exerted by the block $A$ on the block $B$ ?

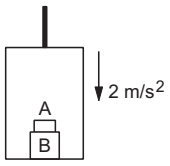



Figure 5-E5

**14.** A pendulum bob of mass 50 g is suspended from the ceiling of an elevator. Find the tension in the string if the elevator (a) goes up with acceleration 1·2 m/s$^2$, (b) goes up with deceleration 1·2 m/s$^2$, (c) goes up with uniform velocity, (d) goes down with acceleration 1·2 m/s$^2$, (e) goes down with deceleration 1·2 m/s$^2$ and (f) goes down with uniform velocity.

**15**. A person is standing on a weighing machine placed on the floor of an elevator. The elevator starts going up with some acceleration, moves with uniform velocity for a while and finally decelerates to stop. The maximum and the minimum weights recorded are 72 kg and 60 kg. Assuming that the magnitudes of the acceleration and the deceleration are the same, find (a) the true weight of the person and (b) the magnitude of the acceleration. Take $g$ = 9·9 m/s$^2$.

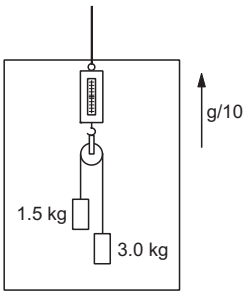



Figure 5-E6

**16.** Find the reading of the spring balance shown in figure (5-E6). The elevator is going up with an acceleration of $g/10$, the pulley and the string are light and the pulley is smooth.

**17.** A block of 2 kg is suspended from the ceiling through a massless spring of spring constant $k$ = 100 N/m. What is the elongation of the spring ? If another 1 kg is added to the block, what would be the further elongation ?

**18.** Suppose the ceiling in the previous problem is that of an elevator which is going up with an acceleration of 2·0 m/s$^2$. Find the elongations.

**19.** The force of buoyancy exerted by the atmosphere on a balloon is $B$ in the upward direction and remains constant. The force of air resistance on the balloon acts opposite to the direction of velocity and is proportional to it. The balloon carries a mass $M$ and is found to fall down near the earth's surface with a constant velocity $v$. How much mass should be removed from the balloon so that it may rise with a constant velocity $v$ ?

**20.** An empty plastic box of mass $m$ is found to accelerate up at the rate of $g/6$ when placed deep inside water. How much sand should be put inside the box so that it may accelerate down at the rate of $g/6$ ?

**21**. A force $\vec{F} = \vec{v} \times \vec{A}$ is exerted on a particle in addition to the force of gravity, where $\vec{v}$ is the velocity of the particle and $\vec{A}$ is a constant vector in the horizontal direction. With what minimum speed a particle of mass $m$ be projected so that it continues to move undeflected with a constant velocity ?

**22.** In a simple Atwood machine, two unequal masses $m_1$ and $m_2$ are connected by a string going over a clamped light smooth pulley. In a typical arrangement (figure 5-E7) $m_1$ = 300 g and $m_2$ = 600 g. The system is released from rest. (a) Find the distance travelled by the first block in the first two seconds. (b) Find the tension in the string. (c) Find the force exerted by the clamp on the pulley.

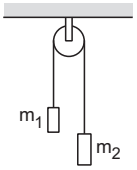



Figure 5-E7

**23.** Consider the Atwood machine of the previous problem. The larger mass is stopped for a moment 2·0 s after the system is set into motion. Find the time elapsed before the string is tight again.

**24**. Figure (5-E8) shows a uniform rod of length 30 cm having a mass of 3·0 kg. The strings shown in the figure are pulled by constant forces of 20 N and 32 N. Find the force exerted by the 20 cm part of the rod on the 10 cm part. All the surfaces are smooth and the strings and the pulleys are light.

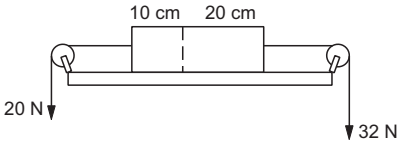



Figure 5-E8

**25.** Consider the situation shown in figure (5-E9). All the surfaces are frictionless and the string and the pulley are light. Find the magnitude of the acceleration of the two blocks.

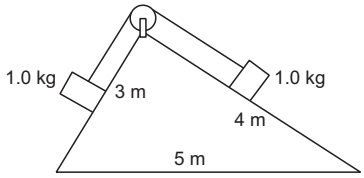



Figure 5-E9

**26.** A constant force $F = m_2 g/2$ is applied on the block of mass $m_1$ as shown in figure (5-E10). The string and the pulley are light and the surface of the table is smooth. Find the acceleration of $m_1$.

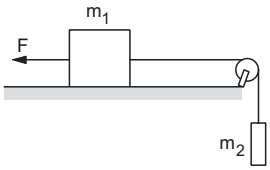



Figure 5-E10

**27.** In figure (5-E11) $m_1 = 5$ kg, $m_2 = 2$ kg and $F = 1$ N. Find the acceleration of either block. Describe the motion of $m_1$ if the string breaks but $F$ continues to act.

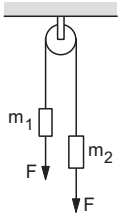



Figure 5-E11

**28.** Let $m_1 = 1$ kg, $m_2 = 2$ kg and $m_3 = 3$ kg in figure (5-E12). Find the accelerations of $m_1$, $m_2$ and $m_3$. The string from the upper pulley to $m_1$ is 20 cm when the system is released from rest. How long will it take before $m_1$ strikes the pulley ?

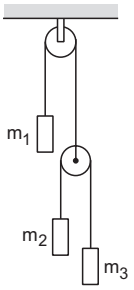



Figure 5-E12

**29.** In the previous problem, suppose $m_2 = 2{\cdot}0$ kg and $m_3 = 3{\cdot}0$ kg. What should be the mass $m$ so that it remains at rest ?

**30.** Calculate the tension in the string shown in figure (5-E13). The pulley and the string are light and all surfaces are frictionless. Take $g = 10$ m/s$^2$.

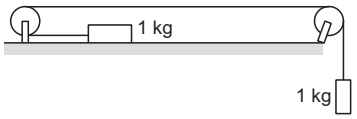



Figure 5-E13

**31.** Consider the situation shown in figure (5-E14). Both the pulleys and the string are light and all the surfaces are frictionless. (a) Find the acceleration of the mass $M$. (b) Find the tension in the string. (c) Calculate the force exerted by the clamp on the pulley $A$ in the figure.

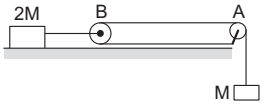



Figure 5-E14

**32.** Find the acceleration of the block of mass $M$ in the situation shown in figure (5-E15). All the surfaces are frictionless and the pulleys and the string are light.

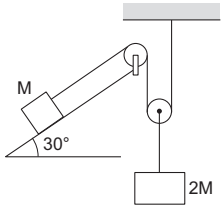



Figure 5-E15

**33.** Find the mass $M$ of the hanging block in figure (5-E16) which will prevent the smaller block from slipping over the triangular block. All the surfaces are frictionless and the strings and the pulleys are light.

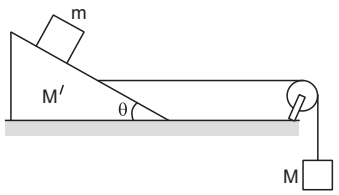



Figure 5-E16

**34.** Find the acceleration of the blocks $A$ and $B$ in the three situations shown in figure (5-E17).

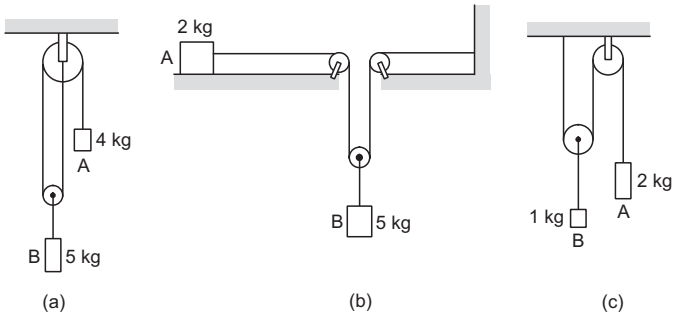



Figure 5-E17

**35.** Find the acceleration of the 500 g block in figure (5-E18).

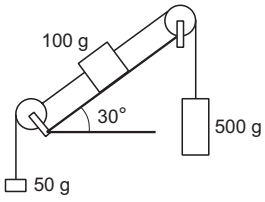



Figure 5-E18

**36.** A monkey of mass 15 kg is climbing on a rope with one end fixed to the ceiling. If it wishes to go up with an acceleration of 1 m/s$^2$, how much force should it apply to the rope ? If the rope is 5 m long and the monkey starts from rest, how much time will it take to reach the ceiling ?

**37.** A monkey is climbing on a rope that goes over a smooth light pulley and supports a block of equal mass at the other end (figure 5-E19). Show that whatever force the monkey exerts on the rope, the monkey and the block move in the same direction with equal acceleration. If initially both were at rest, their separation will not change as time passes.

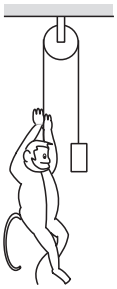


Figure 5-E19

**38.** The monkey $B$ shown in figure (5-E20) is holding on to the tail of the monkey $A$ which is climbing up a rope. The masses of the monkeys $A$ and $B$ are 5 kg and 2 kg respectively. If $A$ can tolerate a tension of 30 N in its tail, what force should it apply on the rope in order to carry the monkey $B$ with it ? Take $g = 10$ m/s$^2$.

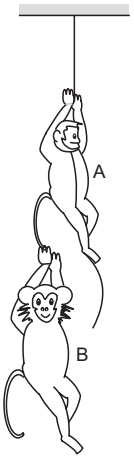



Figure 5-E20

**39.** Figure (5-E21) shows a man of mass 60 kg standing on a light weighing machine kept in a box of mass 30 kg. The box is hanging from a pulley fixed to the ceiling through a light rope, the other end of which is held by the man himself. If the man manages to keep the box at rest, what is the weight shown by the machine ? What force should he exert on the rope to get his correct weight on the machine ?

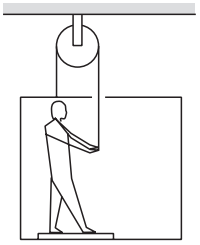


Figure 5-E21

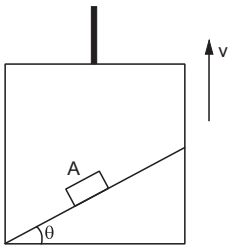



Figure 5-E22

**40.** A block $A$ can slide on a frictionless incline of angle $\theta$ and length $l$, kept inside an elevator going up with uniform velocity $v$ (figure 5-E22). Find the time taken by the block to slide down the length of the incline if it is released from the top of the incline.

**41.** A car is speeding up on a horizontal road with an acceleration $a$. Consider the following situations in the car. (i) A ball is suspended from the ceiling through a string and is maintaining a constant angle with the vertical. Find this angle. (ii) A block is kept on a smooth incline and does not slip on the incline. Find the angle of the incline with the horizontal.

**42.** A block is kept on the floor of an elevator at rest. The elevator starts descending with an acceleration of 12 m/s$^2$. Find the displacement of the block during the first 0·2 s after the start. Take $g = 10$ m/s$^2$.

□

## ANSWERS

### OBJECTIVE I

1. (c) 2. (b) 3. (d) 4. (a) 5. (c) 6. (b)
7. (d) 8. (a) 9. (b) 10. (b) 11. (c) 12. (b)
13. (a) 14. (c)

### OBJECTIVE II

1. (b), (d) 2. (c), (d) 3. (a), (b)
4. (a), (c) 5. (a) 6. (b), (c)
7. (c), (d) 8. (d) 9. (d)

### EXERCISES

1. 10 N
2. $3{\cdot}1 \times 10^4$ N
3. $1{\cdot}1 \times 10^{-15}$ N
4. 5 N and 3 N
5. $F/2$
6. 0·25 N along the motion, zero and 0·25 N opposite to the motion.
7. $F\left(1+\frac{m_A}{m_B}\right)$
8. 1·8 N
9. 10 m/s$^2$
10. $(k_1+k_2)\frac{x}{m}$ opposite to the displacement.
11. 0·45 s.
12. $\frac{mg}{4h}\sqrt{d^2+4h^2}$
13. 4 N
14. (a) 0·55 N (b) 0·43 N (c) 0·49 N
(d) 0·43 N (e) 0·55 N (f) 0·49 N
15. 66 kg and 0·9 m/s$^2$
16. 4·4 kg
17. 0·2 m, 0·1 m
18. 0·24 m, 0·12 m
19. $2\left(M-\frac{B}{g}\right)$
20. $2m/5$
21. $mg/A$
22. (a) 6·5 m (b) 3·9 N (c) 7·8 N
23. 2/3 s
24. 24 N
25. $g/10$
26. $\frac{m_2 g}{2(m_1+m_2)}$ towards right
27. 4·3 m/s$^2$, moves downward with acceleration $g + 0{\cdot}2$ m/s$^2$

28. $\frac{19}{29}g$ (*up*), $\frac{17}{29}g$ (down), $\frac{21}{29}g$ (down), 0·25 s
29. 4·8 kg
30. 5 N
31. (a) $2g/3$ (b) $Mg/3$
    (c) $\sqrt{2}\,Mg/3$ at an angle of 45° with the horizontal
32. $g/3$ up the plane
33. $\frac{M'+m}{\cot\theta - 1}$
34. (a) $\frac{2}{7}g$ downward, $\frac{g}{7}$ upward
    (b) $\frac{10}{13}g$ forward, $\frac{5}{13}g$ downward
    (c) $\frac{2}{3}g$ downward, $\frac{g}{3}$ upward
35. $\frac{8}{13}g$ downward
36. 165 N, $\sqrt{10}$ s
38. between 70 N and 105 N
39. 15 kg, 1800 N
40. $\sqrt{\frac{2l}{g\sin\theta}}$
41. $\tan^{-1}(a/g)$ in each case
42. 20 cm

□

## CHAPTER 6

# FRICTION

### 6.1 FRICTION AS THE COMPONENT OF CONTACT FORCE

When two bodies are kept in contact, electromagnetic forces act between the charged particles at the surfaces of the bodies. As a result, each body exerts a contact force on the other. The magnitudes of the contact forces acting on the two bodies are equal but their directions are opposite and hence the contact forces obey Newton's third law.

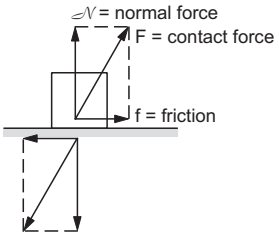



Figure 6.1

The direction of the contact force acting on a particular body is not necessarily perpendicular to the contact surface. We can resolve this contact force into two components, one perpendicular to the contact surface and the other parallel to it (Figure 6.1). The perpendicular component is called the *normal contact force* or *normal force* and the parallel component is called *friction.*

***Example 6.1***

*A body of mass* 400 g *slides on a rough horizontal surface. If the frictional force is* 3·0 N, *find (a) the angle made by the contact force on the body with the vertical and (b) the magnitude of the contact force. Take* $g = 10$ m/s$^2$.

***Solution*** **:** Let the contact force on the block by the surface be $F$ which makes an angle $\theta$ with the vertical (figure 6.2).

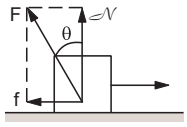



Figure 6.2

The component of $F$ perpendicular to the contact surface is the normal force $\mathcal{N}$ and the component of $F$ parallel to the surface is the friction $f$. As the surface is horizontal, $\mathcal{N}$ is vertically upward. For vertical equilibrium,

$$\mathcal{N} = Mg = (0{\cdot}400 \text{ kg})\ (10 \text{ m/s}^2) = 4{\cdot}0 \text{ N}.$$

The frictional force is $f = 3{\cdot}0$ N.

(a) $$\tan\theta = \frac{f}{\mathcal{N}} = \frac{3}{4}$$

or, $$\theta = \tan^{-1}(3/4) = 37°.$$

(b) The magnitude of the contact force is

$$F = \sqrt{\mathcal{N}^2 + f^2}$$

$$= \sqrt{(4{\cdot}0 \text{ N})^2 + (3{\cdot}0 \text{ N})^2} = 5{\cdot}0 \text{ N}.$$

Friction can operate between a given pair of solids, between a solid and a fluid or between a pair of fluids. Frictional force exerted by fluids is called *viscous force* and we shall study it in a later chapter. Here we shall study about the frictional forces operating between a pair of solid surfaces.

When two solid bodies slip over each other, the force of friction is called *kinetic friction*. When two bodies do not slip on each other, the force of friction is called *static friction.*

It is difficult to work out a reliable theory of friction starting from the electromagnetic interaction between the particles at the surface. However, a wide range of observations can be summarized in a small number of *laws of friction* which we shall discuss.

## 6.2 KINETIC FRICTION

When two bodies in contact move with respect to each other, rubbing the surfaces in contact, the friction between them is called kinetic friction. The directions of the frictional forces are such that the relative slipping is opposed by the friction.

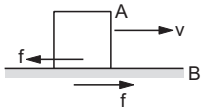



Figure 6.3

Suppose a body $A$ placed in contact with $B$ is moved with respect to it as shown in figure (6.3). The force of friction acting on $A$ due to $B$ will be opposite to the velocity of $A$ with respect to $B$. In figure (6.3) this force is shown towards left. The force of friction on $B$ due to $A$ is opposite to the velocity of $B$ with respect to $A$. In figure (6.3) this force is shown towards right. The force of kinetic friction opposes the relative motion. We can formulate the rules for finding the direction and magnitude of kinetic friction as follows :

### (a) Direction of Kinetic Friction

*The kinetic friction on a body A slipping against another body B is opposite to the velocity of A with respect to B.*

It should be carefully noted that the velocity coming into picture is with respect to the body applying the force of friction.

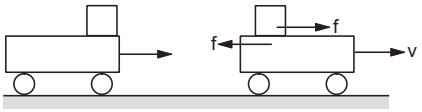



Figure 6.4

As another example, suppose we have a long box having wheels and moving on a horizontal road (figure 6.4). A small block is placed on the box which slips on the box to fall from the rear end. As seen from the road, both the box and the block are moving towards right, of course the velocity of the block is smaller than that of the box. What is the direction of the kinetic friction acting on the block due to the box ? The velocity of the block as seen from the box is towards left. Thus, the friction on the block is towards right. The friction acting on the box due to the block is towards left.

### (b) Magnitude of the Kinetic Friction

*The magnitude of the kinetic friction is proportional to the normal force acting between the two bodies. We can write*

$$f_k = \mu_k \mathcal{N} \qquad \ldots \quad (6.1)$$

*where* $\mathcal{N}$ *is the normal force*. The proportionality constant $\mu_k$ is called the *coefficient of kinetic friction* and its value depends on the nature of the two surfaces in contact. If the surfaces are smooth $\mu_k$ will be small, if the surfaces are rough $\mu_k$ will be large. It also depends on the materials of the two bodies in contact.

According to equation (6.1) the coefficient of kinetic friction does not depend on the speed of the sliding bodies. Once the bodies slip on each other the frictional force is $\mu_k \mathcal{N}$, whatever be the speed. This is approximately true for relative speeds not too large (say for speeds < 10 m/s).

We also see from equation (6.1) that as long as the normal force $\mathcal{N}$ is same, the frictional force is independent of the area of the surface in contact. For example, if a rectangular slab is slid over a table, the frictional force is same whether the slab lies flat on the table or it stands on its face of smaller area (figure 6.5)

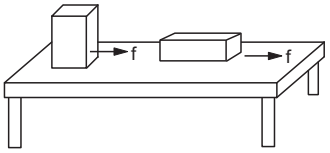



Figure 6.5

***Example 6.2***

*A heavy box of mass* 20 kg *is pulled on a horizontal surface by applying a horizontal force. If the coefficient of kinetic friction between the box and the horizontal surface is* 0·25*, find the force of friction exerted by the horizontal surface on the box.*

***Solution*** **:** The situation is shown in figure (6.6). In the vertical direction there is no acceleration, so

$$\mathcal{N} = Mg.$$

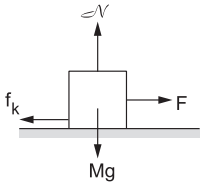



Figure 6.6

As the box slides on the horizontal surface, the surface exerts kinetic friction on the box. The magnitude of the kinetic friction is

$$f_k = \mu_k \mathcal{N} = \mu_k Mg$$

$$= 0{\cdot}25 \times (20 \text{ kg}) \times (9{\cdot}8 \text{ m/s}^2) = 49 \text{ N}.$$

This force acts in the direction opposite to the pull.

### 6.3 STATIC FRICTION

Frictional forces can also act between two bodies which are in contact but are not sliding with respect to each other. The friction in such cases is called *static friction*. For example, suppose several labourers are trying to push a heavy almirah on the floor to take it out of a room (figure 6.7).

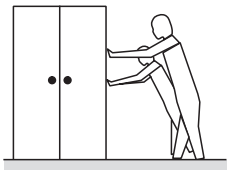


Figure 6.7

The almirah is heavy and even the most sincere effort by them is not able to slide it on the floor even by a millimeter. As the almirah is at rest the resultant force on the almirah should be zero. Thus, something is exerting a force on the almirah in the opposite direction. In this case, it is the floor which exerts a frictional force on the almirah. The labourers push the almirah towards left in figure (6.7) and the floor exerts a frictional force on the almirah towards right. This is an example of static friction.

How strong is this frictional force ? Suppose the almirah is pushed with a small force in the beginning and the force is gradually increased. It does not slide until the force applied is greater than a minimum value say $F$. The force of static friction is equal and opposite to the force exerted by the labourers as long as the almirah is at rest. This means that the magnitude of static friction adjusts its value according to the applied force. As the applied force increases, the frictional force also increases. The static friction is thus, self adjustable. It adjusts its magnitude (and direction) in such a way that together with other forces applied on the body, it maintains 'relative rest' between the two surfaces. However, the frictional force cannot go beyond a maximum. When the applied force exceeds this maximum, friction fails to increase its value and slipping starts. The maximum static friction that a body can exert on the other body in contact with it, is called *limiting friction*. This limiting friction is proportional to the normal contact force between the two bodies. We can write

$$f_{max} = \mu_s \mathcal{N} \qquad \ldots \quad (6.2)$$

where $f_{max}$ is the maximum possible force of static friction and $\mathcal{N}$ is the normal force. The constant of proportionality is called the *coefficient of static friction* and its value again depends on the material and roughness of the two surfaces in contact. In general, $\mu_s$ is slightly greater than $\mu_k$. As long as the normal force is constant, the maximum possible friction does not depend on the area of the surfaces in contact.

Once again we emphasise that $\mu_s \mathcal{N}$ is the **maximum** possible force of static friction that **can** act between the bodies. The actual force of static friction may be smaller than $\mu_s \mathcal{N}$ and its value depends on other forces acting on the body. The magnitude of frictional force is equal to that required to keep the body at relative rest. Thus,

$$f_s \le f_{max} = \mu_s \mathcal{N}. \qquad \ldots \quad (6.3)$$

***Example 6.3***

*A boy* (30 kg) *sitting on his horse whips it. The horse speeds up at an average acceleration of* 2·0 m/s$^2$. *(a) If the boy does not slide back, what is the force of friction exerted by the horse on the boy ? (b) If the boy slides back during the acceleration, what can be said about the coefficient of static friction between the horse and the boy. Take* $g = 10$ m/s$^2$.

***Solution*** : (a) The forces acting on the boy are

(i) the weight $Mg$.

(ii) the normal contact force $\mathcal{N}$ and

(iii) the static friction $f_s$.

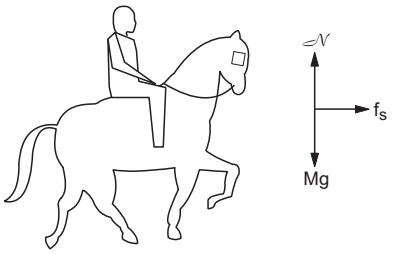



Figure 6.8

As the boy does not slide back, its acceleration $a$ is equal to the acceleration of the horse. As friction is the only horizontal force, it must act along the acceleration and its magnitude is given by Newton's second law

$$f_s = Ma = (30 \text{ kg})\ (2{\cdot}0 \text{ m/s}^2) = 60 \text{ N}.$$

(b) If the boy slides back, the horse could not exert a friction of 60 N on the boy. The maximum force of static friction that the horse may exert on the boy is

$$f_s = \mu_s \mathcal{N} = \mu_s Mg$$

$$= \mu_s\ (30 \text{ kg})\ (10 \text{ m/s}^2) = \mu_s\ 300 \text{ N}$$

where $\mu_s$ is the coefficient of static friction. Thus,

$$\mu_s\ (300 \text{ N}) < 60 \text{ N}$$

or,
$$\mu_s < \frac{60}{300} = 0{\cdot}20.$$

**Finding the Direction of Static Friction**

The direction of static friction on a body is such that the total force acting on it keeps it at rest with respect to the body in contact. Newton's first or second law can often be used to find the direction of static friction. Figure (6.9) shows a block $A$ placed on another block $B$ which is placed on a horizontal table.

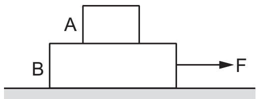



Figure 6.9

The block $B$ is pulled by a force $F$ towards right. Suppose the force is small and the blocks do not move. Let us focus our attention on the upper block. The upper block is at rest with respect to the ground which is assumed to be inertial. Thus, the resultant force on the upper block is zero (Newton's first law). As no other external force acts on the upper block the friction acting on the upper block due to the lower block, must be zero. If the force $F$ is increased, the two blocks move together towards right, with some acceleration. As the upper block accelerates towards right the resultant force on it must be towards right. As friction is the only horizontal force on the upper block it must be towards right.

Notice that it is the friction on the upper block which accelerates it towards right. It is a general misconception that friction always opposes the motion. It is not really true. In many cases friction causes the motion. A vehicle accelerates on the road only because the frictional force on the vehicle due to the road drives it. It is not possible to accelerate a vehicle on a frictionless road. Friction opposes the relative motion between the bodies in contact.

Another way to find the direction of static friction is as follows. For a moment consider the surfaces to be frictionless. In absence of friction the bodies will start slipping against each other. One should then find the direction of friction as opposite to the velocity with respect to the body applying the friction.

## 6.4 LAWS OF FRICTION

We can summarise the laws of friction between two bodies in contact as follows :

(1) If the bodies slip over each other, the force of friction is given by

$$f_k = \mu_k \mathcal{N}$$

where $\mathcal{N}$ is the normal contact force and $\mu_k$ is the coefficient of kinetic friction between the surfaces.

(2) The direction of kinetic friction on a body is opposite to the velocity of this body with respect to the body applying the force of friction.

(3) If the bodies do not slip over each other, the force of friction is given by

$$f_s \le \mu_s \mathcal{N}$$

where $\mu_s$ is the coefficient of static friction between the bodies and $\mathcal{N}$ is the normal force between them. The direction and magnitude of static friction are such that the condition of no slipping between the bodies is ensured.

(4) The frictional force $f_k$ or $f_s$ does not depend on the area of contact as long as the normal force $\mathcal{N}$ is same.

Table (6.1) gives a rough estimate of the values of coefficient of static friction between certain pairs of materials. The actual value depends on the degree of smoothness and other environmental factors. For example, wood may be prepared at various degrees of smoothness and the friction coefficient will varry.

**Table 6.1 : *The Friction Coefficients***

| Material | $\mu_s$ | Material | $\mu_s$ |
|---|---|---|---|
| Steel and steel | 0·58 | Copper and copper | 1·60 |
| Steel and brass | 0·35 | Teflon and teflon | 0·04 |
| Glass and glass | 1·00 | Rubber tyre on dry concrete road | 1·0 |
| Wood and wood | 0·35 | | |
| Wood and metal | 0·40 | Rubber tyre on wet concrete road | 0·7 |
| Ice and ice | 0·10 | | |

Dust, impurities, surface oxidation etc. have a great role in determining the friction coefficient. Suppose we take two blocks of pure copper, clean them carefully to remove any oxide or dust layer at the surfaces, heat them to push out any dissolved gases and keep them in contact with each other in an evacuated chamber at a very low pressure of air. The blocks stick to each other and a large force is needed to slide one over the other. The friction coefficient as defined above, becomes much larger than one. If a small amount of air is allowed to go into the chamber so that some oxidation takes place at the surface, the friction coefficient reduces to usual values.

## 6.5 UNDERSTANDING FRICTION AT ATOMIC LEVEL

It has already been pointed out that friction appears because of the interaction between the charged particles of the two bodies near the surfaces of contact. Any macroscopic object like a steel plate or a wood piece has irregular surface at atomic scale. A polished steel surface may look plane to naked eyes but if seen

under a powerful microscope, its surface is found to be quite irregular. Figure (6.10) shows qualitatively how an apparently plane surface may be at the atomic scale.

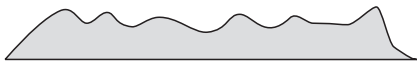


Figure 6.10

When two bodies are kept one over the other, the real area in contact is much smaller than the total surface area of the bodies (figure 6.11) The distance between the particles of the two bodies becomes very small at these actual points of contact and the molecular forces start operating across the surface. Molecular bonds are formed at these contact points. When one of the two bodies is pulled over the other, these bonds are broken, the materials under the bond is deformed and new bonds are formed. The local deformation of the bodies send vibration waves into the bodies. These vibrations finally damp out and the energy appears as the increased random motion of the particles of the bodies. The bodies thus, become heated. A force is, therefore, needed to start the motion or to maintain the motion.

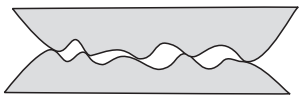


Figure 6.11

## 6.6 A LABORATORY METHOD TO MEASURE FRICTION COEFFICIENT

### (a) Horizontal Table Method

Figure (6.12) shows the apparatus. A wooden plank $A$ is fixed on a wooden frame kept on a table. A frictionless pulley is fixed to one end of the plank. A block $B$ is kept on the plank and is attached to a hanger $H$ by a string which passes over the pulley.

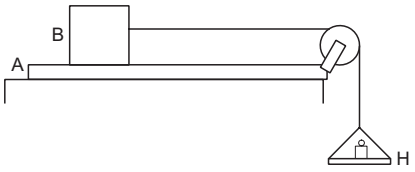



Figure 6.12

The plank is kept in a horizontal position. The friction coefficient between the block $B$ and the plank $A$ can be obtained by using this apparatus.

The weights of the block $B$ and the hanger $H$ are measured. Standard weights are kept on the hanger. The weights are gradually increased and the minimum weight needed to just slide the block is noted.

Suppose the weight of the block is $W_1$ and the weight of the hanger together with the standard weights is $W_2$ when the block just starts to slide. The tension in the string is $W_2$ and same is the force of friction on the block by the plank. Thus, the maximum force of static friction on the block is $f_{\max} = W_2$. The normal force on the block by the plank is equal to the weight of the block itself as the block is in vertical equilibrium. Thus, the normal force is $\mathcal{N} = W_1$.

The coefficient of static friction is

$$\mu_s = \frac{f_{\max}}{\mathcal{N}} = \frac{W_2}{W_1}.$$

To obtain the coefficient of kinetic friction, the weight on the hanger is slightly reduced and the block is gently pushed with a finger to move it on the plank. The weight on the hanger is so adjusted that once pushed, the block continues to move on the plank with uniform speed. In this case, the tension in the string equals the force of kinetic friction. As the hanger also moves with uniform velocity, the tension equals the weight of the hanger plus the standard weights kept in it. For vertical equilibrium of the block, the normal force on the block equals the weight of the block. Thus, if $W_1$ is the weight of the block and $W_2'$ is the weight of the hanger plus the standard weights, the coefficient of kinetic friction is

$$\mu_k = \frac{f_k}{\mathcal{N}} = \frac{W_2'}{W_1}.$$

One can put certain standard weights on the block to increase the normal force and repeat the experiment. It can be verified that the force of friction also increases and $f_k / \mathcal{N}$ comes out to be the same as it should be because the nature of the surfaces is same. If the block is kept on the plank on some other face, the area of contact is changed. It can be verified by repeating the above experiment that the force of friction does not depend on the area of contact for a given value of normal contact force.

### (b) Inclined Table Method

In this method no pulley is needed. A wooden plank $A$ is fixed on a wooden frame. One end of the plank is fixed to the frame on a hinge and the other end can be moved vertically and can be fixed at the desired position. Thus, the plank can be fixed in an inclined position and the angle of incline can be adjusted. A schematic diagram is shown in figure (6.13).

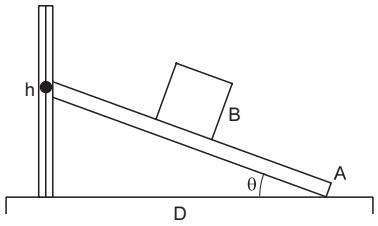



Figure 6.13

Block $B$ is placed on the incline and the angle of the incline is gradually increased. The angle of the incline is so adjusted that the block just starts to slide. The height $h$ and the horizontal distance $D$ between the two ends of the plank are measured. The angle of incline θ satisfies

$$\tan\theta = h/D.$$

Let $m$ be the mass of the block. The forces on the block in case of limiting equilibrium are (figure 6.14)

(i) weight of the block $mg$,

(ii) the normal contact force $\mathcal{N}$, and

(iii) the force of static friction $f_s$.

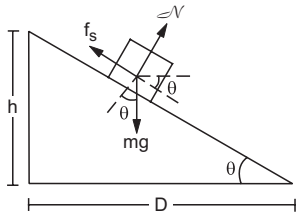



Figure 6.14

Taking components along the incline and applying Newton's first law,

$$f_s = mg\sin\theta.$$

Taking components along the normal to the incline,

$$\mathcal{N} = mg\cos\theta.$$

Thus, the coefficient of static friction between the block and the plank is

$$\mu_s = \frac{f_s}{\mathcal{N}} = \frac{mg\sin\theta}{mg\cos\theta} = \tan\theta = \frac{h}{D}.$$

To obtain the kinetic friction, the inclination is reduced slightly and the block is made to move on the plank by gently pushing it with a finger. The inclination is so adjusted that once started, the block continues with uniform velocity on the plank. The height $h'$ and the distance $D'$ are noted. An identical analysis shows that the force of kinetic friction is

$$f_k = mg\sin\theta$$

and the normal contact force is

$$\mathcal{N} = mg\cos\theta$$

so that the coefficient of kinetic friction between the block and the plank is

$$\mu_k = \frac{f_k}{\mathcal{N}} = \tan\theta = h'/D'.$$

***Example 6.4***

*A wooden block is kept on a polished wooden plank and the inclination of the plank is gradually increased. It is found that the block starts slipping when the plank makes an angle of* 18° *with the horizontal. However, once started the block can continue with uniform speed if the inclination is reduced to* 15°. *Find the coefficients of static and kinetic friction between the block and the plank.*

***Solution*** : The coefficient of static friction is

$$\mu_s = \tan 18°$$

and the coefficient of kinetic friction is

$$\mu_k = \tan 15°$$

**Rolling Friction**

It is quite difficult to pull a heavy iron box on a rough floor. However, if the box is provided with four wheels, also made of iron, it becomes easier to move the box on the same floor.

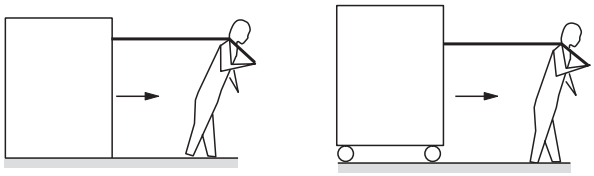


Figure 6.15

The wheel does not slide on the floor rather it rolls on the floor. The surfaces at contact do not rub each other. The velocity of the point of contact of the wheel with respect to the floor remains zero all the time although the centre of the wheel moves forward. The friction in the case of rolling is quite small as compared to kinetic friction. Quite often the rolling friction is negligible in comparison to the static or kinetic friction which may be present simultaneously. To reduce the wear and tear and energy loss against friction, small steel balls are kept between the rotating parts of machines which are known as ball bearings (figure 6.16).

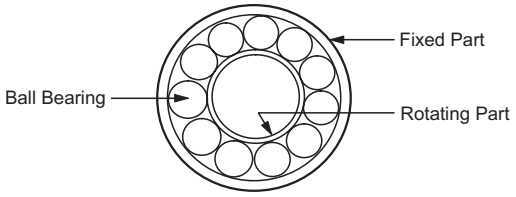



Figure 6.16

As one part moves with respect to the other, the balls roll on the two parts. No kinetic friction is involed and rolling friction being very small causes much less energy loss.

## *Worked Out Examples*

**1.** *The coefficient of static friction between a block of mass m and an incline is* $\mu_s = 0{\cdot}3$. *(a) What can be the maximum angle* $\theta$ *of the incline with the horizontal so that the block does not slip on the plane ? (b) If the incline makes an angle* $\theta/2$ *with the horizontal, find the frictional force on the block.*

***Solution*** **:** The situation is shown in figure (6-W1).

(a) the forces on the block are

(i) the weight $mg$ downward by the earth,

(ii) the normal contact force $\mathcal{N}$ by the incline, and

(iii) the friction $f$ parallel to the incline up the plane, by the incline.

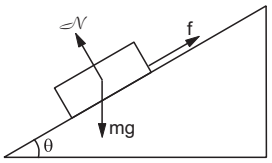



Figure 6-W1

As the block is at rest, these forces should add up to zero. Also, since $\theta$ is the maximum angle to prevent slipping, this is a case of limiting equilibrium and so $f = \mu_s \mathcal{N}$.

Taking components prependicular to the incline,

$$\mathcal{N} - mg\cos\theta = 0$$

or, $$\mathcal{N} = mg\cos\theta. \quad \ldots \text{(i)}$$

Taking components parallel to the incline,

$$f - mg\sin\theta = 0$$

or, $$f = mg\sin\theta$$

or, $$\mu_s \mathcal{N} = mg\sin\theta. \quad \ldots \text{(ii)}$$

Dividing (ii) by (i) $\mu_s = \tan\theta$

or, $$\theta = \tan^{-1}\mu_s = \tan^{-1}(0{\cdot}3).$$

(b) If the angle of incline is reduced to $\theta/2$, the equilibrium is not limiting, and hence the force of static friction $f$ is less than $\mu_s \mathcal{N}$. To know the value of $f$, we proceed as in part (a) and get the equations

$$\mathcal{N} = mg\cos(\theta/2)$$

and $$f = mg\sin(\theta/2).$$

Thus, the force of friction is $mg\sin(\theta/2)$.

**2.** *A horizontal force of* 20 N *is applied to a block of mass* 4 kg *resting on a rough horizontal table. If the block does not move on the table, how much frictional force the table is applying on the block ? What can be said about the coefficient of static friction between the block and the table ? Take* $g = 10$ m/s$^2$.

***Solution*** **:** The situation is shown in figure (6-W2). The forces on the block are

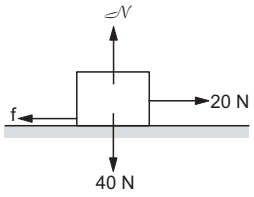



Figure 6-W2

(a) 4 kg $\times$ 10 m/s$^2$ = 40 N downward by the earth,

(b) $\mathcal{N}$ upward by the table,

(c) 20 N towards right by the experimenter and

(d) $f$ towards left by the table (friction).

As the block is at rest, these forces should add up to zero. Taking horizontal and vertical components,

$$f = 20 \text{ N} \quad \text{and} \quad \mathcal{N} = 40 \text{ N}.$$

Thus, the table exerts a frictional (static) force of 20 N on the block in the direction opposite to the applied force. Since it is a case of static friction,

$$f \le \mu_s \mathcal{N}, \quad \text{or,} \quad \mu_s \ge f/\mathcal{N} \quad \text{or,} \quad \mu_s \ge 0{\cdot}5.$$

**3.** *The coefficient of static friction between the block of* 2 kg *and the table shown in figure (6-W3) is* $\mu_s = 0{\cdot}2$. *What should be the maximum value of m so that the blocks do not move ? Take* $g = 10$ m/s$^2$. *The string and the pulley are light and smooth.*

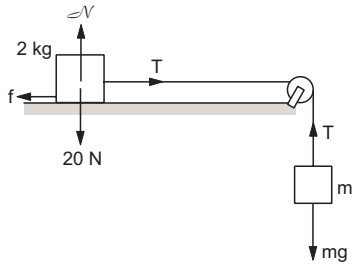



Figure 6-W3

***Solution*** **:** Consider the equilibrium of the block of mass $m$. The forces on this block are

(a) $mg$ downward by the earth and

(b) $T$ upward by the string.

Hence, $T - mg = 0$ or, $T = mg$. … (i)

Now consider the equilibrium of the 2 kg block. The forces on this block are

(a) $T$ towards right by the string,

(b) $f$ towards left (friction) by the table,

(c) 20 N downward (weight) by the earth and

(d) $\mathcal{N}$ upward (normal force) by the table.

For vertical equilibrium of this block,

$$\mathcal{N} = 20 \text{ N}. \quad \text{… (ii)}$$

As $m$ is the largest mass which can be used without moving the system, the friction is limiting.

Thus, $f = \mu_s \mathcal{N}$. … (iii)

For horizontal equilibrium of the 2 kg block,

$$f = T. \quad \text{… (iv)}$$

Using equations (i), (iii) and (iv)

$$\mu_s \mathcal{N} = mg$$

or, $0{\cdot}2 \times 20 \text{ N} = mg$

or, $$m = \frac{0{\cdot}2 \times 20}{10} \text{ kg} = 0{\cdot}4 \text{ kg}.$$

**4.** *The coefficient of static friction between the two blocks shown in figure (6-W4) is* $\mu$ *and the table is smooth. What maximum horizontal force* $F$ *can be applied to the block of mass* $M$ *so that the blocks move together ?*

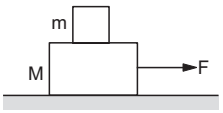



Figure 6-W4

***Solution*** **:** When the maximum force $F$ is applied, both the blocks move together towards right. The only horizontal force on the upper block of mass $m$ is that due to the friction by the lower block of mass $M$. Hence this force on $m$ should be towards right. The force of friction on $M$ by $m$ should be towards left by Newton's third law. As we are talking of the maximum possible force $F$ that can be applied, the friction is limiting and hence $f = \mu \mathcal{N}$, where $\mathcal{N}$ is the normal force between the blocks.

Consider the motion of $m$. The forces on $m$ are (figure 6-W5),

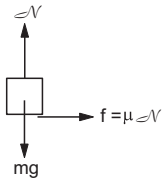



Figure 6-W5

(a) $mg$ downward by the earth (gravity),

(b) $\mathcal{N}$ upward by the block $M$ (normal force) and

(c) $f = \mu \mathcal{N}$ (friction) towards right by the block $M$.

In the vertical direction, there is no acceleration. This gives

$$\mathcal{N} = mg. \quad \text{… (i)}$$

In the horizontal direction, let the acceleration be $a$, then

$$\mu \mathcal{N} = m\, a$$

or, $$\mu\, mg = ma$$

or, $$a = \mu\, g. \quad \text{… (ii)}$$

Next, consider the motion of $M$ (figure 6-W6).

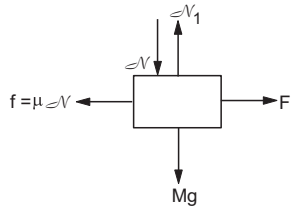



Figure 6-W6

The forces on $M$ are

(a) $Mg$ downward by the earth (gravity),

(b) $\mathcal{N}_1$ upward by the table (normal force),

(c) $\mathcal{N}$ downward by $m$ (normal force),

(d) $f = \mu \mathcal{N}$ (friction) towards left by $m$ and

(e) $F$ (applied force) by the experimenter.

The equation of motion is

$$F - \mu \mathcal{N} = M\, a$$

or, $$F - \mu\, mg = M\, \mu\, g \text{ [Using (i) and (ii)]}$$

or, $$F = \mu\, g\, (M + m).$$

**5.** *A block slides down an incline of angle* 30° *with an acceleration* $g/4$. *Find the kinetic friction coeffcient.*

***Solution*** **:** Let the mass of the block be $m$. The forces on the block are (Figure 6-W7),

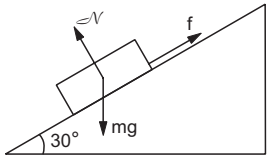



Figure 6-W7

(a) $mg$ downward by the earth (gravity),

(b) $\mathcal{N}$ normal force by the incline and

(c) $f$ up the plane, (friction) by the incline.

Taking components parallel to the incline and writing Newton's second law,

$$mg \sin 30° - f = mg/4$$

or, $$f = mg/4.$$

There is no acceleration perpendicular to the incline. Hence,

$$\mathcal{N} = mg \cos 30° = mg \cdot \frac{\sqrt{3}}{2}.$$

As the block is slipping on the incline, friction is $f = \mu_k \mathcal{N}$.

So, $$\mu_k = \frac{f}{\mathcal{N}} = \frac{mg}{4\, mg\, \sqrt{3}/2} = \frac{1}{2\sqrt{3}}.$$

**6.** *A block of mass* 2·5 kg *is kept on a rough horizontal surface. It is found that the block does not slide if a horizontal force less than* 15 N *is applied to it. Also it is found that it takes* 5 seconds *to slide through the first* 10 m *if a horizontal force of* 15 N *is applied and the block is gently pushed to start the motion. Taking* $g = 10$ m/s$^2$*, calculate the coefficients of static and kinetic friction between the block and the surface.*

***Solution*** **:** The forces acting on the block are shown in figure (6-W8). Here $M$ = 2·5 kg and $F$ = 15 N.

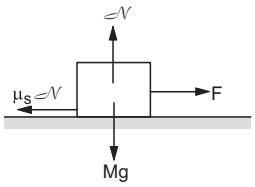



Figure 6-W8

When $F$ = 15 N is applied to the block, the block remains in limiting equilibrium. The force of friction is thus $f = \mu_s \mathcal{N}$. Applying Newton's first law,

$$f = \mu_s \mathcal{N} \quad \text{and} \quad \mathcal{N} = mg$$

so that $$F = \mu_s Mg$$

or, $$\mu_s = \frac{F}{mg} = \frac{15 \text{ N}}{(2{\cdot}5 \text{ kg})\,(10 \text{ m/s}^2)} = 0{\cdot}60.$$

When the block is gently pushed to start the motion, kinetic friction acts between the block and the surface. Since the block takes 5 second to slide through the first 10 m, the acceleration $a$ is given by

$$10 \text{ m} = \frac{1}{2} a\, (5 \text{ s})^2$$

or, $$a = \frac{20}{25} \text{ m/s}^2 = 0{\cdot}8 \text{ m/s}^2.$$

The frictional force is

$$f = \mu_k \mathcal{N} = \mu_k Mg.$$

Applying Newton's second law

$$F - \mu_k Mg = Ma$$

or, $$\mu_k = \frac{F - Ma}{Mg}$$

$$= \frac{15 \text{ N} - (2{\cdot}5 \text{ kg})\,(0{\cdot}8 \text{ m/s}^2)}{(2{\cdot}5 \text{ kg})\,(10 \text{ m/s}^2)} = 0{\cdot}52.$$

**7.** *A block placed on a horizontal surface is being pushed by a force F making an angle* $\theta$ *with the vertical. If the friction coefficient is* $\mu$*, how much force is needed to get the block just started. Discuss the situation when* $\tan\theta < \mu$.

***Solution*** **:** The situation is shown in figure (6-W9). In the limiting equilibrium the frictional force $f$ will be equal to $\mu \mathcal{N}$. For horizontal equilibrium

$$F \sin\theta = \mu \mathcal{N}$$

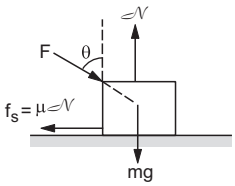



Figure 6-W9

For vertical equilibrium

$$F \cos\theta + mg = \mathcal{N}.$$

Eliminating $\mathcal{N}$ from these equations,

$$F \sin\theta = \mu F \cos\theta + \mu mg$$

or, $$F = \frac{\mu mg}{\sin\theta - \mu \cos\theta}.$$

If $\tan\theta < \mu$ we have $(\sin\theta - \mu \cos\theta) < 0$ and then $F$ is negative. So for angles less than $\tan^{-1}\mu$, one cannot push the block ahead, however large the force may be.

**8.** *Find the maximum value of* $M/m$ *in the situation shown in figure (6-W10) so that the system remains at rest. Friction coefficient at both the contacts is* $\mu$ *. Discuss the situation when* $\tan\theta < \mu$.

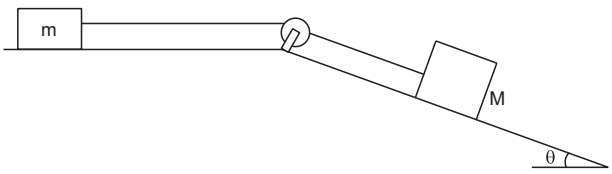



Figure 6-W10

***Solution*** **:** Figure (6-W11) shows the forces acting on the two blocks. As we are looking for the maximum value of $M/m$, the equilibrium is limiting. Hence, the frictional forces are equal to $\mu$ times the corresponding normal forces.

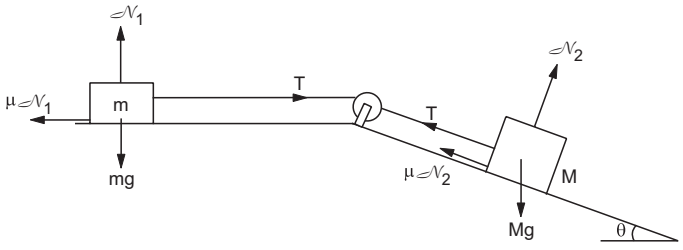



Figure 6-W11

Equilibrium of the block $m$ gives

$T = \mu \mathcal{N}_1$ and $\mathcal{N}_1 = mg$

which gives

$$T = \mu\, mg. \quad \ldots \text{(i)}$$

Next, consider the equilibrium of the block $M$. Taking components parallel to the incline

$$T + \mu \mathcal{N}_2 = Mg \sin\theta.$$

Taking components normal to the incline

$$\mathcal{N}_2 = Mg \cos\theta.$$

These give $T = Mg\,(\sin\theta - \mu \cos\theta)$. … (ii)

From (i) and (ii), $\mu\, mg = Mg\,(\sin\theta - \mu \cos\theta)$

or, $$M/m = \frac{\mu}{\sin\theta - \mu \cos\theta}$$

If $\tan\theta < \mu$, $(\sin\theta - \mu \cos\theta) < 0$ and the system will not slide for any value of $M/m$.

**9.** *Consider the situation shown in figure (6-W12). The horizontal surface below the bigger block is smooth. The coefficient of friction between the blocks is* $\mu$. *Find the minimum and the maximum force F that can be applied in order to keep the smaller block at rest with respect to the bigger block.*

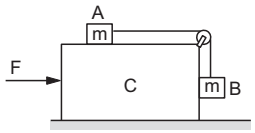



Figure 6-W12

***Solution*** **:** If no force is applied, the block $A$ will slip on $C$ towards right and the block $B$ will move downward. Suppose the minimum force needed to prevent slipping is $F$. Taking $A + B + C$ as the system, the only external horizontal force on the system is $F$. Hence the acceleration of the system is

$$a = \frac{F}{M + 2m}. \quad \ldots \text{(i)}$$

Now take the block $A$ as the system. The forces on $A$ are (figure 6-W13),

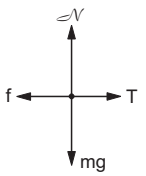



Figure 6-W13

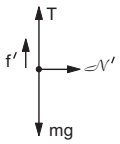



Figure 6-W14

(i) tension $T$ by the string towards right,
(ii) friction $f$ by the block $C$ towards left,
(iii) weight $mg$ downward and
(iv) normal force $\mathcal{N}$ upward.

For vertical equilibrium $\mathcal{N} = mg$.

As the minimum force needed to prevent slipping is applied, the friction is limiting. Thus,

$$f = \mu \mathcal{N} = \mu\, mg.$$

As the block moves towards right with an acceleration $a$,

$$T - f = ma$$

or, $$T - \mu\, mg = ma. \quad \ldots \text{(ii)}$$

Now take the block $B$ as the system. The forces are (figure 6-W14),

(i) tension $T$ upward,
(ii) weight $mg$ downward,
(iii) normal force $\mathcal{N}'$ towards right, and
(iv) friction $f'$ upward.

As the block moves towards right with an acceleration $a$,

$$\mathcal{N}' = ma.$$

As the friction is limiting, $f' = \mu \mathcal{N}' = \mu\, ma$.

For vertical equilibrium

$$T + f' = mg$$

or, $$T + \mu\, ma = mg. \quad \ldots \text{(iii)}$$

Eliminating $T$ from (ii) and (iii)

$$a_{\min} = \frac{1 - \mu}{1 + \mu} g\,.$$

When a large force is applied the block $A$ slips on $C$ towards left and the block $B$ slips on $C$ in the upward direction. The friction on $A$ is towards right and that on $B$ is downwards. Solving as above, the acceleration in this case is

$$a_{\max} = \frac{1 + \mu}{1 - \mu} g\,.$$

Thus, $a$ lies between $\frac{1 - \mu}{1 + \mu} g$ and $\frac{1 + \mu}{1 - \mu} g$.

From (i) the force $F$ should be between

$$\frac{1 - \mu}{1 + \mu}(M + 2m)\, g \text{ and } \frac{1 + \mu}{1 - \mu}(M + 2m)\, g.$$

**10.** *Figure (6-W15) shows two blocks connected by a light string placed on the two inclined parts of a triangular structure. The coefficients of static and kinetic friction are* 0·28 *and* 0·25 *respectively at each of the surfaces. (a) Find the minimum and maximum values of m for which the system remains at rest. (b) Find the acceleration of either block if m is given the minimum value calculated in the first part and is gently pushed up the incline for a short while.*

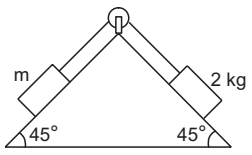



Figure 6-W15

***Solution*** : (a) Take the 2 kg block as the system. The forces on this block are shown in figure (6-W16) with $M$ = 2 kg. It is assumed that $m$ has its minimum value so that the 2 kg block has a tendency to slip down. As the block is in equilibrium, the resultant force should be zero.

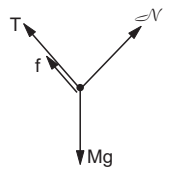



Figure 6-W16

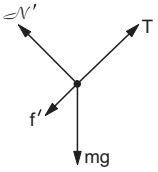



Figure 6-W17

Taking components ⊥ to the incline

$$\mathcal{N} = Mg \cos 45° = Mg/\sqrt{2}.$$

Taking components || to the incline

$$T + f = Mg \sin 45° = Mg/\sqrt{2}$$

or, $$T = Mg/\sqrt{2} - f.$$

As it is a case of limiting equilibrium,

$$f = \mu_s \mathcal{N}$$

or, $$T = \frac{Mg}{\sqrt{2}} - \mu_s \frac{Mg}{\sqrt{2}} = \frac{Mg}{\sqrt{2}} (1 - \mu_s). \quad \ldots \text{(i)}$$

Now consider the other block as the system. The forces acting on this block are shown in figure (6-W17).

Taking components ⊥ to the incline,

$$\mathcal{N}' = mg \cos 45° = mg/\sqrt{2}.$$

Taking components || to the incline

$$T = mg \sin 45° + f' = \frac{mg}{\sqrt{2}} + f'.$$

As it is the case of limiting equilibrium

$$f' = \mu_s \mathcal{N}' = \mu_s \frac{mg}{\sqrt{2}}.$$

Thus, $$T = \frac{mg}{\sqrt{2}} (1 + \mu_s). \quad \ldots \text{(ii)}$$

From (i) and (ii)

$$m(1 + \mu_s) = M (1 - \mu_s) \quad \ldots \text{(iii)}$$

or, $$m = \frac{(1 - \mu_s)}{(1 + \mu_s)} M = \frac{1 - 0{\cdot}28}{1 + 0{\cdot}28} \times 2 \text{ kg}$$

$$= \frac{9}{8} \text{ kg}.$$

When maximum possible value of $m$ is supplied, the directions of friction are reversed because $m$ has the tendency to slip down and 2 kg block to slip up. Thus, the maximum value of $m$ can be obtained from (iii) by putting $\mu = -0{\cdot}28$. Thus, the maximum value of $m$ is

$$m = \frac{1 + 0{\cdot}28}{1 - 0{\cdot}28} \times 2 \text{ kg}$$

$$= \frac{32}{9} \text{ kg}.$$

(b) If $m = 9/8$ kg and the system is gently pushed, kinetic friction will operate. Thus,

$$f = \mu_k \cdot \frac{Mg}{\sqrt{2}} \quad \text{and} \quad f' = \frac{\mu_k \, mg}{\sqrt{2}},$$

where $\mu_k = 0{\cdot}25$. If the acceleration is $a$, Newton's second law for $M$ gives (figure 6-W16).

$$Mg \sin 45° - T - f = Ma$$

or, $$\frac{Mg}{\sqrt{2}} - T - \frac{\mu_k \, Mg}{\sqrt{2}} = Ma. \quad \ldots \text{(iv)}$$

Applying Newton's second law $m$ (figure 6-W17),

$$T - mg \sin 45° - f' = ma$$

or, $$T - \frac{mg}{\sqrt{2}} - \frac{\mu_k \, mg}{\sqrt{2}} = ma. \quad \ldots \text{(v)}$$

Adding (iv) and (v)

$$\frac{Mg}{\sqrt{2}} (1 - \mu_k) - \frac{mg}{\sqrt{2}} (1 + \mu_k) = (M + m) \, a$$

or, $$a = \frac{M(1 - \mu_k) - m \, (1 + \mu_k)}{\sqrt{2} \, (M + m)} g$$

$$= \frac{2 \times 0{\cdot}75 - 9/8 \times 1{\cdot}25}{\sqrt{2} \, (2 + 9/8)} g$$

$$= 0{\cdot}31 \text{ m/s}^2.$$

□

## QUESTIONS FOR SHORT ANSWER

**1.** For most of the surfaces used in daily life, the friction coefficient is less than 1. Is it always necessary that the friction coefficient is less than 1 ?

**2.** Why is it easier to push a heavy block from behind than to press it on the top and push ?

**3.** What is the average friction force when a person has a usual 1 km walk ?

**4**. Why is it difficult to walk on solid ice ?

**5**. Can you accelerate a car on a frictionless horizontal road by putting more petrol in the engine ? Can you stop a car going on a frictionless horizontal road by applying brakes ?

**6**. Spring fitted doors close by themselves when released. You want to keep the door open for a long time, say for an hour. If you put a half kg stone in front of the open door, it does not help. The stone slides with the door and the door gets closed. However, if you sandwitch a

20 g piece of wood in the small gap between the door and the floor, the door stays open. Explain why a much lighter piece of wood is able to keep the door open while the heavy stone fails.

**7**. A classroom demonstration of Newton's first law is as follows : A glass is covered with a plastic card and a coin is placed on the card. The card is given a quick strike and the coin falls in the glass. (a) Should the friction coefficient between the card and the coin be small or large ? (b) Should the coin be light or heavy ? (c) Why does the experiment fail if the card is gently pushed ?

**8**. Can a tug of war be ever won on a frictionless surface ?

**9**. Why do tyres have a better grip of the road while going on a level road than while going on an incline ?

**10.** You are standing with your bag in your hands, on the ice in the middle of a pond. The ice is so slippery that it can offer no friction. How can you come out of the ice ?

**11.** When two surfaces are polished, the friction coefficient between them decreases. But the friction coefficient increases and becomes very large if the surfaces are made highly smooth. Explain.

## OBJECTIVE I

**1.** In a situation the contact force by a rough horizontal surface on a body placed on it has constant magnitude. If the angle between this force and the vertical is decreased, the frictional force between the surface and the body will
(a) increase (b) decrease
(c) remain the same (d) may increase or decrease.

**2.** While walking on ice, one should take small steps to avoid slipping. This is because smaller steps ensure
(a) larger friction (b) smaller friction
(c) larger normal force (d) smaller normal force.

**3.** A body of mass $M$ is kept on a rough horizontal surface (friction coefficient = $\mu$). A person is trying to pull the body by applying a horizontal force but the body is not moving. The force by the surface on $A$ is $F$, where
(a) $F = Mg$ (b) $F = \mu Mg$
(c) $Mg \le F \le Mg\sqrt{1+\mu^2}$ (d) $Mg \ge F \ge Mg\sqrt{1-\mu^2}$.

**4.** A scooter starting from rest moves with a constant acceleration for a time $\Delta t_1$, then with a constant velocity for the next $\Delta t_2$ and finally with a constant deceleration for the next $\Delta t_3$ to come to rest. A 500 N man sitting on the scooter behind the driver manages to stay at rest with respect to the scooter without touching any other part. The force exerted by the seat on the man is
(a) 500 N throughout the journey
(b) less than 500 N throughout the journey
(c) more than 500 N throughout the journey
(d) > 500 N for time $\Delta t_1$ and $\Delta t_3$ and 500 N for $\Delta t_2$.

**5.** Consider the situation shown in figure (6-Q1). The wall is smooth but the surfaces of $A$ and $B$ in contact are rough. The friction on $B$ due to $A$ in equilibrium

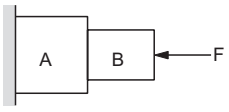



Figure 6-Q1

(a) is upward (b) is downward (c) is zero
(d) the system cannot remain in equilibrium.

**6.** Suppose all the surfaces in the previous problem are rough. The direction of friction on $B$ due to $A$
(a) is upward (b) is downward (c) is zero
(d) depends on the masses of $A$ and $B$.

**7.** Two cars of unequal masses use similar tyres. If they are moving at the same initial speed, the minimum stopping distance
(a) is smaller for the heavier car
(b) is smaller for the lighter car
(c) is same for both cars
(d) depends on the volume of the car.

**8.** In order to stop a car in shortest distance on a horizontal road, one should
(a) apply the brakes very hard so that the wheels stop rotating
(b) apply the brakes hard enough to just prevent slipping
(c) pump the brakes (press and release)
(d) shut the engine off and not apply brakes.

**9.** A block $A$ kept on an inclined surface just begins to slide if the inclination is 30°. The block is replaced by another block $B$ and it is found that it just begins to slide if the inclination is 40°.
(a) mass of $A$ > mass of $B$ (b) mass of $A$ < mass of $B$
(c) mass of $A$ = mass of $B$ (d) all the three are possible.

**10.** A boy of mass $M$ is applying a horizontal force to slide a box of mass $M'$ on a rough horizontal surface. The coefficient of friction between the shoes of the boy and the floor is $\mu$ and that between the box and the floor is $\mu'$. In which of the following cases it is certainly not possible to slide the box ?
(a) $\mu < \mu'$, $M < M'$ (b) $\mu > \mu'$, $M < M'$
(c) $\mu < \mu'$, $M > M'$ (d) $\mu > \mu'$, $M > M'$.

## OBJECTIVE II

**1.** Let $F$, $F_N$ and $f$ denote the magnitudes of the contact force, normal force and the friction exerted by one surface on the other kept in contact. If none of these is zero,
(a) $F > F_N$ (b) $F > f$ (c) $F_N > f$ (d) $F_N - f < F < F_N + f$.

**2.** The contact force exerted by a body $A$ on another body $B$ is equal to the normal force between the bodies. We conclude that
(a) the surfaces must be frictionless
(b) the force of friction between the bodies is zero
(c) the magnitude of normal force equals that of friction
(d) the bodies may be rough but they don't slip on each other.

**3.** Mark the correct statements about the friction between two bodies.
(a) Static friction is always greater than the kinetic friction.
(b) Coefficient of static friction is always greater than the coefficient of kinetic friction.
(c) Limiting friction is always greater than the kinetic friction.
(d) Limiting friction is never less than static friction.

**4.** A block is placed on a rough floor and a horizontal force $F$ is applied on it. The force of friction $f$ by the floor on the block is measured for different values of $F$ and a graph is plotted between them.
(a) The graph is a straight line of slope 45°.
(b) The graph is a straight line parallel to the $F$-axis.
(c) The graph is a straight line of slope 45° for small $F$ and a straight line parallel to the $F$-axis for large $F$.
(d) There is a small kink on the graph.

**5.** Consider a vehicle going on a horizontal road towards east. Neglect any force by the air. The frictional forces on the vehicle by the road
(a) is towards east if the vehicle is accelerating
(b) is zero if the vehicle is moving with a uniform velocity
(c) must be towards east
(d) must be towards west.

## EXERCISES

**1.** A body slipping on a rough horizontal plane moves with a deceleration of 4·0 m/s$^2$. What is the coefficient of kinetic friction between the block and the plane ?

**2.** A block is projected along a rough horizontal road with a speed of 10 m/s. If the coefficient of kinetic friction is 0·10, how far will it travel before coming to rest ?

**3.** A block of mass $m$ is kept on a horizontal table. If the static friction coefficient is $\mu$, find the frictional force acting on the block.

**4.** A block slides down an inclined surface of inclination 30° with the horizontal. Starting from rest it covers 8 m in the first two seconds. Find the coefficient of kinetic friction between the two.

**5.** Suppose the block of the previous problem is pushed down the incline with a force of 4 N. How far will the block move in the first two seconds after starting from rest ? The mass of the block is 4 kg.

6. A body of mass 2 kg is lying on a rough inclined plane of inclination 30°. Find the magnitude of the force parallel to the incline needed to make the block move (a) up the incline (b) down the incline. Coefficient of static friction = 0·2.

**7.** Repeat part (a) of problem 6 if the push is applied horizontally and not parallel to the incline.

**8.** In a children-park an inclined plane is constructed with an angle of incline 45° in the middle part (figure 6-E1). Find the acceleration of a boy sliding on it if the friction coefficient between the cloth of the boy and the incline is 0·6 and $g = 10$ m/s$^2$.

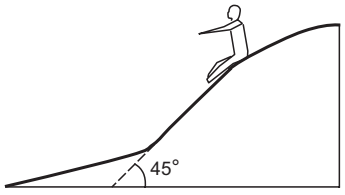



Figure 6-E1

**9.** A body starts slipping down an incline and moves half meter in half second. How long will it take to move the next half meter ?

**10.** The angle between the resultant contact force and the normal force exerted by a body on the other is called the *angle of friction*. Show that, if $\lambda$ be the angle of friction and $\mu$ the coefficient of static friction, $\lambda \leq \tan^{-1}\mu$.

**11.** Consider the situation shown in figure (6-E2). Calculate (a) the acceleration of the 1·0 kg blocks, (b) the tension in the string connecting the 1·0 kg blocks and (c) the tension in the string attached to 0·50 kg.

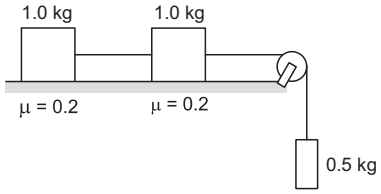



Figure 6-E2

**12.** If the tension in the string in figure (6-E3) is 16 N and the acceleration of each block is 0·5 m/s$^2$, find the friction coefficients at the two contacts with the blocks.

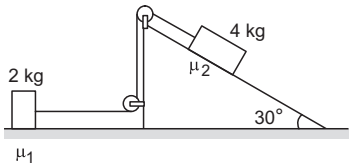



Figure 6-E3

**13.** The friction coefficient between the table and the block shown in figure (6-E4) is 0·2. Find the tensions in the two strings.

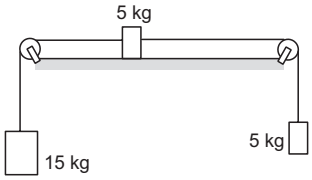



Figure 6-E4

**14.** The friction coefficient between a road and the tyre of a vehicle is 4/3. Find the maximum incline the road may have so that once hard brakes are applied and the wheel starts skidding, the vehicle going down at a speed of 36 km/hr is stopped within 5 m.

**15.** The friction coefficient between an athelete's shoes and the ground is 0·90. Suppose a superman wears these shoes and races for 50 m. There is no upper limit on his capacity of running at high speeds. (a) Find the minimum time that he will have to take in completing the 50 m starting from rest. (b) Suppose he takes exactly this minimum time to complete the 50 m, what minimum time will he take to stop ?

**16.** A car is going at a speed of 21·6 km/hr when it encounters a 12·8 m long slope of angle 30° (figure 6-E5). The friction coefficient between the road and the tyre is $1/2\sqrt{3}$. Show that no matter how hard the driver applies the brakes, the car will reach the bottom with a speed greater than 36 km/hr. Take $g = 10$ m/s$^2$.

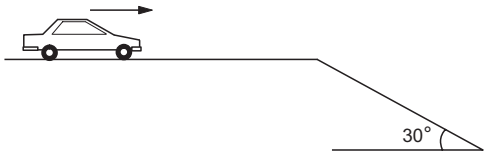



Figure 6-E5

**17.** A car starts from rest on a half kilometer long bridge. The coefficient of friction between the tyre and the road is 1·0. Show that one cannot drive through the bridge in less than 10 s.

**18.** Figure (6-E6) shows two blocks in contact sliding down an inclined surface of inclination 30°. The friction coefficient between the block of mass 2·0 kg and the incline is $\mu_1$, and that between the block of mass 4·0 kg and the incline is $\mu_2$. Calculate the acceleration of the 2·0 kg block if (a) $\mu_1 = 0{\cdot}20$ and $\mu_2 = 0{\cdot}30$, (b) $\mu_1 = 0{\cdot}30$ and $\mu_2 = 0{\cdot}20$. Take $g = 10$ m/s$^2$.

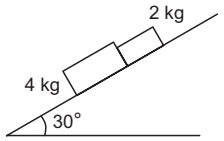



Figure 6-E6

**19.** Two masses $M_1$ and $M_2$ are connected by a light rod and the system is slipping down a rough incline of angle $\theta$ with the horizontal. The friction coefficient at both the contacts is $\mu$. Find the acceleration of the system and the force by the rod on one of the blocks.

**20.** A block of mass $M$ is kept on a rough horizontal surface. The coefficient of static friction between the block and the surface is $\mu$. The block is to be pulled by applying a force to it. What minimum force is needed to slide the block ? In which direction should this force act ?

**21.** The friction coefficient between the board and the floor shown in figure (6-E7) is $\mu$. Find the maximum force that the man can exert on the rope so that the board does not slip on the floor.

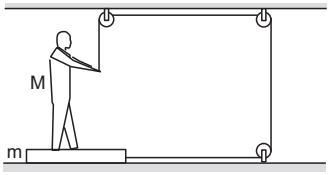



Figure 6-E7

**22.** A 2 kg block is placed over a 4 kg block and both are placed on a smooth horizontal surface. The coefficient of friction between the blocks is 0·20. Find the acceleration of the two blocks if a horizontal force of 12 N is applied to (a) the upper block, (b) the lower block. Take $g = 10$ m/s$^2$.

**23.** Find the accelerations $a_1$, $a_2$, $a_3$ of the three blocks shown in figure (6-E8) if a horizontal force of 10 N is applied on (a) 2 kg block, (b) 3 kg block, (c) 7 kg block. Take $g = 10$ m/s$^2$.

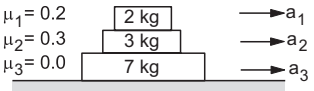



Figure 6-E8

**24.** The friction coefficient between the two blocks shown in figure (6-E9) is $\mu$ but the floor is smooth. (a) What maximum horizontal force $F$ can be applied without disturbing the equilibrium of the system ? (b) Suppose the horizontal force applied is double of that found in part (a). Find the accelerations of the two masses.

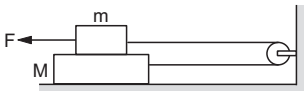



Figure 6-E9

**25.** Suppose the entire system of the previous question is kept inside an elevator which is coming down with an acceleration $a < g$. Repeat parts (a) and (b).

**26.** Consider the situation shown in figure (6-E9). Suppose a small electric field $E$ exists in the space in the vertically upward direction and the upper block carries a positive charge $Q$ on its top surface. The friction coefficient between the two blocks is $\mu$ but the floor is smooth. What maximum horizontal force $F$ can be applied without disturbing the equilibrium ?
[Hint : The force on a charge $Q$ by the electric field $E$ is $F = QE$ in the direction of $E$.]

**27.** A block of mass $m$ slips on a rough horizontal table under the action of a horizontal force applied to it. The coefficient of friction between the block and the table is $\mu$. The table does not move on the floor. Find the total frictional force applied by the floor on the legs of the table. Do you need the friction coefficient between the table and the floor or the mass of the table ?

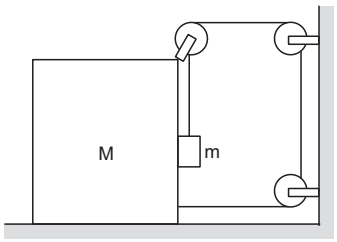



Figure 6-E10

**28.** Find the acceleration of the block of mass $M$ in the situation of figure (6-E10). The coefficient of friction between the two blocks is $\mu_1$ and that between the bigger block and the ground is $\mu_2$.

**29.** A block of mass 2 kg is pushed against a rough vertical wall with a force of 40 N, coefficient of static friction being 0·5. Another horizontal force of 15 N, is applied on the block in a direction parallel to the wall. Will the block move ? If yes, in which direction ? If no, find the frictional force exerted by the wall on the block.

**30.** A person (40 kg) is managing to be at rest between two vertical walls by pressing one wall $A$ by his hands and feet and the other wall $B$ by his back (figure 6-E11). Assume that the friction coefficient between his body and the walls is 0·8 and that limiting friction acts at all the contacts. (a) Show that the person pushes the two walls with equal force. (b) Find the normal force exerted by either wall on the person. Take $g = 10$ m/s$^2$.

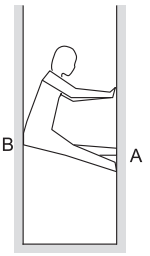



Figure 6-E11

**31.** Figure (6-E12) shows a small block of mass $m$ kept at the left end of a larger block of mass $M$ and length $l$. The system can slide on a horizontal road. The system is started towards right with an initial velocity $v$. The friction coefficient between the road and the bigger block is $\mu$ and that between the block is $\mu/2$. Find the time elapsed before the smaller blocks separates from the bigger block.

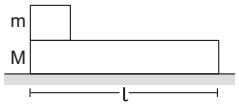



Figure 6-E12

□

## ANSWERS

### OBJECTIVE I

1. (b) 2. (b) 3. (c) 4. (d) 5. (d) 6. (a)
7. (c) 8. (b) 9. (d) 10. (a)

### OBJECTIVE II

1. (a), (b), (d) 2. (b), (d) 3. (b), (c), (d)
4. (c), (d) 5. (a), (b)

### EXERCISES

1. 0·4
2. 50 m
3. zero
4. 0·11
5. 10 m
6. (a) 13 N (b) zero

7. 17·5 N
8. $2\sqrt{2}$ m/s$^2$
9. 0·21 s
11. (a) 0·4 m/s$^2$ (b) 2·4 N (c) 4·8 N
12. $\mu_1 = 0{\cdot}75$, $\mu_2 = 0{\cdot}06$
13. 96 N in the left string and 68 N in the right
14. 16°
15. (a) $\frac{10}{3}$ s (b) $\frac{10}{3}$ s
18. 2·7 m/s$^2$, 2·4 m/s$^2$
19. $a = g(\sin\theta - \mu\cos\theta)$, zero
20. $\frac{\mu mg}{\sqrt{1+\mu^2}}$ at an angle $\tan^{-1}\mu$ with the horizontal
21. $\frac{\mu(M+m)g}{1+\mu}$
22. (a) upper block 4 m/s$^2$, lower block 1 m/s$^2$
    (b) both blocks 2 m/s$^2$
23. (a) $a_1 = 3$ m/s$^2$, $a_2 = a_3 = 0{\cdot}4$ m/s$^2$
    (b) $a_1 = a_2 = a_3 = \frac{5}{6}$ m/s$^2$ (c) same as (b)
24. (a) $2\mu mg$ (b) $\frac{2\mu mg}{M+m}$ in opposite directions
25. (a) $2\mu m(g-a)$ (b) $\frac{2\mu m(g-a)}{m+M}$
26. $2\mu(mg - QE)$
27. $\mu mg$
28. $\frac{[2m - \mu_2(M+m)]g}{M+m[5+2(\mu_1-\mu_2)]}$
29. it will move at an angle of 53° with the 15 N force
30. (b) 250 N
31. $\sqrt{\frac{4Ml}{(M+m)\mu g}}$

□

## CHAPTER 7

# CIRCULAR MOTION

### 7.1 ANGULAR VARIABLES

Suppose a particle $P$ is moving in a circle of radius $r$ (figure 7.1). Let $O$ be the centre of the circle. Let $O$ be the origin and $OX$ the $X$-axis. The position of the particle $P$ at a given instant may be described by the angle $\theta$ between $OP$ and $OX$. We call $\theta$ the *angular position* of the particle. As the particle moves on the circle, its angular position $\theta$ changes. Suppose the particle goes to a nearby point $P'$ in time $\Delta t$ so that $\theta$ increases to $\theta + \Delta\theta$. The rate of change of angular position is called *angular velocity*. Thus, the angular velocity is

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}.$$

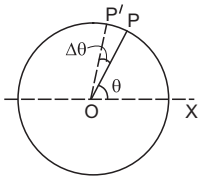



Figure 7.1

The rate of change of angular velocity is called *angular acceleration*. Thus, the angular acceleration is

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}.$$

If the angular acceleration $\alpha$ is constant, we have

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 \qquad \ldots (7.1)$$

$$\omega = \omega_0 + \alpha t \qquad \ldots (7.2)$$

and $$\omega^2 = \omega_0^2 + 2\alpha\theta \qquad \ldots (7.3)$$

where $\omega_0$ and $\omega$ are the angular velocities at $t = 0$ and at time $t$ and $\theta$ is the angular position at time $t$. The linear distance $PP'$ travelled by the particle in time $\Delta t$ is

$$\Delta s = r\,\Delta\theta$$

or, $$\frac{\Delta s}{\Delta t} = r\frac{\Delta\theta}{\Delta t}$$

or, $$v = r\,\omega \qquad \ldots (7.4)$$

where $v$ is the linear speed of the particle. Differentiating equation (7.4) with respect to time, the rate of change of speed is

$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt}$$

or, $$a_t = r\,\alpha. \qquad \ldots (7.5)$$

Remember that $a_t = \frac{dv}{dt}$ is the rate of change of speed and is not the rate of the change of velocity. It is, therefore, not equal to the net acceleration.

We shall show that $a_t$ is the component of acceleration along the tangent and hence we have used the suffix $t$. It is called the *tangential acceleration.*

***Example 7.1***

*A particle moves in a circle of radius* 20 cm *with a linear speed of* 10 m/s. *Find the angular velocity.*

***Solution*** : The angular velocity is

$$\omega = \frac{v}{r} = \frac{10\text{ m/s}}{20\text{ cm}} = 50\text{ rad/s}.$$

***Example 7.2***

*A particle travels in a circle of radius* 20 cm *at a speed that uniformly increases. If the speed changes from* 5·0 m/s *to* 6·0 m/s *in* 2·0 s, *find the angular acceleration.*

***Solution*** : The tangential acceleration is given by

$$a_t = \frac{dv}{dt} = \frac{v_2 - v_1}{t_2 - t_1}$$

$$= \frac{6{\cdot}0 - 5{\cdot}0}{2{\cdot}0}\text{ m/s}^2 = 0{\cdot}5\text{ m/s}^2.$$

The angular acceleration is $\alpha = a_t / r$

$$= \frac{0{\cdot}5\text{ m/s}^2}{20\text{ cm}} = 2{\cdot}5\text{ rad/s}^2.$$

## 7.2 UNIT VECTORS ALONG THE RADIUS AND THE TANGENT

Consider a particle moving in a circle. Suppose the particle is at a point $P$ in the circle at a given instant (figure 7.2). Take the centre of the circle to be the origin, a line $OX$ as the $X$-axis and a perpendicular radius $OY$ as the $Y$-axis. The angular position of the particle at this instant is θ.

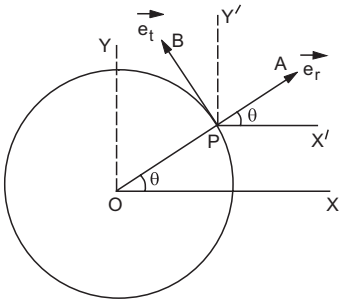



Figure 7.2

Draw a unit vector $\vec{PA} = \vec{e_r}$ along the outward radius and a unit vector $\vec{PB} = \vec{e_t}$ along the tangent in the direction of increasing θ. We call $\vec{e_r}$ the *radial unit vector* and $\vec{e_t}$ the *tangential unit vector*. Draw $PX'$ parallel to the $X$-axis and $PY'$ parallel to the $Y$-axis. From the figure,

$$\vec{PA} = \vec{i}\, PA\cos\theta + \vec{j}\, PA\sin\theta$$

or, $$\vec{e_r} = \vec{i}\cos\theta + \vec{j}\sin\theta, \qquad \ldots (7.6)$$

where $\vec{i}$ and $\vec{j}$ are the unit vectors along the $X$ and $Y$ axes respectively. Similarly,

$$\vec{PB} = -\vec{i}\, PB\sin\theta + \vec{j}\, PB\cos\theta$$

or, $$\vec{e_t} = -\vec{i}\sin\theta + \vec{j}\cos\theta. \qquad \ldots (7.7)$$

## 7.3 ACCELERATION IN CIRCULAR MOTION

Consider the situation shown in figure (7.2). It is clear from the figure that the position vector of the particle at time $t$ is

$$\vec{r} = \vec{OP} = OP\,\vec{e_r}$$
$$= r(\vec{i}\cos\theta + \vec{j}\sin\theta). \qquad \ldots (i)$$

Differentiating equation (i) with respect to time, the velocity of the particle at time $t$ is

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}[r(\vec{i}\cos\theta + \vec{j}\sin\theta)]$$
$$= r\left[\vec{i}\left(-\sin\theta\frac{d\theta}{dt}\right) + \vec{j}\left(\cos\theta\frac{d\theta}{dt}\right)\right]$$
$$= r\,\omega[-\vec{i}\sin\theta + \vec{j}\cos\theta]. \qquad \ldots (ii)$$

The term $r\,\omega$ is the speed of the particle at time $t$ (equation 7.4) and the vector in the square bracket is the unit vector $\vec{e_t}$ along the tangent. Thus, the velocity of the particle at any instant is along the tangent to the circle and its magnitude is $v = r\,\omega$.

The acceleration of the particle at time $t$ is $\vec{a} = \frac{d\vec{v}}{dt}$. From (ii),

$$\vec{a} = r\left[\omega\frac{d}{dt}[-\vec{i}\sin\theta + \vec{j}\cos\theta] + \frac{d\omega}{dt}[-\vec{i}\sin\theta + \vec{j}\cos\theta]\right]$$
$$= \omega r\left[-\vec{i}\cos\theta\frac{d\theta}{dt} - \vec{j}\sin\theta\frac{d\theta}{dt}\right] + r\frac{d\omega}{dt}\vec{e_t}$$
$$= -\omega^2 r[\vec{i}\cos\theta + \vec{j}\sin\theta] + r\frac{d\omega}{dt}\vec{e_t}$$
$$= -\omega^2 r\,\vec{e_r} + \frac{dv}{dt}\vec{e_t}, \qquad \ldots (7.8)$$

where $\vec{e_r}$ and $\vec{e_t}$ are the unit vectors along the radial and tangential directions respectively and $v$ is the speed of the particle at time $t$. We have used

$$r\frac{d\omega}{dt} = \frac{d}{dt}(r\,\omega) = \frac{dv}{dt}.$$

### Uniform Circular Motion

If the particle moves in the circle with a uniform speed, we call it a *uniform circular motion*. In this case, $\frac{dv}{dt} = 0$ and equation (7.8) gives

$$\vec{a} = -\omega^2 r\,\vec{e_r}.$$

Thus, the acceleration of the particle is in the direction of $-\vec{e_r}$, that is, towards the centre. The magnitude of the acceleration is

$$a_r = \omega^2 r$$
$$= \frac{v^2}{r^2}r = \frac{v^2}{r}. \qquad \ldots (7.9)$$

Thus, if a particle moves in a circle of radius $r$ with a constant speed $v$, its acceleration is $v^2/r$ directed towards the centre. This acceleration is called *centripetal acceleration*. Note that the speed remains constant, the direction continuously changes and hence the "velocity" changes and there is an acceleration during the motion.

***Example 7.3***

*Find the magnitude of the linear acceleration of a particle moving in a circle of radius* 10 cm *with uniform speed completing the circle in* 4 s.

***Solution*** : The distance covered in completing the circle is $2\pi r = 2\pi \times 10$ cm. The linear speed is

$$v = 2\pi r/t$$

$$= \frac{2\pi \times 10 \text{ cm}}{4 \text{ s}} = 5\pi \text{ cm/s}.$$

The linear acceleration is

$$a = \frac{v^2}{r} = \frac{(5\pi \text{ cm/s})^2}{10 \text{ cm}} = 2{\cdot}5\ \pi^2 \text{ cm/s}^2.$$

This acceleration is directed towards the centre of the circle.

**Nonuniform Circular Motion**

If the speed of the particle moving in a circle is not constant, the acceleration has both the radial and the tangential components. According to equation (7.8), the radial and the tangential accelerations are

$$\left.\begin{aligned} a_r &= -\omega^2 r = -v^2/r \\ \text{and} \quad a_t &= \frac{dv}{dt} \end{aligned}\right| \quad \ldots \quad (7.10)$$

Thus, the component of the acceleration towards the centre is $\omega^2 r = v^2/r$ and the component along the tangent (along the direction of motion) is $dv/dt$. The magnitude of the acceleration is

$$a = \sqrt{a_r^2 + a_t^2} = \sqrt{\left(\frac{v^2}{r}\right)^2 + \left(\frac{dv}{dt}\right)^2}\ .$$

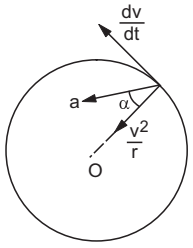



Figure 7.3

The direction of this resultant acceleration makes an angle $\alpha$ with the radius (figure 7.3) where

$$\tan\alpha = \left(\frac{dv}{dt}\right) \Big/ \left(\frac{v^2}{r}\right).$$

***Example 7.4***

*A particle moves in a circle of radius* 20 cm. *Its linear speed is given by* $v = 2t$, *where* $t$ *is in second and* $v$ *in metre/second. Find the radial and tangential acceleration at* $t$ = 3 s.

***Solution*** : The linear speed at $t$ = 3 s is

$$v = 2t = 6 \text{ m/s}.$$

The radial acceleration at $t$ = 3 s is

$$a_r = v^2/r = \frac{36 \text{ m}^2/\text{s}^2}{0{\cdot}20 \text{ m}} = 180 \text{ m/s}^2.$$

The tangential acceleration is

$$a_t = \frac{dv}{dt} = \frac{d(2t)}{dt} = 2 \text{ m/s}^2.$$

## 7.4 DYNAMICS OF CIRCULAR MOTION

If a particle moves in a circle as seen from an inertial frame, a resultant nonzero force must act on the particle. That is because a particle moving in a circle is accelerated and acceleration can be produced in an inertial frame only if a resultant force acts on it. If the speed of the particle remains constant, the acceleration of the particle is towards the centre and its magnitude is $v^2/r$. Here $v$ is the speed of the particle and $r$ is the radius of the circle. The resultant force must act towards the centre and its magnitude $F$ must satisfy

$$a = \frac{F}{m}$$

or,
$$\frac{v^2}{r} = \frac{F}{m}$$

or,
$$F = \frac{mv^2}{r}\ . \quad \ldots \quad (7.11)$$

Since this resultant force is directed towards the centre, it is called *centripetal force*. Thus, a centripetal force of magnitude $mv^2/r$ is needed to keep the particle in uniform circular motion.

It should be clearly understood that "centripetal force" is another word for "force towards the centre". This force must originate from some external source such as gravitation, tension, friction, coulomb force, etc. Centripetal force is not a new kind of force, just as an "upward force" or a "downward force" is not a new kind of force.

***Example 7.5***

*A small block of mass* 100 g *moves with uniform speed in a horizontal circular groove, with vertical side walls, of radius* 25 cm. *If the block takes* 2·0 s *to complete one round, find the normal contact force by the side wall of the groove.*

***Solution*** : The speed of the block is

$$v = \frac{2\pi \times (25 \text{ cm})}{2{\cdot}0 \text{ s}} = 0{\cdot}785 \text{ m/s}\ .$$

The acceleration of the block is

$$a = \frac{v^2}{r} = \frac{(0{\cdot}785 \text{ m/s})^2}{0{\cdot}25 \text{ m}} = 2{\cdot}5 \text{ m/s}^2$$

towards the centre. The only force in this direction is the normal contact force due to the side walls. Thus, from Newton's second law, this force is

$$\mathcal{N} = ma = (0{\cdot}100 \text{ kg})\ (2{\cdot}5 \text{ m/s}^2) = 0{\cdot}25 \text{ N}.$$

## 7.5 CIRCULAR TURNINGS AND BANKING OF ROADS

When vehicles go through turnings, they travel along a nearly circular arc. There must be some force which will produce the required acceleration. If the vehicle goes in a horizontal circular path, this resultant force is also horizontal. Consider the situation as shown in figure (7.4). A vehicle of mass $M$ moving at a speed $v$ is making a turn on the circular path of radius $r$. The external forces acting on the vehicle are

(i) weight $Mg$

(ii) Normal contact force $\mathcal{N}$ and

(iii) friction $f_s$.

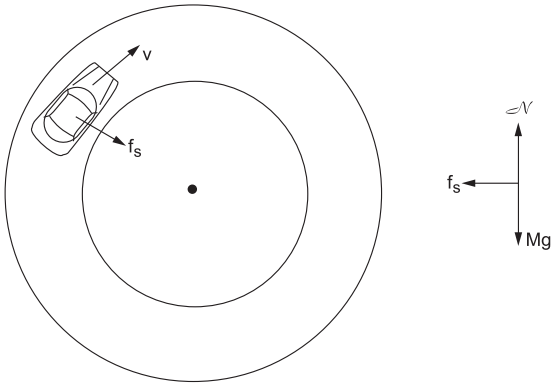



Figure 7.4

If the road is horizontal, the normal force $\mathcal{N}$ is vertically upward. The only horizontal force that can act towards the centre is the friction $f_s$. This is static friction and is self adjustable. The tyres get a tendency to skid outward and the frictional force which opposes this skidding acts towards the centre. Thus, for a safe turn we must have

$$\frac{v^2}{r} = \frac{f_s}{M}$$

or, $$f_s = \frac{Mv^2}{r}.$$

However, there is a limit to the magnitude of the frictional force. If $\mu_s$ is the coefficient of static friction between the tyres and the road, the magnitude of friction $f_s$ cannot exceed $\mu_s \mathcal{N}$. For vertical equilibrium $\mathcal{N} = Mg$, so that

$$f_s \leq \mu_s Mg.$$

Thus, for a safe turn

$$\frac{Mv^2}{r} \leq \mu_s Mg$$

or, $$\mu_s \geq \frac{v^2}{rg}. \quad \ldots (7.12)$$

Friction is not always reliable at circular turns if high speeds and sharp turns are involved. To avoid dependence on friction, the roads are banked at the turn so that the outer part of the road is somewhat lifted up as compared to the inner part (figure 7.5).

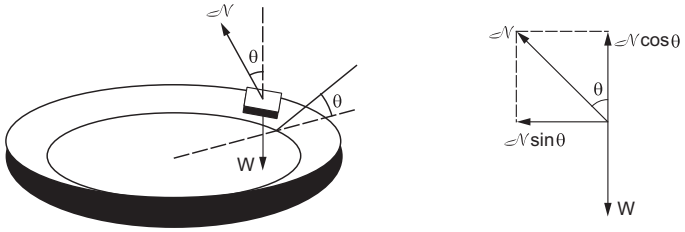



Figure 7.5

The surface of the road makes an angle θ with the horizontal throughout the turn. The normal force $\mathcal{N}$ makes an angle θ with the vertical. At the correct speed, the horizontal component of $\mathcal{N}$ is sufficient to produce the acceleration towards the centre and the self adjustable frictional force keeps its value zero. Applying Newton's second law along the radius and the first law in the vertical direction,

$$\mathcal{N} \sin\theta = \frac{Mv^2}{r}$$

and $$\mathcal{N} \cos\theta = Mg.$$

These equations give

$$\tan\theta = \frac{v^2}{rg}. \quad \ldots (7.13)$$

The angle θ depends on the speed of the vehicle as well as on the radius of the turn. Roads are banked for the average expected speed of the vehicles. If the speed of a particular vehicle is a little less or a little more than the correct speed, the self adjustable static friction operates between the tyres and the road and the vehicle does not skid or slip. If the speed is too different from that given by equation (7.13), even the maximum friction cannot prevent a skid or a slip.

***Example 7.6***

*The road at a circular turn of radius* 10 m *is banked by an angle of* 10°. *With what speed should a vehicle move on the turn so that the normal contact force is able to provide the necessary centripetal force* ?

***Solution*** : If $v$ is the correct speed,

$$\tan\theta = \frac{v^2}{rg}$$

or, $v = \sqrt{rg \tan \theta}$

$= \sqrt{(10 \text{ m}) (9{\cdot}8 \text{ m/s}^2) (\tan 10°)} = 4{\cdot}2 \text{ m/s}.$

## 7.6 CENTRIFUGAL FORCE

We discussed in chapter 5 that Newton's laws of motion are not valid if one is working from a noninertial frame. If the frame translates with respect to an inertial frame with an acceleration $\vec{a}_0$, one must assume the existence of a pseudo force $-m\vec{a}_0$, acting on a particle of mass $m$. Once this pseudo force is included, one can use Newton's laws in their usual form. What pseudo force is needed if the frame of reference rotates at a constant angular velocity ω with respect to an inertial frame ?

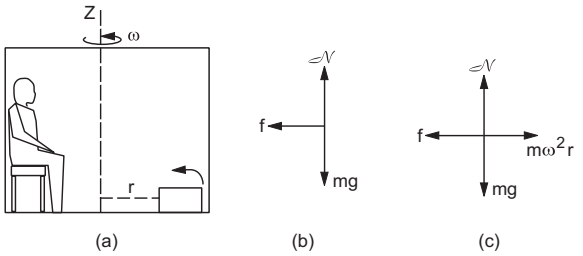



Figure 7.6

Suppose the observer is sitting in a closed cabin which is made to rotate about the vertical $Z$-axis at a uniform angular velocity ω (figure 7.6). The $X$ and $Y$ axes are fixed in the cabin. Consider a heavy box of mass $m$ kept on the floor at a distance $r$ from the $Z$-axis. Suppose the floor and the box are rough and the box does not slip on the floor as the cabin rotates. The box is at rest with respect to the cabin and hence is rotating with respect to the ground at an angular velocity ω. Let us first analyse the motion of the box from the ground frame. In this frame (which is inertial) the box is moving in a circle of radius $r$. It, therefore, has an acceleration $v^2/r = \omega^2 r$ towards the centre. The resultant force on the box must be towards the centre and its magnitude must be $m\omega^2 r$. The forces on the box are

(a) weight $mg$

(b) normal force $\mathcal{N}$ by the floor

(c) friction $f$ by the floor.

Figure (7.6b) shows the free body diagram for the box. Since the resultant is towards the centre and its magnitude is $m\omega^2 r$, we should have

$$f = m\omega^2 r.$$

The floor exerts a force of static friction $f = m\omega^2 r$ towards the origin.

Now consider the same box when observed from the frame of the rotating cabin. The observer there finds that the box is at rest. If he or she applies Newton's laws, the resultant force on the box should be zero. The weight and the normal contact force balance each other but the frictional force $f = m\omega^2 r$ acts on the box towards the origin. To make the resultant zero, a pseudo force must be assumed which acts on the box away from the centre (radially outward) and has a magnitude $m\omega^2 r$. This pseudo force is called the *centrifugal force*. The analysis from the rotating frame is as follows :

The forces on the box are

(a) weight $mg$

(b) normal force $\mathcal{N}$

(c) friction $f$

(d) centrifugal force $m\omega^2 r$.

The free body diagram is shown in figure (7.6c). As the box is at rest, Newton's first law gives

$$f = m\omega^2 r.$$

Note that we get the same equation for friction as we got from the ground frame. But we had to apply Newton's second law from the ground frame and Newton's first law from the rotating frame. Let us now summarise our discussion.

Suppose we are working from a frame of reference that is rotating at a constant angular velocity ω with respect to an inertial frame. If we analyse the dynamics of a particle of mass $m$ kept at a distance $r$ from the axis of rotation, we have to assume that a force $m\omega^2 r$ acts radially outward on the particle. Only then we can apply Newton's laws of motion in the rotating frame. This radially outward pseudo force is called the centrifugal force.

In fact, centrifugal force is a sufficient pseudo force, only if we are analysing the particles at rest in a uniformly rotating frame. If we analyse the motion of a particle that moves in the rotating frame, we may have to assume other pseudo forces, together with the centrifugal force. Such forces are called the *coriolis forces*. The coriolis force is perpendicular to the velocity of the particle and also perpendicular to the axis of rotation of the frame. Once again, we emphasise that all these pseudo forces, centrifugal or coriolis, are needed only if the working frame is rotating. If we work from an inertial frame, there is no need to apply any pseudo force.

It is a common misconception among the beginners that centrifugal force acts on a particle because the

particle goes on a circle. Centrifugal force acts (or is assumed to act) because we describe the particle from a rotating frame which is noninertial and still use Newton's laws.

## 7.7 EFFECT OF EARTH'S ROTATION ON APPARENT WEIGHT

The earth rotates about its axis at an angular speed of one revolution per 24 hours. The line joining the north and the south poles is the axis of rotation. Every point on the earth moves in a circle. A point at equator moves in a circle of radius equal to the radius of the earth and the centre of the circle is same as the centre of the earth. For any other point on the earth, the circle of rotation is smaller than this. Consider a place $P$ on the earth (figure 7.7).

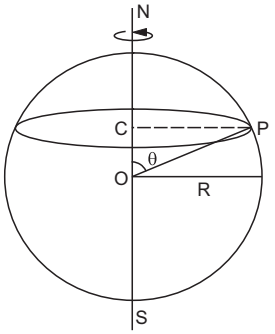



Figure 7.7

Drop a perpendicular $PC$ from $P$ to the axis $SN$. The place $P$ rotates in a circle with the centre at $C$. The radius of this circle is $CP$. The angle between the axis $SN$ and the radius $OP$ through $P$ is called the *colatitude* of the place $P$. We have

$$CP = OP \sin\theta$$

or, $$r = R \sin\theta$$

where $R$ is the radius of the earth.

If we work from the frame of reference of the earth, we shall have to assume the existence of the pseudo forces. In particular, a centrifugal force $m\omega^2 r$ has to be assumed on any particle of mass $m$ placed at $P$. Here $\omega$ is the angular speed of the earth. If we discuss the equilibrium of bodies at rest in the earth's frame, no other pseudo force is needed.

Consider a heavy particle of mass $m$ suspended through a string from the ceiling of a laboratory at colatitude $\theta$ (figure 7.8). Looking from the earth's frame the particle is in equilibrium and the forces on it are

(a) gravitational attraction $mg$ towards the centre of the earth, i.e., vertically downward,

(b) centrifugal force $m\omega^2 r$ towards $CP$ and

(c) the tension in the string $T$ along the string.

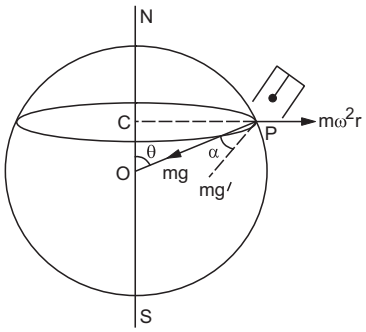



Figure 7.8

As the particle is in equilibrium (in the frame of earth), the three forces on the particle should add up to zero.

The resultant of $mg$ and $m\omega^2 r$

$$= \sqrt{(mg)^2 + (m\omega^2 r)^2 + 2(mg)(m\omega^2 r)\cos(90° + \theta)}$$

$$= m\sqrt{g^2 + \omega^4 R^2 \sin^2\theta - 2g\omega^2 R \sin^2\theta}$$

$$= mg'$$

where $g' = \sqrt{g^2 - \omega^2 R \sin^2\theta\,(2g - \omega^2 R)}$. … (7.14)

Also, the direction of this resultant makes an angle $\alpha$ with the vertical $OP$, where

$$\tan\alpha = \frac{m\omega^2 r \sin(90° + \theta)}{mg + m\omega^2 r \cos(90° + \theta)}$$

$$= \frac{\omega^2 R \sin\theta \cos\theta}{g - \omega^2 R \sin^2\theta}. \quad \ldots (7.15)$$

As the three forces acting on the particle must add up to zero, the force of tension must be equal and opposite to the resultant of the rest two. Thus, the magnitude of the tension in the string must be $mg'$ and the direction of the string should make an angle $\alpha$ with the true vertical.

The direction of $g'$ is the apparent vertical direction, because a plumb line stays in this direction only. The walls of the buildings are constructed by making them parallel to $g'$ and not to $g$. The water surface placed at rest is perpendicular to $g'$.

The magnitude of $g'$ is also different from $g$. As $2g > \omega^2 R$, it is clear from equation (7.14) that $g' < g$. One way of measuring the weight of a body is to suspend it by a string and find the tension in the string. The tension itself is taken as a measure of the weight. As $T = mg'$, the weight so observed is less than the true weight $mg$. This is known as the *apparent weight*. Similarly, if a person stands on the platform of a weighing machine, the platform exerts a normal

force $\mathcal{N}$ which is equal to $mg'$. The reading of the machine responds to the force exerted on it and hence the weight recorded is the apparent weight $mg'$.

At equator, $\theta = 90°$ and equation (7.14) gives

$$g' = \sqrt{g^2 - 2g\omega^2 R + \omega^4 R^2}$$

$$= g - \omega^2 R$$

or, $$mg' = mg - m\omega^2 R. \quad \ldots (7.16)$$

This can be obtained in a more straightforward way. At the equator, $m\omega^2 R$ is directly opposite to $mg$ and the resultant is simply $mg - m\omega^2 R$. Also, this resultant is towards the centre of the earth so that at the equator the plumb line stands along the true vertical.

At poles, $\theta = 0$ and equation (7.14) gives $g' = g$ and equation (7.15) shows that $\alpha = 0$. Thus, there is no apparent change in $g$ at the poles. This is because the poles themselves do not rotate and hence the effect of earth's rotation is not felt there.

***Example 7.7***

*A body weighs* 98 N *on a spring balance at the north pole. What will be its weight recorded on the same scale if it is shifted to the equator ? Use* $g = GM/R^2 = 9{\cdot}8$ m/s$^2$ *and the radius of the earth* $R$ = 6400 km.

***Solution :*** At poles, the apparent weight is same as the true weight.

Thus,

$$98 \text{ N} = mg = m(9{\cdot}8 \text{ m/s}^2)$$

or, $$m = 10 \text{ kg}.$$

At the equator, the apparent weight is

$$mg' = mg - m\omega^2 R.$$

The radius of the earth is 6400 km and the angular speed is

$$\omega = \frac{2\pi \text{ rad}}{24 \times 60 \times 60 \text{ s}} = 7{\cdot}27 \times 10^{-5} \text{ rad/s}.$$

Thus,

$$mg' = 98 \text{ N} - (10 \text{ kg})\,(7{\cdot}27 \times 10^{-5} \text{ s}^{-1})^2\,(6400 \text{ km})$$

$$= 97{\cdot}66 \text{ N}.$$

## *Worked Out Examples*

**1.** *A car has to move on a level turn of radius* 45 m. *If the coefficient of static friction between the tyre and the road is* $\mu_s$ = 2.0, *find the maximum speed the car can take without skidding.*

***Solution :*** Let the mass of the car be $M$. The forces on the car are

(a) weight $Mg$ downward

(b) normal force $\mathcal{N}$ by the road upward

(c) friction $f_s$ by the road towards the centre.

The car is going on a horizontal circle of radius $R$, so it is accelerating. The acceleration is towards the centre and its magnitude is $v^2/R$, where $v$ is the speed. For vertical direction, acceleration = 0. Resolving the forces in vertical and horizontal directions and applying Newton's laws, we have

$$\mathcal{N} = mg$$

and $$f_s = Mv^2/R.$$

As we are looking for the maximum speed for no skidding, it is a case of limiting friction and hence $f_s = \mu_s \mathcal{N} = \mu_s Mg$.

So, we have

$$\mu_s Mg = Mv^2/R$$

or, $$v^2 = \mu_s gR.$$

Putting the values, $v = \sqrt{2 \times 10 \text{ m/s}^2 \times 45 \text{ m}}$

$$= 30 \text{ m/s} = 108 \text{ km/hr}.$$

**2.** *A circular track of radius* 600 m *is to be designed for cars at an average speed of* 180 km/hr. *What should be the angle of banking of the track* ?

***Solution :*** Let the angle of banking be $\theta$. The forces on the car are (figure 7-W1)

(a) weight of the car $Mg$ downward and

(b) normal force $\mathcal{N}$.

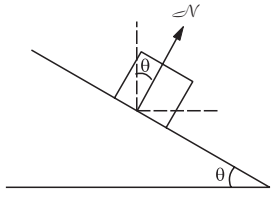



Figure 7-W1

For proper banking, static frictional force is not needed. For vertical direction the acceleration is zero. So,

$$\mathcal{N} \cos\theta = Mg. \quad \ldots (i)$$

For horizontal direction, the acceleration is $v^2/r$ towards the centre, so that

$$\mathcal{N} \sin\theta = Mv^2/r. \quad \ldots (ii)$$

From (i) and (ii),

$$\tan\theta = v^2/rg.$$

Putting the values, $\tan\theta = \dfrac{(180 \text{ km/hr})^2}{(600 \text{ m})(10 \text{ m/s}^2)} = 0{\cdot}4167$

or, $\theta = 22{\cdot}6°.$

**3.** *A particle of mass m is suspended from a ceiling through a string of length L. The particle moves in a horizontal circle of radius r. Find (a) the speed of the particle and (b) the tension in the string. Such a system is called a conical pendulum.*

***Solution*** **:** The situation is shown in figure (7-W2). The angle θ made by the string with the vertical is given by

$$\sin\theta = r/L. \quad \ldots \text{(i)}$$

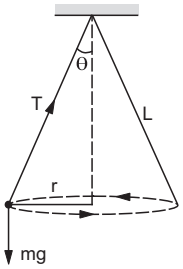



Figure 7-W2

The forces on the particle are

(a) the tension $T$ along the string and

(b) the weight $mg$ vertically downward.

The particle is moving in a circle with a constant speed $v$. Thus, the radial acceleration towards the centre has magnitude $v^2/r$. Resolving the forces along the radial direction and applying Newton's second law,

$$T\sin\theta = m(v^2/r). \quad \ldots \text{(ii)}$$

As there is no acceleration in vertical direction, we have from Newton's first law,

$$T\cos\theta = mg. \quad \ldots \text{(iii)}$$

Dividing (ii) by (iii),

$$\tan\theta = \frac{v^2}{rg}$$

or, $v = \sqrt{rg\tan\theta}.$

And from (iii),

$$T = \frac{mg}{\cos\theta}.$$

Using (i),

$$v = \frac{r\sqrt{g}}{(L^2 - r^2)^{1/4}} \quad \text{and} \quad T = \frac{mgL}{(L^2 - r^2)^{1/2}}.$$

**4.** *One end of a massless spring of spring constant* 100 N/m *and natural length* 0·5 m *is fixed and the other end is connected to a particle of mass* 0·5 kg *lying on a frictionless horizontal table. The spring remains horizontal. If the mass is made to rotate at an angular velocity of* 2 rad/s*, find the elongation of the spring.*

***Solution*** **:** The particle is moving in a horizontal circle, so it is accelerated towards the centre with magnitude $v^2/r$. The horizontal force on the particle is due to the spring and equals $kl$, where $l$ is the elongation and $k$ is the spring constant. Thus,

$$kl = mv^2/r = m\omega^2 r = m\omega^2(l_0 + l).$$

Here ω is the angular velocity, $l_0$ is the natural length (0·5 m) and $l_0 + l$ is the total length of the spring which is also the radius of the circle along which the particle moves.

Thus, $(k - m\omega^2)l = m\omega^2 l_0$

or, $l = \dfrac{m\omega^2 l_0}{k - m\omega^2}.$

Putting the values,

$$l = \frac{0{\cdot}5 \times 4 \times 0{\cdot}5}{100 - 0{\cdot}5 \times 4} \text{ m} \approx \frac{1}{100} \text{ m} = 1 \text{ cm}.$$

**5.** *A simple pendulum is constructed by attaching a bob of mass m to a string of length L fixed at its upper end. The bob oscillates in a vertical circle. It is found that the speed of the bob is v when the string makes an angle* θ *with the vertical. Find the tension in the string at this instant.*

***Solution*** **:** The forces acting on the bob are (figure 7-W3)

(a) the tension $T$

(b) the weight $mg$.

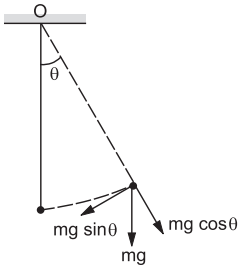



Figure 7-W3

As the bob moves in a vertical circle with centre at $O$, the radial acceleration is $v^2/L$ towards $O$. Taking the components along this radius and applying Newton's second law, we get,

$$T - mg\cos\theta = mv^2/L$$

or, $T = m(g\cos\theta + v^2/L).$

**6.** *A cylindrical bucket filled with water is whirled around in a vertical circle of radius r. What can be the minimum speed at the top of the path if water does not fall out from the bucket ? If it continues with this speed, what normal contact force the bucket exerts on water at the lowest point of the path ?*

***Solution*** : Consider water as the system. At the top of the circle its acceleration towards the centre is vertically downward with magnitude $v^2/r$. The forces on water are (figure 7-W4)

(a) weight $Mg$ downward and

(b) normal force by the bucket, also downward.

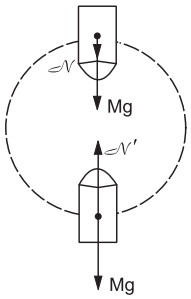



Figure 7-W4

So, from Newton's second law

$$Mg + \mathcal{N} = Mv^2/r.$$

For water not to fall out from the bucket, $\mathcal{N} \geq 0$.

Hence, $Mv^2/r \geq Mg$ or, $v^2 \geq rg$.

The minimum speed at the top must be $\sqrt{rg}$.

If the bucket continues on the circle with this minimum speed $\sqrt{rg}$, the forces at the bottom of the path are

(a) weight $Mg$ downward and

(b) normal contact force $\mathcal{N}'$ by the bucket upward.

The acceleration is towards the centre which is vertically upward, so

$$\mathcal{N}' - Mg = Mv^2/r$$

or, $$\mathcal{N}' = M(g + v^2/r) = 2\,Mg.$$

**7.** *A fighter plane is pulling out for a dive at a speed of* 900 km/hr. *Assuming its path to be a vertical circle of radius* 2000 m *and its mass to be* 16000 kg, *find the force exerted by the air on it at the lowest point. Take* $g = 9{\cdot}8$ m/s$^2$.

***Solution*** : At the lowest point in the path the acceleration is vertically upward (towards the centre) and its magnitude is $v^2/r$.

The forces on the plane are

(a) weight $Mg$ downward and

(b) force $F$ by the air upward.

Hence, Newton's second law of motion gives

$$F - Mg = Mv^2/r$$

or, $$F = M(g + v^2/r).$$

Here $v = 900$ km/hr $= \dfrac{9 \times 10^5}{3600}$ m/s $= 250$ m/s

or, $F = 16000\left(9{\cdot}8 + \dfrac{62500}{2000}\right)\text{N} = 6{\cdot}56 \times 10^5$ N (upward).

**8.** *Figure (7-W5) shows a rod of length* 20 cm *pivoted near an end and which is made to rotate in a horizontal plane with a constant angular speed. A ball of mass m is suspended by a string also of length* 20 cm *from the other end of the rod. If the angle* $\theta$ *made by the string with the vertical is* 30°, *find the angular speed of the rotation. Take* $g = 10$ m/s$^2$.

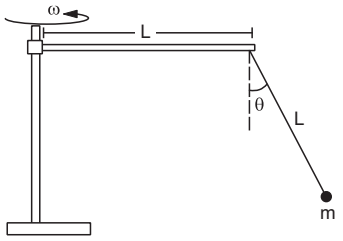



Figure 7-W5

***Solution*** : Let the angular speed be $\omega$. As is clear from the figure, the ball moves in a horizontal circle of radius $L + L\sin\theta$, where $L = 20$ cm. Its acceleration is, therefore, $\omega^2(L + L\sin\theta)$ towards the centre. The forces on the bob are (figure 7-W5)

(a) the tension $T$ along the string and

(b) the weight $mg$.

Resolving the forces along the radius and applying Newton's second law,

$$T\sin\theta = m\omega^2 L\,(1 + \sin\theta). \qquad \ldots \text{(i)}$$

Applying Newton's first law in the vertical direction,

$$T\cos\theta = mg. \qquad \ldots \text{(ii)}$$

Dividing (i) by (ii),

$$\tan\theta = \frac{\omega^2 L(1 + \sin\theta)}{g}$$

or, $$\omega^2 = \frac{g\tan\theta}{L(1 + \sin\theta)} = \frac{(10\text{ m/s}^2)\,(1/\sqrt{3})}{(0{\cdot}20\text{ m})\,(1 + 1/2)}$$

or, $$\omega = 4{\cdot}4 \text{ rad/s}.$$

**9.** *Two blocks each of mass M are connected to the ends of a light frame as shown in figure (7-W6). The frame is rotated about the vertical line of symmetry. The rod breaks if the tension in it exceeds* $T_0$. *Find the maximum frequency with which the frame may be rotated without breaking the rod.*

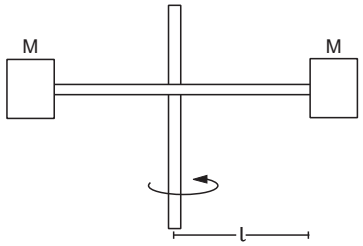



Figure 7-W6

***Solution*** : Consider one of the blocks. If the frequency of revolution is $f$, the angular velocity is $\omega = 2\pi f$. The acceleration towards the centre is $v^2/l = \omega^2 l = 4\pi^2 f^2 l$. The only horizontal force on the block is the tension of the rod. At the point of breaking, this force is $T_0$. So from Newton's second law,

$$T_0 = M \cdot 4\pi^2 f^2 l$$

or, $$f = \frac{1}{2\pi}\left[\frac{T_0}{Ml}\right]^{1/2}.$$

**10**. *In a rotor, a hollow vertical cylindrical structure rotates about its axis and a person rests against the inner wall. At a particular speed of the rotor, the floor below the person is removed and the person hangs resting against the wall without any floor. If the radius of the rotor is* 2 m *and the coefficient of static friction between the wall and the person is* 0·2, *find the minimum speed at which the floor may be removed. Take* $g = 10$ m/s$^2$.

***Solution*** : The situation is shown in figure (7-W7).

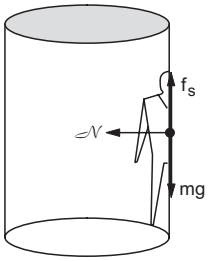



Figure 7-W7

When the floor is removed, the forces on the person are
(a) weight $mg$ downward
(b) normal force $\mathcal{N}$ due to the wall, towards the centre
(c) frictional force $f_s$, parallel to the wall, upward.

The person is moving in a circle with a uniform speed, so its acceleration is $v^2/r$ towards the centre.
Newton's law for the horizontal direction (2nd law) and for the vertical direction (1st law) give

$$\mathcal{N} = mv^2/r \quad \ldots \text{(i)}$$

and $$f_s = mg. \quad \ldots \text{(ii)}$$

For the minimum speed when the floor may be removed, the friction is limiting one and so equals $\mu_s \mathcal{N}$. This gives

$$\mu_s \mathcal{N} = mg$$

or, $$\frac{\mu_s m v^2}{r} = mg \quad \text{[using (i)]}$$

or, $$v = \sqrt{\frac{rg}{\mu_s}} = \sqrt{\frac{2\text{ m} \times 10\text{ m/s}^2}{0{\cdot}2}} = 10\text{ m/s}.$$

**11.** *A hemispherical bowl of radius R is set rotating about its axis of symmetry which is kept vertical. A small block kept in the bowl rotates with the bowl without slipping on its surface. If the surface of the bowl is smooth, and the angle made by the radius through the block with the vertical is* θ, *find the angular speed at which the bowl is rotating.*

***Solution*** : Suppose the angular speed of rotation of the bowl is ω. The block also moves with this angular speed. The forces on the block are (figure 7-W8)
(a) the normal force $\mathcal{N}$ and
(b) the weight $mg$.

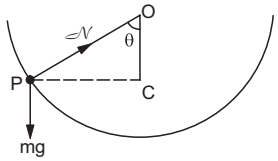



Figure 7-W8

The block moves in a horizontal circle with the centre at $C$, so that the radius is $PC = OP \sin\theta = R \sin\theta$. Its acceleration is, therefore, $\omega^2 R \sin\theta$. Resolving the forces along $PC$ and applying Newton's second law,

$$\mathcal{N} \sin\theta = m\omega^2 R \sin\theta$$

or, $$\mathcal{N} = m\omega^2 R. \quad \ldots \text{(i)}$$

As there is no vertical acceleration,

$$\mathcal{N} \cos\theta = mg. \quad \ldots \text{(ii)}$$

Dividing (i) by (ii),

$$\frac{1}{\cos\theta} = \frac{\omega^2 R}{g}.$$

or, $$\omega = \sqrt{\frac{g}{R\cos\theta}}.$$

**12.** *A metal ring of mass m and radius R is placed on a smooth horizontal table and is set rotating about its own axis in such a way that each part of the ring moves with a speed v. Find the tension in the ring.*

***Solution*** : Consider a small part $ACB$ of the ring that subtends an angle $\Delta\theta$ at the centre as shown in figure (7-W9). Let the tension in the ring be $T$.

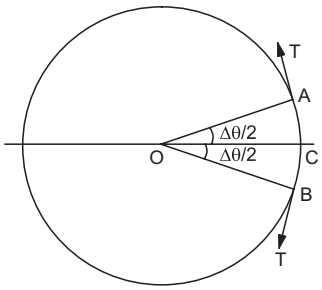



Figure 7-W9

The forces on this small part $ACB$ are
(a) tension $T$ by the part of the ring left to $A$,
(b) tension $T$ by the part of the ring right to $B$,

(c) weight $(\Delta m)g$ and

(d) normal force $\perp N$ by the table.

The tension at $A$ acts along the tangent at $A$ and the tension at $B$ acts along the tangent at $B$. As the small part $ACB$ moves in a circle of radius $R$ at a constant speed $v$, its acceleration is towards the centre (along $CO$) and has a magnitude $(\Delta m)v^2/R$.

Resolving the forces along the radius $CO$,

$$T\cos\left(90°-\frac{\Delta\theta}{2}\right)+T\cos\left(90°-\frac{\Delta\theta}{2}\right)=(\Delta m)\left(\frac{v^2}{R}\right)$$

or, $$2T\sin\frac{\Delta\theta}{2}=(\Delta m)\left(\frac{v^2}{R}\right). \quad \ldots \text{(i)}$$

The length of the part $ACB$ is $R\Delta\theta$. As the total mass of the ring is $m$, the mass of the part $ACB$ will be

$$\Delta m=\frac{m}{2\pi R}R\Delta\theta=\frac{m\Delta\theta}{2\pi}.$$

Putting $\Delta m$ in (i),

$$2T\sin\frac{\Delta\theta}{2}=\frac{m}{2\pi}\Delta\theta\left(\frac{v^2}{R}\right)$$

or, $$T=\frac{mv^2}{2\pi R}\frac{\Delta\theta/2}{\sin(\Delta\theta/2)}$$

As $\Delta\theta$ is very small, $\frac{\Delta\theta/2}{\sin(\Delta\theta/2)}\approx 1$ and $T=\frac{mv^2}{2\pi R}$.

**13.** *A table with smooth horizontal surface is turning at an angular speed $\omega$ about its axis. A groove is made on the surface along a radius and a particle is gently placed inside the groove at a distance a from the centre. Find the speed of the particle as its distance from the centre becomes L.*

***Solution*** **:** The situation is shown in figure (7-W10).

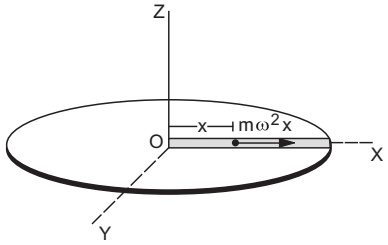



Figure 7-W10

Let us work from the frame of reference of the table. Let us take the origin at the centre of rotation $O$ and the $X$-axis along the groove (figure 7-W10). The $Y$-axis is along the line perpendicular to $OX$ coplanar with the surface of the table and the $Z$-axis is along the vertical.

Suppose at time $t$ the particle in the groove is at a distance $x$ from the origin and is moving along the $X$-axis with a speed $v$. The forces acting on the particle (including the pseudo forces that we must assume because we have taken our frame on the table which is rotating and is noninertial) are

(a) weight $mg$ vertically downward,

(b) normal contact force $\perp N_1$ vertically upward by the bottom surface of the groove,

(c) normal contact force $\perp N_2$ parallel to the $Y$-axis by the side walls of the groove,

(d) centrifugal force $m\omega^2 x$ along the $X$-axis, and

(e) coriolis force along $Y$-axis (coriolis force is perpendicular to the velocity of the particle and the axis of rotation.)

As the particle can only move in the groove, its acceleration is along the $X$-axis. The only force along the $X$-axis is the centrifugal force $m\omega^2 x$. All the other forces are perpendicular to the $X$-axis and have no components along the $X$-axis.

Thus, the acceleration along the $X$-axis is

$$a=\frac{F}{m}=\frac{m\omega^2 x}{m}=\omega^2 x$$

or, $$\frac{dv}{dt}=\omega^2 x$$

or, $$\frac{dv}{dx}\cdot\frac{dx}{dt}=\omega^2 x$$

or, $$\frac{dv}{dx}\cdot v=\omega^2 x$$

or, $$v\,dv=\omega^2 x\,dx$$

or, $$\int_0^v v\,dv=\int_a^L \omega^2 x\,dx$$

or, $$\left[\frac{1}{2}v^2\right]_0^v=\left[\frac{1}{2}\omega^2 x^2\right]_a^L$$

or, $$\frac{v^2}{2}=\frac{1}{2}\omega^2(L^2-a^2)$$

or, $$v=\omega\sqrt{L^2-a^2}.$$

□

## QUESTIONS FOR SHORT ANSWER

**1.** You are driving a motorcycle on a horizontal road. It is moving with a uniform velocity. Is it possible to accelerate the motorcyle without putting higher petrol input rate into the engine ?

**2.** Some washing machines have cloth driers. It contains a drum in which wet clothes are kept. As the drum rotates, the water particles get separated from the cloth. The general description of this action is that "the centrifugal force throws the water particles away from the drum". Comment on this statement from the viewpoint of an observer rotating with the drum and the observer who is washing the clothes.

**3.** A small coin is placed on a record rotating at $33\frac{1}{3}$ rev/minute. The coin does not slip on the record. Where does it get the required centripetal force from ?

**4.** A bird while flying takes a left turn, where does it get the centripetal force from ?

**5.** Is it necessary to express all angles in radian while using the equation $\omega = \omega_0 + \alpha t$ ?

**6.** After a good meal at a party you wash your hands and find that you have forgotten to bring your handkerchief. You shake your hands vigorously to remove the water as much as you can. Why is water removed in this process ?

**7.** A smooth block loosely fits in a circular tube placed on a horizontal surface. The block moves in a uniform circular motion along the tube (figure 7-Q1). Which wall (inner or outer) will exert a nonzero normal contact force on the block ?

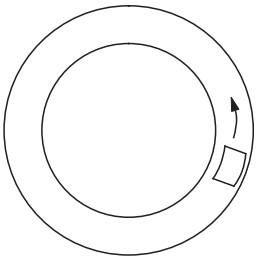


Figure 7-Q1

**8.** Consider the circular motion of the earth around the sun. Which of the following statements is more appropriate ?
(a) Gravitational attraction of the sun on the earth is equal to the centripetal force.
(b) Gravitational attraction of the sun on the earth is the centripetal force.

**9.** A car driver going at some speed $v$ suddenly finds a wide wall at a distance $r$. Should he apply brakes or turn the car in a circle of radius $r$ to avoid hitting the wall ?

**10.** A heavy mass $m$ is hanging from a string in equilibrium without breaking it. When this same mass is set into oscillation, the string breaks. Explain.

## OBJECTIVE I

**1.** When a particle moves in a circle with a uniform speed
(a) its velocity and acceleration are both constant
(b) its velocity is constant but the acceleration changes
(c) its acceleration is constant but the velocity changes
(d) its velocity and acceleration both change.

**2.** Two cars having masses $m_1$ and $m_2$ move in circles of radii $r_1$ and $r_2$ respectively. If they complete the circle in equal time, the ratio of their angular speeds $\omega_1/\omega_2$ is
(a) $m_1/m_2$ (b) $r_1/r_2$ (c) $m_1r_1/m_2r_2$ (d) 1.

**3.** A car moves at a constant speed on a road as shown in figure (7-Q2). The normal force by the road on the car is $N_A$ and $N_B$ when it is at the points $A$ and $B$ respectively.
(a) $N_A = N_B$ (b) $N_A > N_B$ (c) $N_A < N_B$ (d) insufficient information to decide the relation of $N_A$ and $N_B$.

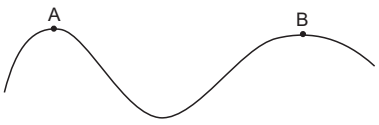



Figure 7-Q2

**4.** A particle of mass $m$ is observed from an inertial frame of reference and is found to move in a circle of radius $r$ with a uniform speed $v$. The centrifugal force on it is
(a) $\frac{mv^2}{r}$ towards the centre
(b) $\frac{mv^2}{r}$ away from the centre
(c) $\frac{mv^2}{r}$ along the tangent through the particle
(d) zero.

**5.** A particle of mass $m$ rotates about the $Z$-axis in a circle of radius $a$ with a uniform angular speed $\omega$. It is viewed from a frame rotating about the $Z$-axis with a uniform angular speed $\omega_0$. The centrifugal force on the particle is
(a) $m\omega^2 a$ (b) $m\omega_0^2 a$ (c) $m\left(\frac{\omega+\omega_0}{2}\right)^2 a$ (d) $m\omega\,\omega_0\, a$.

**6.** A particle is kept fixed on a turntable rotating uniformly. As seen from the ground the particle goes in a circle, its speed is 20 cm/s and acceleration is 20 cm/s$^2$. The particle is now shifted to a new position to make the radius half of the original value. The new values of the speed and acceleration will be
(a) 10 cm/s, 10 cm/s$^2$ (b) 10 cm/s, 80 cm/s$^2$
(c) 40 cm/s, 10 cm/s$^2$ (d) 40 cm/s, 40 cm/s$^2$.

**7.** Water in a bucket is whirled in a vertical circle with a string attached to it. The water does not fall down even when the bucket is inverted at the top of its path. We conclude that in this position

(a) $mg = \frac{mv^2}{r}$ (b) $mg$ is greater than $\frac{mv^2}{r}$

(c) $mg$ is not greater than $\frac{mv^2}{r}$

(d) $mg$ is not less than $\frac{mv^2}{r}$.

**8.** A stone of mass $m$ tied to a string of length $l$ is rotated in a circle with the other end of the string as the centre. The speed of the stone is $v$. If the string breaks, the stone will move
(a) towards the centre (b) away from the centre
(c) along a tangent (d) will stop.

**9.** A coin placed on a rotating turntable just slips if it is placed at a distance of 4 cm from the centre. If the angular velocity of the turntable is doubled, it will just slip at a distance of
(a) 1 cm (b) 2 cm (c) 4 cm (d) 8 cm.

**10.** A motorcyle is going on an overbridge of radius $R$. The driver maintains a constant speed. As the motorcycle is ascending on the overbridge, the normal force on it
(a) increases (b) decreases
(c) remains the same (d) fluctuates.

**11.** Three identical cars $A$, $B$ and $C$ are moving at the same speed on three bridges. The car $A$ goes on a plane bridge, $B$ on a bridge convex upward and $C$ goes on a bridge concave upward. Let $F_A$, $F_B$ and $F_C$ be the normal forces exerted by the cars on the bridges when they are at the middle of bridges.
(a) $F_A$ is maximum of the three forces.
(b) $F_B$ is maximum of the three forces.
(c) $F_C$ is maximum of the three forces.
(d) $F_A = F_B = F_C$.

**12.** A train $A$ runs from east to west and another train $B$ of the same mass runs from west to east at the same speed along the equator. $A$ presses the track with a force $F_1$ and $B$ presses the track with a force $F_2$.
(a) $F_1 > F_2$. (b) $F_1 < F_2$. (c) $F_1 = F_2$.
(d) the information is insufficient to find the relation between $F_1$ and $F_2$.

**13.** If the earth stops rotating, the apparent value of $g$ on its surface will
(a) increase everywhere
(b) decrease everywhere
(c) remain the same everywhere
(d) increase at some places and remain the same at some other places.

**14.** A rod of length $L$ is pivoted at one end and is rotated with a uniform angular velocity in a horizontal plane. Let $T_1$ and $T_2$ be the tensions at the points $L/4$ and $3L/4$ away from the pivoted ends.
(a) $T_1 > T_2$. (b) $T_2 > T_1$. (c) $T_1 = T_2$. (d) The relation between $T_1$ and $T_2$ depends on whether the rod rotates clockwise or anticlockwise.

**15.** A simple pendulum having a bob of mass $m$ is suspended from the ceiling of a car used in a stunt film shooting. The car moves up along an inclined cliff at a speed $v$ and makes a jump to leave the cliff and lands at some distance. Let $R$ be the maximum height of the car from the top of the cliff. The tension in the string when the car is in air is
(a) $mg$ (b) $mg - \frac{mv^2}{R}$ (c) $mg + \frac{mv^2}{R}$ (d) zero.

**16.** Let $\theta$ denote the angular displacement of a simple pendulum oscillating in a vertical plane. If the mass of the bob is $m$, the tension in the string is $mg\cos\theta$
(a) always (b) never
(c) at the extreme positions
(d) at the mean position.

## OBJECTIVE II

**1.** An object follows a curved path. The following quantities may remain constant during the motion
(a) speed (b) velocity
(c) acceleration (d) magnitude of acceleration.

**2.** Assume that the earth goes round the sun in a circular orbit with a constant speed of 30 km/s.
(a) The average velocity of the earth from 1st Jan, 90 to 30th June, 90 is zero.
(b) The average acceleration during the above period is 60 km/s$^2$.
(c) The average speed from 1st Jan, 90 to 31st Dec, 90 is zero.
(d) The instantaneous acceleration of the earth points towards the sun.

**3.** The position vector of a particle in a circular motion about the origin sweeps out equal area in equal time. Its
(a) velocity remains constant
(b) speed remains constant
(c) acceleration remains constant
(d) tangential acceleration remains constant.

**4.** A particle is going in a spiral path as shown in figure (7-Q3) with constant speed.

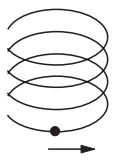


Figure 7-Q3

(a) The velocity of the particle is constant.
(b) The acceleration of the particle is constant.

(c) The magnitude of acceleration is constant.
(d) The magnitude of acceleration is decreasing continuously.

**5.** A car of mass $M$ is moving on a horizontal circular path of radius $r$. At an instant its speed is $v$ and is increasing at a rate $a$.
(a) The acceleration of the car is towards the centre of the path.
(b) The magnitude of the frictional force on the car is greater than $\frac{mv^2}{r}$.
(c) The friction coefficient between the ground and the car is not less than $a/g$.
(d) The friction coefficient between the ground and the car is $\mu = \tan^{-1}\frac{v^2}{rg}$.

**6.** A circular road of radius $r$ is banked for a speed $v$ = 40 km/hr. A car of mass $m$ attempts to go on the circular road. The friction coefficient between the tyre and the road is negligible.
(a) The car cannot make a turn without skidding.
(b) If the car turns at a speed less than 40 km/hr, it will slip down.
(c) If the car turns at the correct speed of 40 km/hr, the force by the road on the car is equal to $\frac{mv^2}{r}$.
(d) If the car turns at the correct speed of 40 km/hr, the force by the road on the car is greater than $mg$ as well as greater than $\frac{mv^2}{r}$.

**7.** A person applies a constant force $\vec{F}$ on a particle of mass $m$ and finds that the particle moves in a circle of radius $r$ with a uniform speed $v$ as seen from an inertial frame of reference.
(a) This is not possible.
(b) There are other forces on the particle.
(c) The resultant of the other forces is $\frac{mv^2}{r}$ towards the centre.
(d) The resultant of the other forces varies in magnitude as well as in direction.

## EXERCISES

**1.** Find the acceleration of the moon with respect to the earth from the following data : Distance between the earth and the moon = $3{\cdot}85 \times 10^5$ km and the time taken by the moon to complete one revolution around the earth = 27·3 days.

**2.** Find the acceleration of a particle placed on the surface of the earth at the equator due to earth's rotation. The diameter of earth = 12800 km and it takes 24 hours for the earth to complete one revolution about its axis.

**3.** A particle moves in a circle of radius 1·0 cm at a speed given by $v$ = 2·0 $t$ where $v$ is in cm/s and $t$ in seconds.
(a) Find the radial acceleration of the particle at $t$ = 1 s.
(b) Find the tangential acceleration at $t$ = 1 s. (c) Find the magnitude of the acceleration at $t$ = 1 s.

**4.** A scooter weighing 150 kg together with its rider moving at 36 km/hr is to take a turn of radius 30 m. What horizontal force on the scooter is needed to make the turn possible ?

**5.** If the horizontal force needed for the turn in the previous problem is to be supplied by the normal force by the road, what should be the proper angle of banking ?

**6.** A park has a radius of 10 m. If a vehicle goes round it at an average speed of 18 km/hr, what should be the proper angle of banking ?

**7.** If the road of the previous problem is horizontal (no banking), what should be the minimum friction coefficient so that a scooter going at 18 km/hr does not skid ?

**8.** A circular road of radius 50 m has the angle of banking equal to 30°. At what speed should a vehicle go on this road so that the friction is not used ?

**9.** In the Bohr model of hydrogen atom, the electron is treated as a particle going in a circle with the centre at the proton. The proton itself is assumed to be fixed in an inertial frame. The centripetal force is provided by the Coloumb attraction. In the ground state, the electron goes round the proton in a circle of radius $5{\cdot}3 \times 10^{-11}$ m. Find the speed of the electron in the ground state. Mass of the electron = $9{\cdot}1 \times 10^{-31}$ kg and charge of the electron = $1{\cdot}6 \times 10^{-19}$ C.

**10.** A stone is fastened to one end of a string and is whirled in a vertical circle of radius $R$. Find the minimum speed the stone can have at the highest point of the circle.

**11.** A ceiling fan has a diameter (of the circle through the outer edges of the three blades) of 120 cm and rpm 1500 at full speed. Consider a particle of mass 1 g sticking at the outer end of a blade. How much force does it experience when the fan runs at full speed ? Who exerts this force on the particle ? How much force does the particle exert on the blade along its surface ?

**12.** A mosquito is sitting on an L.P. record disc rotating on a turn table at $33\frac{1}{3}$ revolutions per minute. The distance of the mosquito from the centre of the turn table is 10 cm. Show that the friction coefficient between the record and the mosquito is greater than $\pi^2/81$. Take $g = 10$ m/s$^2$.

**13.** A simple pendulum is suspended from the ceiling of a car taking a turn of radius 10 m at a speed of 36 km/h. Find the angle made by the string of the pendulum with the vertical if this angle does not change during the turn. Take $g = 10$ m/s$^2$.

**14.** The bob of a simple pendulum of length 1 m has mass 100 g and a speed of 1·4 m/s at the lowest point in its path. Find the tension in the string at this instant.

**15.** Suppose the bob of the previous problem has a speed of 1·4 m/s when the string makes an angle of 0·20 radian with the vertical. Find the tension at this instant. You can use $\cos\theta \approx 1 - \theta^2/2$ and $\sin\theta \approx \theta$ for small $\theta$.

**16.** Suppose the amplitude of a simple pendulum having a bob of mass $m$ is $\theta_0$. Find the tension in the string when the bob is at its extreme position.

**17.** A person stands on a spring balance at the equator. (a) By what fraction is the balance reading less than his true weight ? (b) If the speed of earth's rotation is increased by such an amount that the balance reading is half the true weight, what will be the length of the day in this case ?

**18.** A turn of radius 20 m is banked for the vehicles going at a speed of 36 km/h. If the coefficient of static friction between the road and the tyre is 0·4, what are the possible speeds of a vehicle so that it neither slips down nor skids up ?

**19.** A motorcycle has to move with a constant speed on an overbridge which is in the form of a circular arc of radius $R$ and has a total length $L$. Suppose the motorcycle starts from the highest point. (a) What can its maximum velocity be for which the contact with the road is not broken at the highest point ? (b) If the motorcycle goes at speed $1/\sqrt{2}$ times the maximum found in part (a), where will it lose the contact with the road ? (c) What maximum uniform speed can it maintain on the bridge if it does not lose contact anywhere on the bridge ?

**20.** A car goes on a horizontal circular road of radius $R$, the speed increasing at a constant rate $\frac{dv}{dt} = a$. The friction coefficient between the road and the tyre is $\mu$. Find the speed at which the car will skid.

**21.** A block of mass $m$ is kept on a horizontal ruler. The friction coefficient between the ruler and the block is $\mu$. The ruler is fixed at one end and the block is at a distance $L$ from the fixed end. The ruler is rotated about the fixed end in the horizontal plane through the fixed end. (a) What can the maximum angular speed be for which the block does not slip ? (b) If the angular speed of the ruler is uniformly increased from zero at an angular acceleration $\alpha$, at what angular speed will the block slip ?

**22.** A track consists of two circular parts *ABC* and *CDE* of equal radius 100 m and joined smoothly as shown in figure (7-E1). Each part subtends a right angle at its centre. A cycle weighing 100 kg together with the rider travels at a constant speed of 18 km/h on the track. (a) Find the normal contact force by the road on the cycle when it is at *B* and at *D*. (b) Find the force of friction exerted by the track on the tyres when the cycle is at *B*, *C* and *D*. (c) Find the normal force between the road and the cycle just before and just after the cycle crosses *C*. (d) What should be the minimum friction coefficient between the road and the tyre, which will ensure that the cyclist can move with constant speed ? Take $g = 10$ m/s$^2$.

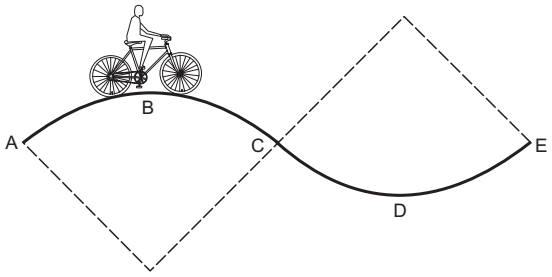



Figure 7-E1

**23.** In a children's park a heavy rod is pivoted at the centre and is made to rotate about the pivot so that the rod always remains horizontal. Two kids hold the rod near the ends and thus rotate with the rod (figure 7-E2). Let the mass of each kid be 15 kg, the distance between the points of the rod where the two kids hold it be 3·0 m and suppose that the rod rotates at the rate of 20 revolutions per minute. Find the force of friction exerted by the rod on one of the kids.

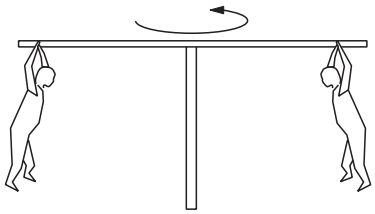


Figure 7-E2

**24.** A hemispherical bowl of radius $R$ is rotated about its axis of symmetry which is kept vertical. A small block is kept in the bowl at a position where the radius makes an angle $\theta$ with the vertical. The block rotates with the bowl without any slipping. The friction coefficient between the block and the bowl surface is $\mu$. Find the range of the angular speed for which the block will not slip.

**25.** A particle is projected with a speed $u$ at an angle $\theta$ with the horizontal. Consider a small part of its path near the highest position and take it approximately to be a circular arc. What is the radius of this circle ? This radius is called the radius of curvature of the curve at the point.

**26.** What is the radius of curvature of the parabola traced out by the projectile in the previous problem at a point where the particle velocity makes an angle $\theta/2$ with the horizontal ?

**27.** A block of mass $m$ moves on a horizontal circle against the wall of a cylindrical room of radius $R$. The floor of the room on which the block moves is smooth but the friction coefficient between the wall and the block is $\mu$. The block is given an initial speed $v_0$. As a function of the speed $v$ write (a) the normal force by the wall on the block, (b) the frictional force by the wall and (c) the tangential acceleration of the block. (d) Integrate the

tangential acceleration $\left(\frac{dv}{dt} = v\frac{dv}{ds}\right)$ to obtain the speed of the block after one revolution.

**28.** A table with smooth horizontal surface is fixed in a cabin that rotates with a uniform angular velocity ω in a circular path of radius $R$ (figure 7-E3). A smooth groove $AB$ of length $L(<<R)$ is made on the surface of the table. The groove makes an angle θ with the radius $OA$ of the circle in which the cabin rotates. A small particle is kept at the point $A$ in the groove and is released to move along $AB$. Find the time taken by the particle to reach the point $B$.

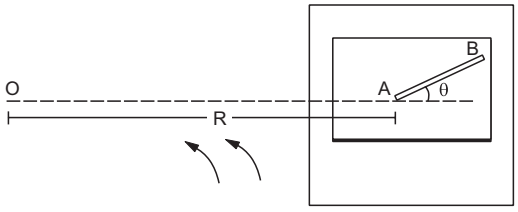



Figure 7-E3

**29.** A car moving at a speed of 36 km/hr is taking a turn on a circular road of radius 50 m. A small wooden plate is kept on the seat with its plane perpendicular to the radius of the circular road (figure 7-E4). A small block of mass 100 g is kept on the seat which rests against the plate. The friction coefficient between the block and the plate is μ = 0·58. (a) Find the normal contact force exerted by the plate on the block. (b) The plate is slowly turned so that the angle between the normal to the plate and the radius of the road slowly increases. Find the angle at which the block will just start sliding on the plate.

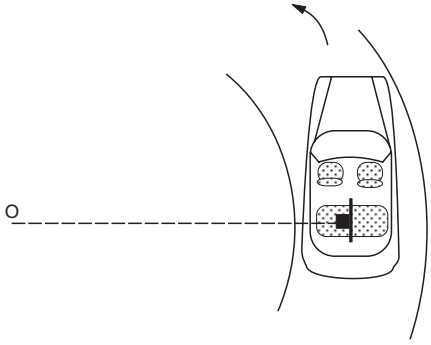



Figure 7-E4

**30.** A table with smooth horizontal surface is placed in a cabin which moves in a circle of a large radius $R$ (figure 7-E5). A smooth pulley of small radius is fastened to the table. Two masses $m$ and $2m$ placed on the table are connected through a string going over the pulley. Initially the masses are held by a person with the strings along the outward radius and then the system is released from rest (with respect to the cabin). Find the magnitude of the initial acceleration of the masses as seen from the cabin and the tension in the string.

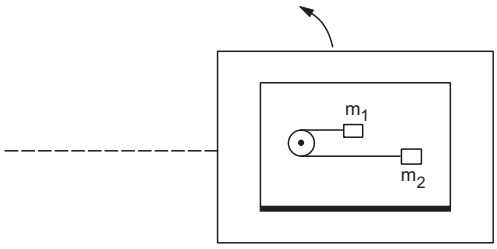



Figure 7-E5

□

## ANSWERS

### OBJECTIVE I

| | | | | | |
|---|---|---|---|---|---|
| 1. (d) | 2. (d) | 3. (c) | 4. (d) | 5. (b) | 6. (a) |
| 7. (c) | 8. (c) | 9. (a) | 10. (a) | 11. (c) | 12. (a) |
| 13. (d) | 14. (a) | 15. (d) | 16. (c) | | |

### OBJECTIVE II

| | | |
|---|---|---|
| 1. (a), (d) | 2. (d) | 3. (b), (d) |
| 4. (c) | 5. (b), (c) | 6. (b), (d) |
| 7. (b), (d) | | |

### EXERCISES

1. $2{\cdot}73 \times 10^{-3}$ m/s$^2$
2. 0·0336 m/s$^2$
3. (a) 4·0 cm/s$^2$ (b) 2·0 cm/s$^2$ (c) $\sqrt{20}$ cm/s$^2$
4. 500 N
5. $\tan^{-1}(1/3)$
6. $\tan^{-1}(1/4)$
7. 0·25
8. 17 m/s
9. $2{\cdot}2 \times 10^6$ m/s
10. $\sqrt{Rg}$
11. 14·8 N, 14·8 N
13. 45°
14. 1·2 N

15. 1·16 N

16. $mg\cos\theta_0$

17. (a) $3{\cdot}5\times10^{-3}$ (b) 2·0 hour

18. Between 14·7 km/h and 54 km/hr

19. (a) $\sqrt{Rg}$,
(b) a distance $\pi R/3$ along the bridge from the highest point,
(c) $\sqrt{gR\cos(L/2R)}$

20. $\left[(\mu^2g^2-a^2)R^2\right]^{1/4}$

21. (a) $\sqrt{\mu g/L}$ (b) $\left[\left(\frac{\mu g}{L}\right)^2-\alpha^2\right]^{1/4}$

22. (a) 975 N, 1025 N (b) 0, 707 N, 0
(c) 682 N, 732 N (d) 1·037

23. $10\pi^2$ N

24. $\left[\frac{g(\sin\theta-\mu\cos\theta)}{R\sin\theta(\cos\theta+\mu\sin\theta)}\right]^{1/2}$ to $\left[\frac{g(\sin\theta+\mu\cos\theta)}{R\sin\theta(\cos\theta-\mu\sin\theta)}\right]^{1/2}$

25. $\frac{u^2\cos^2\theta}{g}$

26. $\frac{u^2\cos^2\theta}{g\cos^3(\theta/2)}$

27. (a) $\frac{mv^2}{R}$ (b) $\frac{\mu mv^2}{R}$ (c) $-\frac{\mu v^2}{R}$ (d) $v_0e^{-2\pi\mu}$

28. $\sqrt{\frac{2L}{\omega^2R\cos\theta}}$

29. (a) 0·2 N (b) 30°

30. $\frac{\omega^2R}{3}, \frac{4}{3}m\omega^2R$

□

## CHAPTER 8

# WORK AND ENERGY

### 8.1 KINETIC ENERGY

A dancing, running man is said to be more energetic compared to a sleeping snoring man. In physics, a moving particle is said to have more energy than an identical particle at rest. Quantitatively the energy of the moving particle (over and above its energy at rest) is defined by

$$K(v) = \frac{1}{2} mv^2 = \frac{1}{2} m\vec{v} \cdot \vec{v} \qquad \ldots \ (8.1)$$

and is called the *kinetic energy* of the particle. The kinetic energy of a system of particles is the sum of the kinetic energies of all its constituent particles, i.e.,

$$K = \sum_i \frac{1}{2} m_i v_i^2 \cdot$$

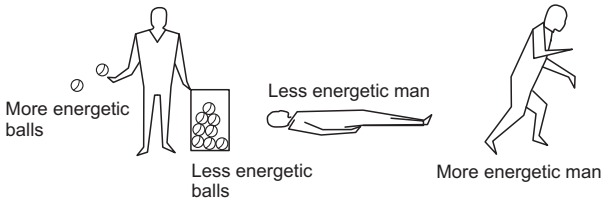



Figure 8.1

The kinetic energy of a particle or a system of particles can increase or decrease or remain constant as time passes.

If no force is applied on the particle, its velocity $v$ remains constant and hence the kinetic energy remains the same. A force is necessary to change the kinetic energy of a particle. If the resultant force acting on a particle is perpendicular to its velocity, the speed of the particle does not change and hence the kinetic energy does not change. Kinetic energy changes only when the speed changes and that happens only when the resultant force has a tangential component. When a particle falls near the earth's surface, the force of gravity is parallel to its velocity. Its kinetic energy increases as time passes. On the other hand, a particle projected upward has the force opposite to the velocity and its kinetic energy decreases.

From the definition of kinetic energy

$$\frac{dK}{dt} = \frac{d}{dt}\left(\frac{1}{2} mv^2\right) = mv \frac{dv}{dt} = F_t v,$$

where $F_t$ is the resultant tangential force. If the resultant force $\vec{F}$ makes an angle $\theta$ with the velocity,

$$F_t = F\cos\theta \text{ and } \frac{dK}{dt} = Fv\cos\theta = \vec{F} \cdot \vec{v} = \vec{F} \cdot \frac{d\vec{r}}{dt}$$

or,
$$dK = \vec{F} \cdot d\vec{r}. \qquad \ldots \ (8.2)$$

### 8.2 WORK AND WORK-ENERGY THEOREM

The quantity $\vec{F} \cdot d\vec{r} = F\,dr\cos\theta$ is called the *work done* by the force $\vec{F}$ on the particle during the small displacement $d\vec{r}$.

The work done on the particle by a force $\vec{F}$ acting on it during a finite displacement is obtained by

$$W = \int \vec{F} \cdot d\vec{r} = \int F\cos\theta\, dr, \qquad \ldots \ (8.3)$$

where the integration is to be performed along the path of the particle. If $\vec{F}$ is the resultant force on the particle we can use equation (8.2) to get

$$W = \int \vec{F} \cdot d\vec{r} = \int dK = K_2 - K_1.$$

Thus, *the work done on a particle by the resultant force is equal to the change in its kinetic energy.* This is called the *work-energy theorem.*

Let $\vec{F}_1, \vec{F}_2, \vec{F}_3, \ldots$ be the individual forces acting on a particle. The resultant force is $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 \ldots$, and the work done by the resultant force on the particle is

$$\begin{aligned} W &= \int \vec{F} \cdot d\vec{r} \\ &= (\vec{F}_1 + \vec{F}_2 + \vec{F}_3 \ldots\ldots) \cdot d\vec{r} \\ &= \int \vec{F}_1 \cdot d\vec{r} + \int \vec{F}_2 \cdot d\vec{r} + \int \vec{F}_3 \cdot d\vec{r} \ldots, \end{aligned}$$

where $= \int \vec{F}_1 \cdot d\vec{r}$ is the work done on the particle by $\vec{F}_1$ and so on. Thus, *the work done by the resultant force is equal to the sum of the work done by the individual forces.* Note that the work done on a particle by an individual force is not equal to the change in its

kinetic energy; the sum of the work done by all the forces acting on the particle (which is equal to the work done by the resultant force) is equal to the change in its kinetic energy.

The rate of doing work is called the *power* delivered. The work done by a force $\vec{F}$ in a small displacement $\vec{dr}$ is $dW = \vec{F} \cdot \vec{dr}$.

Thus, the power delivered by the force is

$$P = \frac{dW}{dt} = \vec{F} \cdot \frac{\vec{dr}}{dt} = \vec{F} \cdot \vec{v}.$$

The SI unit of power is joule/second and is written as "watt". A commonly used unit of power is *horsepower* which is equal to 746 W.

## 8.3 CALCULATION OF WORK DONE

The work done by a force on a particle during a displacement has been defined as

$$W = \int \vec{F} \cdot \vec{dr}.$$

### Constant Force

Suppose, the force is constant (in direction and magnitude) during the displacement. Then $W = \int \vec{F} \cdot \vec{dr} = \vec{F} \cdot \int \vec{dr} = \vec{F} \cdot \vec{r}$, where $\vec{r}$ is the total displacement of the particle during which the work is calculated. If θ be the angle between the constant force $\vec{F}$ and the displacement $\vec{r}$, the work is

$$W = Fr \cos\theta. \qquad \ldots (8.4)$$

In particular, if the displacement is along the force, as is the case with a freely and vertically falling particle, $\theta = 0$ and $W = Fr$.

The force of gravity ($\vec{mg}$) is constant in magnitude and direction if the particle moves near the surface of the earth. Suppose a particle moves from $A$ to $B$ along some curve and that $\vec{AB}$ makes an angle θ with the vertical as shown in figure (8.2). The work done by the force of gravity during the transit from $A$ to $B$ is

$$W = mg\ (AB) \cos\theta = mgh,$$

where $h$ is the height descended by the particle. If a particle ascends a height $h$, the work done by the force of gravity is $-mgh$.

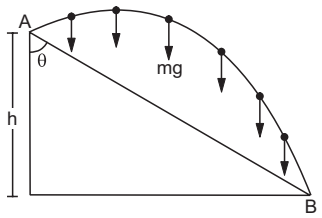



Figure 8.2

If the particle goes from the point $A$ to the point $B$ along some other curve, the work done by the force of gravity is again $mgh$. We see that the work done by a constant force in going from $A$ to $B$ depends only on the positions of $A$ and $B$ and not on the actual path taken. In case of gravity, the work is weight $mg$ times the height descended. If a particle starts from $A$ and reaches to the same point $A$ after some time, the work done by gravity during this round trip is zero, as the height descended is zero. We shall encounter other forces having this property.

### Spring Force

Consider the situation shown in figure (8.3). One end of a spring is attached to a fixed vertical support and the other end to a block which can move on a horizontal table. Let $x = 0$ denote the position of the block when the spring is in its natural length. We shall calculate the work done on the block by the spring-force as the block moves from $x = 0$ to $x = x_1$.

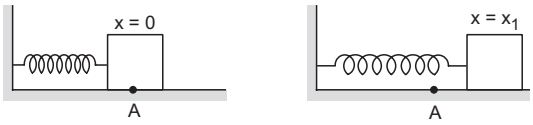



Figure 8.3

The force on the block is $k$ times the elongation of the spring. But the elongation changes as the block moves and so does the force. We cannot take $\vec{F}$ out of the integration $\int \vec{F} \cdot \vec{dr}$. We have to write the work done during a small interval in which the block moves from $x$ to $x + dx$. The force in this interval is $kx$ and the displacement is $dx$. The force and the displacement are opposite in direction.

So, $$\vec{F} \cdot \vec{dr} = -F\,dx = -kx\,dx$$

during this interval. The total work done as the block is displaced from $x = 0$ to $x = x_1$ is

$$W = \int_0^{x_1} -kx\,dx = \left[-\frac{1}{2}kx^2\right]_0^{x_1} = -\frac{1}{2}kx_1^2.$$

If the block moves from $x = x_1$ to $x = x_2$, the limits of integration are $x_1$ and $x_2$ and the work done is

$$W = \left(\frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2\right). \qquad \ldots (8.5)$$

Note that if the block is displaced from $x_1$ to $x_2$ and brought back to $x = x_1$, the work done by the spring-force is zero. The work done during the return journey is negative of the work during the onward journey. The net work done by the spring-force in a round trip is zero.

Three positions of a spring are shown in figure (8.4). In (i) the spring is in its natural length, in (ii) it is compressed by an amount $x$ and in (iii) it is elongated by an amount $x$. Work done by the spring-force on the block in various situations is shown in the following table.

**Table 8.1**

| Initial state of the spring | Final state of the spring | $x_1$ | $x_2$ | $W$ |
|---|---|---|---|---|
| Natural | Compressed | 0 | $-x$ | $-\frac{1}{2}kx^2$ |
| Natural | Elongated | 0 | $x$ | $-\frac{1}{2}kx^2$ |
| Elongated | Natural | $x$ | 0 | $\frac{1}{2}kx^2$ |
| Compressed | Natural | $-x$ | 0 | $\frac{1}{2}kx^2$ |
| Elongated | Compressed | $x$ | $-x$ | 0 |
| Compressed | Elongated | $-x$ | $x$ | 0 |

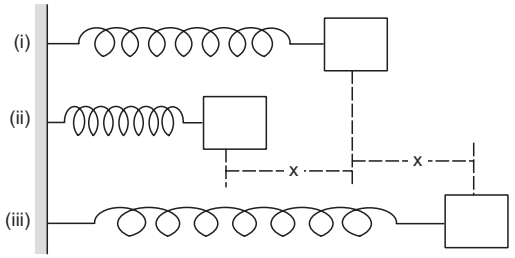



Figure 8.4

**Force Perpendicular to Velocity**

Suppose $\vec{F} \perp \vec{v}$ for all the time. Then $\vec{F} \cdot d\vec{r} = \vec{F} \cdot \vec{v}\, dt$ is zero in any small interval and the work done by this force is zero.

For example, if a particle is fastened to the end of a string and is whirled in a circular path, the tension is always perpendicular to the velocity of the particle and hence the work done by the tension is zero in circular motion.

***Example 8.1***

*A spring of spring constant* 50 N/m *is compressed from its natural position through* 1 cm. *Find the work done by the spring-force on the agency compressing the spring.*

***Solution*** : The magnitude of the work is

$$\frac{1}{2}kx^2 = \frac{1}{2} \times (50\ \text{N/m}) \times (1\ \text{cm})^2$$

$$= (25\ \text{N/m}) \times (1 \times 10^{-2}\text{m})^2 = 2{\cdot}5 \times 10^{-3}\ \text{J}.$$

As the compressed spring will push the agency, the force will be opposite to the displacement of the point of application and the work will be negative. Thus, the work done by the spring-force is –2·5 mJ.

The following three cases occur quite frequently :

(a) The force is perpendicular to the velocity at all the instants. The work done by the force is then zero.

(b) The force is constant (both in magnitude and direction). The work done by the force is $W = Fd\cos\theta$, where $F$ and $d$ are magnitudes of the force and the displacement and $\theta$ is the angle between them. The amount of work done depends only on the end positions and not on the intermediate path. The work in a round trip is zero. Force of gravity on the bodies near the earth's surface is an example.

The work done due to the force of gravity on a particle of mass $m$ is $mgh$, where $h$ is the vertical height 'descended' by the particle.

(c) The force is $F = -kx$ as is the case with an elastic spring. The magnitude of the work done by the force during a displacement $x$ from or to its natural position ($x = 0$) is $\frac{1}{2}kx^2$. The work may be $+\frac{1}{2}kx^2$ or $-\frac{1}{2}kx^2$ depending on whether the force and the displacement are along the same or opposite directions.

***Example 8.2***

*A particle of mass* 20 g *is thrown vertically upwards with a speed of* 10 m/s. *Find the work done by the force of gravity during the time the particle goes up.*

***Solution*** : Suppose the particle reaches a maximum height $h$. As the velocity at the highest point is zero, we have

$$0 = u^2 - 2gh$$

or,
$$h = \frac{u^2}{2g}.$$

The work done by the force of gravity is

$$-mgh = -mg\frac{u^2}{2g} = -\frac{1}{2}mu^2$$

$$= -\frac{1}{2}(0{\cdot}02\ \text{kg}) \times (10\ \text{m/s})^2 = -1{\cdot}0\ \text{J}.$$

## 8.4 WORK-ENERGY THEOREM FOR A SYSTEM OF PARTICLES

So far we have considered the work done on a single particle. The total work done on a particle equals the change in its kinetic energy. In other words, to change the kinetic energy of a particle we have to apply a force on it and the force must do work on it. Next, consider a system containing more than one particle and suppose the particles exert forces on each other. As a simple example, take a system of two

charged particles as shown in figure (8.5) attracting each other (such as a positive and a negative charge).

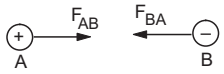



Figure 8.5

Because of mutual attraction, the particles are accelerated towards each other and the kinetic energy of the system increases. We have not applied any external force on the system, yet the kinetic energy has changed. Let us examine this in more detail. The particle $B$ exerts a force $\vec{F}_{AB}$ on $A$. As $A$ moves towards $B$, this force does work. The work done by this force is equal to the increase in the kinetic energy of $A$.

Similarly, $A$ exerts a force $\vec{F}_{BA}$ on $B$. This force does work on $B$ and this work is equal to the increase in the kinetic energy of $B$. The work by $\vec{F}_{AB}$ + the work by $\vec{F}_{BA}$ is equal to the increase in the total kinetic energy of the two particles. Note that $\vec{F}_{AB} = -\vec{F}_{BA}$, so that $\vec{F}_{AB} + \vec{F}_{BA} = 0$. But the work by $\vec{F}_{AB}$ + the work by $\vec{F}_{BA} \neq 0$. The two forces are opposite in direction but the displacements are also opposite. Thus, the work done by both the forces are positive and are added. The total work done on different particles of the system by the internal forces may not be zero. The change in the kinetic energy of a system is equal to the work done on the system by the external as well as the internal forces.

## 8.5 POTENTIAL ENERGY

Consider the example of the two charged particles $A$ and $B$ taken in the previous section. Suppose at some instant $t_1$ the particles are at positions $A$, $B$ and are going away from each other with speeds $v_1$ and $v_2$ (figure 8.6).

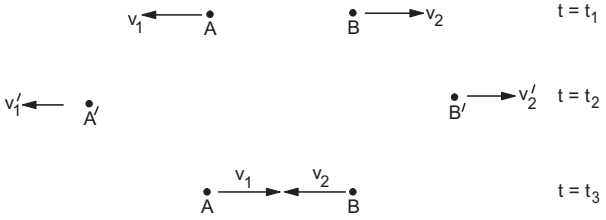



Figure 8.6

The kinetic energy of the system is $K_1$. We call the positions of the particles at time $t_1$ as configuration-1. The particle $B$ attracts $A$ and hence the speed $v_1$ decreases as time passes. Similarly, the speed $v_2$ of $B$ decreases. Thus, the kinetic energy of the two-particle system decreases as time passes. Suppose at a time $t_2$, the particles are at $A'$ and $B'$, the speeds have changed to $v_1'$ and $v_2'$ and the kinetic energy becomes $K_2$. We call the positions of the particles at time $t_2$ as configuration-2. The kinetic energy of the system is decreased by $K_1 - K_2$.

However, if you wait for some more time, the particles return to the original positions $A$ and $B$, i.e., in configuration-1. At this time, say $t_3$, the particles move towards each other with speeds $v_1$ and $v_2$. Their kinetic energy is again $K_1$.

When the particles were in configuration-1 the kinetic energy was $K_1$. When they reached configuration-2 it decreased to $K_2$. The kinetic energy has decreased but is not lost for ever. We just have to wait. When the particles return to configuration-1 at time $t_3$, the kinetic energy again becomes $K_1$. It seems meaningful and reasonable if we think of yet another kind of energy which depends on the configuration. We call this as the *potential energy* of the system. Some kinetic energy was converted into potential energy when the system passed from configuration-1 to configuration-2. As the system returns to configuration-1, this potential energy is converted back into kinetic energy. The sum of the kinetic energy and the potential energy remains constant.

How do we precisely define the potential energy of a system ? Before defining potential energy, let us discuss the idea of conservative and nonconservative forces.

## 8.6 CONSERVATIVE AND NONCONSERVATIVE FORCES

Let us consider the following two examples.

(1) Suppose a block of mass $m$ rests on a rough horizontal table (figure 8.7). It is dragged horizontally towards right through a distance $l$ and then back to its initial position. Let μ be the friction coefficient between the block and the table. Let us calculate the work done by friction during the round trip.

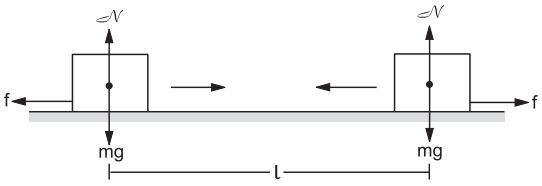



Figure 8.7

The normal force between the table and the block is $N = mg$ and hence the force of friction is $\mu mg$. When

the block moves towards right, friction on it is towards left and the work by friction is $(-\mu mgl)$. When the block moves towards left, friction on it is towards right and the work is again $(-\mu mgl)$.

Hence, the total work done by the force of friction in the round trip is $(-2\mu mgl)$.

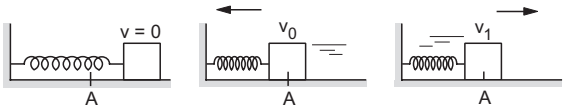



Figure 8.8

(2) Suppose a block connected by a spring is kept on a rough table as shown in figure (8.8). The block is pulled aside and then released. It moves towards the centre $A$ and has some velocity $v_0$ as it passes through the centre. It goes to the other side of $A$ and then comes back. This time it passes through the centre with somewhat smaller velocity $v_1$. Compare these two cases in which the block is at $A$, once going towards left and then towards right. In both the cases the system (table + block + spring) has the same configuration. The spring has the same length. The block is at the same point on the table and the table of course is fixed to the ground. The kinetic energy in the second case is less than the kinetic energy in the first case. This loss in the kinetic energy is a real loss. Every time the block passes through the mean position $A$, the kinetic energy of the system is smaller and in due course, the block stops on the table. We hold friction as the culprit, because in absence of friction the system regains its kinetic energy as it returns to its original configuration. Remember, work done by friction in a round trip is negative and not zero [example (1) above].

We divide the forces in two categories (a) *conservative forces* and (b) *nonconservative forces*. *If the work done by a force during a round trip of a system is always zero, the force is said to be conservative. Otherwise, it is called nonconservative.*

Conservative force can also be defined as follows :

*If the work done by a force depends only on the initial and final states and not on the path taken, it is called a conservative force.*

Thus, the force of gravity, Coulomb force and the force of spring are conservative forces, as the work done by these forces are zero in a round trip. The force of friction is nonconservative because the work done by the friction is not zero in a round trip.

## 8.7 DEFINITION OF POTENTIAL ENERGY AND CONSERVATION OF MECHANICAL ENERGY

We define the *change in potential energy* of a system corresponding to a conservative internal force as

$$U_f - U_i = -W = -\int_i^f \vec{F} \cdot d\vec{r}$$

where $W$ is the work done by the internal force on the system as the system passes from the initial configuration $i$ to the final configuration $f$.

We don't (or can't) define potential energy corresponding to a nonconservative internal force.

Suppose only conservative internal forces operate between the parts of the system and the potential energy $U$ is defined corresponding to these forces. There are either no external forces or the work done by them is zero. We have

$$U_f - U_i = -W = -(K_f - K_i)$$

or, $$U_f + K_f = U_i + K_i . \quad \ldots (8.6)$$

The sum of the kinetic energy and the potential energy is called the total *mechanical energy*. We see from equation (8.6) that *the total mechanical energy of a system remains constant if the internal forces are conservative and the external forces do no work.* This is called the *principle of conservation of energy.*

The total mechanical energy $K + U$ is not constant if nonconservative forces, such as friction, act between the parts of the system. We can't apply the principle of conservation of energy in presence of nonconservative forces. The work-energy theorem is still valid even in the presence of nonconservative forces.

Note that only a change in potential energy is defined above. We are free to choose the zero potential energy in any configuration just as we are free to choose the origin in space anywhere we like.

If nonconservative internal forces operate within the system, or external forces do work on the system, the mechanical energy changes as the configuration changes. According to the work-energy theorem, the work done by all the forces equals the change in the kinetic energy. Thus,

$$W_c + W_{nc} + W_{ext} = K_f - K_i$$

where the three terms on the left denote the work done by the conservative internal forces, nonconservative internal forces and the external forces.

As $$W_c = -(U_f - U_i) ,$$

we get

$$W_{nc} + W_{ext} = (K_f + U_f) - (K_i + U_i)$$

$$= E_f - E_i \quad \ldots (8.7)$$

where $E = K + U$ is the total mechanical energy.

If the internal forces are conservative but external forces also act on the system and they do work, $W_{nc} = 0$ and from (8.7),

$$W_{ext} = E_f - E_i . \quad \ldots (8.8)$$

*The work done by the external forces equals the change in the mechanical energy of the system.*

Let us summarise the concepts developed so far in this chapter.

(1) Work done on a particle is equal to the change in its kinetic energy.

(2) Work done on a system by all the (external and internal) forces is equal to the change in its kinetic energy.

(3) A force is called conservative if the work done by it during a round trip of a system is always zero. The force of gravitation, Coulomb force, force by a spring etc. are conservative. If the work done by it during a round trip is not zero, the force is nonconservative. Friction is an example of nonconservative force.

(4) The change in the potential energy of a system corresponding to conservative internal forces is equal to negative of the work done by these forces.

(5) If no external forces act (or the work done by them is zero) and the internal forces are conservative, the mechanical energy of the system remains constant. This is known as the principle of conservation of mechanical energy.

(6) If some of the internal forces are nonconservative, the mechanical energy of the system is not constant.

(7) If the internal forces are conservative, the work done by the external forces is equal to the change in mechanical energy.

***Example 8.3***

*Two charged particles A and B repel each other by a force $k/r^2$, where k is a constant and r is the separation between them. The particle A is clamped to a fixed point in the lab and the particle B which has a mass m, is released from rest with an initial separation $r_0$ from A. Find the change in the potential energy of the two-particle system as the separation increases to a large value. What will be the speed of the particle B in this situation ?*

***Solution*** : The situation is shown in figure (8.9). Take $A + B$ as the system. The only external force acting on the system is that needed to hold $A$ fixed. (You can imagine the experiment being conducted in a gravity free region or the particles may be kept and allowed to move on a smooth horizontal surface, so that the normal force balances the force of gravity). This force does no work on the system because it acts on the charge $A$ which does not move. Thus, the external forces do no work and internal forces are conservative. The total mechanical energy must, therefore, remain constant. There are two internal forces; $F_{AB}$ acting on $A$ and $F_{BA}$ acting on $B$. The force $F_{AB}$ does no work because it acts on $A$ which does not move. The work done by $F_{BA}$ as the particle $B$ is taken away is,

$$W = \int \vec{F} \cdot d\vec{r} = \int_{r_0}^{\infty} \frac{k}{r^2} dr = \frac{k}{r_0} . \quad \ldots (i)$$

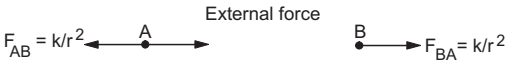



Figure 8.9

The change in the potential energy of the system is

$$U_f - U_i = - W = - \frac{k}{r_0} .$$

As the total mechanical energy is conserved,

$$K_f + U_f = K_i + U_i$$

or, $$K_f = K_i - (U_f - U_i)$$

or, $$\frac{1}{2} mv^2 = \frac{k}{r_0}$$

or, $$v = \sqrt{\frac{2k}{mr_0}} .$$

## 8.8 CHANGE IN THE POTENTIAL ENERGY IN A RIGID-BODY-MOTION

If the separation between the particles do not change during motion, such as in the case of the motion of a rigid body, the internal forces do no work. This is a consequence of Newton's third law. As an example, consider a system of two particles $A$ and $B$. Suppose, the particles move in such a way that the line $AB$ translates parallel to itself. The displacement $d\vec{r}_A$ of the particle $A$ is equal to the displacement $d\vec{r}_B$ of the particle $B$ in any short time interval. The net work done by the internal forces $\vec{F}_{AB}$ and $\vec{F}_{BA}$ is

$$W = \int (\vec{F}_{AB} \cdot d\vec{r}_A + \vec{F}_{BA} \cdot d\vec{r}_B)$$

$$= \int (\vec{F}_{AB} + \vec{F}_{BA}) \cdot d\vec{r}_A = 0.$$

Thus, the work done by $\vec{F}_{AB}$ and $\vec{F}_{BA}$ add up to zero. Even if $AB$ does not translate parallel to itself but rotates, the result is true. The internal forces acting between the particles of a rigid body do no work in its motion and we need not consider the potential energy corresponding to these forces.

The potential energy of a system changes only when the separations between the parts of the system change. In other words, *the potential energy depends only on the separation between the interacting particles.*

## 8.9 GRAVITATIONAL POTENTIAL ENERGY

Consider a block of mass $m$ kept near the surface of the earth and suppose it is raised through a height $h$. Consider "the earth + the block" as the system. The gravitational force between the earth and the block is conservative and we can define a potential energy corresponding to this force. The earth is very heavy as compared to the block and so one can neglect its acceleration. Thus, we take our reference frame attached to the earth, it will still be very nearly an inertial frame. The work done by the gravitational force due to the block on the earth is zero in this frame. The force $mg$ on the block does work $(-mgh)$ if the block ascends through a height $h$ and hence the potential energy is increased by $mgh$. Thus, *if a block of mass m ascends a height h above the earth's surface (h << radius of earth), the potential energy of the "earth + block" system increases by mgh.* If the block descends by a height $h$, the potential energy decreases by $mgh$. Since the earth almost remains fixed, it is customary to call the potential energy of the earth-block system as the potential energy of the block only. We then say that the gravitational potential energy of the "block" is increased by an amount $mgh$ when it is raised through a hieght $h$ above the earth's surface.

We have been talking in terms of the changes in gravitational potential energy. We can choose any position of the block and call the gravitational potential energy to be zero in this position. The potential energy at a height $h$ above this position is $mgh$. The position of the zero potential energy is chosen according to the convenience of the problem.

***Example 8.4***

*A block of mass m slides along a frictionless surface as shown in the figure (8.10). If it is released from rest at A, what is its speed at B ?*

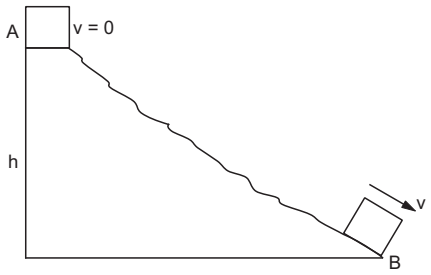



Figure 8.10

***Solution :*** Take the block + the earth as the system. Only the block moves, so only the work done on the block will contribute to the gravitational potential energy. As it descends through a height $h$ between $A$ and $B$, the potential energy decreases by $mgh$. The normal contact force $\mathcal{N}$ on the block by the surface does no work as it is perpendicular to its velocity. No external force does any work on the system. Hence,

increase in kinetic energy = decrease in potential energy

or, $\frac{1}{2}mv^2 = mgh$ or, $v = \sqrt{2gh}$.

***Example 8.5***

*A pendulum bob has a speed* 3 m/s *while passing through its lowest position. What is its speed when it makes an angle of* 60° *with the vertical ? The length of the pendulum is* 0·5 m. *Take* $g = 10$ m/s$^2$.

***Solution :*** Take the bob + earth as the system. The external force acting on the system is that due to the string. But this force is always perpendicular to the velocity of the bob and so the work done by this force is zero. Hence, the total mechanical energy will remain constant. As is clear from figure (8.11), the height ascended by the bob at an angular displacement θ is $l - l\cos\theta = l(1 - \cos\theta)$. The increase in the potential energy is $mgl(1 - \cos\theta)$. This should be equal to the decrease in the kinetic energy of the system. Again, as the earth does not move in the lab frame, this is the decrease in the kinetic energy of the bob. If the speed at an angular displacement θ is $v_1$, the decrease in kinetic energy is

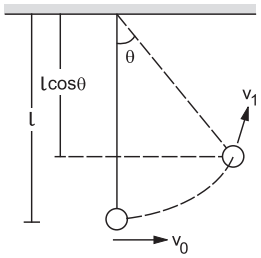



Figure 8.11

$$\frac{1}{2}mv_0^2 - \frac{1}{2}mv_1^2,$$

where $v_0$ is the speed of the block at the lowest position.

Thus, $\frac{1}{2}mv_0^2 - \frac{1}{2}mv_1^2 = mgl(1 - \cos\theta)$

or, $v_1 = \sqrt{v_0^2 - 2gl(1 - \cos\theta)}$

$$= \sqrt{(9\text{ m}^2/\text{s}^2) - 2 \times (10\text{ m/s}^2) \times (0{\cdot}5\text{ m})\left(1 - \frac{1}{2}\right)}$$

$$= 2\text{ m/s}.$$

## 8.10 POTENTIAL ENERGY OF A COMPRESSED OR EXTENDED SPRING

Consider a massless spring of natural length $l$, one end of which is fastened to a wall (figure 8.12). The other end is attached to a block which is slowly pulled

on a smooth horizontal surface to extend the spring. Take the spring as the system. When it is elongated by a distance $x$, the tension in it is $kx$, where $k$ is its spring constant. It pulls the wall towards right and the block towards left by forces of magnitude $kx$. The forces exerted on the spring are (i) $kx$ towards left by the wall and (ii) $kx$ towards right by the block.

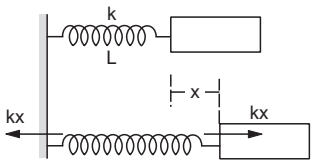



Figure 8.12

How much work has been done on the spring by these two external forces ? The force by the wall does no work as the point of application is fixed. The force by the block does work $\int_0^x kx\,dx = \frac{1}{2}kx^2$. The work is positive as the force is towards right and the particles of the spring, on which this force is acting, also move towards right. Thus, the total external work done on the spring is $\frac{1}{2}kx^2$, when the spring is elongated by an amount $x$ from its natural length. The same is the external work done on the spring if it is compressed by a distance $x$.

We have seen (equation 8.8) that the external work done on a system is equal to the change in its total mechanical energy. The spring is assumed to be massless and hence its kinetic energy remains zero all the time. Thus, its potential energy has increased by $\frac{1}{2}kx^2$.

We conclude that a stretched or compressed spring has a potential energy $\frac{1}{2}kx^2$ larger than its potential energy at its natural length. The potential energy of the spring corresponds to the internal forces between the particles of the spring when it is stretched or compressed. It is called *elastic potential energy* or the *strain energy* of the spring. Again, the calculation gives only the change in the elastic potential energy of the spring and we are free to choose any length of the spring and call the potential energy zero at that length. It is customary to choose the potential energy of a spring in its natural length to be zero. With this choice *the potential energy of a spring is* $\frac{1}{2}kx^2$, *where x is the elongation or the compression of the spring.*

***Example 8.6***

*A block of mass m, attached to a spring of spring constant k, oscillates on a smooth horizontal table. The other end of the spring is fixed to a wall. If it has a speed v when the spring is at its natural length, how far will it move on the table before coming to an instantaneous rest ?*

***Solution*** : Consider the block + the spring as the system. The external forces acting on the system are (a) the force of gravity, (b) the normal force by the table and (c) the force by the wall. None of these do any work on this system and hence the total mechanical energy is conserved. If the block moves a distance $x$ before comming to rest, we have,

$$\frac{1}{2}mv^2 = \frac{1}{2}kx^2$$

or, $$x = v\sqrt{m/k}.$$

***Example 8.7***

*A block of mass m is suspended through a spring of spring constant k and is in equilibrium. A sharp blow gives the block an initial downward velocity v. How far below the equilibrium position, the block comes to an instantaneous rest ?*

***Solution*** : Let us consider the block + the spring + the earth as the system. The system has gravitational potential energy corresponding to the force between the block and the earth as well as the elastic potential energy corresponding to the spring-force. The total mechanical energy includes kinetic energy, gravitational potential energy and elastic potential energy.

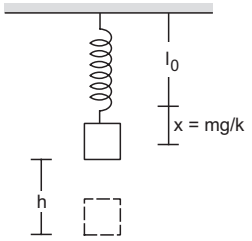



Figure 8.13

When the block is in equilibrium, it is acted upon by two forces, (a) the force of gravity $mg$ and (b) the tension in the spring $T = kx$, where $x$ is the elongation. For equilibrium, $mg = kx$, so that the spring is stretched by a length $x = mg/k$. The potential energy of the spring in this position is

$$\frac{1}{2}k(mg/k)^2 = \frac{m^2g^2}{2k}.$$

Take the gravitational potential energy to be zero in this position. The total mechanical energy of the system just after the blow is

$$\frac{1}{2}mv^2 + \frac{m^2g^2}{2k}.$$

The only external force on this system is that due to the ceiling which does no work. Hence, the mechanical

energy of this system remains constant. If the block descends through a height $h$ before coming to an instantaneous rest, the elastic potential energy becomes $\frac{1}{2}k\,(mg/k+h)^2$ and the gravitational potential energy $-mgh$. The kinetic energy is zero in this state. Thus, we have

$$\frac{1}{2}mv^2+\frac{m^2g^2}{2k}=\frac{1}{2}k\,(mg/k+h)^2-mgh.$$

Solving this we get,

$$h=v\sqrt{m/k}.$$

Compare this with the result obtained in Example (8.6). If we neglect gravity and consider the length of the spring in equilibrium position as the natural length, the answer is same. This simplification is often used while dealing with vertical springs.

## 8.11 DIFFERENT FORMS OF ENERGY : MASS ENERGY EQUIVALENCE

The kinetic energy and the potential energy of a system, taken together, form mechanical energy. Energy can exist in many other forms. In measuring kinetic energy of an extended body, we use the speed of the body as a whole. Even if we keep the body at rest, the particles in it are continuously moving inside the body. These particles also exert forces on each other and there is a potential energy corresponding to these forces. The total energy corresponding to the internal motion of molecules and their interaction, is called *internal energy* or *thermal energy* of the body. Light and sound are other forms of energy. When a source emits light or sound, it loses energy. Chemical energy is significant if there are chemical reactions.

Einstein's special theory of relativity shows that a material particle itself is a form of energy. Thus, about $8{\cdot}18\times10^{-14}$ J of energy may be converted to form an electron and equal amount of energy may be obtained by destroying an electron. The ralation between the mass of a particle $m$ and its equivalent energy $E$ is given as

$$E=mc^2,$$

where $c=3\times10^8$ m/s is the speed of light in vacuum.

When all forms of energy are taken into account, we arrive at the generalised law of conservation of energy.

*Energy can never be created or destroyed, it can only be changed from one form into another.*

### *Worked Out Examples*

**1.** *A porter lifts a suitcase weighing* 20 kg *from the platform and puts it on his head* 2·0 m *above the platform. Calculate the work done by the porter on the suitcase.*

***Solution*** **:** The kinetic energy of the suitcase was zero when it was at the platform and it again became zero when it was put on the head. The change in kinetic energy is zero and hence the total work done on the suitcase is zero. Two forces act on the suitcase, one due to gravity and the other due to the porter. Thus, the work done by the porter is negative of the work done by gravity. As the suitcase is lifted up, the work done by gravity is

$$W=-mgh$$

$$=-(20\text{ kg})(9{\cdot}8\text{ m/s}^2)(2\text{ m})=-392\text{ J}$$

The work done by the porter is 392 J ≈ 390 J.

**2.** *An elevator weighing* 500 kg *is to be lifted up at a constant velocity of* 0·20 m/s. *What would be the minimum horsepower of the motor to be used* ?

***Solution*** **:** As the elevator is going up with a uniform velocity, the total work done on it is zero in any time interval. The work done by the motor is, therefore, equal to the work done by the force of gravity in that interval (in magnitude). The rate of doing work, i.e., the power delivered is

$$P=F\,v=mgv$$

$$=(500\text{ kg})(9{\cdot}8\text{ m/s}^2)(0{\cdot}2\text{ m/s})=980\text{ W}$$

Assuming no loss against friction etc., in the motor, the minimum horsepower of the motor is

$$P=980\text{ W}=\frac{980}{746}\text{ hp}=1{\cdot}3\text{ hp}.$$

**3.** *A block of mass* 2·0 kg *is pulled up on a smooth incline of angle* 30° *with the horizontal. If the block moves with an acceleration of* 1·0 m/s$^2$, *find the power delivered by the pulling force at a time* 4·0 s *after the motion starts. What is the average power delivered during the* 4·0 s *after the motion starts* ?

***Solution*** **:** The forces acting on the block are shown in figure (8-W1). Resolving the forces parallel to the incline, we get

$$F-mg\sin\theta=ma$$

or, $$F=mg\sin\theta+ma$$

$$=(2{\cdot}0\text{ kg})[(9{\cdot}8\text{ m/s}^2)(1/2)+1{\cdot}0\text{ m/s}^2]=11{\cdot}8\text{ N}.$$

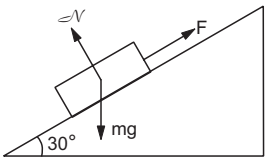



Figure 8-W1

The velocity at $t = 4{\cdot}0$ s is

$$v = at = (1{\cdot}0 \text{ m/s}^2)\,(4{\cdot}0 \text{ s}) = 4{\cdot}0 \text{ m/s}.$$

The power delivered by the force at $t$ = 4·0 s is

$$P = \vec{F} \cdot \vec{v} = (11{\cdot}8 \text{ N})\,(4{\cdot}0 \text{ m/s}) \approx 47 \text{ W}.$$

The displacement during the first four seconds is

$$x = \frac{1}{2} at^2 = \frac{1}{2}(1{\cdot}0 \text{ m/s}^2)\,(16 \text{ s}^2) = 8{\cdot}0 \text{ m}.$$

The work done in these four seconds is, therefore,

$$W = \vec{F} \cdot \vec{d} = (11{\cdot}8 \text{ N})\,(8{\cdot}0 \text{ m}) = 94{\cdot}4 \text{ J}.$$

The average power delivered $= \dfrac{94{\cdot}4 \text{ J}}{4{\cdot}0 \text{ s}}$

$$= 23{\cdot}6 \text{ W} \approx 24 \text{ W}.$$

**4.** *A force $F = (10 + 0{\cdot}50\,x)$ acts on a particle in the x direction, where F is in newton and x in meter. Find the work done by this force during a displacement from $x = 0$ to $x$ = 2·0 m.*

***Solution*** **:** As the force is variable, we shall find the work done in a small displacement $x$ to $x + dx$ and then integrate it to find the total work. The work done in this small displacement is

$$dW = \vec{F} \cdot \vec{dx} = (10 + 0{\cdot}5\,x)\,dx.$$

Thus, $W = \int_0^{2\cdot0} (10 + 0{\cdot}50\,x)\,dx$

$$= \left[10\,x + 0{\cdot}50\,\frac{x^2}{2}\right]_0^{2\cdot0} = 21 \text{ J}.$$

**5.** *A body dropped from a height H reaches the ground with a speed of $1{\cdot}2\sqrt{gH}$. Calculate the work done by air-friction.*

***Solution*** **:** The forces acting on the body are the force of gravity and the air-friction. By work-energy theorem, the total work done on the body is

$$W = \frac{1}{2} m(1{\cdot}2\sqrt{gH})^2 - 0 = 0{\cdot}72\,mgH.$$

The work done by the force of gravity is $mgH$. Hence, the work done by the air-friction is

$$0{\cdot}72\,mgH - mgH = -\,0{\cdot}28\,mgH.$$

**6.** *A block of mass M is pulled along a horizontal surface by applying a force at an angle θ with the horizontal. The friction coefficient between the block and the surface is μ. If the block travels at a uniform velocity, find the work done by this applied force during a displacement d of the block.*

***Solution*** **:** Forces on the block are

(i) its weight $Mg$,
(ii) the normal force $\mathcal{N}$,
(iii) the applied force $F$ and
(iv) the kinetic friction $\mu\mathcal{N}$.

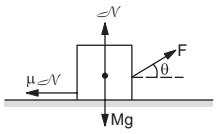



Figure 8-W2

The forces are shown in figure (8-W2). As the block moves with a uniform velocity, the forces add up to zero. Taking horizontal and vertical components,

$$F\cos\theta = \mu\mathcal{N}$$

and $$F\sin\theta + \mathcal{N} = Mg.$$

Eliminating $\mathcal{N}$ from these equations,

$$F\cos\theta = \mu\,(Mg - F\sin\theta)$$

or, $$F = \frac{\mu\,Mg}{\cos\theta + \mu\sin\theta}.$$

The work done by this force during a displacement $d$ is

$$W = F\,d\cos\theta = \frac{\mu\,Mgd\cos\theta}{\cos\theta + \mu\sin\theta}.$$

**7.** *Two cylindrical vessels of equal cross-sectional area A contain water upto heights $h_1$ and $h_2$. The vessels are interconnected so that the levels in them become equal. Calculate the work done by the force of gravity during the process. The density of water is ρ.*

***Solution*** **:** Since the total volume of the water is constant, the height in each vessel after interconnection will be $(h_1 + h_2)/2$. The level in the left vessel shown in the figure, drops from $A$ to $C$ and that in the right vessel rises from $B$ to $D$. Effectively, the water in the part $AC$ has dropped down to $DB$.

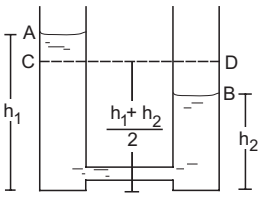



Figure 8-W3

The mass of this volume of water is

$$m = \rho A\left(h_1 - \frac{h_1 + h_2}{2}\right)$$

$$= \rho A \left( \frac{h_1 - h_2}{2} \right).$$

The height descended by this water is $AC = (h_1 - h_2)/2$. The work done by the force of gravity during this process is, therefore,

$$= \rho A \left( \frac{h_1 - h_2}{2} \right)^2 g.$$

**8.** *What minimum horizontal speed should be given to the bob of a simple pendulum of length l so that it describes a complete circle ?*

***Solution*** **:** Suppose the bob is given a horizontal speed $v_0$ at the bottom and it describes a complete vertical circle. Let its speed at the highest point be $v$. Taking the gravitational potential energy to be zero at the bottom, the conservation of energy gives,

$$\frac{1}{2} m v_0^2 = \frac{1}{2} m v^2 + 2mgl$$

or, $$m v^2 = m v_0^2 - 4\, mgl. \quad \ldots \text{(i)}$$

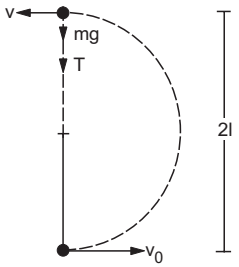



Figure 8-W4

The forces acting on the bob at the highest point are $mg$ due to the gravity and $T$ due to the tension in the string. The resultant force towards the centre is, therefore, $mg + T$. As the bob is moving in a circle, its acceleration towards the centre is $v^2/l$. Applying Newton's second law and using (i),

$$mg + T = m\frac{v^2}{l} = \frac{1}{l}(m v_0^2 - 4mgl)$$

or, $$m v_0^2 = 5\, mgl + T\, l.$$

Now, for $v_0$ to be minimum, $T$ should be minimum. As the minimum value of $T$ can be zero, for minimum speed,

$$m v_0^2 = 5\, mgl \quad \text{or,} \quad v_0 = \sqrt{5\, gl}.$$

**9.** *A uniform chain of length l and mass m overhangs a smooth table with its two third part lying on the table. Find the kinetic energy of the chain as it completely slips off the table.*

***Solution*** **:** Let us take the zero of potential energy at the table. Consider a part $dx$ of the chain at a depth $x$ below the surface of the table. The mass of this part is $dm = m/l \;\; dx$ and hence its potential energy is $-(m/l\, dx)gx$.

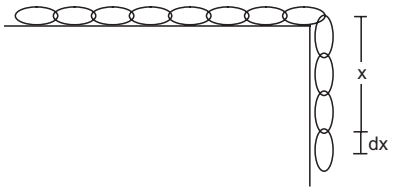



Figure 8-W5

The potential energy of the $l/3$ of the chain that overhangs is $U_1 = \int_0^{l/3} -\frac{m}{l} gx\, dx$

$$= -\left[ \frac{m}{l} g \left( \frac{x^2}{2} \right) \right]_0^{l/3} = -\frac{1}{18} mgl.$$

This is also the potential energy of the full chain in the initial position because the part lying on the table has zero potential energy. The potential energy of the chain when it completely slips off the table is

$$U_2 = \int_0^{l} -\frac{m}{l} gx\, dx = -\frac{1}{2} mgl.$$

The loss in potential energy $= \left( -\frac{1}{18} mgl \right) - \left( -\frac{1}{2} mgl \right)$

$$= \frac{4}{9} mgl.$$

This should be equal to the gain in the kinetic energy. But the initial kinetic enegry is zero. Hence, the kinetic energy of the chain as it completely slips off the table is $\frac{4}{9} mgl$.

**10.** *A block of mass m is pushed against a spring of spring constant k fixed at one end to a wall. The block can slide on a frictionless table as shown in figure (8-W6). The natural length of the spring is $L_0$ and it is compressed to half its natural length when the block is released. Find the velocity of the block as a function of its distance x from the wall.*

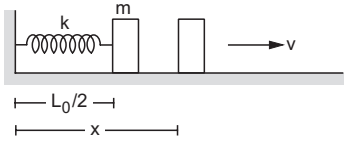



Figure 8-W6

***Solution*** **:** When the block is released, the spring pushes it towards right. The velocity of the block increases till the spring acquires its natural length. Thereafter, the block loses contact with the spring and moves with constant velocity.

Initially, the compression of the spring is $L_0/2$. When the distance of the block from the wall becomes $x$, where

$x < L_0$, the compression is $(L_0 - x)$. Using the principle of conservation of energy,

$$\frac{1}{2}k\left(\frac{L_0}{2}\right)^2 = \frac{1}{2}k\,(L_0 - x)^2 + \frac{1}{2}mv^2.$$

Solving this,

$$v = \sqrt{\frac{k}{m}}\left[\frac{L_0^2}{4} - (L_0 - x)^2\right]^{1/2}.$$

When the spring acquires its natural length, $x = L_0$ and $v = \sqrt{\frac{k}{m}}\,\frac{L_0}{2}$. Thereafter, the block continues with this velocity.

**11.** *A particle is placed at the point A of a frictionless track ABC as shown in figure (8-W7). It is pushed slightly towards right. Find its speed when it reaches the point B. Take* $g = 10$ m/s$^2$.

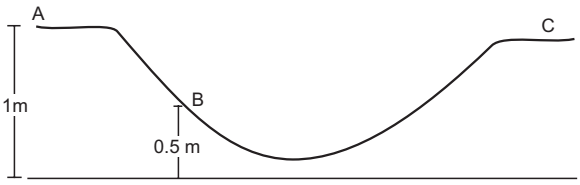



Figure 8-W7

***Solution*** : Let us take the gravitational potential energy to be zero at the horizontal surface shown in the figure. The potential energies of the particle at $A$ and $B$ are

$$U_A = Mg\ (1\text{ m})$$

and $$U_B = Mg\ (0{\cdot}5\text{ m}).$$

The kinetic energy at the point $A$ is zero. As the track is frictionless, no energy is lost. The normal force on the particle does no work. Applying the principle of conservation of energy,

$$U_A + K_A = U_B + K_B$$

or, $$Mg(1\text{ m}) = Mg(0{\cdot}5\text{ m}) + \frac{1}{2}Mv_B^2$$

or, $$\frac{1}{2}v_B^2 = g(1\text{ m} - 0{\cdot}5\text{ m})$$

$$= (10\text{ m/s}^2) \times 0{\cdot}5\text{ m}$$

$$= 5\text{ m}^2/\text{s}^2$$

or, $$v_B = \sqrt{10}\text{ m/s}.$$

**12.** *Figure (8-W8) shows a smooth curved track terminating in a smooth horizontal part. A spring of spring constant* 400 N/m *is attached at one end to a wedge fixed rigidly with the horizontal part. A* 40 g *mass is released from rest at a height* of 4·9 m *on the curved track. Find the maximum compression of the spring.*

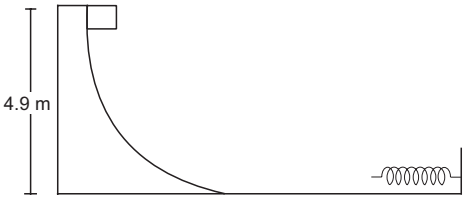



Figure 8-W8

***Solution*** : At the instant of maximum compression the speed of the 40 g mass reduces to zero. Taking the gravitational potential energy to be zero at the horizontal part, the conservation of energy shows,

$$mgh = \frac{1}{2}kx^2$$

where $m$ = 0·04 kg, $h$ = 4·9 m, $k$ = 400 N/m and $x$ is the maximum compression.

Thus, $$x = \sqrt{\frac{2mgh}{k}}$$

$$= \sqrt{\frac{2 \times (0{\cdot}04\text{ kg}) \times (9{\cdot}8\text{ m/s}^2) \times (4{\cdot}9\text{ m})}{(400\text{ N/m})}}$$

$$= 9{\cdot}8\text{ cm}.$$

**13.** *Figure (8-W9) shows a loop-the-loop track of radius R. A car (without engine) starts from a platform at a distance h above the top of the loop and goes around the loop without falling off the track. Find the minimum value of h for a successful looping. Neglect friction.*

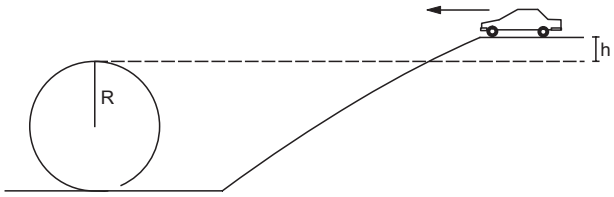



Figure 8-W9

***Solution*** : Suppose the speed of the car at the topmost point of the loop is $v$. Taking the gravitational potential energy to be zero at the platform and assuming that the car starts with a negligible speed, the conservation of energy shows,

$$0 = -mgh + \frac{1}{2}mv^2$$

or, $$mv^2 = 2\,mgh, \qquad \ldots\ (i)$$

where $m$ is the mass of the car. The car moving in a circle must have radial acceleration $v^2/R$ at this instant. The forces on the car are, $mg$ due to gravity and $\mathcal{N}$ due to the contact with the track. Both these forces are in radial direction at the top of the loop. Thus, from Newton's Law

$$mg + \mathcal{N} = \frac{mv^2}{R}$$

or, $mg + \mathcal{N} = 2\,mgh/R.$

For $h$ to be minimum, $\mathcal{N}$ should assume the minimum value which can be zero. Thus,

$$2\,mg\,\frac{h_{\min}}{R} = mg \quad \text{or,} \quad h_{\min} = R/2.$$

**14.** *A heavy particle is suspended by a string of length l. The particle is given a horizontal velocity $v_0$. The string becomes slack at some angle and the particle proceeds on a parabola. Find the value of $v_0$ if the particle passes through the point of suspension.*

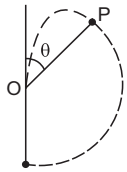



Figure 8-W10

***Solution*** **:** Suppose the string becomes slack when the particle reaches the point $P$ (figure 8-W10). Suppose the string $OP$ makes an angle θ with the upward vertical. The only force acting on the particle at $P$ is its weight $mg$. The radial component of the force is $mg\cos\theta$. As the particle moves on the circle upto $P$,

$$mg\cos\theta = m\left(\frac{v^2}{l}\right)$$

or, $$v^2 = gl\cos\theta \qquad \ldots \text{(i)}$$

where $v$ is its speed at $P$. Using conservation of energy,

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + mgl\,(1+\cos\theta)$$

or, $$v^2 = v_0^2 - 2gl(1+\cos\theta). \qquad \ldots \text{(ii)}$$

From (i) and (ii), $v_0^2 - 2\,g\,l\,(1+\cos\theta) = g\,l\cos\theta$

or, $$v_0^2 = gl\,(2+3\cos\theta). \qquad \ldots \text{(iii)}$$

Now onwards the particle goes in a parabola under the action of gravity. As it passes through the point of suspension $O$, the equations for horizontal and vertical motions give,

$$l\sin\theta = (v\cos\theta)\,t$$

and $$-l\cos\theta = (v\sin\theta)\,t - \frac{1}{2}gt^2$$

or, $$-l\cos\theta = (v\sin\theta)\left(\frac{l\sin\theta}{v\cos\theta}\right) - \frac{1}{2}g\left(\frac{l\sin\theta}{v\cos\theta}\right)^2$$

or, $$-\cos^2\theta = \sin^2\theta - \frac{1}{2}g\,\frac{l\sin^2\theta}{v^2\cos\theta}$$

or, $$-\cos^2\theta = 1 - \cos^2\theta - \frac{1}{2}\frac{gl\sin^2\theta}{gl\cos^2\theta}$$ [From (i)]

or, $$1 = \frac{1}{2}\tan^2\theta$$

or, $$\tan\theta = \sqrt{2}.$$

From (iii), $v_0 = [gl\,(2+\sqrt{3})]^{1/2}$.

□

## QUESTIONS FOR SHORT ANSWER

**1.** When you lift a box from the floor and put it on an almirah the potential energy of the box increases, but there is no change in its kinetic energy. Is it a violation of conservation of energy ?

**2.** A particle is released from the top of an incline of height $h$. Does the kinetic energy of the particle at the bottom of the incline depend on the angle of incline ? Do you need any more information to answer this question in Yes or No ?

**3.** Can the work by kinetic friction on an object be positive ? Zero ?

**4.** Can static friction do nonzero work on an object ? If yes, give an example. If no, give reason.

**5.** Can normal force do a nonzero work on an object. If yes, give an example. If no, give reason.

**6.** Can kinetic energy of a system be increased without applying any external force on the system ?

**7.** Is work-energy theorem valid in noninertial frames ?

**8.** A heavy box is kept on a smooth inclined plane and is pushed up by a force $F$ acting parallel to the plane. Does the work done by the force $F$ as the box goes from $A$ to $B$ depend on how fast the box was moving at $A$ and $B$ ? Does the work by the force of gravity depend on this ?

**9.** One person says that the potential energy of a particular book kept in an almirah is 20 J and the other says it is 30 J. Is one of them necessarily wrong ?

**10.** A book is lifted from the floor and is kept in an almirah. One person says that the potential energy of the book is increased by 20 J and the other says it is increased by 30 J. Is one of them necessarily wrong ?

**11.** In one of the exercises to strengthen the wrist and fingers, a person squeezes and releases a soft rubber ball. Is the work done on the ball positive, negative or zero during compression ? During expansion ?

**12.** In tug of war, the team that exerts a larger tangential force on the ground wins. Consider the period in which a team is dragging the opposite team by applying a larger tangential force on the ground. List which of the

following works are positive, which are negative and which are zero ?
(a) work by the winning team on the losing team
(b) work by the losing team on the winning team
(c) work by the ground on the winning team
(d) work by the ground on the losing team
(e) total external work on the two teams.

**13.** When an apple falls from a tree what happens to its gravitational potential energy just as it reaches the ground ? After it strikes the ground ?

**14.** When you push your bicycle up on an incline the potential energy of the bicyle and yourself increases. Where does this energy come from ?

**15.** The magnetic force on a charged particle is always perpendicular to its velocity. Can the magnetic force change the velocity of the particle ? Speed of the particle ?

**16.** A ball is given a speed $v$ on a rough horizontal surface. The ball travels through a distance $l$ on the surface and stops. (a) What are the initial and final kinetic energies of the ball ? (b) What is the work done by the kinetic friction ?

**17.** Consider the situation of the previous question from a frame moving with a speed $v_0$ parallel to the initial velocity of the block. (a) What are the initial and final kinetic energies ? (b) What is the work done by the kinetic friction ?

## OBJECTIVE I

**1.** A heavy stone is thrown from a cliff of height $h$ with a speed $v$. The stone will hit the ground with maximum speed if it is thrown
(a) vertically downward (b) vertically upward
(c) horizontally
(d) the speed does not depend on the initial direction.

**2.** Two springs $A$ and $B(k_A = 2k_B)$ are stretched by applying forces of equal magnitudes at the four ends. If the energy stored in $A$ is $E$, that in $B$ is
(a) $E/2$ (b) $2E$ (c) $E$ (d) $E/4$.

**3.** Two equal masses are attached to the two ends of a spring of spring constant $k$. The masses are pulled out symmetrically to stretch the spring by a length $x$ over its natural length. The work done by the spring on each mass is
(a) $\frac{1}{2}kx^2$ (b) $-\frac{1}{2}kx^2$ (c) $\frac{1}{4}kx^2$ (d) $-\frac{1}{4}kx^2$.

**4.** The negative of the work done by the conservative internal forces on a system equals the change in
(a) total energy (b) kinetic energy
(c) potential energy (d) none of these.

**5.** The work done by the external forces on a system equals the change in
(a) total energy (b) kinetic energy
(c) potential energy (d) none of these.

**6.** The work done by all the forces (external and internal) on a system equals the change in
(a) total energy (b) kinetic energy
(c) potential energy (d) none of these.

**7.** _____ of a two particle system depends only on the separation between the two particles. The most appropriate choice for the blank space in the above sentence is
(a) Kinetic energy (b) Total mechanical energy
(c) Potential energy (d) Total energy.

**8.** A small block of mass $m$ is kept on a rough inclined surface of inclination $\theta$ fixed in an elevator. The elevator goes up with a uniform velocity $v$ and the block does not slide on the wedge. The work done by the force of friction on the block in time $t$ will be
(a) zero (b) $mgvt\cos^2\theta$
(c) $mgvt\sin^2\theta$ (d) $mgvt\sin 2\theta$.

**9.** A block of mass $m$ slides down a smooth vertical circular track. During the motion, the block is in
(a) vertical equilibrium (b) horizontal equilibrium
(c) radial equilibrium (d) none of these.

**10.** A particle is rotated in a vertical circle by connecting it to a string of length $l$ and keeping the other end of the string fixed. The minimum speed of the particle when the string is horizontal for which the particle will complete the circle is
(a) $\sqrt{gl}$ (b) $\sqrt{2gl}$ (c) $\sqrt{3gl}$ (d) $\sqrt{5gl}$.

## OBJECTIVE II

**1.** A heavy stone is thrown from a cliff of height $h$ in a given direction. The speed with which it hits the ground
(a) must depend on the speed of projection
(b) must be larger than the speed of projection
(c) must be independent of the speed of projection
(d) may be smaller than the speed of projection.

**2.** The total work done on a particle is equal to the change in its kinetic energy
(a) always

(b) only if the forces acting on it are conservative
(c) only if gravitational force alone acts on it
(d) only if elastic force alone acts on it.

**3.** A particle is acted upon by a force of constant magnitude which is always perpendicular to the velocity of the particle. The motion of the particle takes place in a plane. It follows that
(a) its velocity is constant
(b) its acceleration is constant
(c) its kinetic energy is constant
(d) it moves in a circular path.

**4.** Consider two observers moving with respect to each other at a speed $v$ along a straight line. They observe a block of mass $m$ moving a distance $l$ on a rough surface. The following quantities will be same as observed by the two observers
(a) kinetic energy of the block at time $t$
(b) work done by friction
(c) total work done on the block
(d) acceleration of the block.

**5.** You lift a suitcase from the floor and keep it on a table. The work done by you on the suitcase does not depend on
(a) the path taken by the suitcase
(b) the time taken by you in doing so
(c) the weight of the suitcase
(d) your weight.

**6.** No work is done by a force on an object if
(a) the force is always perpendicular to its velocity
(b) the force is always perpendicular to its acceleration
(c) the object is stationary but the point of application of the force moves on the object
(d) the object moves in such a way that the point of application of the force remains fixed.

**7.** A particle of mass $m$ is attached to a light string of length $l$, the other end of which is fixed. Initially the string is kept horizontal and the particle is given an upward velocity $v$. The particle is just able to complete a circle.
(a) The string becomes slack when the particle reaches its highest point.
(b) The velocity of the particle becomes zero at the highest point.
(c) The kinetic energy of the ball in initial position was $\frac{1}{2}mv^2 = mgl$.
(d) The particle again passes through the initial position.

**8.** The kinetic energy of a particle continuously increases with time.
(a) The resultant force on the particle must be parallel to the velocity at all instants.
(b) The resultant force on the particle must be at an angle less than 90° all the time.
(c) Its height above the ground level must continuously decrease.
(d) The magnitude of its linear momentum is increasing continuously.

**9.** One end of a light spring of spring constant $k$ is fixed to a wall and the other end is tied to a block placed on a smooth horizontal surface. In a displacement, the work done by the spring is $\frac{1}{2}kx^2$. The possible cases are
(a) the spring was initially compressed by a distance $x$ and was finally in its natural length
(b) it was initially stretched by a distance $x$ and finally was in its natural length
(c) it was initially in its natural length and finally in a compressed position
(d) it was initially in its natural length and finally in a stretched position.

**10.** A block of mass $M$ is hanging over a smooth and light pulley through a light string. The other end of the string is pulled by a constant force $F$. The kinetic energy of the block increases by 20 J in 1 s.
(a) The tension in the string is $Mg$.
(b) The tension in the string is $F$.
(c) The work done by the tension on the block is 20 J in the above 1 s.
(d) The work done by the force of gravity is –20 J in the above 1 s.

## EXERCISES

**1.** The mass of cyclist together with the bike is 90 kg. Calculate the increase in kinetic energy if the speed increases from 6·0 km/h to 12 km/h.

**2.** A block of mass 2·00 kg moving at a speed of 10·0 m/s accelerates at 3·00 m/s$^2$ for 5·00 s. Compute its final kinetic energy.

**3.** A box is pushed through 4·0 m across a floor offering 100 N resistance. How much work is done by the resisting force ?

**4.** A block of mass 5·0 kg slides down an incline of inclination 30° and length 10 m. Find the work done by the force of gravity.

**5.** A constant force of 2·50 N accelerates a stationary particle of mass 15 g through a displacement of 2·50 m. Find the work done and the average power delivered.

**6.** A particle moves from a point $\vec{r}_1 = (2\text{ m})\,\vec{i} + (3\text{ m})\,\vec{j}$ to another point $\vec{r}_2 = (3\text{ m})\,\vec{i} + (2\text{ m})\,\vec{j}$ during which a certain force $\vec{F} = (5\text{ N})\,\vec{i} + (5\text{ N})\,\vec{j}$ acts on it. Find the work done by the force on the particle during the displacement.

**7.** A man moves on a straight horizontal road with a block of mass 2 kg in his hand. If he covers a distance of 40 m with an acceleration of 0·5 m/s$^2$, find the work done by the man on the block during the motion.

**8.** A force $F = a + bx$ acts on a particle in the $x$-direction, where $a$ and $b$ are constants. Find the work done by this force during a displacement from $x = 0$ to $x = d$.

**9.** A block of mass 250 g slides down an incline of inclination 37° with a uniform speed. Find the work done against the friction as the block slides through 1·0 m.

**10.** A block of mass $m$ is kept over another block of mass $M$ and the system rests on a horizontal surface (figure 8-E1). A constant horizontal force $F$ acting on the lower block produces an acceleration $\frac{F}{2(m+M)}$ in the system, the two blocks always move together. (a) Find the coefficient of kinetic friction between the bigger block and the horizontal surface. (b) Find the frictional force acting on the smaller block. (c) Find the work done by the force of friction on the smaller block by the bigger block during a displacement $d$ of the system.

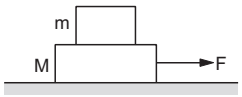



Figure 8-E1

**11.** A box weighing 2000 N is to be slowly slid through 20 m on a straight track having friction coefficient 0·2 with the box. (a) Find the work done by the person pulling the box with a chain at an angle θ with the horizontal. (b) Find the work when the person has chosen a value of θ which ensures him the minimum magnitude of the force.

**12.** A block of weight 100 N is slowly slid up on a smooth incline of inclination 37° by a person. Calculate the work done by the person in moving the block through a distance of 2·0 m, if the driving force is (a) parallel to the incline and (b) in the horizontal direction.

**13.** Find the average frictional force needed to stop a car weighing 500 kg in a distance of 25 m if the initial speed is 72 km/h.

**14.** Find the average force needed to accelerate a car weighing 500 kg from rest to 72 km/h in a distance of 25 m.

**15.** A particle of mass $m$ moves on a straight line with its velocity varying with the distance travelled according to the equation $v = a\sqrt{x}$, where $a$ is a constant. Find the total work done by all the forces during a displacement from $x = 0$ to $x = d$.

**16.** A block of mass 2·0 kg kept at rest on an inclined plane of inclination 37° is pulled up the plane by applying a constant force of 20 N parallel to the incline. The force acts for one second. (a) Show that the work done by the applied force does not exceed 40 J. (b) Find the work done by the force of gravity in that one second if the work done by the applied force is 40 J. (c) Find the kinetic energy of the block at the instant the force ceases to act. Take $g = 10$ m/s$^2$.

**17.** A block of mass 2·0 kg is pushed down an inclined plane of inclination 37° with a force of 20 N acting parallel to the incline. It is found that the block moves on the incline with an acceleration of 10 m/s$^2$. If the block started from rest, find the work done (a) by the applied force in the first second, (b) by the weight of the block in the first second and (c) by the frictional force acting on the block in the first second. Take $g = 10$ m/s$^2$.

**18.** A 250 g block slides on a rough horizontal table. Find the work done by the frictional force in bringing the block to rest if it is initially moving at a speed of 40 cm/s. If the friction coefficient between the table and the block is 0·1, how far does the block move before coming to rest ?

**19.** Water falling from a 50 m high fall is to be used for generating electric energy. If $1{\cdot}8 \times 10^5$ kg of water falls per hour and half the gravitational potential energy can be converted into electric energy, how many 100 W lamps can be lit ?

**20.** A person is painting his house walls. He stands on a ladder with a bucket containing paint in one hand and a brush in other. Suddenly the bucket slips from his hand and falls down on the floor. If the bucket with the paint had a mass of 6·0 kg and was at a height of 2·0 m at the time it slipped, how much gravitational potential energy is lost together with the paint ?

**21.** A projectile is fired from the top of a 40 m high cliff with an initial speed of 50 m/s at an unknown angle. Find its speed when it hits the ground.

**22.** The 200 m free style women's swimming gold medal at Seol Olympic 1988 went to Heike Friendrich of East Germany when she set a new Olympic record of 1 minute and 57·56 seconds. Assume that she covered most of the distance with a uniform speed and had to exert 460 W to maintain her speed. Calculate the average force of resistance offered by the water during the swim.

**23.** The US athlete Florence Griffith-Joyner won the 100 m sprint gold medal at Seol Olympic 1988 setting a new Olympic record of 10·54 s. Assume that she achieved her maximum speed in a very short-time and then ran the race with that speed till she crossed the line. Take her mass to be 50 kg. (a) Calculate the kinetic energy of Griffith-Joyner at her full speed. (b) Assuming that the track, the wind etc. offered an average resistance of one tenth of her weight, calculate the work done by the resistance during the run. (c) What power Griffith-Joyner had to exert to maintain uniform speed ?

**24.** A water pump lifts water from a level 10 m below the ground. Water is pumped at a rate of 30 kg/minute with negligible velocity. Calculate the minimum horsepower the engine should have to do this.

**25.** An unruly demonstrator lifts a stone of mass 200 g from the ground and throws it at his opponent. At the time of projection, the stone is 150 cm above the ground and has a speed of 3·00 m/s. Calculate the work done by the demonstrator during the process. If it takes one second for the demonstrator to lift the stone and throw, what horsepower does he use ?

**26.** In a factory it is desired to lift 2000 kg of metal through a distance of 12 m in 1 minute. Find the minimum horsepower of the engine to be used.

**27.** A scooter company gives the following specifications about its product.

Weight of the scooter — 95 kg
Maximum speed — 60 km/h
Maximum engine power — 3·5 hp
Pick up time to get the maximum speed — 5 s
Check the validity of these specifications.

**28.** A block of mass 30·0 kg is being brought down by a chain. If the block acquires a speed of 40·0 cm/s in dropping down 2·00 m, find the work done by the chain during the process.

**29.** The heavier block in an Atwood machine has a mass twice that of the lighter one. The tension in the string is 16·0 N when the system is set into motion. Find the decrease in the gravitational potential energy during the first second after the system is released from rest.

**30.** The two blocks in an Atwood machine have masses 2·0 kg and 3·0 kg. Find the work done by gravity during the fourth second after the system is released from rest.

**31.** Consider the situation shown in figure (8-E2). The system is released from rest and the block of mass 1·0 kg is found to have a speed 0·3 m/s after it has descended through a distance of 1 m. Find the coefficient of kinetic friction between the block and the table.

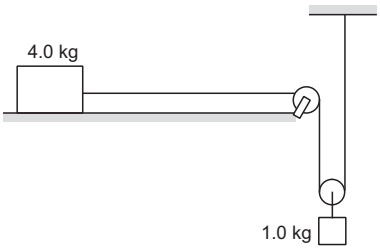



Figure 8-E2

**32.** A block of mass 100 g is moved with a speed of 5·0 m/s at the highest point in a closed circular tube of radius 10 cm kept in a vertical plane. The cross-section of the tube is such that the block just fits in it. The block makes several oscillations inside the tube and finally stops at the lowest point. Find the work done by the tube on the block during the process.

**33.** A car weighing 1400 kg is moving at a speed of 54 km/h up a hill when the motor stops. If it is just able to reach the destination which is at a height of 10 m above the point, calculate the work done against friction (negative of the work done by the friction).

**34.** A small block of mass 200 g is kept at the top of a frictionless incline which is 10 m long and 3·2 m high. How much work was required (a) to lift the block from the ground and put it at the top, (b) to slide the block up the incline ? What will be the speed of the block when it reaches the ground, if (c) it falls off the incline and drops vertically on the ground (d) it slides down the incline ? Take $g = 10\ m/s^2$.

**35.** In a children's park, there is a slide which has a total length of 10 m and a height of 8·0 m (figure 8-E3). Vertical ladder are provided to reach the top. A boy weighing 200 N climbs up the ladder to the top of the slide and slides down to the ground. The average friction offered by the slide is three tenth of his weight. Find (a) the work done by the boy on the ladder as he goes up, (b) the work done by the slide on the boy as he comes down, (c) the work done by the ladder on the boy as he goes up. Neglect any work done by forces inside the body of the boy.

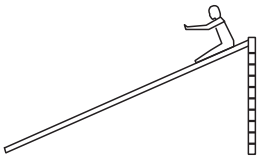


Figure 8-E3

**36.** Figure (8-E4) shows a particle sliding on a frictionless track which terminates in a straight horizontal section. If the particle starts slipping from the point *A*, how far away from the track will the particle hit the ground ?

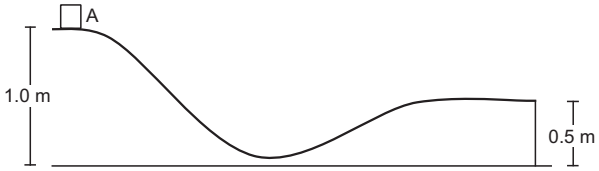



Figure 8-E4

**37.** A block weighing 10 N travels down a smooth curved track *AB* joined to a rough horizontal surface (figure 8-E5). The rough surface has a friction coefficient of 0·20 with the block. If the block starts slipping on the track from a point 1·0 m above the horizontal surface, how far will it move on the rough surface ?

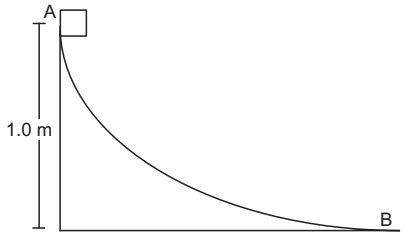



Figure 8-E5

**38.** A uniform chain of mass $m$ and length $l$ overhangs a table with its two third part on the table. Find the work to be done by a person to put the hanging part back on the table.

**39.** A uniform chain of length $L$ and mass $M$ overhangs a horizontal table with its two third part on the table. The friction coefficient between the table and the chain is $\mu$. Find the work done by the friction during the period the chain slips off the table.

**40.** A block of mass 1 kg is placed at the point *A* of a rough track shown in figure (8-E6). If slightly pushed towards right, it stops at the point *B* of the track. Calculate the work done by the frictional force on the block during its transit from *A* to *B*.

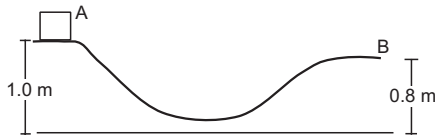



Figure 8-E6

**41.** A block of mass 5·0 kg is suspended from the end of a vertical spring which is stretched by 10 cm under the load of the block. The block is given a sharp impulse from below so that it acquires an upward speed of 2·0 m/s. How high will it rise ? Take $g = 10$ m/s$^2$.

**42.** A block of mass 250 g is kept on a vertical spring of spring constant 100 N/m fixed from below. The spring is now compressed to have a length 10 cm shorter than its natural length and the system is released from this position. How high does the block rise ? Take $g = 10$ m/s$^2$.

**43.** Figure (8-E7) shows a spring fixed at the bottom end of an incline of inclination 37°. A small block of mass 2 kg starts slipping down the incline from a point 4·8 m away from the spring. The block compresses the spring by 20 cm, stops momentarily and then rebounds through a distance of 1 m up the incline. Find (a) the friction coefficient between the plane and the block and (b) the spring constant of the spring. Take $g = 10$ m/s$^2$.

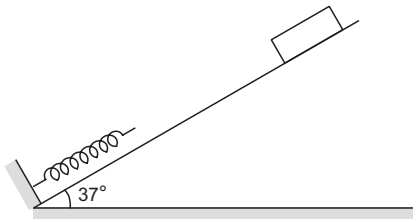



Figure 8-E7

**44.** A block of mass $m$ moving at a speed $v$ compresses a spring through a distance $x$ before its speed is halved. Find the spring constant of the spring.

**45.** Consider the situation shown in figure (8-E8). Initially the spring is unstretched when the system is released from rest. Assuming no friction in the pulley, find the maximum elongation of the spring.

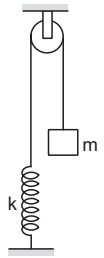



Figure 8-E8

**46.** A block of mass $m$ is attached to two unstretched springs of spring constants $k_1$ and $k_2$ as shown in figure (8-E9). The block is displaced towards right through a distance $x$ and is released. Find the speed of the block as it passes through the mean position shown.

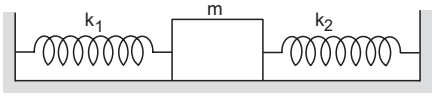



Figure 8-E9

**47.** A block of mass $m$ sliding on a smooth horizontal surface with a velocity $\vec{v}$ meets a long horizontal spring fixed at one end and having spring constant $k$ as shown in figure (8-E10). Find the maximum compression of the spring. Will the velocity of the block be the same as $\vec{v}$ when it comes back to the original position shown ?

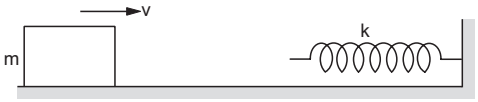



Figure 8-E10

**48.** A small block of mass 100 g is pressed against a horizontal spring fixed at one end to compress the spring through 5·0 cm (figure 8-E11). The spring constant is 100 N/m. When released, the block moves horizontally till it leaves the spring. Where will it hit the ground 2 m below the spring ?

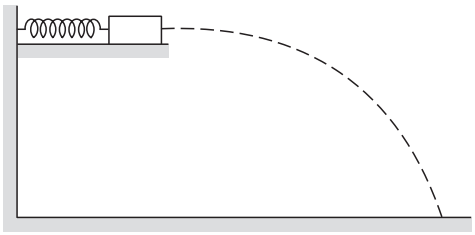


Figure 8-E11

**49.** A small heavy block is attached to the lower end of a light rod of length $l$ which can be rotated about its clamped upper end. What minimum horizontal velocity should the block be given so that it moves in a complete vertical circle ?

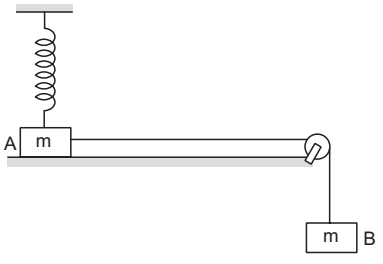



Figure 8-E12

**50.** Figure (8-E12) shows two blocks $A$ and $B$, each having a mass of 320 g connected by a light string passing over a smooth light pulley. The horizontal surface on which the block $A$ can slide is smooth. The block $A$ is attached

to a spring of spring constant 40 N/m whose other end is fixed to a support 40 cm above the horizontal surface. Initially, the spring is vertical and unstretched when the system is released to move. Find the velocity of the block $A$ at the instant it breaks off the surface below it. Take $g = 10\ \text{m/s}^2$.

**51.** One end of a spring of natural length $h$ and spring constant $k$ is fixed at the ground and the other is fitted with a smooth ring of mass $m$ which is allowed to slide on a horizontal rod fixed at a height $h$ (figure 8-E13). Initially, the spring makes an angle of 37° with the vertical when the system is released from rest. Find the speed of the ring when the spring becomes vertical.

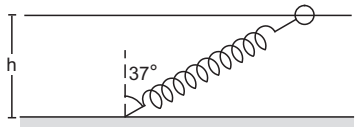



Figure 8-E13

**52.** Figure (8-E14) shows a light rod of length $l$ rigidly attached to a small heavy block at one end and a hook at the other end. The system is released from rest with the rod in a horizontal position. There is a fixed smooth ring at a depth $h$ below the initial position of the hook and the hook gets into the ring as it reaches there. What should be the minimum value of $h$ so that the block moves in a complete circle about the ring ?

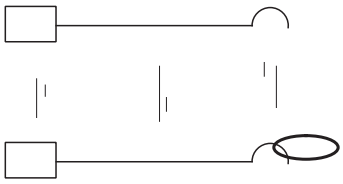


Figure 8-E14

**53.** The bob of a pendulum at rest is given a sharp hit to impart a horizontal velocity $\sqrt{10\,gl}$, where $l$ is the length of the pendulum. Find the tension in the string when (a) the string is horizontal, (b) the bob is at its highest point and (c) the string makes an angle of 60° with the upward vertical.

**54.** A simple pendulum consists of a 50 cm long string connected to a 100 g ball. The ball is pulled aside so that the string makes an angle of 37° with the vertical and is then released. Find the tension in the string when the bob is at its lowest position.

Figure 8-E15

**55.** Figure (8-E15) shows a smooth track, a part of which is a circle of radius $R$. A block of mass $m$ is pushed against a spring of spring constant $k$ fixed at the left end and is then released. Find the initial compression of the spring so that the block presses the track with a force $mg$ when it reaches the point $P$, where the radius of the track is horizontal.

**56.** The bob of a stationary pendulum is given a sharp hit to impart it a horizontal speed of $\sqrt{3\,gl}$. Find the angle rotated by the string before it becomes slack.

**57.** A heavy particle is suspended by a 1·5 m long string. It is given a horizontal velocity of $\sqrt{57}$ m/s. (a) Find the angle made by the string with the upward vertical, when it becomes slack. (b) Find the speed of the particle at this instant. (c) Find the maximum height reached by the particle over the point of suspension. Take $g = 10\ \text{m/s}^2$.

**58.** A simple pendulum of length $L$ having a bob of mass $m$ is deflected from its rest position by an angle $\theta$ and released (figure 8-E16). The string hits a peg which is fixed at a distance $x$ below the point of suspension and the bob starts going in a circle centred at the peg. (a) Assuming that initially the bob has a height less than the peg, show that the maximum height reached by the bob equals its initial height. (b) If the pendulum is released with $\theta = 90°$ and $x = L/2$ find the maximum height reached by the bob above its lowest position before the string becomes slack. (c) Find the minimum value of $x/L$ for which the bob goes in a complete circle about the peg when the pendulum is released from $\theta = 90°$.

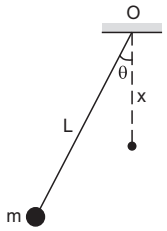



Figure 8-E16

**59.** A particle slides on the surface of a fixed smooth sphere starting from the topmost point. Find the angle rotated by the radius through the particle, when it leaves contact with the sphere.

**60.** A particle of mass $m$ is kept on a fixed, smooth sphere of radius $R$ at a position, where the radius through the particle makes an angle of 30° with the vertical. The particle is released from this position. (a) What is the force exerted by the sphere on the particle just after the release ? (b) Find the distance travelled by the particle before it leaves contact with the sphere.

**61.** A particle of mass $m$ is kept on the top of a smooth sphere of radius $R$. It is given a sharp impulse which imparts it a horizontal speed $v$. (a) Find the normal force between the sphere and the particle just after the impulse. (b) What should be the minimum value of $v$ for which the particle does not slip on the sphere ? (c) Assuming the velocity $v$ to be half the minimum calculated in part (b) find the angle made by the radius

through the particle with the vertical when it leaves the sphere.

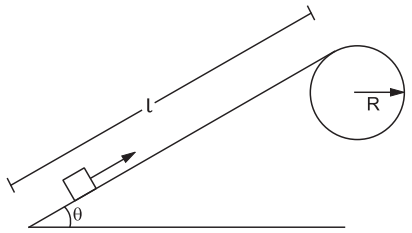



Figure 8-E17

**62.** Figure (8-E17) shows a smooth track which consists of a straight inclined part of length $l$ joining smoothly with the circular part. A particle of mass $m$ is projected up the incline from its bottom. (a) Find the minimum projection-speed $v_0$ for which the particle reaches the top of the track. (b) Assuming that the projection-speed is $2v_0$ and that the block does not lose contact with the track before reaching its top, find the force acting on it when it reaches the top. (c) Assuming that the projection-speed is only slightly greater than $v_0$, where will the block lose contact with the track ?

**63.** A chain of length $l$ and mass $m$ lies on the surface of a smooth sphere of radius $R > l$ with one end tied to the top of the sphere. (a) Find the gravitational potential energy of the chain with reference level at the centre of the sphere. (b) Suppose the chain is released and slides down the sphere. Find the kinetic energy of the chain, when it has slid through an angle θ. (c) Find the tangential acceleration $\frac{dv}{dt}$ of the chain when the chain starts sliding down.

**64.** A smooth sphere of radius $R$ is made to translate in a straight line with a constant acceleration $a$. A particle kept on the top of the sphere is released from there at zero velocity with respect to the sphere. Find the speed of the particle with respect to the sphere as a function of the angle θ it slides.

□

## ANSWERS

### OBJECTIVE I

1. (d) 2. (b) 3. (d) 4. (c) 5. (a) 6. (b)
7. (c) 8. (c) 9. (d) 10. (c)

### OBJECTIVE II

1. (a), (b) 2. (a) 3. (c), (d)
4. (d) 5. (a), (b), (d) 6. (a), (c), (d)
7. (a), (d) 8. (b), (d) 9. (a), (b)
10. (b)

### EXERCISES

1. 375 J
2. 625 J
3. – 400 J
4. 245 J
5. 6·25 J, 36·1 W
6. zero
7. 40 J
8. $\left(a + \frac{1}{2} bd\right) d$
9. 1·5 J
10. (a) $\frac{F}{2(M+m)g}$ (b) $\frac{mF}{2(M+m)}$ (c) $\frac{mFd}{2(M+m)}$
11. (a) $\frac{40000\ \text{J}}{5 + \tan\theta}$ (b) 7690 J
12. (a) 120 J (b) 120 J
13. 4000 N
14. 4000 N
15. $ma^2d/2$
16. (b) – 24 J (c) 16 J
17. (a) 100 J (b) 60 J (c) – 60 J
18. – 0·02 J, 8·2 cm
19. 122
20. 118 J
21. 58 m/s
22. 270 N
23. (a) 2250 J (b) – 4900 J (c) 465 W
24. $6{\cdot}6 \times 10^{-2}$ hp
25. 3·84 J, $5{\cdot}14 \times 10^{-3}$ hp
26. 5·3 hp
27. Seems to be somewhat overclaimed.
28. – 586 J
29. 19·6 J
30. 67 J
31. 0·12
32. – 1·45 J
33. 20300 J
34. (a) 6·4 J (b) 6·4 J (c) 8·0 m/s (d) 8·0 m/s
35. (a) zero (b) – 600 J (c) 1600 J
36. At a horizontal distance of 1 m from the end of the track.
37. 5·0 m

38. $mgl/18$
39. $-2\mu MgL/9$
40. $-2$ J
41. 20 cm
42. 20 cm
43. (a) 0·5 (b) 1000 N/m
44. $\frac{3mv^2}{4x^2}$
45. $2mg/k$
46. $\sqrt{\frac{k_1+k_2}{m}}\,x$
47. $v\sqrt{m/k}$, No
48. At a horizontal distance of 1 m from the free end of the spring.
49. $2\sqrt{gl}$
50. 1·5 m/s
51. $\frac{h}{4}\sqrt{k/m}$
52. $l$
53. (a) $8mg$ (b) $5mg$ (c) $6{\cdot}5mg$
54. 1·4 N
55. $\sqrt{\frac{3mgR}{k}}$
56. $\cos^{-1}(-1/3)$
57. (a) 53° (b) 3·0 m/s (c) 1·2 m
58. (b) $5L/6$ above the lowest point (c) 0·6
59. $\cos^{-1}(2/3)$
60. $\sqrt{3}\,mg/2$ (b) $0{\cdot}43\,R$
61. (a) $mg-\frac{mv^2}{R}$ (b) $\sqrt{Rg}$ (c) $\cos^{-1}(3/4)$
62. (a) $\sqrt{2g[R(1-\cos\theta)+l\sin\theta]}$ (b) $6mg\left(1-\cos\theta+\frac{l}{R}\sin\theta\right)$
    (c) The radius through the particle makes an angle $\cos^{-1}(2/3)$ with the vertical.
63. (a) $\frac{mR^2g}{l}\sin(l/R)$
    (b) $\frac{mR^2g}{l}\left[\sin\left(\frac{l}{R}\right)+\sin\theta-\sin\left(\theta+\frac{l}{R}\right)\right]$
    (c) $\frac{Rg}{l}[1-\cos(l/R)]$
64. $[2R(a\sin\theta+g-g\cos\theta)]^{1/2}$

□

## CHAPTER 9

# CENTRE OF MASS, LINEAR MOMENTUM, COLLISION

### 9.1 CENTRE OF MASS

Suppose a spin bowler throws a cricket ball vertically upward. Being a spinner, his fingers turn while throwing the ball and the ball goes up spinning rapidly. Focus your attention to a particular point on the surface of the ball. How does it move in the space ? Because the ball is spinning as well as rising up, in general, the path of a particle at the surface is complicated, not confined to a straight line or to a plane. The centre of the ball, however, still goes on the vertical straight line and the spinner's fingers could not make its path complicated. If he does not throw the ball vertically up, rather passes it to his fellow fielder, the centre of the ball goes in a parabola.

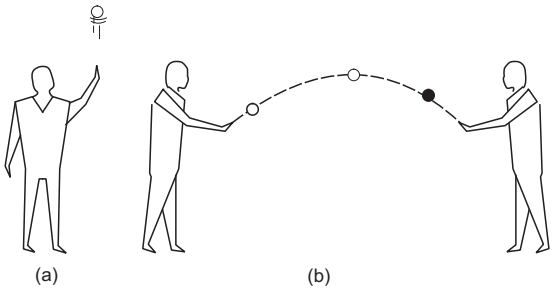



Figure 9.1

All the points of the ball do not go in parabolic paths. If the ball is spinning, the paths of most of the particles of the ball are complicated. But the centre of the ball always goes in a parabola irrespective of how the ball is thrown. (In fact the presence of air makes the path of the centre slightly different from a parabola and bowlers utilise this deviation. This effect will be discussed in a later chapter. At present we neglect it.)

The centre of the ball is a very special point which is called the *centre of mass* of the ball. Its motion is just like the motion of a single particle thrown.

**Definition of Centre of Mass**

Let us consider a collection of $N$ particles (Figure 9.2). Let the mass of the $i$th particle be $m_i$ and its coordinates with reference to the chosen axes be $x_i$ , $y_i$ , $z_i$ . Write the product $m_i x_i$ for each of the particles and add them to get $\sum m_i x_i$ . Similarly get $\sum_i m_i y_i$ and $\sum_i m_i z_i$. Then find

$$X = \frac{1}{M}\sum_i m_i x_i\,,\; Y = \frac{1}{M}\sum_i m_i y_i \text{ and } Z = \frac{1}{M}\sum_i m_i z_i$$

where $M = \sum_i m_i$ is the total mass of the system.

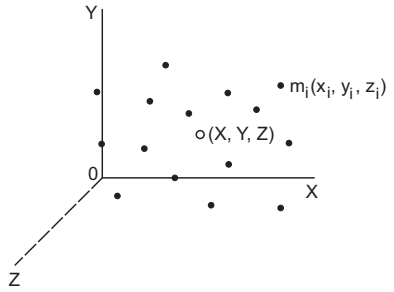



Figure 9.2

Locate the point with coordinates $(X, Y, Z)$. This point is called the *centre of mass* of the given collection of the particles. If the position vector of the $i$ th particle is $\vec{r_i}$ , the centre of mass is defined to have the position vector

$$\vec{R}_{CM} = \frac{1}{M}\sum_i m_i \vec{r_i}\,. \qquad \ldots \quad (9.1)$$

Taking $x, y, z$ components of this equation, we get the coordinates of centre of mass as defined above

$$X = \frac{1}{M}\sum_i m_i x_i\,,\; Y = \frac{1}{M}\sum_i m_i y_i\,,\; Z = \frac{1}{M}\sum_i m_i z_i \quad \ldots \quad (9.2)$$

***Example 9.1***

*Four particles A, B, C and D having masses m, 2m, 3m and 4m respectively are placed in order at the corners of a square of side a. Locate the centre of mass.*

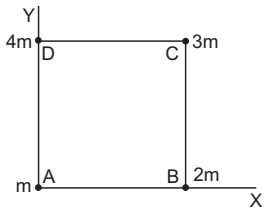



Figure 9.3

***Solution*** **:** Take the axes as shown in figure (9.3). The coordinates of the four particles are as follows :

| Particle | mass | $x$-coordinate | $y$-coordinate |
|---|---|---|---|
| $A$ | $m$ | 0 | 0 |
| $B$ | $2m$ | $a$ | 0 |
| $C$ | $3m$ | $a$ | $a$ |
| $D$ | $4m$ | 0 | $a$ |

Hence, the coordinates of the centre of mass of the four-particle system are

$$X = \frac{m \cdot 0 + 2\,m\,a + 3\,m\,a + 4\,m \cdot 0}{m + 2\,m + 3\,m + 4\,m} = \frac{a}{2}$$

$$Y = \frac{m \cdot 0 + 2\,m \cdot 0 + 3\,m\,a + 4\,m\,a}{m + 2\,m + 3\,m + 4\,m} = \frac{7a}{10}$$

The centre of mass is at $\left(\frac{a}{2}, \frac{7a}{10}\right)$.

**Centre of Mass of Two Particles**

As the simplest example, consider a system of two particles of masses $m_1$ and $m_2$ separated by a distance $d$ (figure 9.4). Where is the centre of mass of this system ?

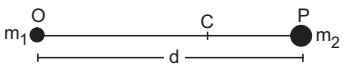



Figure 9.4

Take the origin at $m_1$ and the $X$-axis along the line joining $m_1$ and $m_2$. The coordinates of $m_1$ are (0, 0, 0) and of $m_2$ are ($d$, 0, 0). So,

$$\sum_i m_i x_i = m_1\,0 + m_2\,d = m_2\,d,\ \sum_i m_i y_i = 0,\ \sum_i m_i z_i = 0.$$

The total mass is $M = m_1 + m_2$. By definition, the centre of mass will be at $\left(\frac{m_2\,d}{m_1 + m_2}, 0, 0\right)$. We find that the centre of mass of a system of two particles is situated on the line joining the particles. If $O, C, P$ be the positions of $m_1$, the centre of mass and $m_2$ respectively, we have

$$OC = \frac{m_2\,d}{m_1 + m_2} \quad \text{and} \quad CP = \frac{m_1\,d}{m_1 + m_2}$$

so that $m_1(OC) = m_2(CP)$ … (9.3)

*The centre of mass divides internally the line joining the two particles in inverse ratio of their masses.*

**Centre of Mass of Several Groups of Particles**

Consider a collection of $N_1 + N_2$ particles. We call the group of $N_1$ particles as the first part and the other group of $N_2$ particles as the second part. Suppose the first part has its centre of mass at $C_1$ and the total mass $M_1$ (figure 9.5). Similarly the second part has its centre of mass at $C_2$ and the total mass $M_2$ . Where is the centre of mass of the system of $N_1 + N_2$ particles ?

Figure 9.5

The $x$-coordinate of the centre of mass is

$$X = \frac{\sum_{i=1}^{N_1+N_2} m_i x_i}{M_1 + M_2} = \frac{\sum_{i=1}^{N_1} m_i x_i + \sum_{i=N_1+1}^{N_1+N_2} m_i x_i}{M_1 + M_2}. \quad \text{… (9.4)}$$

If $X_1$ , $X_2$ are the $x$-coordinates of $C_1$ and $C_2$ , then by the definition of centre of mass, $\sum m_i x_i$ for the first part is $M_1 X_1$ and $\sum m_i x_i$ for the second part is $M_2 X_2$. Hence equation (9.4) becomes,

$$X = \frac{M_1 X_1 + M_2 X_2}{M_1 + M_2}$$

Similarly, $Y = \frac{M_1 Y_1 + M_2 Y_2}{M_1 + M_2}$

and $Z = \frac{M_1 Z_1 + M_2 Z_2}{M_1 + M_2}$.

But this is also the centre of mass of two point particles of masses $M_1$ and $M_2$ placed at $C_1$ and $C_2$ respectively. Thus, we obtain a very useful result. *If we know the centres of mass of parts of the system and their masses, we can get the combined centre of mass by treating the parts as point particles placed at their respective centres of mass.*

***Example 9.2***

*Two identical uniform rods AB and CD, each of length L are jointed to form a T-shaped frame as shown in figure (9.6). Locate the centre of mass of the frame. The centre of mass of a uniform rod is at the middle point of the rod.*

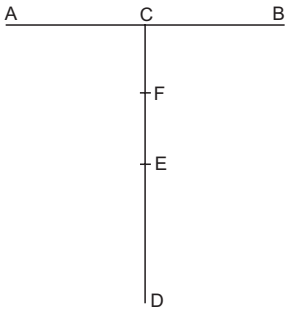



Figure 9.6

***Solution*** **:** Let the mass of each rod be $m$. Take the centre $C$ of the rod $AB$ as the origin and $CD$ as the $Y$-axis. The rod $AB$ has mass $m$ and its centre of mass is at $C$. For the calculation of the centre of mass of the combined system, $AB$ may be replaced by a point particle of mass $m$ placed at the point $C$. Similarly the rod $CD$ may be replaced by a point particle of mass m placed at the centre $E$ of the rod $CD$. Thus, the frame is equivalent to a system of two particles of equal masses m each, placed at $C$ and $E$. The centre of mass of this pair of particles will be at the middle point $F$ of $CE$.

The centre of mass of the frame is, therefore, on the rod $CD$ at a distance $L/4$ from $C$.

## 9.2 CENTRE OF MASS OF CONTINUOUS BODIES

If we consider the body to have continuous distribution of matter, the summation in the formula of centre of mass should be replaced by integration. So, we do not talk of the $i$th particle, rather we talk of a small element of the body having a mass $dm$. If $x, y, z$ are the coordinates of this small mass $dm$, we write the coordinates of the centre of mass as

$$X = \frac{1}{M}\int x\, dm\ ,\ Y = \frac{1}{M}\int y\, dm\ ,\ Z = \frac{1}{M}\int z\, dm. \quad \ldots \quad (9.5)$$

The integration is to be performed under proper limits so that as the integration variable goes through the limits, the elements cover the entire body. We illustrate the method with three examples.

### (a) Centre of Mass of a Uniform Straight Rod

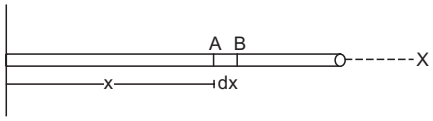



Figure 9.7

Let $M$ and $L$ be the mass and the length of the rod respectively. Take the left end of the rod as the origin and the $X$-axis along the rod (figure 9.7). Consider an element of the rod between the positions $x$ and $x + dx$. If $x = 0$, the element is at the left end of the rod. If $x = L$, the element is at its right end. So as $x$ varies from 0 through $L$, the elements cover the entire rod. As the rod is uniform, the mass per unit length is $M/L$ and hence the mass of the element is $dm = (M/L)\, dx$. The coordinates of the element are $(x, 0, 0)$. (The coordinates of different points of the element differ, but the difference is less than $dx$ and that much is harmless as integration will automatically correct it. So $x$-coordinate of the left end of the element may be called the "$x$-coordinate of the element.")

The $x$-coordinate of the centre of mass of the rod is

$$X = \frac{1}{M}\int x\, dm = \frac{1}{M}\int_0^L x\left(\frac{M}{L}\, dx\right)$$

$$= \frac{1}{L}\left[\frac{x^2}{2}\right]_0^L = \frac{L}{2}\,.$$

The $y$-coordinate is

$$Y = \frac{1}{M}\int y\, dm = 0$$

and similarly $Z = 0$. The centre of mass is at $\left(\frac{L}{2}, 0, 0\right)$, i.e., at the middle point of the rod.

### (b) Centre of Mass of a Uniform Semicircular Wire

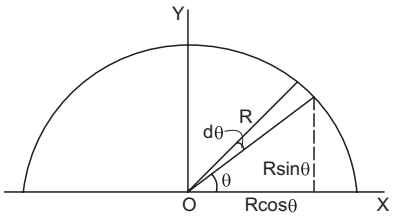



Figure 9.8

Let $M$ be the mass and $R$ the radius of a uniform semicircular wire. Take its centre as the origin, the line joining the ends as the $X$-axis, and the $Y$-axis in

the plane of the wire (figure 9.8). The centre of mass must be in the plane of the wire i.e., in the $X$-$Y$ plane.

How do we choose a small element of the wire ? First, the element should be so defined that we can vary the element to cover the whole wire. Secondly, if we are interested in $\int x\,dm$, the $x$-coordinates of different parts of the element should only infinitesimally differ in range. We select the element as follows. Take a radius making an angle $\theta$ with the $X$-axis and rotate it further by an angle $d\theta$. Note the points of intersection of the radius with the wire during this rotation. This gives an element of length $R\,d\theta$. When we take $\theta = 0$, the element is situated near the right edge of the wire. As $\theta$ is gradually increased to $\pi$, the element takes all positions on the wire i.e., the whole wire is covered. The "coordinates of the element" are $(R\cos\theta\,,\ R\sin\theta)$. Note that the coordinates of different parts of the element differ only by an infinitesimal amount.

As the wire is uniform, the mass per unit length of the wire is $\frac{M}{\pi R}$. The mass of the element is, therefore,

$$dm = \left(\frac{M}{\pi R}\right)(R\,d\theta) = \frac{M}{\pi}\,d\theta.$$

The coordinates of the centre of mass are

$$X = \frac{1}{M}\int x\,dm = \frac{1}{M}\int_0^{\pi}(R\cos\theta)\left(\frac{M}{\pi}\right)d\theta = 0$$

and

$$Y = \frac{1}{M}\int y\,dm = \frac{1}{M}\int_0^{\pi}(R\sin\theta)\left(\frac{M}{\pi}\right)d\theta = \frac{2R}{\pi}$$

The centre of mass is at $\left(0, \frac{2R}{\pi}\right)$.

**(c) Centre of Mass of a Uniform Semicircular Plate**

This problem can be worked out using the result obtained for the semicircular wire and that any part of the system (semicircular plate) may be replaced by a point particle of the same mass placed at the centre of mass of that part.

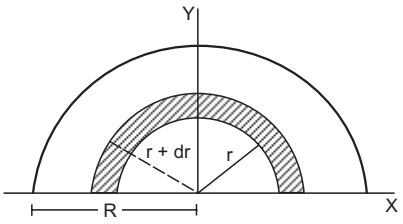



Figure 9.9

Figure (9.9) shows the semicircular plate. We take the origin at the centre of the semicircular plate, the $X$-axis along the straight edge and the $Y$-axis in the plane of the plate. Let $M$ be the mass and $R$ be its radius. Let us draw a semicircle of radius $r$ on the plate with the centre at the origin. We increase $r$ to $r + dr$ and draw another semicircle with the same centre. Consider the part of the plate between the two semicircles of radii $r$ and $r + dr$. This part, shown shaded in figure (9.9), may be considered as a semicircular wire.

If we take $r = 0$, the part will be formed near the centre and if $r = R$, it will be formed near the edge of the plate. Thus, if $r$ is varied from 0 to $R$, the elemental parts will cover the entire semicircular plate.

We can replace the semicircular shaded part by a point particle of the same mass at its centre of mass for the calculation of the centre of mass of the plate.

The area of the shaded part $= \pi\,r dr$. The area of the plate is $\pi R^2/2$. As the plate is uniform, the mass per unit area is $\frac{M}{\pi R^2/2}$. Hence the mass of the semicircular element

$$\frac{M}{\pi R^2/2}(\pi\,r\,dr) = \frac{2\,M\,r dr}{R^2}.$$

The $y$-coordinate of the centre of mass of this wire is $2r/\pi$. The $y$-coordinate of the centre of mass of the plate is, therefore,

$$Y = \frac{1}{M}\int_0^{R}\left(\frac{2r}{\pi}\right)\left(\frac{2\,M\,r}{R^2}dr\right) = \frac{1}{M}\cdot\frac{4\,M}{\pi R^2}\frac{R^3}{3} = \frac{4R}{3\pi}.$$

The $x$-coordinate of the centre of mass is zero by symmetry.

## 9.3 MOTION OF THE CENTRE OF MASS

Consider two particles $A$ and $B$ of masses $m_1$ and $m_2$ respectively. Take the line joining $A$ and $B$ as the $X$-axis. Let the coordinates of the particles at time $t$ be $x_1$ and $x_2$. Suppose no external force acts on the two-particle-system. The particles $A$ and $B$, however, exert forces on each other and the particles accelerate along the line joining them. Suppose the particles are initially at rest and the force between them is attractive. The particles will then move along the line $AB$ as shown in figure (9.10).

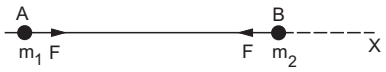



Figure 9.10

The centre of mass at time $t$ is situated at

$$X = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}.$$

As time passes, $x_1$, $x_2$ change and hence $X$ changes and the centre of mass moves along the $X$-axis. Velocity of the centre of mass at time $t$ is

$$V_{CM} = \frac{dx}{dt} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} . \quad \ldots (9.6)$$

The acceleration of the centre of mass is

$$a_{CM} = \frac{dV_{CM}}{dt} = \frac{m_1 a_1 + m_2 a_2}{m_1 + m_2} . \quad \ldots (9.7)$$

Suppose the magnitude of the forces between the particles is $F$. As the only force acting on $A$ is $F$ towards $B$, its acceleration is $a_1 = F/m_1$. The force on $B$ is $(-F)$ and hence $a_2 = -F/m_2$

Putting in (9.7),

$$a_{CM} = \frac{m_1(F/m_1) + m_2(-F/m_2)}{m_1 + m_2} = 0.$$

That means, the velocity of the centre of mass does not change with time. But as we assumed, initially the particles are at rest. Thus, at this instant, $v_1 = v_2 = 0$ and from (9.5) $V_{CM} = 0$. Hence the centre of mass remains fixed and does not change with time.

Thus, *if no external force acts on a two-particle-system and its centre of mass is at rest (say in the inertial frame A) initially, it remains fixed (in the inertial frame A) even when the particles individually move and accelerate*. Let us now generalise this result.

Consider a system of $N$ particles, the $i$th particle having a mass $m_i$ and the position vector $\vec{r}_i$ with respect to an inertial frame. Each particle is acted upon by forces due to all other $(N-1)$ particles and forces due to the sources outside the system. The acceleration of the $i$th particle is

$$\vec{a}_i = \frac{1}{m_i}\left( \sum_{j \neq i} \vec{F}_{ij} + \vec{F}_i^{\,ext} \right) \text{ (Newton's second law)}$$

$$\text{or, } m_i \vec{a}_i = \left( \sum_{j \neq i} \vec{F}_{ij} + \vec{F}_i^{\,ext} \right).$$

Here $\vec{F}_{ij}$ is the force on the $i$th particle due to the $j$th particle and $\vec{F}_i^{\,ext}$ is the vector sum of the forces acting on the $i$th particle by the external sources. Summing over all the particles

$$\sum_i m_i \vec{a}_i = \sum_{i \neq j} \sum_j \vec{F}_{ij} + \sum_i \vec{F}_i^{\,ext} = \vec{F}^{\,ext}. \quad \ldots (9.8)$$

The internal forces $\vec{F}_{ij}$ add up to zero as they cancel in pairs $(\vec{F}_{ij} + \vec{F}_{ji} = 0)$ by Newton's third law. $\vec{F}^{\,ext}$ is the sum of all the forces acting on all the particles by the external sources.

Now $\sum_i m_i \vec{r}_i = M \vec{R}_{CM}$ giving $\sum_i m_i \vec{a}_i = M \vec{a}_{CM}$

Putting in (9.8),

$$M \vec{a}_{CM} = \vec{F}^{ext} . \quad \ldots (9.9)$$

If the external forces acting on the system add to zero, $\vec{a}_{CM} = 0$ and hence the velocity of the centre of mass is constant. If initially the centre of mass was at rest with respect to an inertial frame, it will continue to be at rest with respect to that frame. The individual particles may go on complicated paths changing their positions, but the centre of mass will be obtained at the same position.

If the centre of mass was moving with respect to the inertial frame at a speed $v$ along a particular direction, it will continue its motion along the same straight line with the same speed. Thus, *the motion of the centre of mass of the system is not affected by the internal forces. If the external forces add up to zero, the centre of mass has no acceleration*.

***Example 9.3***

*Two charged particles of masses m and 2m are placed a distance d apart on a smooth horizontal table. Because of their mutual attraction, they move towards each other and collide. Where will the collision occur with respect to the initial positions* ?

***Solution*** **:** As the table is smooth, there is no friction. The weight of the particles and the normal force balance each other as there is no motion in the vertical direction. Thus, taking the two particles as constituting the system, the sum of the external forces acting on the system is zero. The forces of attraction between the particles are the internal forces as we have included both the particles in the system. Therefore, the centre of mass of the system will have no acceleration.

Initially, the two particles are placed on the table and their velocities are zero. The velocity of the centre of mass is, therefore, zero. As time passes, the particles move, but the centre of mass will continue to be at the same place. At the time of collision, the two particles are at one place and the centre of mass will also be at that place. As the centre of mass does not move, the collision will take place at the centre of mass.

The centre of mass will be at a distance $2d/3$ from the initial position of the particle of mass $m$ towards the other particle and the collision will take place there.

When the external forces do not add up to zero, the centre of mass is accelerated and the acceleration is given by equation (9.9)

$$\vec{a}_{CM} = \frac{\vec{F}^{\,ext}}{M} .$$

If we have a single particle of mass $M$ on which a force $\vec{F}^{\,ext}$ acts, its acceleration would be the same as

$\frac{\vec{F}^{ext}}{M}$. Thus *the motion of the centre of mass of a system is identical to the motion of a single particle of mass equal to the mass of the given system, acted upon by the same external forces that act on the system.*

To explain this statement, once again consider the spinning ball of figure (9.1b). The ball is spinning and at the same time moving under gravity. To find the motion of the centre of mass of the ball, which is actually the centre of the ball, we imagine a particle of mass equal to that of the ball. We throw this particle with the velocity $v$, which the centre of mass had at the time of projection. What is the motion of this single particle of mass $M$ subjected to the force $Mg$ downward, thrown initially with velocity $v$ ? It is a parabolic motion, given by,

$$x = v_x t, \quad y = v_y t - \frac{1}{2} g t^2. \qquad \ldots \ (9.10)$$

The centre of the ball exactly traces this curve with coordinates given by this equation only.

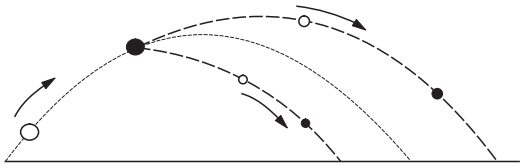


Figure 9.11

Next, suppose the ball breaks up into two parts (figure 9.11) because of some internal stress, while moving along the parabola. The two parts go on two different parabolae because the velocities of the parts change at the instant of breaking. Locate the two parts at some instant $t$ and calculate the position of the centre of mass of the combination at that instant. It will be found at the same point on the original parabola where the centre would have been at the instant $t$ according to equation (9.10).

## 9.4 LINEAR MOMENTUM AND ITS CONSERVATION PRINCIPLE

The (linear) momentum of a particle is defined as $\vec{p} = m\vec{v}$. The momentum of an $N$-particle system is the (vector) sum of the momenta of the $N$ particles i.e.,

$$\vec{P} = \sum_i \vec{p_i} = \sum_i m_i \vec{v_i}.$$

But $\sum_i m_i \vec{v_i} = \frac{d}{dt} \sum_i m_i \vec{r_i} = \frac{d}{dt} M \vec{R}_{CM} = M \vec{V}_{CM}$.

Thus, $$\vec{P} = M \vec{V}_{CM}. \qquad \ldots \ (9.11)$$

As we have seen, if the external forces acting on the system add up to zero, the centre of mass moves with constant velocity, which means $\vec{P}$ = constant. Thus the linear momentum of a system remains constant (in magnitude and direction) if the external forces acting on the system add up to zero. This is known as *the principle of conservation of linear momentum*.

Consider a trivial example of a single particle on which no force acts (imagine a practical situation where this can be achieved). Looking from an inertial frame, the particle is moving with uniform velocity and so its momentum remains constant as time passes.

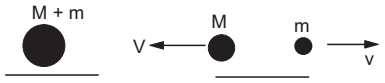



Figure 9.12

As a different example, consider a radioactive nucleus at rest which emits an alpha particle along the $X$-axis. Let $m$ and $M$ be the masses of the alpha particle and the residual nucleus respectively. Take the entire nucleus as the system. The alpha particle is ejected from the nucleus because of the forces between the neutrons and protons of the nucleus (this is the nuclear force and not gravitational or electromagnetic). There is no external force acting on the system and hence its linear momentum should not change. The linear momentum before the emission was zero as the nucleus was at rest. After the emission, the system is broken up into two parts, the alpha particle and the residual nucleus. If the alpha particle is emitted with a speed $v$, the residual nucleus must recoil in the opposite direction with a speed $V$, so that

$$M\vec{V} + m\vec{v} = 0 \quad \text{or,} \quad \vec{V} = -\frac{m}{M}\vec{v}.$$

## 9.5 ROCKET PROPULSION

In a rocket, the fuel burns and produces gases at high temperatures. These gases are ejected out of the rocket from a nozzle at the backside of the rocket. The ejecting gas exerts a forward force on the rocket which helps it in accelerating.

Suppose, a rocket together with its fuel has a mass $M_0$ at $t = 0$. Let the gas be ejected at a constant rate $r = -\frac{dM}{dt}$. Also suppose, the gas is ejected at a constant velocity $u$ with respect to the rocket.

At time $t$, the mass of the rocket together with the remaining fuel is

$$M = M_0 - rt.$$

If the velocity of the rocket at time $t$ is $v$, the linear momemtum of this mass $M$ is

$$P = Mv = (M_0 - rt)v. \qquad \ldots \ (i)$$

Consider a small time interval $\Delta t$. A mass $\Delta M = r\Delta t$ of the gas is ejected in this time and the

velocity of the rocket becomes $v + \Delta v$. The velocity of the gas with respect to ground is

$$\vec{v}_{\text{gas, ground}} = \vec{v}_{\text{gas, rocket}} + \vec{v}_{\text{rocket, ground}}$$
$$= -u + v$$

in the forward direction.

The linear momentum of the mass $M$ at $t + \Delta t$ is,

$$(M - \Delta M)(v + \Delta v) + \Delta M(v - u). \quad \ldots \text{(ii)}$$

Assuming no external force on the rocket-fuel system, from (i) and (ii),

$$(M - \Delta M)(v + \Delta v) + \Delta M(v - u) = Mv$$

or, $$(M - \Delta M)(\Delta v) = (\Delta M)u$$

or, $$\Delta v = \frac{(\Delta M)u}{M - \Delta M}$$

or, $$\frac{\Delta v}{\Delta t} = \frac{\Delta M}{\Delta t}\frac{u}{M - \Delta M} = \frac{ru}{M - r\Delta t}$$

Taking the limit as $\Delta t \to 0$,

$$\frac{dv}{dt} = \frac{ru}{M} = \frac{ru}{M_0 - rt}.$$

This gives the acceleration of the rocket. We see that the acceleration keeps on increasing as time passes. If the rocket starts at $t = 0$ and we neglect any external force such as gravity,

$$\int_0^v dv = ru\int_0^t \frac{dt}{M_0 - rt}$$

or, $$v = ru\left(-\frac{1}{r}\right)\ln\frac{M_0 - rt}{M_0}$$

or, $$v = u\ln\frac{M_0}{M_0 - rt}.$$

## 9.6 COLLISION

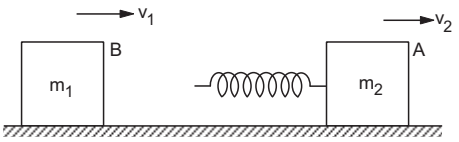


Figure 9.13

Consider the situation shown in figure (9.13). Two blocks of masses $m_1$ and $m_2$ are moving on the same straight line on a frictionless horizontal table. The block $m_2$, which is ahead of $m_1$, is going with a speed $v_2$ smaller than the speed $v_1$ of $m_1$. A spring is attached to the rear end of $m_2$. Since $v_1 > v_2$, the block $m_1$ will touch the rear of the spring at some instant, say $t_1$. Then onwards, the velocity of the left end of the spring will be equal to the velocity of $m_1$ (as they are in contact). The velocity of the right end of the spring will be same as that of $m_2$ (as they are in contact). Since $m_1$ moves faster than $m_2$, the length of the spring will decrease. The spring will be compressed. As it is compressed, it pushes back both the blocks with forces $kx$ where $x$ is the compression and $k$, the spring constant. This force is in the direction of the velocity of $m_2$, hence $m_2$ will accelerate. However, this is opposite to the velocity of $m_1$ and so $m_1$ will decelerate. The velocity of the front block $A$ (which was slower initially) will gradually increase, and the velocity of the rear block $B$ (which was faster initially) will gradually decrease. The spring will continue to become more and more compressed as long as the rear block $B$ is faster than the front block $A$. There will be an instant $t_1 + \Delta t_1$, when the two blocks will have equal velocities. At this instant, both the ends of the spring will move with the same velocity and no further compression will take place. This corresponds to the maximum compression of the spring. Thus, "the spring-compression is maximum when the two blocks attain equal velocities".

Now, the spring being already compressed, it continues to push back the two blocks. Thus, the front block $A$ will still be accelerated and the rear block $B$ will still be decelerated. At $t_1 + \Delta t_1$ the velocities were equal and hence, after $t_1 + \Delta t_1$ the front block will move faster than the rear block. And so do the ends of the spring as they are in contact with the blocks. The spring will thus increase its length. This process will continue till the spring acquires its natural length, say at a time $t_1 + \Delta t_1 + \Delta t_2$. Once the spring regains its natural length, it stops exerting any force on the blocks. As the two blocks are moving with different velocities by this time, the rear one slower, the rear block will leave contact with the spring and the blocks will move with constant velocities. Their separation will go on increasing.

During the whole process, the momentum of the two-blocks system remains constant. The momentum before the instant $t_1$ was $m_1v_1 + m_2v_2 = P$. At time $t_1 + \Delta t_1$, the two blocks have equal velocities say $V$ and we have $m_1V + m_2V = P$. After the contact is broken, the blocks finally attain constant velocities $v_1'$ and $v_2'(v_2' > v_1')$ and the momentum will be $m_1v_1' + m_2v_2' = P$. In fact, take the velocities of the blocks at any instant, before the collision, during the collision or after the collision; the momentum will be equal to $P$. This is because there is no resultant external force acting on the system. Note that the spring being massless, exerts equal and opposite forces on the blocks.

Next, consider the energy of the system. As there is no friction anywhere, the sum of the kinetic energy and the elastic potential energy remains constant. The gravitational potential energy does not come into the

picture, as the motion is horizontal. The elastic potential energy is $\frac{1}{2}kx^2$ when the spring is compressed by $x$. If $u_1$ and $u_2$ are the speeds at this time, we have,

$$\frac{1}{2}m_1u_1^2+\frac{1}{2}m_2u_2^2+\frac{1}{2}kx^2=E$$

where $E$ is the total energy of the system.

At and before $t=t_1$, the spring is at its natural length so that,

$$\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2=E. \quad \ldots \text{(i)}$$

At time $t=t_1+\Delta t_1$, $u_1=u_2=V$ and the compression of the spring is maximum. Thus,

$$\frac{1}{2}(m_1+m_2)V^2+\frac{1}{2}kx_{max}^2=E.$$

At and after $t=t_1+\Delta t_1+\Delta t_2$, the spring acquires its natural length, so that,

$$\frac{1}{2}m_1v_1'^2+\frac{1}{2}m_2v_2'^2=E. \quad \ldots \text{(ii)}$$

From (i) and (ii),

$$\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2=\frac{1}{2}m_1v_1'^2+\frac{1}{2}m_2v_2'^2.$$

The kinetic energy before the collision is the same as the kinetic energy after the collision. However, we can not say that the kinetic energy remains constant because it changes as a function of time, during the interval $t_1$ to $t_1+\Delta t_1+\Delta t_2$.

***Example 9.4***

*Each of the blocks shown in figure (9.14) has mass* 1 kg. *The rear block moves with a speed of* 2 m/s *towards the front block kept at rest. The spring attached to the front block is light and has a spring constant* 50 N/m. *Find the maximum compression of the spring.*

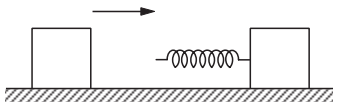


Figure 9.14

***Solution*** : Maximum compression will take place when the blocks move with equal velocity. As no net external force acts on the system of the two blocks, the total linear momentum will remain constant. If $V$ is the common speed at maximum compression, we have,

$$(1\text{ kg})(2\text{ m/s})=(1\text{ kg})\text{ V}+(1\text{ kg})V$$

or, $V=1$ m/s.

Initial kinetic energy $=\frac{1}{2}(1\text{ kg})(2\text{ m/s})^2=2$ J

Final kinetic energy

$$=\frac{1}{2}(1\text{ kg})(1\text{ m/s})^2+\frac{1}{2}(1\text{ kg})(1\text{ m/s})^2$$

$=1$ J.

The kinetic energy lost is stored as the elastic energy in the spring.

Hence, $\frac{1}{2}(50\text{ N/m})x^2=2\text{ J}-1\text{ J}=1\text{ J}$

or, $x=0{\cdot}2$ m.

Almost similar is the situation when two balls collide with each other and no spring is put between them (figure 9.15). At the instant they come into contact, the rear ball has a larger velocity $v_1$ and the front ball has a smaller velocity $v_2$. But the surfaces in contact must move equal distance in any time interval as long as they remain in contact. The balls have to be deformed at the contact.

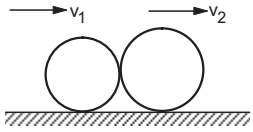



Figure 9.15

The deformed balls push each other and the velocities of the two balls change. The total kinetic energy of the two balls decreases as some energy is converted into the elastic potential energy of the deformed balls. The deformation is maximum (and the kinetic energy minimum) when the two balls attain equal velocities. Total momentum of the balls remains constant. The behaviour of the balls after this depends on the nature of the materials of the balls. If the balls are perfectly elastic, forces may develop inside them so that the balls try to regain their original shapes. In this case, the balls continue to push each other, the velocity of the front ball increases while that of the rear ball decreases and thus the balls separate. After separation, the balls regain their original shapes so that the elastic potential energy is completely converted back into kinetic energy. Thus, although the kinetic energy is not constant, the initial kinetic energy is equal to the final kinetic energy. Such a collision is called an *elastic collision*.

On the contrary, if the materials of the balls are *perfectly inelastic,* the balls have no tendency to regain their original shapes after maximum deformation. As a result, they do not push each other and continue to move with the common velocity with their deformed shapes. The kinetic energy decreases at the time of deformation and thereafter remains constant at this decreased value. Such a collision is called an *inelastic collision*.

If the material is *partially elastic,* the balls try to regain their original shapes, they push each other, even after maximum deformation. The velocities further change, the balls separate but the shapes are not completely recovered. Some energy remains inside the deformed ball. The final kinetic energy is, therefore, less than the initial kinetic energy. But the loss of kinetic energy is not as large as that in the case of a perfectly inelastic collision.

Thus, for an elastic collision,

$$m_1\vec{v_1} + m_2\vec{v_2} = m_1\vec{v_1}' + m_2\vec{v_2}'$$

and $$\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1 v_1'^2 + \frac{1}{2}m_2 v_2'^2 \quad \ldots (9.12)$$

i.e., $K_f = K_i$.

For an inelastic collision, $\vec{v_1}' = \vec{v_2}' = \vec{V}$,

$$m_1\vec{v_1} + m_2\vec{v_2} = m_1\vec{V} + m_2\vec{V} \quad \ldots (9.13)$$

and $K_f < K_i$.

For a partially elastic collision,

$$m_1\vec{v_1} + m_2\vec{v_2} = m_1\vec{v_1}' + m_2\vec{v_2}'$$

$K_f < K_i$, $\Delta K = K_i - K_f$

is the loss of kinetic energy. It is less than that in the case of a perfectly inelastic collision. For one dimensional collision (head-on collision) the vector sign may be removed.

## 9.7 ELASTIC COLLISION IN ONE DIMENSION

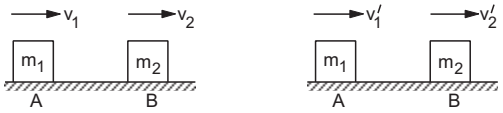



Figure 9.16

Consider two elastic bodies $A$ and $B$ moving along the same line (figure 9.16). The body $A$ has a mass $m_1$ and moves with a velocity $v_1$ towards right and the body $B$ has a mass $m_2$ and moves with a velocity $v_2$ in the same direction. We assume $v_1 > v_2$ so that the two bodies may collide. Let $v_1'$ and $v_2'$ be the final velocities of the bodies after the collision. The total linear momentum of the two bodies remains constant, so that,

$$m_1v_1 + m_2v_2 = m_1v_1' + m_2v_2' \quad \ldots (i)$$

or, $$m_1v_1 - m_1v_1' = m_2v_2' - m_2v_2$$

or, $$m_1(v_1 - v_1') = m_2(v_2' - v_2). \quad \ldots (ii)$$

Also, since the collision is elastic, the kinetic energy before the collision is equal to the kinetic energy after the collision. Hence,

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2$$

or, $$m_1v_1^2 - m_1v_1'^2 = m_2v_2'^2 - m_2v_2^2$$

or, $$m_1(v_1^2 - v_1'^2) = m_2(v_2'^2 - v_2^2). \quad \ldots (iii)$$

Dividing (iii) by (ii),

$$v_1 + v_1' = v_2' + v_2$$

or, $$v_1 - v_2 = v_2' - v_1'. \quad \ldots (iv)$$

Now, $(v_1 - v_2)$ is the rate at which the separation between the bodies decreases before the collision. Similarly, $(v_2' - v_1')$ is the rate of increase of separation after the collision. So the equation (iv) may be written as

*Velocity of separation (after collision)*

= *Velocity of approach (before collision).* ... (9.14)

This result is very useful in solving problems involving elastic collision. The final velocities $v_1'$ and $v_2'$ may be obtained from equation (i) and (iv). Multiply equation (iv) by $m_2$ and subtract from equation (i).

$$2m_2v_2 + (m_1 - m_2)v_1 = (m_1 + m_2)v_1'$$

or, $$v_1' = \frac{(m_1 - m_2)}{m_1 + m_2}v_1 + \frac{2m_2}{m_1 + m_2}v_2. \quad \ldots (9.15)$$

Now multiply equation (iv) by $m_1$ and add to equation (i),

$$2m_1v_1 - (m_1 - m_2)v_2 = (m_2 + m_1)v_2'$$

or, $$v'_2 = \frac{2m_1v_1}{m_1 + m_2} - \frac{(m_1 - m_2)v_2}{m_1 + m_2}. \quad \ldots (9.16)$$

Equations (9.15) and (9.16) give the final velocities in terms of the initial velocities and the masses.

**Special cases :**

**(a) Elastic collision between a heavy body and a light body :**

Let $m_1 >> m_2$. A heavy body hits a light body from behind.

We have,

$$\frac{m_1 - m_2}{m_1 + m_2} \approx 1, \quad \frac{2m_2}{m_1 + m_2} \approx 0$$

and $$\frac{2m_1}{m_1 + m_2} \approx 2.$$

With these approximations the final velocities of the bodies are, from (9.15) and (9.16),

$$v_1' \approx v_1 \text{ and } v_2' \approx 2v_1 - v_2.$$

The heavier body continues to move with almost the same velocity. If the lighter body were kept at rest $v_2 = 0$, $v_2' = 2v_1$ which means the lighter body, after getting a push from the heavier body will fly away with a velocity double the velocity of the heavier body.

Next suppose $m_2 >> m_1$. A light body hits a heavy body from behind.

We have,

$$\frac{m_1 - m_2}{m_1 + m_2} \approx -1$$

$$\frac{2m_2}{m_1 + m_2} \approx 2$$

and $$\frac{2m_1}{m_1 + m_2} \approx 0$$

The final velocities of the bodies are, from (9.15) and (9.16),

$$v_1' \approx -v_1 + 2v_2 \quad \text{and} \quad v_2' \approx v_2 .$$

The heavier body continues to move with almost the same velocity, the velocity of the lighter body changes. If the heavier body were at rest, $v_2 = 0$ then $v_1' = -v_1$, the lighter body returns after collision with almost the same speed. This is the case when a ball collides elastically with a fixed wall and returns with the same speed.

**(b) Elastic collision of two bodies of equal mass :**

Putting $m_1 = m_2$ in equation (9.15) and (9.16)

$$v_1' = v_2 \quad \text{and} \quad v_2' = v_1 .$$

*When two bodies of equal mass collide elastically, their velocities are mutually interchanged.*

## 9.8 PERFECTLY INELASTIC COLLISION IN ONE DIMENSION

### Final Velocity

When perfectly inelastic bodies moving along the same line collide, they stick to each other. Let $m_1$ and $m_2$ be the masses, $v_1$ and $v_2$ be their velocities before the collision and $V$ be the common velocity of the bodies after the collision. By the conservation of linear momentum,

$$m_1 v_1 + m_2 v_2 = m_1 V + m_2 V$$

or, $$V = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} . \qquad \ldots \text{(i)}$$

### Loss in Kinetic Energy

The kinetic energy before the collision is

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

and that after the collision is $\frac{1}{2}(m_1 + m_2)V^2$. Using equation (i), the loss in kinetic energy due to the collision is

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 - \frac{1}{2}(m_1 + m_2) V^2$$

$$= \frac{1}{2}\left[ m_1 v_1^2 + m_2 v_2^2 - \frac{(m_1 v_1 + m_2 v_2)^2}{m_1 + m_2} \right]$$

$$= \frac{1}{2}\left[ \frac{m_1 m_2 (v_1^2 + v_2^2 - 2 v_1 v_2)}{m_1 + m_2} \right]$$

$$= \frac{m_1 m_2 (v_1 - v_2)^2}{2(m_1 + m_2)} .$$

We see that the loss in kinetic energy is positive.

***Example 9.5***

*A cart A of mass* 50 kg *moving at a speed of* 20 km/h *hits a lighter cart B of mass* 20 kg *moving towards it at a speed of* 10 km/h. *The two carts cling to each other. Find the speed of the combined mass after the collision.*

***Solution*** : This is an example of inelastic collision. As the carts move towards each other, their momenta have opposite sign. If the common speed after the collision is $V$, momentum conservation gives

(50 kg) (20 km/h) – (20 kg) (10 km/h) = (70 kg) $V$

or $$V = \frac{80}{7} \text{ km/h}.$$

## 9.9 COEFFICIENT OF RESTITUTION

We have seen that for a perfectly elastic collision

*velocity of separation = velocity of approach*

and for a perfectly inelastic collision

*velocity of separation* = 0.

In general, the bodies are neither perfectly elastic nor perfectly inelastic. In that case we can write

*velocity of separation = e (velocity of approach),*

where $0 < e < 1$. The constant $e$ depends on the materials of the colliding bodies. This constant is known as *coefficient of restitution*. If $e = 1$, the collision is perfectly elastic and if $e = 0$, the collision is perfectly inelastic.

***Example 9.6***

*A block of mass m moving at speed v collides with another block of mass* 2 m *at rest. The lighter block comes to rest after the collision. Find the coefficient of restitution.*

***Solution*** : Suppose the second block moves at speed $v'$ towards right after the collision. From the principle of conservation of momentum,

$mv = 2mv'$ or $v' = v/2$.

Hence, the velocity of separation $= v/2$ and the velocity of approach $= v$. By definition,

$$e = \frac{\text{velocity of the separation}}{\text{velocity of approach}} = \frac{v/2}{v} = \frac{1}{2} .$$

## 9.10 ELASTIC COLLISION IN TWO DIMENSIONS

Consider two objects $A$ and $B$ of mass $m_1$ and $m_2$ kept on the $X$-axis (figure 9.17). Initially, the object

$B$ is at rest and $A$ moves towards $B$ with a speed $u_1$. If the collision is not *head-on* (the force during the collision is not along the initial velocity), the objects move along different lines. Suppose the object $A$ moves with a velocity $\vec{v}_1$ making an angle θ with the $X$-axis and the object $B$ moves with a velocity $\vec{v}_2$ making an angle Φ with the same axis. Also, suppose $\vec{v}_1$ and $\vec{v}_2$ lie in $X$-$Y$ plane. Using conservation of momentum in $X$ and $Y$ directions, we get

$$m_1u_1 = m_1v_1\cos\theta + m_2v_2\cos\Phi \quad \ldots \text{(i)}$$

and $$0 = m_1v_1\sin\theta - m_2v_2\sin\Phi. \quad \ldots \text{(ii)}$$

If the collision is elastic, the final kinetic energy is equal to the initial kinetic energy. Thus,

$$\frac{1}{2}m_1u_1^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2. \quad \ldots \text{(iii)}$$

We have four unknowns $v_1$, $v_2$, θ and Φ to describe the final motion whereas there are only three relations. Thus, the final motion cannot be uniquely determined with this information.

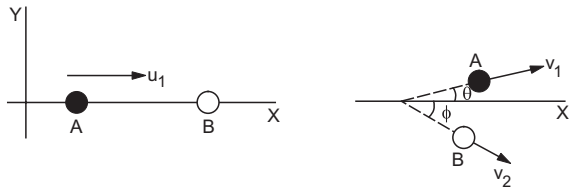



Figure 9.17

In fact, the final motion depends on the angle between the line of force during the collision and the direction of initial velocity. The momentum of each object must be individually conserved in the direction perpendicular to the force. The motion along the line of force may be treated as a one-dimensional collision.

## 9.11 IMPULSE AND IMPULSIVE FORCE

When two bodies collide, they exert forces on each other while in contact. The momentum of each body is changed due to the force on it exerted by the other. On an ordinary scale, the time duration of this contact is very small and yet the change in momentum is sizeable. This means that the magnitude of the force must be large on an ordinary scale. Such large forces acting for a very short duration are called *impulsive forces*. The force may not be uniform while the contact lasts.

The change in momentum produced by such an implusive force is

$$\vec{P}_f - \vec{P}_i = \int_{P_i}^{P_f} d\vec{P} = \int_{t_i}^{t_f} \frac{d\vec{P}}{dt}\,dt = \int_{t_i}^{t_f} \vec{F}\,dt. \quad \ldots \text{(9.17)}$$

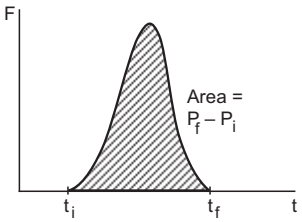



Figure 9.18

This quantity $\int_{t_i}^{t_f} \vec{F}\,dt$ is known as the impluse of the force $\vec{F}$ during the time interval $t_i$ to $t_f$ and is equal to the change in the momentum of the body on which it acts. Obviously, it is the area under the $F - t$ curve for one-dimensional motion (figure 9.18).

### *Worked Out Examples*

**1.** *Three particles of masses* 0·50 kg, 1·0 kg *and* 1·5 kg *are placed at the three corners of a right-angled triangle of sides* 3·0 cm, 4·0 cm *and* 5·0 cm *as shown in figure (9-W1). Locate the centre of mass of the system.*

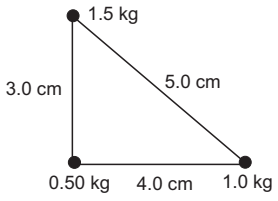



Figure 9-W1.

***Solution*** **:** Let us take the 4·0 cm line as the $X$-axis and the 3·0 cm line as the $Y$-axis. The coordinates of the three particles are as follows :

| $m$ | $x$ | $y$ |
|---|---|---|
| 0·50 kg | 0 | 0 |
| 1·0 kg | 4·0 cm | 0 |
| 1·5 kg | 0 | 3·0 cm |

The $x$-coordinate of the centre of mass is

$$X = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

$$= \frac{(0{\cdot}50\text{ kg})\,.\,0 + (1{\cdot}0\text{ kg})\,.\,(4{\cdot}0\text{ cm}) + (1{\cdot}5\text{ kg})\,.\,0}{0{\cdot}50\text{ kg} + 1{\cdot}0\text{ kg} + 1{\cdot}5\text{ kg}}$$

$$= \frac{4 \text{ kg-cm}}{3 \text{ kg}} = 1{\cdot}3 \text{ cm}.$$

The $y$-coordinate of the centre of mass is

$$Y = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3}$$

$$= \frac{(0{\cdot}50 \text{ kg}) \,.\, 0 + (1{\cdot}0 \text{ kg}) \,.\, 0 + (1{\cdot}5 \text{ kg})\,(3{\cdot}0 \text{ cm})}{0{\cdot}50 \text{ kg} + 1{\cdot}0 \text{ kg} + 1{\cdot}5 \text{ kg}}$$

$$= \frac{4{\cdot}5 \text{ kg-cm}}{3 \text{ kg}} = 1{\cdot}5 \text{ cm}.$$

Thus, the centre of mass is 1·3 cm right and 1·5 cm above the 0·5 kg particle.

**2.** *Half of the rectangular plate shown in figure (9-W2) is made of a material of density $\rho_1$ and the other half of density $\rho_2$. The length of the plate is L. Locate the centre of mass of the plate.*

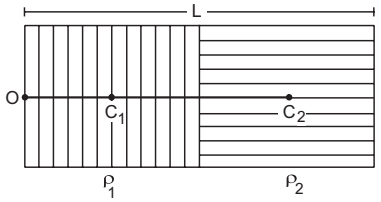



Figure 9-W2

***Solution*** **:** The centre of mass of each half is located at the geometrical centre of that half. Thus, the left half may be replaced by a point particle of mass $K\rho_1$ placed at $C_1$ and the right half may be replaced by a point particle of mass $K\rho_2$ placed at $C_2$. This replacement is for the specific purpose of locating the combined centre of mass. Take the middle point of the left edge to be the origin. The $x$-coordinate of $C_1$ is $L/4$ and that of $C_2$ is $3L/4$. Hence, the x-coordinate of the centre of mass is

$$X = \frac{(K\rho_1)\frac{L}{4} + (K\rho_2)\frac{3L}{4}}{K\rho_1 + K\rho_2}$$

$$= \frac{(\rho_1 + 3\rho_2)}{4(\rho_1 + \rho_2)} L.$$

The combined centre of mass is this much to the right of the assumed origin.

**3.** *The density of a linear rod of length L varies as $\rho = A + Bx$ where x is the distance from the left end. Locate the centre of mass.*

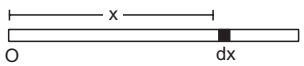



Figure 9-W3

***Solution*** **:** Let the cross-sectional area be $\alpha$. The mass of an element of length $dx$ located at a distance $x$ away from the left end is $(A + Bx)\,\alpha\,dx$. The $x$-coordinate of the centre of mass is given by

$$X_{CM} = \frac{\int x\,dm}{\int dm} = \frac{\int_0^L x\,(A + Bx)\,\alpha\,dx}{\int_0^L (A + Bx)\,\alpha\,dx}$$

$$= \frac{A\frac{L^2}{2} + B\frac{L^3}{3}}{AL + B\frac{L^2}{2}} = \frac{3AL + 2BL^2}{3(2A + BL)}.$$

**4.** *A cubical block of ice of mass m and edge L is placed in a large tray of mass M. If the ice melts, how far does the centre of mass of the system "ice plus tray" come down ?*

***Solution*** **:** Consider figure (9-W4). Suppose the centre of mass of the tray is a distance $x_1$ above the origin and that of the ice is a distance $x_2$ above the origin. The height of the centre of mass of the ice-tray system is

$$x = \frac{m x_2 + M x_1}{m + M}.$$

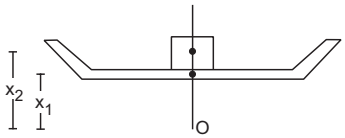



Figure 9-W4

When the ice melts, the water of mass $m$ spreads on the surface of the tray. As the tray is large, the height of water is negligible. The centre of mass of the water is then on the surface of the tray and is at a distance $x_2 - L/2$ above the origin. The new centre of mass of the ice-tray system will be at the height

$$x' = \frac{m\left(x_2 - \frac{L}{2}\right) + M x_1}{m + M}.$$

The shift in the centre of mass $= x - x' = \dfrac{mL}{2(m + M)}$.

**5.** *Consider a two-particle system with the particles having masses $m_1$ and $m_2$. If the first particle is pushed towards the centre of mass through a distance d, by what distance should the second particle be moved so as to keep the centre of mass at the same position ?*

***Solution*** **:** Consider figure (9-W5). Suppose the distance of $m_1$ from the centre of mass $C$ is $x_1$ and that of $m_2$ from $C$ is $x_2$. Suppose the mass $m_2$ is moved through a distance $d'$ towards $C$ so as to keep the centre of mass at $C$.

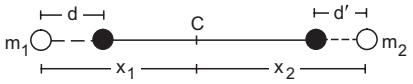



Figure 9-W5

Then,

$$m_1 x_1 = m_2 x_2 \quad \ldots \text{(i)}$$

and $$m_1 (x_1 - d) = m_2 (x_2 - d'). \quad \ldots \text{(ii)}$$

Subtracting (ii) from (i)

$$m_1 d = m_2 d'$$

or, $$d' = \frac{m_1}{m_2} d.$$

**6.** *A body of mass* 2·5 kg *is subjected to the forces shown in figure (9-W6). Find the acceleration of the centre of mass.*

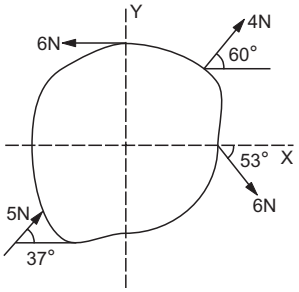



Figure 9-W6

***Solution*** **:** Take the $X$ and $Y$ axes as shown in the figure. The $x$-component of the resultant force is

$$F_x = -6\ \text{N} + (5\ \text{N}) \cos 37° + (6\ \text{N}) \cos 53° + (4\ \text{N}) \cos 60°$$

$$= -6\ \text{N} + (5\ \text{N}) \cdot (4/5) + (6\ \text{N}) \cdot (3/5) + (4\ \text{N}) \cdot (1/2) = 3{\cdot}6\ \text{N}.$$

Similarly, the $y$-component of the resultant force is

$$F_y = 5\ \text{N} \sin 37° - (6\ \text{N}) \sin 53° + 4\ \text{N} \sin 60°$$

$$= (5\ \text{N}) \cdot (3/5) - (6\ \text{N}) \cdot (4/5) + (4\ \text{N}) \cdot (\sqrt{3}/2) = 1{\cdot}7\ \text{N} \cdot$$

The magnitude of the resultant force is

$$F = \sqrt{F_x^{\,2} + F_y^{\,2}} = \sqrt{(3{\cdot}6\ \text{N})^2 + (1{\cdot}7\ \text{N})^2} \approx 4{\cdot}0\ \text{N}.$$

The direction of the resultant force makes an angle θ with the $X$-axis where

$$\tan\theta = \frac{F_y}{F_x} = \frac{1{\cdot}7}{3{\cdot}6} = 0{\cdot}47.$$

The acceleration of the centre of mass is

$$a_{CM} = \frac{F}{M} = \frac{4{\cdot}0\ \text{N}}{2{\cdot}5\ \text{kg}} = 1{\cdot}6\ \text{m/s}^2$$

in the direction of the resultant force.

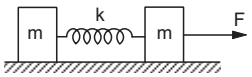



Figure 9-W7

**7.** *Two blocks of equal mass m are connected by an unstretched spring and the system is kept at rest on a frictionless horizontal surface. A constant force F is applied on one of the blocks pulling it away from the other as shown in figure (9-W7). (a) Find the position of the centre of mass at time t. (b) If the extension of the spring is $x_0$ at time t, find the displacement of the two blocks at this instant.*

***Solution*** **:** (a) The acceleration of the centre of mass is given by

$$a_{CM} = \frac{F}{M} = \frac{F}{2m}.$$

The position of the centre of mass at time $t$ is

$$x = \frac{1}{2} a_{CM} t^2 = \frac{F t^2}{4 m}.$$

(b) Suppose the displacement of the first block is $x_1$ and that of the second is $x_2$. As the centre of mass is at $x$, we should have

$$x = \frac{m x_1 + m x_2}{2 m}$$

or, $$\frac{F t^2}{4 m} = \frac{x_1 + x_2}{2}$$

or, $$x_1 + x_2 = \frac{F t^2}{2 m}. \quad \ldots \text{(i)}$$

The extension of the spring is $x_2 - x_1$. Therefore,

$$x_2 - x_1 = x_0. \quad \ldots \text{(ii)}$$

from (i) and (ii), $$x_1 = \frac{1}{2}\left(\frac{F t^2}{2 m} - x_0\right)$$

and $$x_2 = \frac{1}{2}\left(\frac{F t^2}{2 m} + x_0\right).$$

**8.** *A projectile is fired at a speed of* 100 m/s *at an angle of* 37° *above the horizontal. At the highest point, the projectile breaks into two parts of mass ratio* 1 : 3*, the smaller coming to rest. Find the distance from the launching point to the point where the heavier piece lands.*

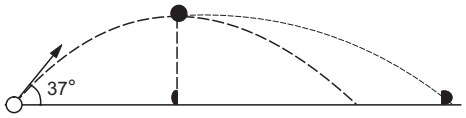



Figure 9-W8

***Solution*** **:** See figure (9-W8). At the highest point, the projectile has horizontal velocity. The lighter part comes to rest. Hence the heavier part will move with increased horizontal velocity. In vertical direction, both parts have zero velocity and undergo same acceleration, hence they will cover equal vertical displacements in a given time. Thus, both will hit the ground together. As internal forces do not affect the motion of the centre of mass, the centre of mass hits the ground at the position where the original projectile would have landed. The range of the original projectile is

$$x_{CM} = \frac{2 u^2 \sin\theta \cos\theta}{g} = \frac{2 \times 10^4 \times \frac{3}{5} \times \frac{4}{5}}{10}\ \text{m}$$

$$= 960\ \text{m}.$$

The centre of mass will hit the ground at this position. As the smaller block comes to rest after breaking, it falls down vertically and hits the ground at half of the range i.e., at $x = 480$ m. If the heavier block hits the ground at $x_2$, then

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

or, $$960 \text{ m} = \frac{\frac{M}{4} \times 480 \text{ m} + \frac{3M}{4} \times x_2}{M}$$

or, $$x_2 = 1120 \text{ m}.$$

**9.** *A block of mass M is placed on the top of a bigger block of mass* 10 *M as shown in figure (9-W9). All the surfaces are frictionless. The system is released from rest. Find the distance moved by the bigger block at the instant the smaller block reaches the ground.*

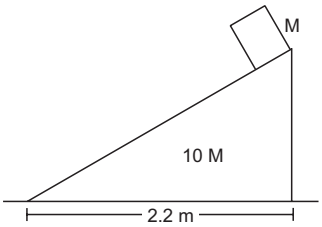



Figure 9-W9

***Solution*** **:** If the bigger block moves towards right by a distance $X$, the smaller block will move towards left by a distance $(2{\cdot}2 \text{ m} - X)$. Taking the two blocks together as the system, there is no horizontal external force on it. The centre of mass, which was at rest initially, will remain at the same horizontal position.

Thus,

$$M(2{\cdot}2 \text{ m} - X) = 10\,MX$$

or, $$2{\cdot}2 \text{ m} = 11\,X$$

or, $$X = 0{\cdot}2 \text{ m}.$$

**10.** *The hero of a stunt film fires* 50 g *bullets from a machine gun, each at a speed of* 1·0 km/s. *If he fires* 20 *bullets in* 4 seconds, *what average force does he exert against the machine gun during this period* ?

***Solution*** **:** The momentum of each bullet

$$= (0{\cdot}050 \text{ kg})(1000 \text{ m/s}) = 50 \text{ kg-m/s}.$$

The gun is imparted this much of momentum by each bullet fired. Thus, the rate of change of momentum of the gun

$$= \frac{(50 \text{ kg-m/s}) \times 20}{4 \text{ s}} = 250 \text{ N}$$

In order to hold the gun, the hero must exert a force of 250 N against the gun.

**11.** *A block moving horizontally on a smooth surface with a speed of* 20 m/s *bursts into two equal parts continuing in the same direction. If one of the parts moves at* 30 m/s, *with what speed does the second part move and what is the fractional change in the kinetic energy* ?

***Solution*** **:** There is no external force on the block. Internal forces break the block in two parts. The linear momentum of the block before the break should, therefore, be equal to the linear momentum of the two parts after the break. As all the velocities are in same direction, we get,

$$M(20 \text{ m/s}) = \frac{M}{2}(30 \text{ m/s}) + \frac{M}{2} v$$

where $v$ is the speed of the other part. From this equation $v = 10$ m/s. The change in kinetic energy is

$$\frac{1}{2}\frac{M}{2}(30 \text{ m/s})^2 + \frac{1}{2}\frac{M}{2}(10 \text{ m/s})^2 - \frac{1}{2}M(20 \text{ m/s})^2$$

$$= \frac{M}{2}(450 + 50 - 400)\frac{\text{m}^2}{\text{s}^2} = \left(50 \frac{\text{m}^2}{\text{s}^2}\right)M.$$

Hence, the fractional change in the kinetic energy

$$= \frac{M\left(50 \frac{\text{m}^2}{\text{s}^2}\right)}{\frac{1}{2}M(20 \text{ m/s})^2} = \frac{1}{4}.$$

**12.** *A car of mass M is moving with a uniform velocity v on a horizontal road when the hero of a Hindi film drops himself on it from above. Taking the mass of the hero to be m, what will be the velocity of the car after the event* ?

***Solution*** **:** Consider the car plus the hero as the system. In the horizontal direction, there is no external force. Since the hero has fallen vertically, so his initial horizontal momentum = 0.

Initial horizontal momemtum of the system $= Mv$ towards right.

Finally the hero sticks to the roof of the car, so they move with equal horizontal velocity say $V$. Final horizontal momentum of the system

$$= (M + m)\,V$$

Hence, $$Mv = (M + m)\,V$$

or, $$V = \frac{Mv}{M + m}.$$

**13.** *A space shuttle, while travelling at a speed of* 4000 km/h *with respect to the earth, disconnects and ejects a module backward, weighing one fifth of the residual part. If the shuttle ejects the disconnected module at a speed of* 100 km/h *with respect to the state of the shuttle before the ejection, find the final velocity of the shuttle.*

***Solution*** **:** Suppose the mass of the shuttle including the module is $M$. The mass of the module will be $M/6$. The total linear momentum before disconnection

$$= M\,(4000 \text{ km/h}).$$

The velocity of the ejected module with respect to the earth = its velocity with respect to the shuttle + the velocity of the shuttle with respect to the earth

$$= -100 \text{ km/h} + 4000 \text{ km/h} = 3900 \text{ km/h}.$$

If the final velocity of the shuttle is $V$ then the total final linear momentum

$$= \frac{5M}{6} V + \frac{M}{6} \times 3900 \text{ km/h}.$$

By the principle of conservation of linear momentum,

$$M\,(4000 \text{ km/h}) = \frac{5M}{6} V + \frac{M}{6} \times 3900 \text{ km/h}$$

or, $$V = 4020 \text{ km/h}.$$

**14.** *A boy of mass* 25 kg *stands on a board of mass* 10 kg *which in turn is kept on a frictionless horizontal ice surface. The boy makes a jump with a velocity component* 5 m/s *in a horizontal direction with respect to the ice. With what velocity does the board recoil ? With what rate are the boy and the board separating from each other ?*

***Solution*** **:** Consider the "board + boy" as a system. The external forces on this system are (a) weight of the system and (b) normal contact force by the ice surface. Both these forces are vertical and there is no external force in horizontal direction. The horizontal component of linear momentum of the "board + boy" system is, therefore, constant.

If the board recoils at a speed $v$,

$$0 = (25 \text{ kg}) \times (5 \text{ m/s}) - (10 \text{ kg}) v$$

or, $$v = 12{\cdot}5 \text{ m/s}.$$

The boy and the board are separating with a rate

$$5 \text{ m/s} + 12{\cdot}5 \text{ m/s} = 17{\cdot}5 \text{ m/s}.$$

**15.** *A man of mass m is standing on a platform of mass M kept on smooth ice. If the man starts moving on the platform with a speed v relative to the platform, with what velocity relative to the ice does the platform recoil ?*

***Solution*** **:** Consider the situation shown in figure ( 9-W10). Suppose the man moves at a speed $w$ towards right and the platform recoils at a speed $V$ towards left, both relative to the ice. Hence, the speed of the man relative to the platform is $V + w$. By the question,

$$V + w = v, \text{ or } w = v - V. \quad \ldots \text{ (i)}$$

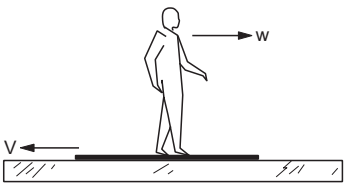



Figure 9-W10

Taking the platform and the man to be the system, there is no external horizontal force on the system. The linear momentum of the system remains constant. Initially, both the man and the platform were at rest. Thus,

$$0 = MV - mw$$

or, $$MV = m\,(v - V) \quad \text{[Using (i)]}$$

or, $$V = \frac{m\,v}{M + m}.$$

**16.** *A ball of mass m, moving with a velocity v along X-axis, strikes another ball of mass* 2*m kept at rest. The first ball comes to rest after collision and the other breaks into two equal pieces. One of the pieces starts moving along Y-axis with a speed* $v_1$*. What will be the velocity of the other piece ?*

***Solution*** **:** The total linear momentum of the balls before the collision is $mv$ along the $X$-axis. After the collision, momentum of the first ball = 0, momentum of the first piece $= m\,v_1$ along the $Y$-axis and momentum of the second piece $= m\,v_2$ along its direction of motion where $v_2$ is the speed of the second piece. These three should add to $mv$ along the $X$-axis, which is the initial momentum of the system.

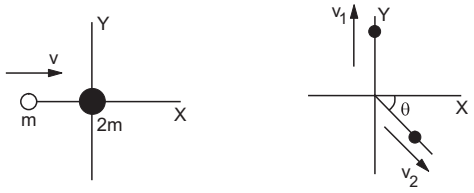



Figure 9-W11

Taking components along the $X$-axis,

$$m\,v_2 \cos\theta = m\,v \quad \ldots \text{ (i)}$$

and taking components along the $Y$-axis,

$$m\,v_2 \sin\theta = m\,v_1. \quad \ldots \text{ (ii)}$$

From (i) and (ii),

$$v_2 = \sqrt{v^2 + v_1^2} \text{ and } \tan\theta = v_1 / v.$$

**17.** *A bullet of mass* 50 g *is fired from below into the bob of mass* 450 g *of a long simple pendulum as shown in figure (9-W12). The bullet remains inside the bob and the bob rises through a height of* 1·8 m. *Find the speed of the bullet. Take* $g = 10 \text{ m/s}^2$.

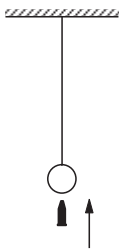


Figure 9-W12

***Solution*** : Let the speed of the bullet be $v$. Let the common velocity of the bullet and the bob, after the bullet is embedded into the bob, is $V$. By the principle of conservation of linear momentum,

$$V = \frac{(0{\cdot}05\ \text{kg})v}{0{\cdot}45\ \text{kg} + 0{\cdot}05\ \text{kg}} = \frac{v}{10}.$$

The string becomes loose and the bob will go up with a deceleration of $g = 10\ \text{m/s}^2$. As it comes to rest at a height of 1·8 m, using the equation $v^2 = u^2 + 2ax$,

$$1{\cdot}8\ \text{m} = \frac{(v/10)^2}{2 \times 10\ \text{m/s}^2}$$

or, $$v = 60\ \text{m/s}.$$

**18.** *A light spring of spring constant k is kept compressed between two blocks of masses m and M on a smooth horizontal surface (figure 9-W13). When released, the blocks acquire velocities in opposite directions. The spring loses contact with the blocks when it acquires natural length. If the spring was initially compressed through a distance x, find the final speeds of the two blocks.*

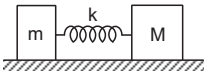



Figure 9-W13

***Solution*** : Consider the two blocks plus the spring to be the system. No external force acts on this system in horizontal direction. Hence, the linear momentum will remain constant. As the spring is light, it has no linear momentum. Suppose the block of mass $M$ moves with a speed $V$ and the other block with a speed $v$ after losing contact with the spring. As the blocks are released from rest, the initial momentum is zero. The final momentum is $MV - mv$ towards right. Thus,

$$MV - mv = 0 \quad \text{or,} \quad V = \frac{m}{M}v. \qquad \ldots \text{(i)}$$

Initially, the energy of the system $= \frac{1}{2}kx^2$.

Finally, the energy of the system $= \frac{1}{2}mv^2 + \frac{1}{2}MV^2$.

As there is no friction,

$$\frac{1}{2}mv^2 + \frac{1}{2}MV^2 = \frac{1}{2}kx^2. \qquad \ldots \text{(ii)}$$

Using (i) and (ii),

$$mv^2\left(1 + \frac{m}{M}\right) = k\,x^2$$

or, $$v = \sqrt{\frac{kM}{m\,(M+m)}}\,x$$

and $$V = \sqrt{\frac{km}{M\,(M+m)}}\,x.$$

**19.** *A block of mass m is connected to another block of mass M by a massless spring of spring constant k. The blocks are kept on a smooth horizontal plane. Initially, the blocks are at rest and the spring is unstretched when a constant force F starts acting on the block of mass M to pull it. Find the maximum extension of the spring.*

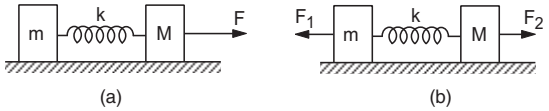



Figure 9-W14

***Solution*** : Let us take the two blocks plus the spring as the system. The centre of mass of the system moves with an acceleration $a = \frac{F}{m+M}$. Let us work from a reference frame with its origin at the centre of mass. As this frame is accelerated with respect to the ground we have to apply a pseudo force $ma$ towards left on the block of mass $m$ and $Ma$ towards left on the block of mass $M$. The net external force on $m$ is

$$F_1 = ma = \frac{mF}{m+M} \text{ towards left}$$

and the net external force on $M$ is

$$F_2 = F - Ma = F - \frac{MF}{m+M} = \frac{mF}{m+M} \text{ towards right.}$$

The situation from this frame is shown in figure (9-W14b). As the centre of mass is at rest in this frame, the blocks move in opposite directions and come to instantaneous rest at some instant. The extension of the spring will be maximum at this instant. Suppose the left block is displaced through a distance $x_1$ and the right block through a distance $x_2$ from the initial positions. The total work done by the external forces $F_1$ and $F_2$ in this period are

$$W = F_1 x_1 + F_2 x_2 = \frac{mF}{m+M}(x_1 + x_2).$$

This should be equal to the increase in the potential energy of the spring as there is no change in the kinetic energy. Thus,

$$\frac{mF}{m+M}(x_1 + x_2) = \frac{1}{2}k\,(x_1 + x_2)^2$$

or, $$x_1 + x_2 = \frac{2\,mF}{k\,(m+M)}.$$

This is the maximum extension of the spring.

**20.** *The two balls shown in figure (9-W15) are identical, the first moving at a speed v towards right and the second staying at rest. The wall at the extreme right is fixed. Assume all collisions to be elastic. Show that the speeds of the balls remain unchanged after all the collisions have taken place.*

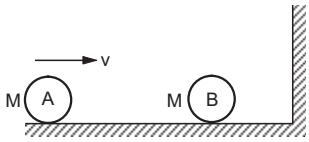



Figure 9-W15

***Solution*** **:** **1st collision :** As the balls have equal mass and make elastic collision, the velocities are interchanged.

Hence, after the first collision, the ball $A$ comes to rest and the ball $B$ moves towards right at a speed $v$.

**2nd collision :** The ball $B$ moving with a speed $v$, collides with the wall and rebounds. As the wall is rigid and may be taken to be of infinite mass, momentum conservation gives no useful result. Velocity of separation should be equal to the velocity of approach. Hence, the ball rebounds at the same speed $v$ towards left.

**3rd collision :** The ball $B$ moving towards left at the speed $v$ again collides with the ball $A$ kept at rest. As the masses are equal and the collision is elastic, the velocities are interchanged. Thus, the ball $B$ comes to rest and the ball $A$ moves towards left at a speed $v$. No further collision takes place. Thus, the speeds of the balls remain the same as their initial values.

**21.** *A block of mass m moving at a velocity v collides head on with another block of mass* 2*m at rest. If the coefficient of restitution is* 1/2, *find the velocities of the blocks after the collision.*

***Solution*** **:** Suppose after the collision the block of mass $m$ moves at a velocity $u_1$ and the block of mass $2m$ moves at a velocity $u_2$. By conservation of momentum,

$$mv = mu_1 + 2mu_2. \qquad \ldots \text{(i)}$$

The velocity of separation is $u_2 - u_1$ and the velocity of approach is $v$.

So, $$u_2 - u_1 = v/2. \qquad \ldots \text{(ii)}$$

From (i) and (ii), $u_1 = 0$ and $u_2 = v/2$.

**22.** *A block of mass* 1·2 kg *moving at a speed of* 20 cm/s *collides head-on with a similar block kept at rest. The coefficient of restitution is* 3/5. *Find the loss of kinetic energy during the collision.*

***Solution*** **:** Suppose the first block moves at a speed $v_1$ and the second at $v_2$ after the collision. Since the collision is head-on, the two blocks move along the original direction of motion of the first block.

By conservation of linear momentum,

$$(1{\cdot}2 \text{ kg})\,(20 \text{ cm/s}) = (1{\cdot}2 \text{ kg})\,v_1 + (1{\cdot}2 \text{ kg})\,v_2$$

or, $$v_1 + v_2 = 20 \text{ cm/s}. \qquad \ldots \text{(i)}$$

The velocity of separation is $v_2 - v_1$ and the velocity of approach is 20 cm/s. As the coefficient of restitution is 3/5, we have,

$$v_2 - v_1 = (3/5) \times 20 \text{ cm/s} = 12 \text{ cm/s}. \qquad \ldots \text{(ii)}$$

By (i) and (ii),

$$v_1 = 4 \text{ cm/s} \quad \text{and} \quad v_2 = 16 \text{ cm/s}.$$

The loss in kinetic energy is

$$\frac{1}{2}(1{\cdot}2 \text{ kg})[(20 \text{ cm/s})^2 - (4 \text{ cm/s})^2 - (16 \text{ cm/s})^2]$$

$$= (0{\cdot}6 \text{ kg})\,[0{\cdot}04 \text{ m}^2/\text{s}^2 - 0{\cdot}0016 \text{ m}^2/\text{s}^2 - 0{\cdot}0256 \text{ m}^2/\text{s}^2]$$

$$= (0{\cdot}6 \text{ kg})\,(0{\cdot}0128 \text{ m}^2/\text{s}^2) = 7{\cdot}7 \times 10^{-3} \text{ J}.$$

**23.** *A ball of mass m hits the floor with a speed v making an angle of incidence* $\theta$ *with the normal. The coefficient of restitution is e. Find the speed of the reflected ball and the angle of reflection of the ball.*

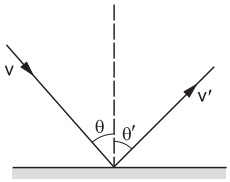



Figure 9-W16

***Solution*** **:** See figure (9-W16). Suppose the angle of reflection is $\theta'$ and the speed after the collision is $v'$. The floor exerts a force on the ball along the normal during the collision. There is no force parallel to the surface. Thus, the parallel component of the velocity of the ball remains unchanged. This gives

$$v' \sin\theta' = v \sin\theta. \qquad \ldots \text{(i)}$$

For the components normal to the floor,

the velocity of separation $= v' \cos\theta'$

and the velocity of approach $= v \cos\theta$.

Hence, $$v' \cos\theta' = e\, v \cos\theta. \qquad \ldots \text{(ii)}$$

From (i) and (ii),

$$v' = v\sqrt{\sin^2\theta + e^2 \cos^2\theta}$$

and $$\tan\theta' = \frac{\tan\theta}{e}.$$

For elastic collision, $e = 1$ so that $\theta' = \theta$ and $v' = v$.

**24.** *A block of mass m and a pan of equal mass are connected by a string going over a smooth light pulley as shown in figure (9-W17). Initially the system is at rest when a particle of mass m falls on the pan and sticks to it. If the particle strikes the pan with a speed v find the speed with which the system moves just after the collision.*

***Solution*** **:** Let the required speed be $V$.

As there is a sudden change in the speed of the block, the tension must change by a large amount during the collision.

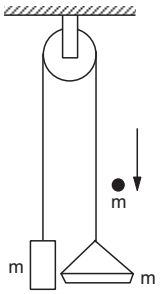



Figure 9-W17

Let $N$ = magnitude of the contact force between the particle and the pan

$T$ = tension in the string

Consider the impulse imparted to the particle. The force is $N$ in upward direction and the impulse is $\int N\,dt$. This should be equal to the change in its momentum.

Thus, $$\int N\,dt = mv - mV. \quad \ldots \text{(i)}$$

Similarly considering the impulse imparted to the pan,

$$\int (N - T)dt = mV \quad \ldots \text{(ii)}$$

and that to the block,

$$\int T\,dt = mV. \quad \ldots \text{(iii)}$$

Adding (ii) and (iii),

$$\int N\,dt = 2\,mV.$$

Comparing with (i),

$$mv - mV = 2mV$$

or, $$V = v/3.$$

□

## QUESTIONS FOR SHORT ANSWER

**1.** Can the centre of mass of a body be at a point outside the body ?

**2.** If all the particles of a system lie in $X$-$Y$ plane, is it necessary that the centre of mass be in $X$-$Y$ plane ?

**3.** If all the particle of a system lie in a cube, is it neccesary that the centre of mass be in the cube ?

**4.** The centre of mass is defined as $\vec{R} = \frac{1}{M}\sum_i m_i \vec{r_i}$. Suppose we define "centre of charge" as $\vec{R}_C = \frac{1}{Q}\sum_i q_i \vec{r_i}$ where $q_i$ represents the $i$th charge placed at $\vec{r_i}$ and $Q$ is the total charge of the system.
(a) Can the centre of charge of a two-charge system be outside the line segment joining the charges ?
(b) If all the charges of a system are in $X$-$Y$ plane, is it necessary that the centre of charge be in $X$-$Y$ plane ?
(c) If all the charges of a system lie in a cube, is it necessary that the centre of charge be in the cube ?

**5.** The weight $Mg$ of an extended body is generally shown in a diagram to act through the centre of mass. Does it mean that the earth does not attract other particles ?

**6.** A bob suspended from the ceiling of a car which is accelerating on a horizontal road. The bob stays at rest with respect to the car with the string making an angle $\theta$ with the vertical. The linear momentum of the bob as seen from the road is increasing with time. Is it a violation of conservation of linear momentum ? If not, where is the external force which changes the linear momentum ?

**7.** You are waiting for a train on a railway platform. Your three-year-old niece is standing on your iron trunk containing the luggage. Why does the trunk not recoil as she jumps off on the platform ?

**8.** In a head-on collision between two particles, is it necessary that the particles will acquire a common velocity at least for one instant ?

**9.** A collision experiment is done on a horizontal table kept in an elevator. Do you expect a change in the results if the elevator is accelerated up or down because of the noninertial character of the frame ?

**10.** Two bodies make an elastic head-on collision on a smooth horizontal table kept in a car. Do you expect a change in the result if the car is accelerated on a horizontal road because of the noninertial character of the frame ? Does the equation "Velocity of separation = Velocity of approach" remain valid in an accelerating car ? Does the equation "final momentum = initial momentum" remain valid in the accelerating car ?

**11.** If the total mechanical energy of a particle is zero, is its linear momentum necessarily zero ? Is it necessarily nonzero ?

**12.** If the linear momentum of a particle is known, can you find its kinetic energy ? If the kinetic energy of a particle is known can you find its linear momentum ?

**13.** What can be said about the centre of mass of a uniform hemisphere without making any calculation ? Will its distance from the centre be more than $r/2$ or less than $r/2$ ?

**14.** You are holding a cage containing a bird. Do you have to make less effort if the bird flies from its position in the cage and manages to stay in the middle without touching the walls of the cage ? Does it make a difference whether the cage is completely closed or it has rods to let air pass ?

**15.** A fat person is standing on a light plank floating on a calm lake. The person walks from one end to the other on the plank. His friend sitting on the shore watches him and finds that the person hardly moves any distance because the plank moves backward about the same distance as the person moves on the plank. Explain.

**16.** A high-jumper successfully clears the bar. Is it possible that his centre of mass crossed the bar from below it ? Try it with appropriate figures.

**17.** Which of the two persons shown in figure (9-Q1) is more likely to fall down ? Which external force is responsible for his falling down ?

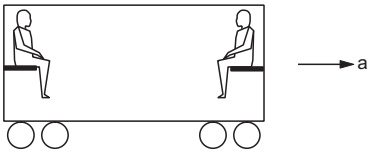



Figure 9-Q1

**18.** Suppose we define a quantity 'Linear Momentum' as

linear momentum = mass × speed.

The linear momentum of a system of particles is the sum of linear momenta of the individual particles. Can we state a principle of conservation of linear momentum as "linear momentum of a system remains constant if no external force acts on it" ?

**19.** Use the definition of linear momentum from the previous question. Can we state the principle of conservation of linear momentum for a single particle ?

**20.** To accelerate a car we ignite petrol in the engine of the car. Since only an external force can accelerate the centre of mass, is it proper to say that "the force generated by the engine accelerates the car" ?

**21.** A ball is moved on a horizontal table with some velocity. The ball stops after moving some distance. Which external force is responsible for the change in the momentum of the ball ?

**22.** Consider the situation of the previous problem. Take "the table plus the ball" as the system. Friction between the table and the ball is then an internal force. As the ball slows down, the momentum of the system decreases. Which external force is responsible for this change in the momentum ?

**23.** When a nucleus at rest emits a beta particle, it is found that the velocities of the recoiling nucleus and the beta particle are not along the same straight line. How can this be possible in view of the principle of conservation of momentum ?

**24.** A van is standing on a frictionless portion of a horizontal road. To start the engine, the vehicle must be set in motion in the forward direction. How can the persons sitting inside the van do it without coming out and pushing from behind ?

**25.** In one-dimensional elastic collision of equal masses, the velocities are interchanged. Can velocities in a one-dimensional collision be interchanged if the masses are not equal ?

## OBJECTIVE I

**1.** Consider the following two equations :

(A) $\vec{R} = \frac{1}{M}\sum_i m_i \vec{r_i}$

and (B) $\vec{a}_{CM} = \frac{\vec{F}}{M}$.

In a noninertial frame

(a) both are correct (b) both are wrong

(c) $A$ is correct but $B$ is wrong

(d) $B$ is correct but $A$ is wrong.

**2.** Consider the following two statements :

(A) Linear momentum of the system remains constant.

(B) Centre of mass of the system remains at rest.

(a) $A$ implies $B$ and $B$ implies $A$.

(b) $A$ does not imply $B$ and $B$ does not imply $A$.

(c) $A$ implies $B$ but $B$ does not imply $A$.

(d) $B$ implies $A$ but $A$ does not imply $B$.

**3.** Consider the following two statements :

(A) Linear momentum of a system of particles is zero.

(B) Kinetic energy of a system of particles is zero.

(a) $A$ implies $B$ and $B$ implies $A$.

(b) $A$ does not imply $B$ and $B$ does not imply $A$.

(c) $A$ implies $B$ but $B$ does not imply $A$.

(d) $B$ implies $A$ but $A$ does not imply $B$.

**4.** Consider the following two statements :

(A) The linear momentum of a particle is independent of the frame of reference.

(B) The kinetic energy of a particle is independent of the frame of reference.

(a) Both $A$ and $B$ are true. (b) $A$ is true but $B$ is false.

(c) $A$ is false but $B$ is true. (d) both $A$ and $B$ are false.

**5.** All the particles of a body are situated at a distance $R$ from the origin. The distance of the centre of mass of the body from the origin is

(a) $= R$ (b) $\leq R$ (c) $> R$ (d) $\geq R$.

**6.** A circular plate of diameter $d$ is kept in contact with a square plate of edge $d$ as shown in figure (9-Q2). The density of the material and the thickness are same

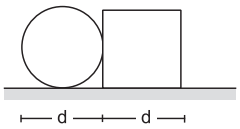



Figure 9-Q2

everywhere. The centre of mass of the composite system will be

(a) inside the circular plate (b) inside the square plate
(c) at the point of contact (d) outside the system.

**7.** Consider a system of two identical particles. One of the particles is at rest and the other has an acceleration $\vec{a}$. The centre of mass has an acceleration

(a) zero (b) $\frac{1}{2}\vec{a}$ (c) $\vec{a}$ (d) $2\vec{a}$.

**8.** Internal forces can change
(a) the linear momentum but not the kinetic energy
(b) the kinetic energy but not the linear momentum
(c) linear momentum as well as kinetic energy
(d) neither the linear momentum nor the kinetic energy.

**9.** A bullet hits a block kept at rest on a smooth horizontal surface and gets embedded into it. Which of the following does not change ?
(a) linear momentum of the block
(b) kinetic energy of the block
(c) gravitational potential energy of the block
(d) temperature of the block.

**10.** A uniform sphere is placed on a smooth horizontal surface and a horizontal force $F$ is applied on it at a distance $h$ above the surface. The acceleration of the centre
(a) is maximum when $h = 0$
(b) is maximum when $h = R$
(c) is maximum when $h = 2R$
(d) is independent of $h$.

**11.** A body falling vertically downwards under gravity breaks in two parts of unequal masses. The centre of mass of the two parts taken together shifts horizontally towards

(a) heavier piece (b) lighter piece
(c) does not shift horizontally
(d) depends on the vertical velocity at the time of breaking.

**12.** A ball kept in a closed box moves in the box making collisions with the walls. The box is kept on a smooth surface. The velocity of the centre of mass
(a) of the box remains constant
(b) of the box plus the ball system remains constant
(c) of the ball remains constant
(d) of the ball relative to the box remains constant.

**13.** A body at rest breaks into two pieces of equal masses. The parts will move
(a) in same direction (b) along different lines
(c) in opposite directions with equal speeds
(d) in opposite directions with unequal speeds.

**14.** A heavy ring of mass $m$ is clamped on the periphery of a light circular disc. A small particle having equal mass is clamped at the centre of the disc. The system is rotated in such a way that the centre moves in a circle of radius $r$ with a uniform speed $v$. We conclude that an external force
(a) $\frac{mv^2}{r}$ must be acting on the central particle
(b) $\frac{2mv^2}{r}$ must be acting on the central particle
(c) $\frac{2mv^2}{r}$ must be acting on the system
(d) $\frac{2mv^2}{r}$ must be acting on the ring.

**15.** The quantities remaining constant in a collision are
(a) momentum, kinetic energy and temperature
(b) momentum and kinetic energy but not temperature
(c) momentum and temperature but not kinetic energy
(d) momentum, but neither kinetic energy nor temperature.

**16.** A nucleus moving with a velocity $\vec{v}$ emits an α-particle. Let the velocities of the α-particle and the remaining nucleus be $\vec{v}_1$ and $\vec{v}_2$ and their masses be $m_1$ and $m_2$.
(a) $\vec{v}$, $\vec{v}_1$ and $\vec{v}_2$ must be parallel to each other.
(b) None of the two of $\vec{v}$, $\vec{v}_1$ and $\vec{v}_2$ should be parallel to each other.
(c) $\vec{v}_1 + \vec{v}_2$ must be parallel to $\vec{v}$.
(d) $m_1\vec{v}_1 + m_2\vec{v}_2$ must be parallel to $\vec{v}$.

**17.** A shell is fired from a cannon with a velocity $V$ at an angle θ with the horizontal direction. At the highest point in its path, it explodes into two pieces of equal masses. One of the pieces retraces its path to the cannon. The speed of the other piece immediately after the explosion is

(a) $3V\cos\theta$ (b) $2V\cos\theta$ (c) $\frac{3}{2}V\cos\theta$ (d) $V\cos\theta$.

**18.** In an elastic collision
(a) the initial kinetic energy is equal to the final kinetic energy
(b) the final kinetic energy is less than the initial kinetic energy
(c) the kinetic energy remains constant
(d) the kinetic energy first increases then decreases.

**19.** In an inelastic collsion
(a) the initial kinetic energy is equal to the final kinetic energy
(b) the final kinetic energy is less than the initial kinetic energy
(c) the kinetic energy remains the constant
(d) the kinetic energy first increases then decreases.

## OBJECTIVE II

**1.** The centre of mass of a system of particles is at the origin. It follows that
(a) the number of particles to the right of the origin is equal to the number of particles to the left
(b) the total mass of the particles to the right of the origin is same as the total mass to the left of the origin

(c) the number of particles on $X$-axis should be equal to the number of particles on $Y$-axis
(d) if there is a particle on the positive $X$-axis, there must be at least one particle on the negative $X$-axis.

**2.** A body has its centre of mass at the origin. The $x$-coordinates of the particles
(a) may be all positive (b) may be all negative
(c) may be all non-negative
(d) may be positive for some case and negative in other cases.

**3.** In which of the following cases the centre of mass of a rod is certainly not at its centre ?
(a) the density continuously increases from left to right
(b) the density continuously decreases from left to right
(c) the density decreases from left to right upto the centre and then increases
(d) the density increases from left to right upto the centre and then decreases.

**4.** If the external forces acting on a system have zero resultant, the centre of mass
(a) must not move (b) must not accelerate
(c) may move (d) may accelerate.

**5.** A nonzero external force acts on a system of particles. The velocity and the acceleration of the centre of mass are found to be $v_0$ and $a_0$ at an instant $t$. It is possible that
(a) $v_0 = 0$, $a_0 = 0$ (b) $v_0 = 0$, $a_0 \neq 0$,
(c) $v_0 \neq 0$, $a_0 = 0$ (d) $v_0 \neq 0$, $a_0 \neq 0$.

**6.** Two balls are thrown simultaneously in air. The acceleration of the centre of mass of the two balls while in air
(a) depends on the direction of the motion of the balls
(b) depends on the masses of the two balls
(c) depends on the speeds of the two balls
(d) is equal to $g$.

**7.** A block moving in air breaks in two parts and the parts separate
(a) the total momentum must be conserved
(b) the total kinetic energy must be conserved
(c) the total momentum must change
(d) the total kinetic energy must change.

**8.** In an elastic collision
(a) the kinetic energy remains constant
(b) the linear momentum remains constant
(c) the final kinetic energy is equal to the initial kinetic energy
(d) the final linear momentum is equal to the initial linear momentum.

**9.** A ball hits a floor and rebounds after an inelastic collision. In this case
(a) the momentum of the ball just after the collision is same as that just before the collision
(b) the mechanical energy of the ball remains the same during the collision
(c) the total momentum of the ball and the earth is conserved
(d) the total energy of the ball and the earth remains the same.

**10.** A body moving towards a finite body at rest collides with it. It is possible that
(a) both the bodies come to rest
(b) both the bodies move after collision
(c) the moving body comes to rest and the stationary body starts moving
(d) the stationary body remains stationary, the moving body changes its velocity.

**11.** In a head-on elastic collision of two bodies of equal masses
(a) the velocities are interchanged
(b) the speeds are interchanged
(c) the momenta are interchanged
(d) the faster body slows down and the slower body speeds up.

## EXERCISES

**1.** Three particles of masses 1·0 kg, 2·0 kg and 3·0 kg are placed at the corners $A$, $B$ and $C$ respectively of an equilateral triangle $ABC$ of edge 1 m. Locate the centre of mass of the system.

**2.** The structure of a water molecule is shown in figure (9-E1). Find the distance of the centre of mass of the molecule from the centre of the oxygen atom.

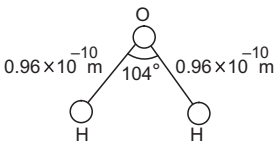



Figure 9-E1

**3.** Seven homogeneous bricks, each of length $L$, are arranged as shown in figure (9-E2). Each brick is displaced with respect to the one in contact by $L/10$. Find the $x$-coordinate of the centre of mass relative to the origin shown.

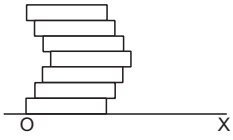



Figure 9-E2

**4.** A uniform disc of radius $R$ is put over another uniform disc of radius $2R$ of the same thickness and density. The

peripheries of the two discs touch each other. Locate the centre of mass of the system.

**5.** A disc of radius $R$ is cut out from a larger disc of radius $2R$ in such a way that the edge of the hole touches the edge of the disc. Locate the centre of mass of the residual disc.

**6.** A square plate of edge $d$ and a circular disc of diameter $d$ are placed touching each other at the midpoint of an edge of the plate as shown in figure (9-Q2). Locate the centre of mass of the combination, assuming same mass per unit area for the two plates.

**7.** Calculate the velocity of the centre of mass of the system of particles shown in figure (9-E3).

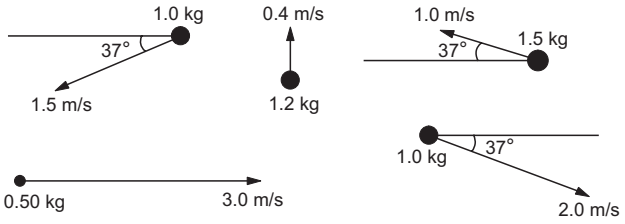



Figure 9-E3

**8.** Two blocks of masses 10 kg and 20 kg are placed on the $X$-axis. The first mass is moved on the axis by a distance of 2 cm. By what distance should the second mass be moved to keep the position of the centre of mass unchanged ?

**9.** Two blocks of masses 10 kg and 30 kg are placed along a vertical line. The first block is raised through a height of 7 cm. By what distance should the second mass be moved to raise the centre of mass by 1 cm ?

**10.** Consider a gravity-free hall in which a tray of mass $M$, carrying a cubical block of ice of mass $m$ and edge $L$, is at rest in the middle (figure 9-E4). If the ice melts, by what distance does the centre of mass of "the tray plus the ice" system descend ?

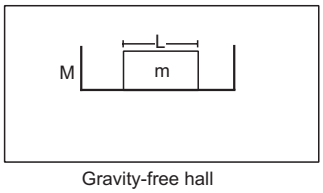



Figure 9-E4

**11.** Find the centre of mass of a uniform plate having semicircular inner and outer boundaries of radii $R_1$ and $R_2$ (figure 9-E5).

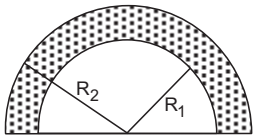



Figure 9-E5

**12.** Mr. Verma (50 kg) and Mr. Mathur (60 kg) are sitting at the two extremes of a 4 m long boat (40 kg) standing still in water. To discuss a mechanics problem, they come to the middle of the boat. Neglecting friction with water, how far does the boat move on the water during the process ?

**13.** A cart of mass $M$ is at rest on a frictionless horizontal surface and a pendulum bob of mass $m$ hangs from the roof of the cart (figure 9-E6). The string breaks, the bob falls on the floor, makes several collisions on the floor and finally lands up in a small slot made in the floor. The horizontal distance between the string and the slot is $L$. Find the displacement of the cart during this process.

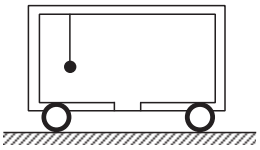


Figure 9-E6

**14.** The balloon, the light rope and the monkey shown in figure (9-E7) are at rest in the air. If the monkey reaches the top of the rope, by what distance does the balloon descend ? Mass of the balloon = $M$, mass of the monkey = $m$ and the length of the rope ascended by the monkey = $L$.

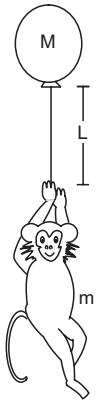



Figure 9-E7

**15.** Find the ratio of the linear momenta of two particles of masses 1·0 kg and 4·0 kg if their kinetic energies are equal.

**16.** A uranium-238 nucleus, initially at rest, emits an alpha particle with a speed of $1{\cdot}4 \times 10^7$ m/s. Calculate the recoil speed of the residual nucleus thorium-234. Assume that the mass of a nucleus is proportional to the mass number.

**17.** A man of mass 50 kg starts moving on the earth and acquires a speed of 1·8 m/s. With what speed does the earth recoil ? Mass of earth = $6 \times 10^{24}$ kg.

**18.** A neutron initially at rest, decays into a proton, an electron and an antineutrino. The ejected electron has a momentum of $1{\cdot}4 \times 10^{-26}$ kg-m/s and the antineutrino

$6{\cdot}4 \times 10^{-27}$ kg-m/s. Find the recoil speed of the proton (a) if the electron and the antineutrino are ejected along the same direction and (b) if they are ejected along perpendicular directions. Mass of the proton $= 1{\cdot}67 \times 10^{-27}$ kg.

**19.** A man of mass $M$ having a bag of mass $m$ slips from the roof of a tall building of height $H$ and starts falling vertically (figure 9-E8). When at a height $h$ from the ground, he notices that the ground below him is pretty hard, but there is a pond at a horizontal distance $x$ from the line of fall. In order to save himself he throws the bag horizontally (with respect to himself) in the direction opposite to the pond. Calculate the minimum horizontal velocity imparted to the bag so that the man lands in the water. If the man just succeeds to avoid the hard ground, where will the bag land ?

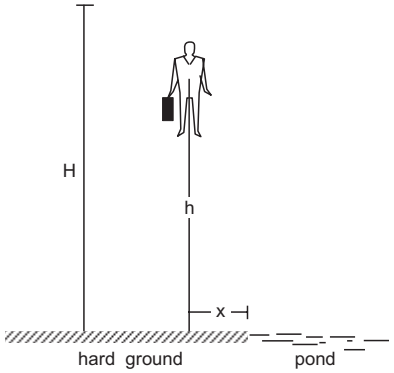



Figure 9-E8

**20.** A ball of mass 50 g moving at a speed of 2·0 m/s strikes a plane surface at an angle of incidence 45°. The ball is reflected by the plane at equal angle of reflection with the same speed. Calculate (a) the magnitude of the change in momentum of the ball (b) the change in the magnitude of the momentum of the ball.

**21.** Light in certain cases may be considered as a stream of particles called photons. Each photon has a linear momentum $h/\lambda$ where $h$ is the Planck's constant and $\lambda$ is the wavelength of the light. A beam of light of wavelength $\lambda$ is incident on a plane mirror at an angle of incidence $\theta$. Calculate the change in the linear momentum of a photon as the beam is reflected by the mirror.

**22.** A block at rest explodes into three equal parts. Two parts start moving along $X$ and $Y$ axes respectively with equal speeds of 10 m/s. Find the initial velocity of the third part.

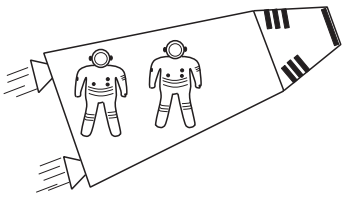


Figure 9-E9

**23.** Two fat astronauts each of mass 120 kg are travelling in a closed spaceship moving at a speed of 15 km/s in the outer space far removed from all other material objects. The total mass of the spaceship and its contents including the astronauts is 660 kg. If the astronauts do slimming exercise and thereby reduce their masses to 90 kg each, with what velocity will the spaceship move ?

**24.** During a heavy rain, hailstones of average size 1·0 cm in diameter fall with an average speed of 20 m/s. Suppose 2000 hailstones strike every square meter of a $10\text{ m} \times 10\text{ m}$ roof perpendicularly in one second and assume that the hailstones do not rebound. Calculate the average force exerted by the falling hailstones on the roof. Density of a hailstone is 900 kg/m$^3$.

**25.** A ball of mass $m$ is dropped onto a floor from a certain height. The collision is perfectly elastic and the ball rebounds to the same height and again falls. Find the average force exerted by the ball on the floor during a long time interval.

**26.** A railroad car of mass $M$ is at rest on frictionless rails when a man of mass $m$ starts moving on the car towards the engine. If the car recoils with a speed $v$ backward on the rails, with what velocity is the man approaching the engine ?

**27.** A gun is mounted on a railroad car. The mass of the car, the gun, the shells and the operator is $50\,m$ where $m$ is the mass of one shell. If the velocity of the shell with respect to the gun (in its state before firing) is 200 m/s, what is the recoil speed of the car after the second shot ? Neglect friction.

**28.** Two persons each of mass $m$ are standing at the two extremes of a railroad car of mass $M$ resting on a smooth track (figure 9-E10). The person on left jumps to the left with a horizontal speed $u$ with respect to the state of the car before the jump. Thereafter, the other person jumps to the right, again with the same horizontal speed $u$ with respect to the state of the car before his jump. Find the velocity of the car after both the persons have jumped off.

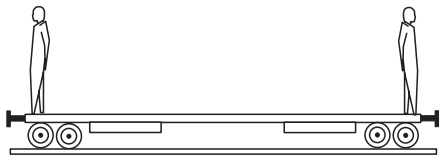


Figure 9-E10

**29.** Figure (9-E11) shows a small block of mass $m$ which is started with a speed $v$ on the horizontal part of the bigger block of mass $M$ placed on a horizontal floor. The curved part of the surface shown is semicircular. All the surfaces are frictionless. Find the speed of the bigger block when the smaller block reaches the point $A$ of the surface.

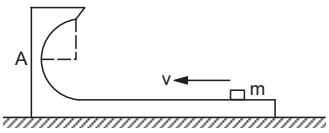



Figure 9-E11

**30.** In a typical Indian *Bugghi* (a luxury cart drawn by horses), a wooden plate is fixed on the rear on which one person can sit. A bugghi of mass 200 kg is moving at a speed of 10 km/h. As it overtakes a school boy walking at a speed of 4 km/h, the boy sits on the wooden plate. If the mass of the boy is 25 kg, what will be the new velocity of the bugghi ?

**31.** A ball of mass 0·50 kg moving at a speed of 5·0 m/s collides with another ball of mass 1·0 kg. After the collision the balls stick together and remain motionless. What was the velocity of the 1·0 kg block before the collision ?

**32.** A 60 kg man skating with a speed of 10 m/s collides with a 40 kg skater at rest and they cling to each other. Find the loss of kinetic energy during the collision.

**33.** Consider a head-on collision between two particles of masses $m_1$ and $m_2$. The initial speeds of the particles are $u_1$ and $u_2$ in the same direction. The collision starts at $t = 0$ and the particles interact for a time interval $\Delta t$. During the collision, the speed of the first particle varies as

$$v(t) = u_1 + \frac{t}{\Delta t}(v_1 - u_1).$$

Find the speed of the second particle as a function of time during the collision.

**34.** A bullet of mass $m$ moving at a speed $v$ hits a ball of mass $M$ kept at rest. A small part having mass $m'$ breaks from the ball and sticks to the bullet. The remaining ball is found to move at a speed $v_1$ in the direction of the bullet. Find the velocity of the bullet after the collision.

**35.** A ball of mass $m$ moving at a speed $v$ makes a head-on collision with an identical ball at rest. The kinetic energy of the balls after the collision is three fourths of the original. Find the coefficient of restitution.

**36.** A block of mass 2·0 kg moving at 2·0 m/s collides head on with another block of equal mass kept at rest. (a) Find the maximum possible loss in kinetic energy due to the collision. (b) If the actual loss in kinetic energy is half of this maximum, find the coefficient of restitution.

**37.** A particle of mass 100 g moving at an initial speed $u$ collides with another particle of same mass kept initially at rest. If the total kinetic energy becomes 0·2 J after the collision, what could be the minimum and the maximum value of $u$.

**38.** Two friends $A$ and $B$ (each weighing 40 kg) are sitting on a frictionless platform some distance $d$ apart. $A$ rolls a ball of mass 4 kg on the platform towards $B$ which $B$ catches. Then $B$ rolls the ball towards $A$ and $A$ catches it. The ball keeps on moving back and forth between $A$ and $B$. The ball has a fixed speed of 5 m/s on the platform. (a) Find the speed of $A$ after he rolls the ball for the first time. (b) Find the speed of $A$ after he catches the ball for the first time. (c) Find the speeds of $A$ and $B$ after the ball has made 5 round trips and is held by $A$. (d) How many times can $A$ roll the ball ? (e) Where is the centre of mass of the system "$A$ + $B$ + ball" at the end of the $n$th trip ?

**39.** A ball falls on the ground from a height of 2·0 m and rebounds up to a height of 1·5 m. Find the coefficient of restitution.

**40.** In a gamma decay process, the internal energy of a nucleus of mass $M$ decreases, a gamma photon of energy $E$ and linear momentum $E/c$ is emitted and the nucleus recoils. Find the decrease in internal energy.

**41.** A block of mass 2·0 kg is moving on a frictionless horizontal surface with a velocity of 1·0 m/s (figure 9-E12) towards another block of equal mass kept at rest. The spring constant of the spring fixed at one end is 100 N/m. Find the maximum compression of the spring.

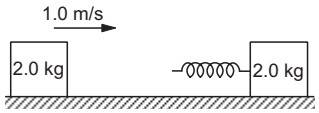



Figure 9-E12

**42.** A bullet of mass 20 g travelling horizontally with a speed of 500 m/s passes through a wooden block of mass 10·0 kg initially at rest on a level surface. The bullet emerges with a speed of 100 m/s and the block slides 20 cm on the surface before coming to rest. Find the friction coefficient between the block and the surface (figure 9-E13).

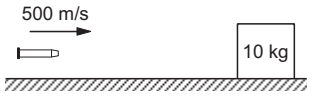



Figure 9-E13

**43.** A projectile is fired with a speed $u$ at an angle $\theta$ above a horizontal field. The coefficient of restitution of collision between the projectile and the field is $e$. How far from the starting point, does the projectile makes its second collision with the field ?

**44.** A ball falls on an inclined plane of inclination $\theta$ from a height $h$ above the point of impact and makes a perfectly elastic collision. Where will it hit the plane again ?

**45.** Solve the previous problem if the coefficient of restitution is $e$. Use $\theta = 45°$, $e = \frac{3}{4}$ and $h = 5$ m.

**46.** A block of mass 200 g is suspended through a vertical spring. The spring is stretched by 1·0 cm when the block is in equilibrium. A particle of mass 120 g is dropped on the block from a height of 45 cm. The particle sticks to the block after the impact. Find the maximum extension of the spring. Take $g = 10$ m/s$^2$.

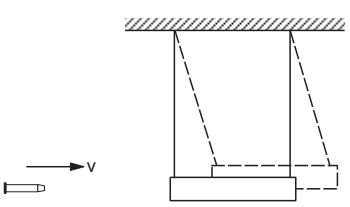



Figure 9-E14

**47.** A bullet of mass 25 g is fired horizontally into a ballistic pendulum of mass 5·0 kg and gets embedded in it (figure 9-E14). If the centre of the pendulum rises by a distance of 10 cm, find the speed of the bullet.

**48.** A bullet of mass 20 g moving horizontally at a speed of 300 m/s is fired into a wooden block of mass 500 g suspended by a long string. The bullet crosses the block and emerges on the other side. If the centre of mass of the block rises through a height of 20·0 cm, find the speed of the bullet as it emerges from the block.

**49.** Two masses $m_1$ and $m_2$ are connected by a spring of spring constant $k$ and are placed on a frictionless horizontal surface. Initially the spring is stretched through a distance $x_0$ when the system is released from rest. Find the distance moved by the two masses before they again come to rest.

**50.** Two blocks of masses $m_1$ and $m_2$ are connected by a spring of spring constant $k$ (figure 9-E15). The block of mass $m_2$ is given a sharp impulse so that it acquires a velocity $v_0$ towards right. Find (a) the velocity of the centre of mass, (b) the maximum elongation that the spring will suffer.

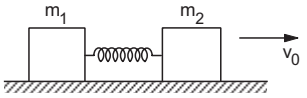



Figure 9-E15

**51.** Consider the situation of the previous problem. Suppose each of the blocks is pulled by a constant force $F$ instead of any impulse. Find the maximum elongation that the spring will suffer and the distances moved by the two blocks in the process.

**52.** Consider the situation of the previous problem. Suppose the block of mass $m_1$ is pulled by a constant force $F_1$ and the other block is pulled by a constant force $F_2$. Find the maximum elongation that the spring will suffer.

**53.** Consider a gravity-free hall in which an experimenter of mass 50 kg is resting on a 5 kg pillow, 8 ft above the floor of the hall. He pushes the pillow down so that it starts falling at a speed of 8 ft/s. The pillow makes a perfectly elastic collision with the floor, rebounds and reaches the experimenter's head. Find the time elapsed in the process.

**54.** The track shown in figure (9-E16) is frictionless. The block $B$ of mass $2m$ is lying at rest and the block $A$ of mass $m$ is pushed along the track with some speed. The collision between $A$ and $B$ is perfectly elastic. With what velocity should the block $A$ be started to get the sleeping man awakened ?

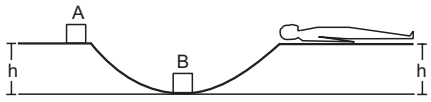



Figure 9-E16

**55.** A bullet of mass 10 g moving horizontally at a speed of $50\sqrt{7}$ m/s strikes a block of mass 490 g kept on a frictionless track as shown in figure (9-E17). The bullet remains inside the block and the system proceeds towards the semicircular track of radius 0·2 m. Where will the block strike the horizontal part after leaving the semicircular track ?

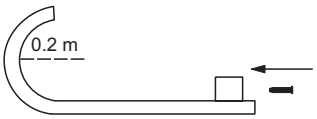



Figure 9-E17

**56.** Two balls having masses $m$ and $2m$ are fastened to two light strings of same length $l$ (figure 9-E18). The other ends of the strings are fixed at $O$. The strings are kept in the same horizontal line and the system is released from rest. The collision between the balls is elastic. (a) Find the velocities of the balls just after their collision. (b) How high will the balls rise after the collision ?

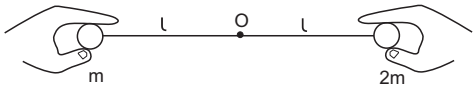



Figure 9-E18

**57.** A uniform chain of mass $M$ and length $L$ is held vertically in such a way that its lower end just touches the horizontal floor. The chain is released from rest in this position. Any portion that strikes the floor comes to rest. Assuming that the chain does not form a heap on the floor, calculate the force exerted by it on the floor when a length $x$ has reached the floor.

**58.** The blocks shown in figure (9-E19) have equal masses. The surface of $A$ is smooth but that of $B$ has a friction coefficient of 0·10 with the floor. Block $A$ is moving at a speed of 10 m/s towards $B$ which is kept at rest. Find the distance travelled by $B$ if (a) the collision is perfectly elastic and (b) the collision is perfectly inelastic. Take $g$ = 10 m/s$^2$.

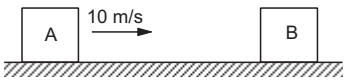



Figure 9-E19

**59.** The friction coefficient between the horizontal surface and each of the blocks shown in figure (9-E20) is 0·20. The collision between the blocks is perfectly elastic. Find the separation between the two blocks when they come to rest. Take $g$ = 10 m/s$^2$.

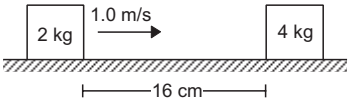



Figure 9-E20

**60.** A block of mass $m$ is placed on a triangular block of mass $M$, which in turn is placed on a horizontal surface as shown in figure (9-E21). Assuming frictionless

surfaces find the velocity of the triangular block when the smaller block reaches the bottom end.

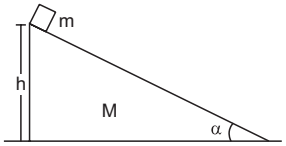



Figure 9-E21

**61.** Figure (9-E22) shows a small body of mass $m$ placed over a larger mass $M$ whose surface is horizontal near the smaller mass and gradually curves to become vertical. The smaller mass is pushed on the longer one at a speed $v$ and the system is left to itself. Assume that all the surfaces are frictionless. (a) Find the speed of the larger block when the smaller block is sliding on the vertical part. (b) Find the speed of the smaller mass when it breaks off the larger mass at height $h$. (c) Find the maximum height (from the ground) that the smaller mass ascends. (d) Show that the smaller mass will again land on the bigger one. Find the distance traversed by the bigger block during the time when the smaller block was in its flight under gravity.

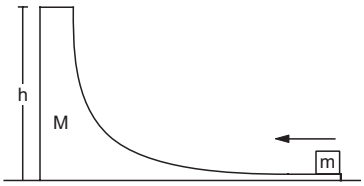



Figure 9-E22

**62.** A small block of superdense material has a mass of $3 \times 10^{24}$ kg. It is situated at a height $h$ (much smaller than the earth's radius) from where it falls on the earth's surface. Find its speed when its height from the earth's surface has reduced to $h/2$. The mass of the earth is $6 \times 10^{24}$ kg.

**63.** A body of mass $m$ makes an elastic collision with another identical body at rest. Show that if the collision is not head-on, the bodies go at right angle to each other after the collision.

**64.** A small particle travelling with a velocity $v$ collides elastically with a spherical body of equal mass and of radius $r$ initially kept at rest. The centre of this spherical body is located a distance $\rho(< r)$ away from the direction of motion of the particle (figure 9-E23). Find the final velocities of the two particles.

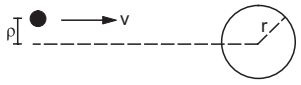



Figure 9-E23

[**Hint :** The force acts along the normal to the sphere through the contact. Treat the collision as one-dimensional for this direction. In the tangential direction no force acts and the velocities do not change].

□

## ANSWERS

### OBJECTIVE I

| | | | | | |
|---|---|---|---|---|---|
| 1. (c) | 2. (d) | 3. (d) | 4. (d) | 5. (b) | 6. (b) |
| 7. (b) | 8. (b) | 9. (c) | 10. (d) | 11. (c) | 12. (b) |
| 13. (c) | 14. (c) | 15. (d) | 16. (d) | 17. (a) | 18. (a) |
| 19. (b) | | | | | |

### OBJECTIVE II

| | | |
|---|---|---|
| 1. none | 2. (c), (d) | 3. (a), (b) |
| 4. (b), (c) | 5. (b), (d) | 6. (d) |
| 7. (a), (d) | 8. (b), (c), (d) | 9. (c), (d) |
| 10. (b), (c) | 11. all | |

### EXERCISES

1. Taking $AB$ as the $x$-axis and $A$ as the origin, the centre of mass is at $(7/12\ \text{m},\ \sqrt{3}/4\ \text{m})$

2. $6{\cdot}6 \times 10^{-12}$ m

3. 22 $L/35$

4. At $R/5$ from the centre of the bigger disc towards the centre of the smaller disc

5. At $R/3$ from the centre of the original disc away from the centre of the hole

6. $\dfrac{4d}{4+\pi}$ right to the centre of the disc

7. 0·20 m/s at 45° below the direction towards right

8. 1 cm

9. 1 cm downward

10. zero

11. $\dfrac{4(R_1^2 + R_1R_2 + R_2^2)}{3\pi(R_1 + R_2)}$ above the centre

12. 13 cm

13. $mL/(m + M)$

14. $mL/(m + M)$

15. 1 : 2

16. $2{\cdot}4 \times 10^5$ m/s

17. $1{\cdot}5 \times 10^{-23}$ m/s
18. (a) 12·2 m/s (b) 9·2 m/s
19. $\frac{Mx\sqrt{g}}{m\,[\sqrt{2H} - \sqrt{2(H-h)}]}$, $Mx/m$ left to the line of fall
20. (a) 0·14 kg-m/s (b) zero
21. $2h\cos\theta/\lambda$
22. $10\sqrt{2}$ m/s 135° below the $X$-axis
23. 15 km/s
24. 1900 N
25. $mg$
26. $\left(1 + \frac{M}{m}\right)v$
27. $200\left(\frac{1}{49} + \frac{1}{48}\right)$m/s
28. $\frac{m^2u}{M(M+m)}$ towards left
29. $\frac{mv}{M+m}$
30. $\frac{28}{3}$ km/h
31. 2·5 m/s opposite to the direction of motion of the first ball
32. 1200 J
33. $u_2 - \frac{m_1}{m_2}\frac{t}{\Delta t}(v_1 - u_1)$
34. $\frac{mv - (M-m')v_1}{m+m'}$ in the initial direction
35. $1/\sqrt{2}$ 36. 2 J, $1/\sqrt{2}$
37. 2 m/s, $2\sqrt{2}$ m/s
38. (a) 0·5 m/s (b) $\frac{10}{11}$ m/s (c) $\frac{50}{11}$ m/s, 5 m/s (d) 6 (e) $\frac{10}{21}d$ away from the initial position of $A$ towards $B$
39. $\sqrt{3}/2$
40. $E + \frac{E^2}{2Mc^2}$
41. 10 cm
42. 0·16
43. $\frac{(1+e)u^2\sin 2\theta}{g}$
44. $8h\sin\theta$ along the incline
45. 18·5 m along the incline
46. 6·1 cm
47. 280 m/s
48. 250 m/s
49. $\frac{2m_2x_0}{m_1+m_2}$, $\frac{2m_1x_0}{m_1+m_2}$
50. (a) $\frac{m_2v_0}{m_1+m_2}$ (b) $v_0\left[\frac{m_1m_2}{(m_1+m_2)k}\right]^{1/2}$
51. $2F/k$, $\frac{2Fm_2}{k(m_1+m_2)}$, $\frac{2Fm_1}{k(m_1+m_2)}$
52. $\frac{2(m_1F_2 + m_2F_1)}{k(m_1+m_2)}$
53. 2·22 s
54. Greater than $\sqrt{2{\cdot}5gh}$
55. At the junction of the straight and the curved parts
56. (a) Light ball $\frac{\sqrt{50gl}}{3}$ towards left, heavy ball $\frac{\sqrt{2gl}}{3}$ towards right (b) Light ball $2l$ and heavy ball $\frac{l}{9}$
57. $3Mgx/L$
58. (a) 50 m (b) 25 m
59. 5 cm
60. $\left[\frac{2m^2gh\cos^2\alpha}{(M+m)(M+m\sin^2\alpha)}\right]^{1/2}$
61. (a) $\frac{mv}{M+m}$ (b) $\left[\frac{(M^2+Mm+m^2)}{(M+m)^2}v^2 - 2gh\right]^{1/2}$ (c) $\frac{Mv^2}{2g(M+m)}$ (d) $\frac{2mv[Mv^2 - 2(M+m)gh]^{1/2}}{g(M+m)^{3/2}}$
62. $\sqrt{\frac{2gh}{3}}$
64. The small particle goes along the tangent with a speed of $v\rho/r$ and the spherical body goes perpendicular to the smaller particle with a speed of $\frac{v}{r}\sqrt{r^2 - \rho^2}$

□

## CHAPTER 10

# ROTATIONAL MECHANICS

Consider a pulley fixed at a typical Indian well on which a rope is wound with one end attached to a bucket. When the bucket is released, the pulley starts rotating. As the bucket goes down, the pulley rotates more rapidly till the bucket goes into the water.

Take the pulley as the system. The centre of mass of the pulley is at its geometrical centre which remains at rest. However, the other particles of the pulley move and are accelerated. The pulley is said to be executing rotational motion. Also, the rotational motion is not uniform. Since $\vec{a}_{CM} = 0$, the resultant external force $\vec{F}$ acting on the pulley must be zero. Even then the pulley is not in rotational equilibrium. We shall now study this type of motion.

### 10.1 ROTATION OF A RIGID BODY ABOUT A GIVEN FIXED LINE

Take a rigid body like a plate or a ball or your tennis racket or anything else present nearby and hold it between your fingers at two points. Now keep these two points fixed and then displace the body (try it with any rigid body at hand).

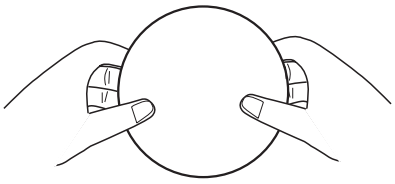


Figure 10.1

Notice the kind of displacement you can produce. In particular, notice that each particle of the rigid body goes in a circle, the centre being on the line joining the two fixed points between your finger tips. Let us call this line the axis of rotation. In fact, the centre of the circular path of a particle is at the foot of the perpendicular from the particle to this axis. Different particles move in different circles, the planes of all these circles are parallel to each other, and the radii depend on the distances of the particles from this axis. The particles on the axis remain stationary, those close to this line move on smaller circles and those far away from this line move in larger circles. However, each particle takes equal time to complete its circle.

Such a displacement of a rigid body in which a given line is held fixed, is called *rotation of the rigid body about the given line.* The line itself is called the *axis of rotation.*

Examples : (1) Consider the door of your almirah. When you open the door, the vertical line passing through the hinges is held fixed and that is the axis of rotation. Each point of the door describes a circle with the centre at the foot of the perpendicular from the particle on the axis. All these circles are horizontal and thus perpendicular to the axis.

(2) Consider the ceiling fan in your room. When it is on, each point on its body goes in a circle. Locate the centres of the circles traced by different particles on the three blades of the fan and the body covering the motor. All these centres lie on a vertical line through the centre of the body. The fan rotates about this vertical line.

(3) Look at the on–off switch on the front panel of your gas stove in the kitchen. To put the gas on, you push the switch a little, and then you rotate it. While rotating, each particle of the switch moves on a circle. Think about the centres of all such circles. They lie on a straight line (generally horizontal, towards the operator). This is the axis of rotation and the switch rotates about this axis.

Sometimes the axis may not pass through the body. Consider a record rotating on the turntable of a record player. Suppose a fly is sitting on the record near the rim. Look at the path of any particle of the fly. It is a circle with the centre on the vertical line through the centre of the record. The fly is “rotating about this vertical line” (can you consider the fly as a rigid body ?). The axis of rotation is lying completely outside the fly.

If each particle of a rigid body moves in a circle, with centres of all the circles on a straight line and

with planes of the circles perpendicular to this line, we say that the body is rotating about this line. The straight line itself is called the *axis of rotation.*

## 10.2 KINEMATICS

Consider a rigid body rotating about a given fixed line. Take this line as the $Z$-axis. Consider a particle $P$ of the body (figure 10.2). Look at its position $P_0$ at $t = 0$. Draw a perpendicular $P_0Q$ to the axis of rotation. At time $t$, the particle moves to $P$. Let $\angle PQP_0 = \theta$. We say that the particle $P$ has rotated through an angle $\theta$. In fact, all the particles of the body have also rotated through the same angle $\theta$ and so we say that the whole rigid body has rotated through an angle $\theta$. The "angular position" of the body at time $t$ is said to be $\theta$. If $P$ has made a complete revolution on its circle, every particle has done so and we say that the body has rotated through an angle of $2\pi$. So the rotation of a rigid body is measured by the rotation of the line $QP$ from its initial position.

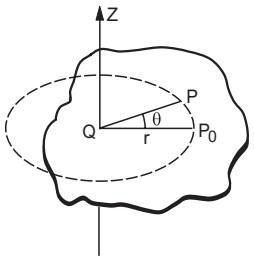



Figure 10.2

Now, suppose the angular position of the body at time $t$ is $\theta$. During a time $\Delta t$, it further rotates through $\Delta\theta$, so that its angular position becomes $\theta + \Delta\theta$. The average angular velocity during the time interval $\Delta t$ is

$$\omega = \frac{\Delta\theta}{\Delta t}.$$

The instantaneous angular velocity at time $t$ is

$$\omega = \frac{d\theta}{dt}.$$

We associate the direction of the axis of rotation with the angular velocity. If the body rotates anticlockwise as seen through the axis, the angular velocity is towards the viewer. If it rotates clockwise, the angular velocity is away from the reader. It turns out that the angular velocity adds like a vector and hence it becomes a vector quantity. The magnitude of angular velocity is called *angular speed* . However, we shall continue to use the word angular velocity if the direction of the axis is clear from the context. The SI unit for angular velocity is radian/sec (rad/s). Quite often the angular velocity is given in revolutions per second (rev/s). The conversion in radian per second may be made using 1 rev = 2 $\pi$ radian.

If the body rotates through equal angles in equal time intervals (irrespective of the smallness of the intervals), we say that it rotates with uniform angular velocity. In this case $\omega = d\theta/dt =$ constant and thus $\theta = \omega t$. If it is not the case, the body is said to be rotationally "accelerated". The angular acceleration is defined as

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2}.$$

If the angular acceleration $\alpha$ is constant, we have

$$\omega = \omega_0 + \alpha t \quad \ldots (10.1)$$

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 \quad \ldots (10.2)$$

and $$\omega^2 = \omega_0^2 + 2\alpha\theta \quad \ldots (10.3)$$

where $\omega_0$ is the angular velocity of the body at $t = 0$.

As an example, think of your ceiling fan. Switch on the fan. The fan rotates about a vertical line (axis). The angle rotated by the fan in the first second is small, that in the second second is larger, that in the third second is still larger and so on. The fan, thus, has an angular acceleration. The angular velocity $\omega = d\theta/dt$ increases with time. Wait for about a couple of minutes. The fan has now attained full speed. The angle rotated in any time interval is now equal to the angle rotated in the successive equal time interval. The fan is rotating uniformly about the vertical axis. Now switch off the fan. The angle rotated in any second is smaller than the angle rotated in the previous second. The angular velocity $d\omega/dt$ decreases as time passes, and finally it becomes zero when the fan stops. The fan has an angular deceleration.

Given the axis of rotation, the body can rotate in two directions. Looking through the axis, it may be clockwise or anticlockwise. One has to define the "positive" rotation. This may be defined according to the convenience of the problem, but once decided, one has to stick to the choice. The angular displacement, angular velocity and the angular acceleration may accordingly be positive or negative.

Notice the similarity between the motion of a particle on a straight line and the rotation of a rigid body about a fixed axis. The position of the particle was decided by a single variable $x$, which could be positive or negative according to the choice of the positive direction of the $X$-axis. The rate of change of position gave the velocity and the rate of change of velocity gave the acceleration.

***Example 10.1***

*The motor of an engine is rotating about its axis with an angular velocity of* 100 rev/minute. *It comes to rest*

*in* 15 s, *after being switched off. Assuming constant angular deceleration, calculate the number of revolutions made by it before coming to rest.*

***Solution*** : The initial angular velocity = 100 rev/minute $= (10\,\pi/3)$ rad/s.

Final angular velocity = 0.

Time interval = 15 s.

Let the angular acceleration be $\alpha$. Using the equation $\omega = \omega_0 + \alpha t$, we obtain $\alpha = (-2\pi/9)$ rad/s$^2$.

The angle rotated by the motor during this motion is

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$= \left(\frac{10\pi}{3}\frac{\text{rad}}{\text{s}}\right)(15\text{ s}) - \frac{1}{2}\left(\frac{2\pi}{9}\frac{\text{rad}}{\text{s}^2}\right)(15\text{ s})^2$$

$$= 25\pi \text{ rad} = 12{\cdot}5 \text{ revolutions.}$$

Hence the motor rotates through 12·5 revolutions before coming to rest.

***Example 10.2***

*Starting from rest, a fan takes five seconds to attain the maximum speed of* 400 rpm*(revolutions per minute). Assuming constant acceleration, find the time taken by the fan in attaining half the maximum speed.*

***Solution*** : Let the angular acceleration be $\alpha$. According to the question,

$$400 \text{ rev/min} = 0 + \alpha\, 5 \text{ s} \quad \ldots \text{ (i)}$$

Let $t$ be the time taken in attaining the speed of 200 rev/min which is half the maximum.

Then, $$200 \text{ rev/min} = 0 + \alpha t \quad \ldots \text{ (ii)}$$

Dividing (i) by (ii), we get,

$$2 = 5 \text{ s}/t \text{ or, } \quad t = 2{\cdot}5 \text{ s.}$$

**Relation between the Linear Motion of a Particle of a Rigid Body and its Rotation**

Consider a point $P$ of the rigid body rotating about a fixed axis as shown in figure (10.2). As the body rotates, this point moves on a circle. The radius of this circle is the perpendicular distance of the particle from the axis of rotation. Let it be $r$. If the body rotates through an angle $\theta$, so does the radius joining the particle with the centre of its circle. The linear distance moved by the particle is $s = r\theta$ along the circle.

The linear speed along the tangent is

$$v = \frac{ds}{dt} = r \cdot \frac{d\theta}{dt} = r\omega \quad \ldots \text{ (10.4)}$$

and the linear acceleration along the tangent, i.e., the tangential acceleration, is

$$a = \frac{dv}{dt} = r \cdot \frac{d\omega}{dt} = r\alpha. \quad \ldots \text{ (10.5)}$$

The relations $v = r\omega$ and $a = r\alpha$ are very useful and their meanings should be clearly understood. For different particles of the rigid body, the radius $r$ of their circles has different values, but $\omega$ and $\alpha$ are same for all the particles. Thus, the linear speed and the tangential acceleration of different particles are different. For $r = 0$, i.e., for the particles on the axis, $v = r\omega = 0$ and $a = r\alpha = 0$, consistent with the fact that the particles on the axis do not move at all.

***Example 10.3***

*A bucket is being lowered down into a well through a rope passing over a fixed pulley of radius* 10 cm. *Assume that the rope does not slip on the pulley. Find the angular velocity and angular acceleration of the pulley at an instant when the bucket is going down at a speed of* 20 cm/s *and has an acceleration of* 4·0 m/s$^2$.

***Solution*** : Since the rope does not slip on the pulley, the linear speed $v$ of the rim of the pulley is same as the speed of the bucket.

The angular velocity of the pulley is then

$$\omega = v/r = \frac{20 \text{ cm/s}}{10 \text{ cm}} = 2 \text{ rad/s}$$

and the angular acceleration of the pulley is

$$\alpha = a/r = \frac{4{\cdot}0 \text{ m/s}^2}{10 \text{ cm}} = 40 \text{ rad/s}^2.$$

## 10.3 ROTATIONAL DYNAMICS

When one switches a fan on, the centre of the fan remains unmoved while the fan rotates with an angular acceleration. As the centre of mass remains at rest, the external forces acting on the fan must add to zero. This means that one can have angular acceleration even if the resultant external force is zero. But then why do we need to switch on the fan in order to start it ? If an angular acceleration may be achieved with zero total external force, why does not a wheel chair start rotating on the floor as soon as one wishes it to do so. Why are we compelled to use our muscles to set it into rotation ? In fact, one cannot have angular acceleration without external forces.

What is then the relation between the force and the angular acceleration ? We find that even if the resultant external force is zero, we may have angular acceleration. We also find that without applying an external force we cannot have an angular acceleration. What is responsible for producing angular acceleration ? The answer is *torque* which we define below.

## 10.4 TORQUE OF A FORCE ABOUT THE AXIS OF ROTATION

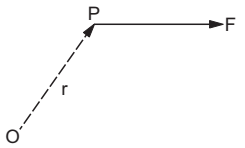



Figure 10.3

Consider a force $\vec{F}$ acting on a particle $P$. Choose an origin $O$ and let $\vec{r}$ be the position vector of the particle experiencing the force. We define *the torque of the force F about O* as

$$\vec{\Gamma} = \vec{r} \times \vec{F} \qquad \ldots \ (10.6)$$

This is a vector quantity having its direction perpendicular to $\vec{r}$ and $\vec{F}$ according to the rule of cross product. Now consider a rigid body rotating about a given axis of rotation $AB$ (figure 10.4). Let $F$ be a force acting on the particle $P$ of the body. $F$ may not be in the plane $ABP$. Take the origin $O$ somewhere on the axis of rotation.

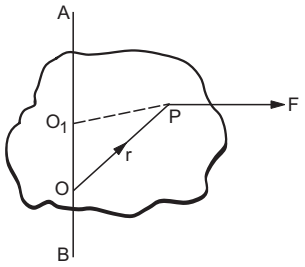



Figure 10.4

The torque of $F$ about $O$ is $\vec{\Gamma} = \vec{r} \times \vec{F}$. Its component along $OA$ is called the torque of $\vec{F}$ *about* $OA$. To calculate it, we should find the vector $\vec{r} \times \vec{F}$ and then find out the angle $\theta$ it makes with $OA$. The torque about $OA$ is then $|\vec{r} \times \vec{F}| \cos\theta$. The torque of a force about a line is independent of the choice of the origin as long as it is chosen on the line. This can be shown as given below. Let $O_1$ be any point on the line $AB$ (figure 10.4). The torque of $F$ about $O_1$ is

$$\vec{O_1P} \times \vec{F} = (\vec{O_1O} + \vec{OP}) \times \vec{F} = \vec{O_1O} \times \vec{F} + \vec{OP} \times \vec{F}.$$

As $\vec{O_1O} \times \vec{F} \perp \vec{O_1O}$, this term will have no component along $AB$.

Thus, the component of $\vec{O_1P} \times \vec{F}$ is equal to that of $\vec{OP} \times \vec{F}$.

There are some special cases which occur frequently.

**Case I**

$$\vec{F} \parallel \vec{AB}.$$

$\vec{r} \times \vec{F}$ is perpendicular to $\vec{F}$, but $\vec{F} \parallel \vec{AB}$, hence $\vec{r} \times \vec{F}$ is perpendicular to $\vec{AB}$. The component of $\vec{r} \times \vec{F}$ along $\vec{AB}$ is, therefore, zero.

**Case II**

$F$ intersects $AB$ (say at O)

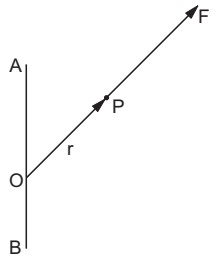



Figure 10.5

Taking the point of intersection as the origin, we see that $\vec{r}(= \vec{OP})$ and $\vec{F}$ are in the same line. The torque about $O$ is $\vec{r} \times \vec{F} = 0$. Hence the component along $OA$ is zero.

**Case III**

$\vec{F} \perp \vec{AB}$ but $\vec{F}$ and $AB$ do not intersect.

In three dimensions, two lines may be perpendicular without intersecting each other. For example, a vertical line on the surface of a wall of your room and a horizontal line on the opposite wall are mutually perpendicular but they never intersect. Two nonparallel and nonintersecting lines are called *skew* lines.

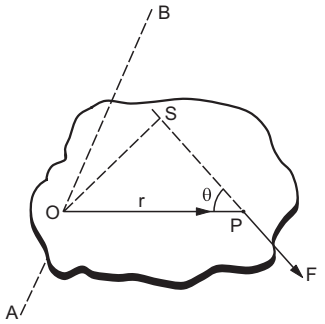



Figure 10.6

Figure (10.6) shows the plane through the particle $P$ that is perpendicular to the axis of rotation $AB$. Suppose the plane intersects the axis at the point $O$. The force $F$ is in this plane. Taking the origin at $O$,

$$\vec{\Gamma} = \vec{r} \times \vec{F} = \vec{OP} \times \vec{F}.$$

Thus, $\Gamma = rF \sin\theta = F.\ (OS)$

where $OS$ is the perpendicular from $O$ to the line of action of the force $\vec{F}$. The line $OS$ is also perpendicular to the axis of rotation. It is thus the length of the common perpendicular to the force and the axis of rotation.

The direction of $\vec{\Gamma} = \vec{OP} \times \vec{F}$ is along the axis $AB$ because $\vec{AB} \perp \vec{OP}$ and $\vec{AB} \perp \vec{F}$. The torque about $AB$ is, therefore, equal to the magnitude of $\vec{\Gamma}$ that is $F.(OS)$.

Thus, the torque of $F$ about $AB$ = magnitude of the force $F \times$ length of the common perpendicular to the force and the axis. The common perpendicular $OS$ is called the *lever arm or moment arm* of this torque.

The torque may try to rotate the body clockwise or anticlockwise about $AB$. Depending on the convenience of the problem one may be called positive and the other negative. It is conventional to take the torque positive if the body rotates anticlockwise as viewed through the axis.

**Case IV**

$\vec{F}$ and $\vec{OA}$ are skew but not perpendicular.

Take components of $\vec{F}$ parallel and perpendicular to the axis.

The torque of the parallel part is zero from case I and that of the perpendicular part may be found as in case III.

In most of the applications that we shall see, cases I, II or III will apply.

***Example 10.4***

*Consider a pulley fixed at its centre of mass by a clamp. A light rope is wound over it and the free end is tied to a block. The tension in the rope is T. (a) Write the forces acting on the pulley. How are they related ? (b) Locate the axis of rotation. (c) Find the torque of the forces about the axis of rotation.*

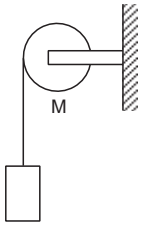



Figure 10.7

***Solution*** **:** (a) The forces on the pulley are (figure 10.7)

(i) attraction by the earth, $Mg$ vertically downward,

(ii) tension $T$ by the rope, along the rope,

(iii) contact force $\mathscr{N}$ by the support at the centre.

$\mathscr{N} = T + Mg$ (centre of mass of the pulley is at rest, so Newton's 1st law applies).

(b) The axis of rotation is the line through the centre of the pulley and perpendicular to the plane of the pulley.

(c) Let us take the positive direction of the axis towards the reader.

The force $Mg$ passes through the centre of mass and it intersects the axis of rotation. Hence the torque of $Mg$ about the axis is zero (Case II). Similarly, the torque of the contact force $\mathscr{N}$ is also zero.

The tension $T$ is along the tangent of the rim in the vertically downward direction. The tension and the axis of rotation are perpendicular but never intersect. Case III applies. Join the point where the rope leaves the rim to the centre. This line is the common perpendicular to the tension and the axis. Hence the torque is $T.r$ (positive, since it will try to rotate the pulley anticlockwise).

If there are more than one forces $\vec{F}_1, \vec{F}_2, \vec{F}_3, \ldots$ acting on a body, one can define the *total torque* acting on the body about a given line.

To obtain the total torque, we have to get separately the torques of the individual forces and then add them.

$$\vec{\Gamma} = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \ldots$$

You may be tempted to add the forces $\vec{F}_1$, $\vec{F}_2$, $\vec{F}_3, \ldots$ vectorially and then obtain the torque of resultant force about the axis. But that won't always work. Even if $\vec{F}_1 + \vec{F}_2 + \ldots = 0$, $\vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \ldots$ may not. However, if the forces act on the same particle, one can add the forces and then take the torque of the resultant.

## 10.5 $\Gamma = I\alpha$

We are now in a position to tell how the angular acceleration is produced when the resultant force on the body is zero. It is the total torque that decides the angular acceleration. Although the resultant force on the fan in our example is zero, the total torque is not. Whereas, if one does not apply any force, the torque is also zero and no angular acceleration is produced. For angular acceleration, there must be a torque.

To have linear acceleration of a particle, the total force on the particle should be nonzero. The acceleration of the particle is proportional to the force applied on it. To have angular acceleration about an axis you must have a nonzero torque on the body about the axis of rotation. Do we also have the relation that the angular acceleration is proportional to the total torque on the body ? Let us hope so.

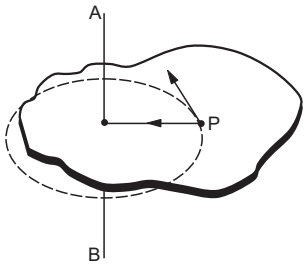



Figure 10.8

Consider a rigid body rotating about a fixed axis *AB* (figure 10.8). Consider a particle *P* of mass *m* rotating in a circle of radius *r*.

The radial acceleration of the particle $= \frac{v^2}{r} = \omega^2 r$.

Thus, the radial force on it $= m\omega^2 r$.

The tangential acceleration of the particle $= \frac{dv}{dt}$.

Thus, the tangential force on it

$$= m\frac{dv}{dt} = mr\frac{d\omega}{dt} = m\,r\,\alpha\,.$$

The torque of $m\omega^2 r$ about *AB* is zero as it intersects the axis and that of $mr\alpha$ is $mr^2\alpha$ as the force and the axis are skew and perpendicular. Thus, the torque of the resultant force acting on *P* is $mr^2\alpha$. Summing over all the particles, the total torque of all the forces acting on all the particles of the body is

$$\Gamma^{total} = \sum_i m_i r_i^2\,\alpha = I\alpha \qquad \ldots \text{ (i)}$$

where

$$I = \sum_i m_i r_i^2. \qquad \ldots (10.7)$$

The quantity *I* is called the *moment of inertia* of the body about the axis of rotation. Note that $m_i$ is the mass of the *i*th particle and $r_i$ is its perpendicular distance from the axis.

We have $\Gamma^{total} = \sum_i (\vec{r_i} \times \vec{F_i})$ where $\vec{F_i}$ is the resultant force on the *i*th particle. This resultant force consists of forces by all the other particles as well as other external forces applied on the *i*th particle. Thus,

$$\Gamma^{total} = \sum_i \vec{r_i} \times \left( \sum_{j \neq i} \vec{F}_{ij} + \vec{F}_i^{\,ext} \right)$$

where $\vec{F}_{ij}$ is the force on the *i*th particle by the *j*th particle and $\vec{F}_i^{\,ext}$ is the external force applied on the *i*th particle. When summation is made on both *i* and *j*, the first summation contains pairs like $\vec{r_i} \times \vec{F}_{ij} + \vec{r_j} \times \vec{F}_{ji}$. Newton's third law tells us that $\vec{F}_{ij} = -\vec{F}_{ji}$ so that such pairs become $(\vec{r_i} - \vec{r_j}) \times \vec{F}_{ij}$. Also the force $\vec{F}_{ij}$ is along the line joining the particles so that $(\vec{r_i} - \vec{r_j}) \parallel \vec{F}_{ij}$ and the cross product is zero. Thus, it is necessary to consider only the torques of the external forces applied on the rigid body and (i) becomes

$$\Gamma^{ext} = I\alpha \qquad \ldots (10.8)$$

where the torque and the moment of inertia are both evaluated about the axis of rotation.

Note the similarity between $\Gamma = I\alpha$ and $F = Ma$. Also, note the dissimilarity between the behaviour of *M* and *I*. The mass *M* is a property of the body and does not depend on the choice of the origin or the axes or the kind of motion it undergoes (as long as we are dealing with velocities much less than $3 \times 10^8$ m/s). But the moment of inertia $I = \sum_i m_i r_i^2$ depends on the choice of the axis about which it is calculated. The quantity $r_i$ is the perpendicular distance of the *i*th particle from the "axis". Changing the axis changes $r_i$ and hence *I*.

Moment of inertia of bodies of simple geometrical shapes may be calculated using the techniques of integration. We shall discuss the calculation for bodies of different shapes in somewhat greater detail in a later section.

Note that $\Gamma = I\alpha$ is not an independent rule of nature. It is derived from the more basic Newton's laws of motion.

***Example 10.5***

*A wheel of radius* 10 cm *can rotate freely about its centre as shown in figure (10.9). A string is wrapped over its rim and is pulled by a force of* 5·0 N. *It is found that the torque produces an angular acceleration* 2·0 rad/s$^2$ *in the wheel. Calculate the moment of inertia of the wheel.*

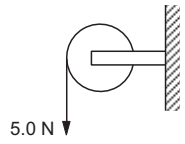



Figure 10.9

***Solution*** : The forces acting on the wheel are (i) *W* due to gravity, (ii) $\mathcal{N}$ due to the support at the centre and (iii) *F* due to tension. The torque of *W* and $\mathcal{N}$ are separately zero and that of *F* is *F.r*. The net torque is

$$\Gamma = (5{\cdot}0 \text{ N}).(10 \text{ cm}) = 0{\cdot}50 \text{ N-m}.$$

The moment of inertia is

$$I = \frac{\Gamma}{\alpha} = \frac{0{\cdot}50 \text{ N-m}}{2 \text{ rad/s}^2} = 0{\cdot}25 \text{ kg-m}^2.$$

## 10.6 BODIES IN EQUILIBRIUM

The centre of mass of a body remains in equilibrium if the total external force acting on the body is zero. This follows from the equation $F = Ma$. Similarly, a body remains in rotational equilibrium if the total external torque acting on the body is zero. This follows from the equation $\Gamma = I\alpha$. Thus, if a body remains at rest in an inertial frame, the total external force acting on the body should be zero in any direction and the total external torque should be zero about any line.

We shall often find situations in which all the forces acting on a body lie in a single plane as shown in figure (10.10).

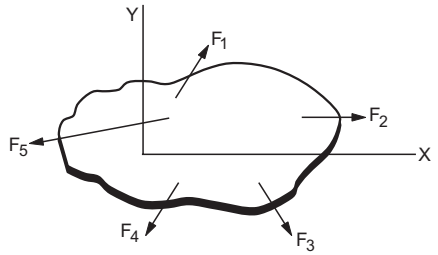



Figure 10.10

Let us take this plane as the $X$-$Y$ plane. For translational equilibrium

$$\sum F_x = 0 \qquad \ldots \text{(i)}$$

and

$$\sum F_y = 0. \qquad \ldots \text{(ii)}$$

As all the forces are in the $X$-$Y$ plane, $F_z$ is identically zero for each force and so $\sum F_z = 0$ is automatically satisfied. Now consider rotational equilibrium. The torque of each force about the $X$-axis is identically zero because either the force intersects the axis or it is parallel to it. Similarly, the torque of each force about the $Y$-axis is identically zero. In fact, the torque about any line in the $X$-$Y$ plane is zero.

Thus, the condition of rotational equilibrium is

$$\sum \Gamma_z = 0. \qquad \ldots \text{(iii)}$$

While taking torque about the $Z$-axis, the origin can be chosen at any point in the plane of the forces. That is, the torque can be taken about any line perpendicular to the plane of the forces. In general, the torque is different about different lines but it can be shown that if the resultant force is zero, the total torque about any line perpendicular to the plane of the forces is equal. If it is zero about one such line, it will be zero about all such lines.

If a body is placed on a horizontal surface, the torque of the contact forces about the centre of mass should be zero to maintain the equilibrium. This may happen only if the vertical line through the centre of mass cuts the base surface at a point within the contact area or the area bounded by the contact points. That is why a person leans in the opposite direction when he or she lifts a heavy load in one hand.

The equilibrium of a body is called *stable* if the body tries to regain its equilibrium position after being slightly displaced and released. It is called *unstable* if it gets further displaced after being slightly displaced and released. If it can stay in equilibrium even after being slightly displaced and released, it is said to be in *neutral equilibrium.*

In the case of stable equilibrium, the centre of mass goes higher on being slightly displaced. For unstable equilibrium it goes lower and for neutral equilibrium it stays at the same height.

## 10.7 BENDING OF A CYCLIST ON A HORIZONTAL TURN

Suppose a cyclist is going at a speed $v$ on a circular horizontal road of radius $r$ which is not banked. Consider the cycle and the rider together as the system. The centre of mass $C$ (figure 10.11a) of the system is going in a circle with the centre at $O$ and radius $r$.

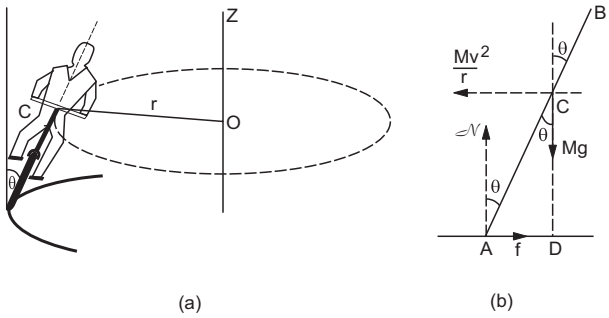



Figure 10.11

Let us choose $O$ as the origin, $OC$ as the $X$-axis and vertically upward as the $Z$-axis. This frame is rotating at an angular speed $\omega = v/r$ about the $Z$-axis. In this frame the system is at rest. Since we are working from a rotating frame of reference, we will have to apply a centrifugal force on each particle. The net centrifugal force on the system will be $M\omega^2 r = Mv^2/r$, where $M$ is the total mass of the system. This force will act through the centre of mass. Since the system is at rest in this frame, no other pseudo force is needed.

Figure (10.11b) shows the forces. The cycle is bent at an angle $\theta$ with the vertical. The forces are

(i) weight $Mg$,

(ii) normal force $\mathcal{N}$,

(iii) friction $f$ and,

(iv) centrifugal force $Mv^2/r$.

In the frame considered, the system is at rest. Thus, the total external force and the total external torque must be zero. Let us consider the torques of all the forces about the point $A$. The torques of $\mathcal{N}$ and $f$ about $A$ are zero because these forces pass through $A$. The torque of $Mg$ about $A$ is $Mg(AD)$ in the clockwise direction and that of $\frac{Mv^2}{r}$ is $\frac{Mv^2}{r}$ $(CD)$ in the anti-clockwise direction. For rotational equilibrium,

$$Mg\,(AD) = \frac{Mv^2}{r}(CD)$$

or, $$\frac{AD}{CD} = \frac{v^2}{rg}$$

or, $$\tan\theta = \frac{v^2}{rg}. \qquad \ldots (10.9)$$

Thus the cyclist bends at an angle $\tan^{-1}\left(\frac{v^2}{rg}\right)$ with the vertical.

## 10.8 ANGULAR MOMENTUM

Angular momentum of a particle about a point $O$ is defined as

$$\vec{l} = \vec{r} \times \vec{p} \qquad \ldots (10.10)$$

where $\vec{p}$ is the linear momentum and $\vec{r}$ is the position vector of the particle from the given point $O$. The angular momentum of a system of particles is the vector sum of the angular momenta of the particles of the system. Thus,

$$\vec{L} = \sum_i \vec{l_i} = \sum (\vec{r_i} \times \vec{p_i}).$$

Suppose a particle $P$ of mass $m$ moves at a velocity $\vec{v}$ (figure 10.12). Its angular momentum about a point $O$ is,

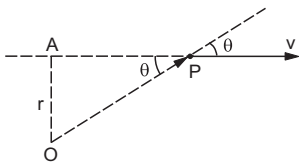



Figure 10.12

$$\vec{l} = \vec{OP} \times (m\vec{v})$$

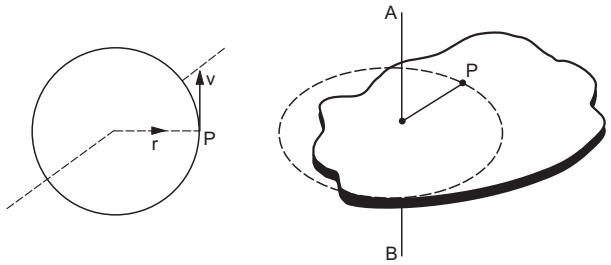
or, $$l = mv\ OP \sin\theta = mvr \qquad \ldots (10.11)$$

where $r = OA = OP \sin\theta$ is the perpendicular distance of the line of motion from $O$.

As in the case of torque, we define the angular momentum of a particle "about a line" say $AB$. Take any point $O$ on the line $AB$ and obtain the angular momentum $\vec{r} \times \vec{p}$ of the particle about $O$. The component of $\vec{r} \times \vec{p}$ along the line $AB$ is called the angular momentum of the particle "about $AB$". The point $O$ may be chosen anywhere on the line $AB$.

## 10.9 $L = I\omega$

Suppose a particle is going in a circle of radius $r$ and at some instant the speed of the particle is $v$ (figure 10.13a). What is the angular momentum of the particle about the axis of the circle ?




(a) (b)

Figure 10.13

As the origin may be chosen anywhere on the axis, we choose it at the centre of the circle. Then $\vec{r}$ is along a radius and $\vec{v}$ is along the tangent so that $\vec{r}$ is perpendicular to $\vec{v}$ and $l = |\vec{r} \times \vec{p}| = mvr$. Also $\vec{r} \times \vec{p}$ is perpendicular to $\vec{r}$ and $\vec{p}$ and hence is along the axis. Thus, the component of $\vec{r} \times \vec{p}$ along the axis is $mvr$ itself.

Next consider a rigid body rotating about an axis $AB$ (figure 10.13b). Let the angular velocity of the body be $\omega$. Consider the $i$th particle going in a circle of radius $r_i$ with its plane perpendicular to $AB$. The linear velocity of this particle at this instant is $v_i = r_i\omega$. The angular momentum of this particle about $AB = m_i v_i r_i = m_i r_i^2 \omega$. The angular momentum of the whole body about $AB$ is the sum of these components, i.e.,

$$L = \sum m_i r_i^2\, \omega = I\omega \qquad \ldots (10.12)$$

where $I$ is the moment of inertia of the body about $AB$.

## 10.10 CONSERVATION OF ANGULAR MOMENTUM

We have defined the angular momentum of a body as $\vec{L} = \sum (\vec{r_i} \times \vec{p_i})$. Differentiating with respect to time,

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}\sum (\vec{r_i} \times \vec{p_i})$$

$$= \sum_i \left[\frac{d\vec{r_i}}{dt} \times \vec{p_i} + \vec{r_i} \times \frac{d\vec{p_i}}{dt}\right]$$

$$= \sum_i \left[\vec{v_i} \times m\vec{v_i} + \vec{r_i} \times \vec{F_i}\right]$$

$$= \sum_i (\vec{r_i} \times \vec{F_i}) = \vec{\Gamma}^{total} \qquad \ldots (i)$$

where $\vec{F}_i$ is the total force acting on the $i$th particle. This includes any external force as well as the forces on the $i$th particle by all the other particles. When summation is taken over all the particles, the internal torques add to zero. Thus, (i) becomes

$$\frac{d\vec{L}}{dt} = \vec{\Gamma}^{ext} \qquad \ldots \quad (10.13)$$

where $\vec{\Gamma}^{ext}$ is the total torque on the system due to all the external forces acting on the system.

For a rigid body rotating about a fixed axis, we can arrive at equation (10.13) in a simpler manner. We have

$$L = I\omega$$

or, $$\frac{dL}{dt} = I\frac{d\omega}{dt} = I\alpha$$

or, $$\frac{dL}{dt} = \Gamma^{ext}.$$

Equation (10.13) shows that

*If the total external torque on a system is zero, its angular momentum remains constant.*

This is known as the *principle of conservation of angular momentum.*

***Example 10.6***

*A wheel is rotating at an angular speed* ω *about its axis which is kept vertical. An identical wheel initially at rest is gently dropped into the same axle and the two wheels start rotating with a common angular speed. Find this common angular speed.*

***Solution*** **:** Let the moment of inertia of the wheel about the axis be $I$. Initially the first wheel is rotating at the angular speed ω about the axle and the second wheel is at rest. Take both the wheels together as the system. The total angular momentum of the system before the coupling is $I\omega + 0 = I\omega$. When the second wheel is dropped into the axle, the two wheels slip on each other and exert forces of friction. The forces of friction have torques about the axis of rotation but these are torques of internal forces. No external torque is applied on the two-wheel system and hence the angular momentum of the system remains unchanged. If the common angular speed is ω′, the total angular momentum of the two-wheel system is $2I\omega'$ after the coupling. Thus,

$$I\omega = 2I\omega'$$

or, $$\omega' = \omega/2.$$

## 10.11 ANGULAR IMPULSE

The angular impulse of a torque in a given time interval is defined as

$$J = \int_{t_1}^{t_2} \Gamma\, dt.$$

If Γ be the resultant torque acting on a body

$$\Gamma = \frac{dL}{dt}, \text{ or, } \Gamma dt = dL.$$

Integrating this

$$J = L_2 - L_1.$$

Thus, the change in angular momentum is equal to the angular impulse of the resultant torque.

## 10.12 KINETIC ENERGY OF A RIGID BODY ROTATING ABOUT A GIVEN AXIS

Consider a rigid body rotating about a line $AB$ with an angular speed ω. The $i$th particle is going in a circle of radius $r_i$ with a linear speed $v_i = \omega r_i$. The kinetic energy of this particle is $\frac{1}{2} m_i(\omega r_i)^2$. The kinetic energy of the whole body is

$$\sum \frac{1}{2} m_i \omega^2 r_i^2 = \frac{1}{2}\sum (m_i r_i^2)\omega^2 = \frac{1}{2} I\omega^2.$$

Sometimes it is called rotational kinetic energy. It is not a new kind of kinetic energy as is clear from the derivation. It is the sum of $\frac{1}{2} mv^2$ of all the particles.

***Example 10.7***

*A wheel of moment of inertia I and radius r is free to rotate about its centre as shown in figure* (10.14). *A string is wrapped over its rim and a block of mass m is attached to the free end of the string. The system is released from rest. Find the speed of the block as it descends through a height h.*

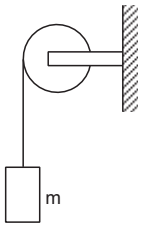



Figure 10.14

***Solution*** **:** Let the speed of the block be $v$ when it descends through a height $h$. So is the speed of the string and hence of a particle at the rim of the wheel. The angular velocity of the wheel is $v/r$ and its kinetic energy at this instant is $\frac{1}{2} I(v/r)^2$. Using the principle of conservation of energy, the gravitational potential energy lost by the block must be equal to the kinetic energy gained by the block and the wheel. Thus,

$$mgh = \frac{1}{2} m v^2 + \frac{1}{2} I \frac{v^2}{r^2}$$

or, $$v = \left[ \frac{2mgh}{m + I/r^2} \right]^{1/2}.$$

## 10.13 POWER DELIVERED AND WORK DONE BY A TORQUE

Consider a rigid body rotating about a fixed axis on which a torque acts. The torque produces angular acceleration and the kinetic energy increases. The rate of increase of the kinetic energy equals the rate of doing work on it, i.e., the power delivered by the torque.

$$P = \frac{dW}{dt} = \frac{dK}{dt}$$

$$= \frac{d}{dt}\left(\frac{1}{2} I\omega^2\right) = I\omega \frac{d\omega}{dt} = I\alpha\omega = \Gamma\omega.$$

The work done in an infinitesimal angular displacement $d\theta$ is

$$dW = Pdt = \Gamma\omega\, dt = \Gamma\, d\theta.$$

The work done in a finite angular displacement $\theta_1$ to $\theta_2$ is

$$W = \int_{\theta_1}^{\theta_2} \Gamma\, d\theta. \qquad \ldots (10.14)$$

## 10.14 CALCULATION OF MOMENT OF INERTIA

We have defined the moment of inertia of a system about a given line as

$$I = \sum_i m_i r_i^2$$

where $m_i$ is the mass of the $i$th particle and $r_i$ is its perpendicular distance from the given line. If the system is considered to be a collection of discrete particles, this definition may directly be used to calculate the moment of inertia.

***Example 10.8***

*Consider a light rod with two heavy mass particles at its ends. Let AB be a line perpendicular to the rod as shown in figure* (10.15). *What is the moment of inertia of the system about AB* ?

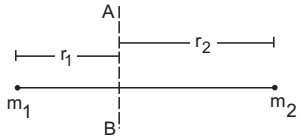



Figure 10.15

***Solution*** : Moment of inertia of the particle on the left is $m_1 r_1^2$.

Moment of inertia of the particle on the right is $m_2 r_2^2$.

Moment of inertia of the rod is negligible as the rod is light.

Thus, the moment of inertia of the system about *AB* is

$$m_1 r_1^2 + m_2 r_2^2.$$

***Example 10.9***

*Three particles, each of mass m, are situated at the vertices of an equilateral triangle ABC of side L* (*figure* 10.16). *Find the moment of inertia of the system about the line AX perpendicular to AB in the plane of ABC.*

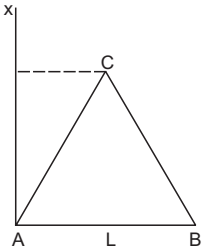



Figure 10.16

***Solution*** : Perpendicular distance of $A$ from $AX = 0$

,, ,, $B$ ,, ,, $= L$

,, ,, $C$ ,, ,, $= L/2.$

Thus, the moment of inertia of the particle at $A = 0$, of the particle at $B = mL^2$, and of the particle at $C = m(L/2)^2$. The moment of inertia of the three-particle system about $AX$ is

$$0 + mL^2 + m(L/2)^2 = \frac{5mL^2}{4}.$$

Note that the particles on the axis do not contribute to the moment of inertia.

### Moment of Inertia of Continuous Mass Distributions

If the body is assumed to be continuous, one can use the technique of integration to obtain its moment of inertia about a given line. Consider a small element of the body. The element should be so chosen that the perpendiculars from different points of the element to the given line differ only by infinitesimal amounts. Let its mass be $dm$ and its perpendicular distance from the given line be $r$. Evaluate the product $r^2 dm$ and integrate it over the appropriate limits to cover the whole body. Thus,

$$I = \int r^2 dm$$

under proper limits.

We can call $r^2 dm$ the moment of inertia of the small element. Moment of inertia of the body about the given line is the sum of the moments of inertia of its constituent elements about the same line.

**(A) Uniform rod about a perpendicular bisector**

Consider a uniform rod of mass $M$ and length $l$ (figure 10.17) and suppose the moment of inertia is to be calculated about the bisector $AB$. Take the origin at the middle point $O$ of the rod. Consider the element of the rod between a distance $x$ and $x + dx$ from the origin. As the rod is uniform,

Mass per unit length of the rod $= M/l$

so that the mass of the element $= (M/l)dx$.

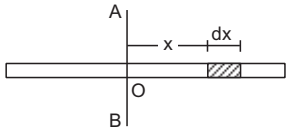



Figure 10.17

The perpendicular distance of the element from the line $AB$ is $x$. The moment of inertia of this element about $AB$ is

$$dI = \frac{M}{l} dx\, x^2.$$

When $x = -l/2$, the element is at the left end of the rod. As $x$ is changed from $-l/2$ to $l/2$, the elements cover the whole rod.

Thus, the moment of inertia of the entire rod about $AB$ is

$$I = \int_{-l/2}^{l/2} \frac{M}{l} x^2\, dx = \left[\frac{M}{l}\frac{x^3}{3}\right]_{-l/2}^{l/2} = \frac{Ml^2}{12}.$$

**(B) Moment of inertia of a rectangular plate about a line parallel to an edge and passing through the centre**

The situation is shown in figure (10.18). Draw a line parallel to $AB$ at a distance $x$ from it and another at a distance $x + dx$. We can take the strip enclosed between the two lines as the small element.

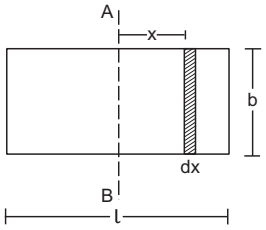



Figure 10.18

It is "small" because the perpendiculars from different points of the strip to $AB$ differ by not more than $dx$. As the plate is uniform,

its mass per unit area $= \frac{M}{bl}$.

Mass of the strip $= \frac{M}{bl} b\, dx = \frac{M}{l} dx$.

The perpendicular distance of the strip from $AB = x$. The moment of inertia of the strip about $AB = dI = \frac{M}{l} dx\, x^2$. The moment of inertia of the given plate is, therefore,

$$I = \int_{-l/2}^{l/2} \frac{M}{l} x^2\, dx = \frac{Ml^2}{12}.$$

The moment of inertia of the plate about the line parallel to the other edge and passing through the centre may be obtained from the above formula by replacing $l$ by $b$ and thus,

$$I = \frac{Mb^2}{12}.$$

**(C) Moment of inertia of a circular ring about its axis (the line perpendicular to the plane of the ring through its centre)**

Suppose the radius of the ring is $R$ and its mass is $M$. As all the elements of the ring are at the same perpendicular distance $R$ from the axis, the moment of inertia of the ring is

$$I = \int r^2 dm = \int R^2 dm = R^2 \int dm = MR^2.$$

**(D) Moment of inertia of a uniform circular plate about its axis**

Let the mass of the plate be $M$ and its radius $R$ (figure 10.19). The centre is at $O$ and the axis $OX$ is perpendicular to the plane of the plate.

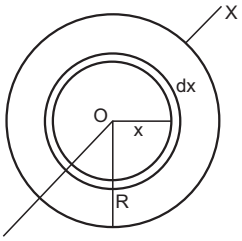



Figure 10.19

Draw two concentric circles of radii $x$ and $x + dx$, both centred at $O$ and consider the area of the plate in between the two circles.

This part of the plate may be considered to be a circular ring of radius $x$. As the periphery of the ring is $2\pi x$ and its width is $dx$, the area of this elementary ring is $2\pi x dx$. The area of the plate is $\pi R^2$. As the plate is uniform,

its mass per unit area $= \frac{M}{\pi R^2}$.

Mass of the ring $= \frac{M}{\pi R^2} 2\pi x\, dx = \frac{2Mx\, dx}{R^2}$.

Using the result obtained above for a circular ring, the moment of inertia of the elementary ring about $OX$ is

$$dI = \left[\frac{2\,Mxdx}{R^2}\right] x^2.$$

The moment of inertia of the plate about $OX$ is

$$I = \int_0^R \frac{2\,M}{R^2} x^3\,dx = \frac{MR^2}{2}.$$

**(E) Moment of inertia of a hollow cylinder about its axis**

Suppose the radius of the cylinder is $R$ and its mass is $M$. As every element of this cylinder is at the same perpendicular distance $R$ from the axis, the moment of inertia of the hollow cylinder about its axis is

$$I = \int R^2 dm = R^2 \int dm = MR^2.$$

**(F) Moment of inertia of a uniform solid cylinder about its axis**

Let the mass of the cylinder be $M$ and its radius $R$. Draw two cylindrical surfaces of radii $x$ and $x + dx$ coaxial with the given cylinder. Consider the part of the cylinder in between the two surfaces (figure 10.20). This part of the cylinder may be considered to be a hollow cylinder of radius $x$. The area of cross-section of the wall of this hollow cylinder is $2\pi\,x\,dx$. If the length of the cylinder is $l$, the volume of the material of this elementary hollow cylinder is $2\pi\,x\,dx\,l$.

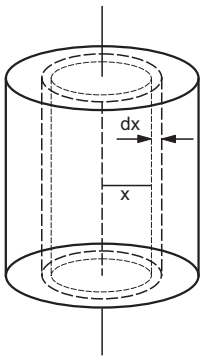



Figure 10.20

The volume of the solid cylinder is $\pi R^2 l$ and it is uniform, hence its mass per unit volume is

$$\rho = \frac{M}{\pi R^2 l}.$$

The mass of the hollow cylinder considered is

$$\frac{M}{\pi R^2 l} 2\pi x\,dx\,l = \frac{2\,M}{R^2}\,xdx.$$

As its radius is $x$, its moment of inertia about the given axis is

$$dI = \left[\frac{2\,M}{R^2} x\,dx\right] x^2.$$

The moment of inertia of the solid cylinder is, therefore,

$$I = \int_0^R \frac{2\,M}{R^2} x^3 dx = \frac{MR^2}{2}.$$

Note that the formula does not depend on the length of the cylinder.

**(G) Moment of inertia of a uniform hollow sphere about a diameter**

Let $M$ and $R$ be the mass and the radius of the sphere, $O$ its centre and $OX$ the given axis (figure 10.21). The mass is spread over the surface of the sphere and the inside is hollow.

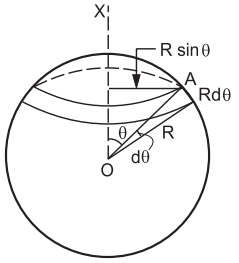



Figure 10.21

Let us consider a radius $OA$ of the sphere at an angle $\theta$ with the axis $OX$ and rotate this radius about $OX$. The point $A$ traces a circle on the sphere. Now change $\theta$ to $\theta + d\theta$ and get another circle of somewhat larger radius on the sphere. The part of the sphere between these two circles, shown in the figure, forms a ring of radius $R\sin\theta$. The width of this ring is $Rd\theta$ and its periphery is $2\pi R\sin\theta$. Hence,

the area of the ring = $(2\pi R\sin\theta)\,(Rd\theta)$.

Mass per unit area of the sphere = $\frac{M}{4\pi R^2}$.

The mass of the ring

$$= \frac{M}{4\pi R^2}(2\pi R\sin\theta)\,(Rd\theta) = \frac{M}{2}\sin\theta\,d\theta.$$

The moment of inertia of this elemental ring about $OX$ is

$$dI = \left(\frac{M}{2}\sin\theta\,d\theta\right)(R\sin\theta)^2.$$

$$= \frac{M}{2} R^2 \sin^3\theta\,d\theta$$

As $\theta$ increases from 0 to $\pi$, the elemental rings cover the whole spherical surface. The moment of inertia of the hollow sphere is, therefore,

$$I = \int_0^{\pi} \frac{M}{2} R^2 \sin^3\theta \, d\theta = \frac{MR^2}{2}\left[\int_0^{\pi} (1 - \cos^2\theta) \sin\theta \, d\theta\right]$$

$$= \frac{MR^2}{2}\left[\int_{\theta=0}^{\pi} -(1 - \cos^2\theta)\, d(\cos\theta)\right]$$

$$= \frac{-MR^2}{2}\left[\cos\theta - \frac{\cos^3\theta}{3}\right]_0^{\pi} = \frac{2}{3} MR^2.$$

**Alternative method**

Consider any particle $P$ of the surface, having coordinates $(x_i, y_i, z_i)$ with respect to the centre $O$ as the origin (figure 10.22) and $OX$ as the $X$-axis. Let $PQ$ be the perpendicular to $OX$. Then $OQ = x_i$. That is the definition of $x$-coordinate.

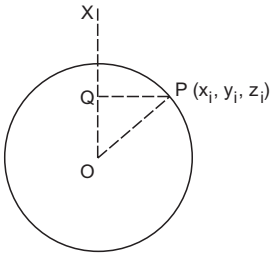



Figure 10.22

Thus, $PQ^2 = OP^2 - OQ^2$

$$= \left(x_i^2 + y_i^2 + z_i^2\right) - x_i^2 = y_i^2 + z_i^2.$$

The moment of inertia of the particle $P$ about the $X$-axis

$$= m_i (y_i^2 + z_i^2).$$

The moment of inertia of the hollow sphere about the $X$-axis is, therefore,

$$I_x = \sum_i m_i \left(y_i^2 + z_i^2\right).$$

Similarly, the moment of inertia of the hollow sphere about the $Y$-axis is

$$I_y = \sum_i m_i \left(z_i^2 + x_i^2\right)$$

and about the $Z$-axis it is

$$I_z = \sum_i m_i \left(x_i^2 + y_i^2\right)$$

Adding these three equations we get

$$I_x + I_y + I_z = \sum_i 2\, m_i \left(x_i^2 + y_i^2 + z_i^2\right)$$

$$= \sum_i 2\, m_i R^2 = 2\, MR^2.$$

As the mass is uniformly distributed over the entire surface of the sphere, all diameters are equivalent. Hence $I_x, I_y$ and $I_z$ must be equal.

Thus, $$I = \frac{I_x + I_y + I_z}{3} = \frac{2}{3} MR^2.$$

**(H) Moment of inertia of a uniform solid sphere about a diameter**

Let $M$ and $R$ be the mass and radius of the given solid sphere. Let $O$ be the centre and $OX$ the given axis. Draw two spheres of radii $x$ and $x + dx$ concentric with the given solid sphere. The thin spherical shell trapped between these spheres may be treated as a hollow sphere of radius $x$.

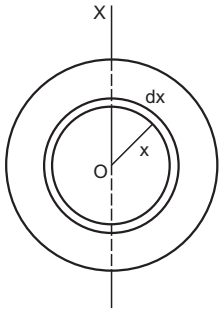



Figure 10.23

The mass per unit volume of the solid sphere

$$= \frac{M}{\frac{4}{3}\pi R^3} = \frac{3\,M}{4\,\pi R^3}.$$

The thin hollow sphere considered above has a surface area $4\,\pi x^2$ and thickness $dx$. Its volume is $4\,\pi\, x^2 dx$ and hence its mass is

$$= \left(\frac{3\,M}{4\,\pi R^3}\right)(4\,\pi x^2 dx)$$

$$= \frac{3\,M}{R^3} x^2 dx.$$

Its moment of inertia about the diameter $OX$ is, therefore,

$$dI = \frac{2}{3}\left[\frac{3\,M}{R^3} x^2 dx\right] x^2 = \frac{2\,M}{R^3} x^4 dx.$$

If $x = 0$, the shell is formed at the centre of the solid sphere. As $x$ increases from 0 to $R$, the shells cover the whole solid sphere.

The moment of inertia of the solid sphere about $OX$ is, therefore,

$$I = \int_0^R \frac{2\,M}{R^3} x^4 dx = \frac{2}{5} MR^2.$$

## 10.15 TWO IMPORTANT THEOREMS ON MOMENT OF INERTIA

**Theorem of Parallel Axes**

Suppose we have to obtain the moment of inertia of a body about a given line $AB$ (figure 10.24). Let $C$

be the centre of mass of the body and let $CZ$ be the line parallel to $AB$ through $C$. Let $I$ and $I_0$ be the moments of inertia of the body about $AB$ and $CZ$ respectively. The parallel axes theorem states that

$$I = I_0 + Md^2$$

where $d$ is the perpendicular distance between the parallel lines $AB$ and $CZ$ and $M$ is the mass of the body.

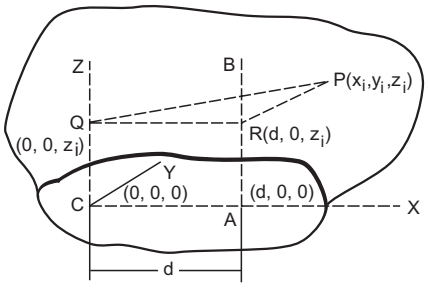



Figure 10.24

Take $C$ to be the origin and $CZ$ the $Z$-axis. Let $CA$ be the perpendicular from $C$ to $AB$. Take $CA$ to be the $X$-axis. As $CA = d$, the coordinates of $A$ are $(d, 0, 0)$.

Let $P$ be an arbitrary particle of the body with the coordinates $(x_i, y_i, z_i)$. Let $PQ$ and $PR$ be the perpendiculars from $P$ to $CZ$ and $AB$ respectively. Note that $P$ may not be in the plane containing $CZ$ and $AB$. We have $CQ = z_i$. Also $AR = CQ = z_i$. Thus, the point $Q$ has coordinates $(0, 0, z_i)$ and the point $R$ has coordinates$(d, 0, z_i)$.

$$I = \sum_i m_i (PR)^2$$

$$= \sum_i m_i \left[ (x_i - d)^2 + (y_i - 0)^2 + (z_i - z_i)^2 \right]$$

$$= \sum_i m_i (x_i^2 + y_i^2 + d^2 - 2 x_i d)$$

$$= \sum_i m_i (x_i^2 + y_i^2) + \sum_i m_i d^2 - 2d \sum_i m_i x_i \quad \ldots \text{(i)}$$

We have

$$\sum_i m_i x_i = MX_{CM} = 0.$$

The moment of inertia about $CZ$ is,

$$I_0 = \sum_i m_i (PQ)^2$$

$$= \sum_i m_i [(x_i - 0)^2 + (y_i - 0)^2 + (z_i - z_i)^2]$$

$$= \sum_i m_i (x_i^2 + y_i^2)$$

From (i),

$$I = I_0 + \sum m_i d^2 = I_0 + Md^2.$$

**Theorem of Perpendicular Axes**

This theorem is applicable only to the plane bodies. Let $X$ and $Y$-axes be chosen in the plane of the body and $Z$-axis perpendicular to this plane, three axes being mutually perpendicular. Then the theorem states that

$$I_z = I_x + I_y .$$

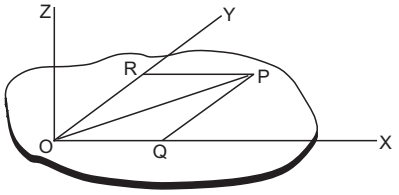



Figure 10.25

Consider an arbitrary particle $P$ of the body (figure 10.25). Let $PQ$ and $PR$ be the perpendiculars from $P$ on the $X$ and the $Y$-axes respectively. Also $PO$ is the perpendicular from $P$ to the $Z$-axis. Thus, the moment of inertia of the body about the Z-axis is

$$I_z = \sum_i m_i (PO)^2 = \sum_i m_i (PQ^2 + OQ^2)$$

$$= \sum_i m_i (PQ^2 + PR^2)$$

$$= \sum_i m_i (PQ)^2 + \sum_i m_i (PR)^2$$

$$= I_x + I_y .$$

***Example 10.10***

*Find the moment of inertia of a uniform ring of mass M and radius R about a diameter.*

***Solution :***

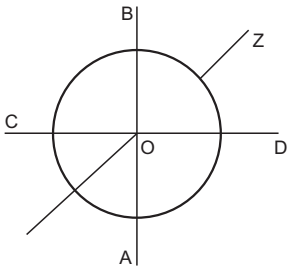



Figure 10.26

Let $AB$ and $CD$ be two mutually perpendicular diameters of the ring. Take them as $X$ and $Y$-axes and the line perpendicular to the plane of the ring through the centre as the $Z$-axis. The moment of inertia of the ring about the $Z$-axis is $I = MR^2$. As the ring is uniform, all of its diameters are equivalent and so $I_x = I_y$. From

perpendicular axes theorem,

$$I_z = I_x + I_y. \text{ Hence } I_x = \frac{I_z}{2} = \frac{MR^2}{2}.$$

Similarly, the moment of inertia of a uniform disc about a diameter is $MR^2/4$.

***Example 10.11***

*Find the moment of inertia of a solid cylinder of mass M and radius R about a line parallel to the axis of the cylinder and on the surface of the cylinder.*

***Solution*** **:** The moment of inertia of the cylinder about its axis $= \frac{MR^2}{2}$.

Using parallel axes theorem

$$I = I_0 + MR^2 = \frac{MR^2}{2} + MR^2 = \frac{3}{2}MR^2.$$

Similarly, the moment of inertia of a solid sphere about a tangent is

$$\frac{2}{5}MR^2 + MR^2 = \frac{7}{5}MR^2.$$

**Radius of Gyration**

The radius of gyration $k$ of a body about a given line is defined by the equation

$$I = Mk^2$$

where $I$ is its moment of inertia about the given line and $M$ is its total mass. It is the radius of a ring with the given line as the axis such that if the total mass of the body is distributed on the ring, it will have the same moment of inertia $I$. For example, the radius of gyration of a uniform disc of radius $r$ about its axis is $r/\sqrt{2}$.

## 10.16 COMBINED ROTATION AND TRANSLATION

We now consider the motion of a rigid body which is neither pure translational nor pure rotational as seen from a lab. Suppose instead, there is a frame of reference $A$ in which the motion of the rigid body is a pure rotation about a fixed line. If the frame $A$ is also inertial, the motion of the body with respect to $A$ is governed by the equations developed above. The motion of the body in the lab may then be obtained by adding the motion of $A$ with respect to the lab to the motion of the body in $A$.

If the frame $A$ is noninertial, we do not hope $\Gamma^{ext} = I\alpha$ to hold. In the derivation of this equation we used $F = m\,a$ for each particle and this holds good only if $a$ is measured from an inertial frame. If the frame $A$ has an acceleration $\vec{a}$ in a fixed direction with respect to an inertial frame, we have to apply a pseudo force $-m\vec{a}$ to each particle. These pseudo forces produce a pseudo torque about the axis.

Pleasantly, there exists a very special and very useful case where $\Gamma^{ext} = I\alpha$ does hold even if the angular acceleration $\alpha$ is measured from a noninertial frame $A$. And that special case is, when the axis of rotation in the frame $A$ passes through the centre of mass.

Take the origin at the centre of mass. The total torque of the pseudo forces is

$$\sum \vec{r_i} \times (-m_i \vec{a}) = -\left(\sum m_i \vec{r_i}\right) \times \vec{a} = -M\left(\frac{\sum m_i \vec{r_i}}{M}\right) \times \vec{a}$$

where $\vec{r_i}$ is the position vector of the $i$th particle as measured from the centre of mass.

But $\frac{\sum m_i \vec{r_i}}{M}$ is the position vector of the centre of mass and that is zero as the centre of mass is at the origin. Hence the pseudo torque is zero and we get $\Gamma^{ext} = I\alpha$. To make the point more explicit, we write $\Gamma_{cm} = I_{cm}\alpha$, reminding us that the equation is valid in a noninertial frame, only if the axis of rotation passes through the centre of mass and the torques and the moment of inertia are evaluated about the axis through the centre of mass.

So, the working rule for discussing combined rotation and translation is as follows. List the external forces acting on the body. The vector sum divided by the mass of the body gives the acceleration of the centre of mass. Then find the torque of the external forces and the moment of inertia of the body about a line through the centre of mass and perpendicular to the plane of motion of the particles. Note that this line may not be the axis of rotation in the lab frame. Still calculate $\Gamma$ and $I$ about this line. The angular acceleration $\alpha$ about the centre of mass will be obtained by $\alpha = \Gamma/I$.

$$\left.\begin{aligned} &\text{Thus} && \vec{a}_{cm} = \vec{F}^{ext}/M \\ &\text{and} && \alpha = \Gamma_{cm}^{ext}/I_{cm} \end{aligned}\right\}. \qquad \ldots (10.15)$$

These equations together with the initial conditions completely determine the motion.

## 10.17 ROLLING

When you go on a bicycle on a straight road what distance on the road is covered during one full pedal ? Suppose a particular spoke of the bicycle is painted black and is vertical at some instant pointing downward. After one full pedal the spoke is again vertical in the similar position. We say that the wheel has made one full rotation. During this period the bicycle has moved through a distance $2\pi R$ in normal

cycling on a good, free road. Here $R$ is the radius of the wheel. The wheels are said to be 'rolling' on the road.

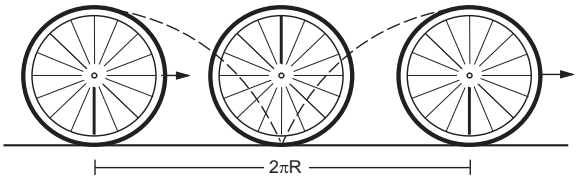



Figure 10.27

Looking from the road frame, the wheel is not making pure rotation about a fixed line. The particles of the wheel do not go on circles. The path of a particle at the rim will be something like that shown in figure (10.27), whereas the centre of the wheel goes in a straight line. But we still say that during one pedal the wheel has made one rotation, i.e., it has rotated through an angle of $2\pi$. By this we mean that the spoke that was vertical (pointing downward from the centre) again became vertical in the similar position. In this period the centre of the wheel has moved through a distance $2\pi R$. In half of this period, the wheel has moved through a distance $\pi R$ and the spoke makes an angle of $\pi$ with its original direction. During a short time-interval $\Delta t$, the wheel moves through a distance $\Delta x$ and the spoke rotates by $\Delta\theta$. Thus the wheel rotates and at the same time moves forward. The relation between the displacement of (the centre of) the wheel and the angle rotated by (a spoke of) the wheel is $\Delta x = R\Delta\theta$. Dividing by $\Delta t$ and taking limits, we get

$$v = R\omega,$$

where $v$ is the linear speed of the centre of mass and $\omega$ is the angular velocity of the wheel.

This type of motion of a wheel (or any other object with circular boundary) in which the centre of the wheel moves in a straight line and the wheel rotates in its plane about its centre with $v = R\omega$, is called pure *rolling*.

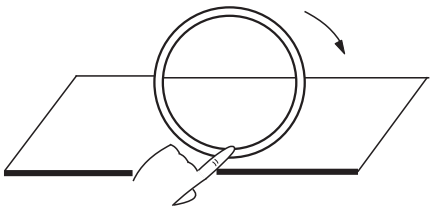


Figure 10.28

Place a ring on a horizontal surface as shown in figure (10.28) and put your finger on the lowest part. Use other hand to rotate the ring a little while the finger is kept on the lowest point. This is approximately a small part of rolling motion. Note the displacements of different particles of the ring. The centre has moved forward a little, say $\Delta x$. The topmost point has moved approximately double of this distance. The part in contact with the horizontal surface below the finger has almost been in the same position.

In pure rolling, the velocity of the contact point is zero. The velocity of the centre of mass is $v_{cm} = R\omega$ and that of the topmost point is $v_{top} = 2R\omega = 2v_{cm}$.

Next, consider another type of combination of rotation and translation, in which the wheel moves through a distance greater than $2\pi R$ in one full rotation. Hold the ring of figure (10.28) between three fingers, apply a forward force to move it fast on the table and rotate it slowly through the fingers. Its angular velocity $\omega = d\theta/dt$ is small and $v_{cm} > R\omega$. This is a case of rolling with forward slipping. This type of motion occurs when you apply sudden brakes to the bicycle on a road which is fairly smooth after rain. The cycle stops after a long distance and the wheel rotates only little during this period. If you look at the particles in contact, these will be found rubbing the road in the forward direction. The particles in contact have a velocity in the forward direction. In this case $v_{cm} > R\omega$. An extreme example of this type occurs when the wheel does not rotate at all and translates with linear velocity $v$. Then $v_{cm} = v$ and $\omega = 0$.

Yet another type of rolling with slipping occurs when the wheel moves a distance shorter than $2\pi R$ while making one rotation. In this case, the velocity $v_{cm} < R\omega$. Hold the ring of figure (10.28) between three fingers, rotate it fast and translate it slowly. It will move a small distance on the table and rotate fast. If you drive a bicycle on a road on which a lot of mud is present, sometimes the wheel rotates fast but moves a little. If you look at the particles in contact, they rub the road in the backward direction. The centre moves less than $2\pi R$ during one full rotation and $v_{cm} < R\omega$.

These situations may be visualised in a different manner which gives another interpretation of rolling. Consider a wheel of radius $r$ with its axis fixed in a second-hand car. The wheel may rotate freely about this axis. Suppose the floor of the second-hand car has a hole in it and the wheel just touches the road through the hole. Suppose the person sitting on the back seat rotates the wheel at a uniform angular velocity $\omega$ and the driver drives the car at a uniform velocity $v$ on the road which is parallel to the plane of the wheel as shown in figure (10.29). The two motions are independent. The backseater is rotating the wheel at an angular velocity according to his will and the driver is driving the car at a velocity according to his will.

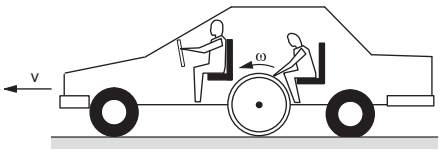



Figure 10.29

Look at the wheel from the road. If the persons inside the car agree to choose $v$ and ω in such a way that $v = \omega r$, the wheel is in pure rolling on the road. Looking from the road, the centre of the wheel is moving forward at a speed $v$ and the wheel is rotating at an angular velocity ω with $v = \omega r$. The velocity of the lowest particle with respect to the road = its velocity with respect to the car + velocity of the car with respect to the road. So,

$$v_{contact,\,road} = v_{contact,\,car} + v_{car,\,road} = -\omega r + v = 0.$$

If the driver drives the car at a higher speed, $v > \omega r$, the wheel rubs the road and we have rolling with forward slipping. In this case

$$v_{contact,\,road} = v_{contact,\,car} + v_{car,\,road} = -\omega r + v > 0$$

Similarly, if $v < \omega r$, we have rolling with backward slipping,

$$v_{contact,\,road} = -\omega r + v < 0,$$

the particles at contact rub the road backward.

## 10.18 KINETIC ENERGY OF A BODY IN COMBINED ROTATION AND TRANSLATION

Consider a body in combined translational and rotational motion in the lab frame. Suppose in the frame of the centre of mass, the body is making a pure rotation with an angular velocity ω. The centre of mass itself is moving in the lab frame at a velocity $\vec{v}_0$. The velocity of a particle of mass $m_i$ is $\vec{v}_{i,\,cm}$ with respect to the centre-of-mass frame and $\vec{v}_i$ with respect to the lab frame. We have,

$$\vec{v}_i = \vec{v}_{i,\,cm} + \vec{v}_0$$

The kinetic energy of the particle in the lab frame is

$$\frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i (\vec{v}_{i,\,cm} + \vec{v}_0) \cdot (\vec{v}_{i,\,cm} + \vec{v}_0)$$
$$= \frac{1}{2} m_i v_{i,\,cm}^2 + \frac{1}{2} m_i v_0^2 + \frac{1}{2} m_i (2\vec{v}_{i,\,cm} \cdot \vec{v}_0).$$

Summing over all the particles, the total kinetic energy of the body in the lab frame is

$$K = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i v_{i,\,cm}^2 + \frac{1}{2} \sum_i m_i v_0^2 + \left( \sum_i m_i \vec{v}_{i,\,cm} \right) \cdot \vec{v}_0.$$

Now $\sum_i \frac{1}{2} m_i v_{i,\,cm}^2$ is the kinetic energy of the body in the centre of mass frame. In this frame, the body is making pure rotation with an angular velocity ω. Thus, this term is equal to $\frac{1}{2} I_{cm} \omega^2$. Also $\frac{\sum m_i \vec{v}_{i,\,cm}}{M}$ is the velocity of the centre of mass in the centre of mass frame which is obviously zero. Thus,

$$K = \frac{1}{2} I_{cm} \omega^2 + \frac{1}{2} M v_0^2.$$

In the case of pure rolling, $v_0 = R\omega$ so that

$$K = \frac{1}{2} (I_{cm} + MR^2) \omega^2.$$

Using the parallel axes theorem, $I_{cm} + MR^2 = I$, which is the moment of inertia of the wheel about the line through the point of contact and parallel to the axis. Thus, $K = \frac{1}{2} I \omega^2$.

This gives another interpretation of rolling. At any instant a rolling body may be considered to be in pure rotation about an axis through the point of contact. This axis translates forward with a speed $v_0$.

***Example 10.12***

*A uniform sphere of mass* 200 g *rolls without slipping on a plane surface so that its centre moves at a speed of* 2·00 cm/s. *Find its kinetic energy.*

***Solution*** : As the sphere rolls without slipping on the plane surface, its angular speed about the center is $\omega = \frac{v_{cm}}{r}$. The kinetic energy is

$$K = \frac{1}{2} I_{cm} \omega^2 + \frac{1}{2} M v_{cm}^2$$
$$= \frac{1}{2} \cdot \frac{2}{5} M r^2 \omega^2 + \frac{1}{2} M v_{cm}^2$$
$$= \frac{1}{5} M v_{cm}^2 + \frac{1}{2} M v_{cm}^2 = \frac{7}{10} M v_{cm}^2$$
$$= \frac{7}{10} (0{\cdot}200 \text{ kg}) (0{\cdot}02 \text{ m/s})^2 = 5{\cdot}6 \times 10^{-5} \text{ J}.$$

## 10.19 ANGULAR MOMENTUM OF A BODY IN COMBINED ROTATION AND TRANSLATION

Consider the situation described in the previous section. Let $O$ be a fixed point in the lab which we take as the origin. Angular momentum of the body about $O$ is

$$\vec{L} = \sum_i m_i \vec{r}_i \times \vec{v}_i$$
$$= \sum_i m_i (\vec{r}_{i,\,cm} + \vec{r}_0) \times (\vec{v}_{i,\,cm} + \vec{v}_0).$$

Here, $\vec{r}_0$ is the position vector of the centre of mass. Thus,

$$\vec{L} = \sum_i m_i (\vec{r}_{i,cm} \times \vec{v}_{i,cm}) + \left( \sum_i m_i \vec{r}_{i,cm} \right) \times \vec{v}_0 + \vec{r}_0 \times \left( \sum_i m_i \vec{v}_{i,cm} \right) + \left( \sum_i m_i \right) \vec{r}_0 \times \vec{v}_0 .$$

Now, $$\sum_i m_i \vec{r}_{i,cm} = M \vec{R}_{cm,cm} = 0$$

and $$\sum_i m_i \vec{v}_{i,cm} = M \vec{V}_{cm,cm} = 0.$$

Thus, $$\vec{L} = \sum_i m_i (\vec{r}_{i,cm} \times \vec{v}_{i,cm}) + M \vec{r}_0 \times \vec{v}_0$$

$$= \vec{L}_{cm} + M \vec{r}_0 \times \vec{v}_0 .$$

The first term $\vec{L}_{cm}$ represents the angular momentum of the body as seen from the centre-of-mass frame. The second term $M \vec{r}_0 \times \vec{v}_0$ equals the angular momentum of the body if it is assumed to be concentrated at the centre of mass translating with the velocity $\vec{v}_0$.

## 10.20 WHY DOES A ROLLING SPHERE SLOW DOWN ?

When a sphere is rolled on a horizontal table it slows down and eventually stops. Figure (10.30) shows the situation. The forces acting on the sphere are (a) weight *mg*, (b) friction at the contact and (c) the normal force. As the centre of the sphere decelerates, the friction should be opposite to its velocity, that is towards left in figure (10.30). But this friction will have a clockwise torque that should increase the angular velocity of the sphere. There must be an anticlockwise torque that causes the decrease in the angular velocity.

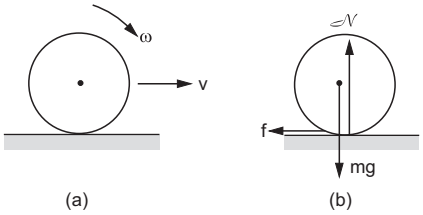



Figure 10.30

In fact, when the sphere rolls on the table, both the sphere and the surface deform near the contact. The contact is not at a single point as we normally assume, rather there is an area of contact. The front part pushes the table a bit more strongly than the back part. As a result, the normal force does not pass through the centre, it is shifted towards the right. This force, then, has an anticlockwise torque. The net torque causes an angular deceleration.

### *Worked Out Examples*

**1.** *A wheel rotates with a constant acceleration of* 2·0 rad/s$^2$. *If the wheel starts from rest, how many revolutions will it make in the first* 10 seconds ?

***Solution*** : The angular displacement in the first 10 seconds is given by

$$\theta = \omega_0 t + \frac{1}{2} \alpha t^2 = \frac{1}{2} (2{\cdot}0 \text{ rad/s}^2) (10 \text{ s})^2 = 100 \text{ rad}.$$

As the wheel turns by $2\pi$ radian in each revolution, the number of revolutions in 10 s is

$$n = \frac{100}{2\pi} = 16.$$

**2.** *The wheel of a motor, accelerated uniformly from rest, rotates through* 2·5 radian *during the first second. Find the angle rotated during the next second.*

***Solution*** : As the angular acceleration is constant, we have

$$\theta = \omega_0 t + \frac{1}{2} \alpha t^2 = \frac{1}{2} \alpha t^2 .$$

Thus, $$2{\cdot}5 \text{ rad} = \frac{1}{2} \alpha (1 \text{ s})^2$$

$$\alpha = 5 \text{ rad/s}^2$$

or, $$\alpha = 5 \text{ rad/s}^2 .$$

The angle rotated during the first two seconds is

$$= \frac{1}{2} \times (5 \text{ rad/s}^2) (2 \text{ s})^2 = 10 \text{ rad}.$$

Thus, the angle rotated during the 2nd second is

$$10 \text{ rad} - 2{\cdot}5 \text{ rad} = 7{\cdot}5 \text{ rad}.$$

**3.** *A wheel having moment of inertia* 2 kg-m$^2$ *about its axis, rotates at* 50 rpm *about this axis. Find the torque that can stop the wheel in one minute.*

***Solution*** : The initial angular velocity

$$= 50 \text{ rpm} = \frac{5\pi}{3} \text{ rad/s}.$$

Using $\omega = \omega_0 + \alpha t$,

$$\alpha = \frac{\omega - \omega_0}{t} = \frac{0 - \frac{5\pi}{3}}{60} \text{ rad/s}^2 = -\frac{\pi}{36} \text{ rad/s}^2 .$$

The torque that can produce this deceleration is

$$\Gamma = I \mid \alpha \mid = (2 \text{ kg-m}^2) \left( \frac{\pi}{36} \text{ rad/s}^2 \right) = \frac{\pi}{18} \text{ N-m}.$$

**4.** *A string is wrapped around the rim of a wheel of moment of inertia* 0·20 kg-m$^2$ *and radius* 20 cm. *The wheel is free to rotate about its axis. Initially, the wheel is at rest. The string is now pulled by a force of* 20 N. *Find the angular velocity of the wheel after* 5·0 seconds.

***Solution*** :

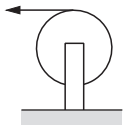


Figure 10-W1

The torque applied to the wheel is

$$\Gamma = F.\, r = (20\text{ N})\,(0{\cdot}20\text{ m}) = 4{\cdot}0\text{ N-m}.$$

The angular acceleration produced is

$$\alpha = \frac{\Gamma}{I} = \frac{4{\cdot}0\text{ N-m}}{0{\cdot}20\text{ kg-m}^2} = 20\text{ rad/s}^2.$$

The angular velocity after 5·0 seconds is

$$\omega = \omega_0 + \alpha\, t = (20\text{ rad/s}^2)\,(5{\cdot}0\text{ s}) = 100\text{ rad/s}.$$

**5.** *A wheel of radius r and moment of inertia I about its axis is fixed at the top of an inclined plane of inclination* θ *as shown in figure (10-W2). A string is wrapped round the wheel and its free end supports a block of mass M which can slide on the plane. Initially, the wheel is rotating at a speed* ω *in a direction such that the block slides up the plane. How far will the block move before stopping* ?

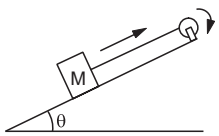



Figure 10-W2

***Solution*** : Suppose the deceleration of the block is $a$. The linear deceleration of the rim of the wheel is also $a$. The angular deceleration of the wheel is $\alpha = a/r$. If the tension in the string is $T$, the equations of motion are as follows:

$$Mg\sin\theta - T = Ma$$

and $$Tr = I\alpha = I\,a/r.$$

Eliminating $T$ from these equations,

$$Mg\sin\theta - I\frac{a}{r^2} = Ma$$

giving, $$a = \frac{Mg\,r^2\sin\theta}{I + Mr^2}.$$

The initial velocity of the block up the incline is $v = \omega\, r$. Thus, the distance moved by the block before stopping is

$$x = \frac{v^2}{2a} = \frac{\omega^2 r^2 (I + Mr^2)}{2\,Mg\,r^2\sin\theta} = \frac{(I + Mr^2)\omega^2}{2\,Mg\sin\theta}.$$

**6.** *The pulley shown in figure (10-W3) has a moment of inertia I about its axis and its radius is R. Find the magnitude of the acceleration of the two blocks. Assume that the string is light and does not slip on the pulley.*

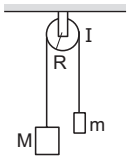



Figure 10-W3

***Solution*** : Suppose the tension in the left string is $T_1$ and that in the right string is $T_2$. Suppose the block of mass $M$ goes down with an acceleration $a$ and the other block moves up with the same acceleration. This is also the tangential acceleration of the rim of the wheel as the string does not slip over the rim. The angular acceleration of the wheel is, therefore, $\alpha = a/R$. The equations of motion for the mass $M$, the mass $m$ and the pulley are as follows :

$$Mg - T_1 = Ma \quad \ldots \text{(i)}$$

$$T_2 - mg = ma \quad \ldots \text{(ii)}$$

$$T_1 R - T_2 R = I\alpha = Ia/R. \quad \ldots \text{(iii)}$$

Putting $T_1$ and $T_2$ from (i) and (ii) into (iii),

$$[M(g - a) - m(g + a)]\,R = I\frac{a}{R}$$

which gives $a = \frac{(M - m)gR^2}{I + (M + m)R^2}$.

**7.** *Two small kids weighing* 10 kg *and* 15 kg *respectively are trying to balance a seesaw of total length* 5·0 m, *with the fulcrum at the centre. If one of the kids is sitting at an end, where should the other sit* ?

***Solution*** :

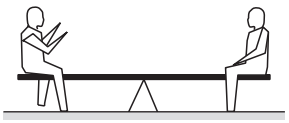


Figure 10-W4

It is clear that the 10 kg kid should sit at the end and the 15 kg kid should sit closer to the centre. Suppose his distance from the centre is $x$. As the kids are in equilibrium, the normal force between a kid and the seesaw equals the weight of that kid. Considering the rotational equilibrium of the seesaw, the torques of the forces acting on it should add to zero. The forces are
(a) (15 kg)$g$ downward by the 15 kg kid,

(b) (10 kg)$g$ downward by the 10 kg kid,
(c) weight of the seesaw and
(d) the normal force by the fulcrum.

Taking torques about the fulcrum,

$$(15 \text{ kg})g\,x = (10 \text{ kg})g\,(2{\cdot}5 \text{ m})$$

or, $$x = 1{\cdot}7 \text{ m}.$$

**8.** *A uniform ladder of mass* 10 kg *leans against a smooth vertical wall making an angle of* 53° *with it. The other end rests on a rough horizontal floor. Find the normal force and the frictional force that the floor exerts on the ladder.*

***Solution*** **:**

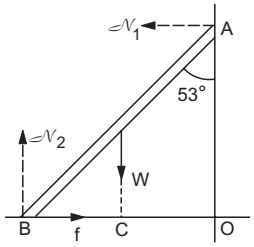



Figure 10-W5

The forces acting on the ladder are shown in figure (10-W5). They are
(a) its weight $W$,
(b) normal force $\mathcal{N}_1$ by the vertical wall,
(c) normal force $\mathcal{N}_2$ by the floor and
(d) frictional force $f$ by the floor.

Taking horizontal and vertical components

$$\mathcal{N}_1 = f \qquad \ldots \text{(i)}$$

and $$\mathcal{N}_2 = W. \qquad \ldots \text{(ii)}$$

Taking torque about $B$,

$$\mathcal{N}_1\,(AO) = W(CB)$$

or, $$\mathcal{N}_1\,(AB)\cos 53° = W\frac{AB}{2}\sin 53°$$

or, $$\mathcal{N}_1\frac{3}{5} = \frac{W}{2}\frac{4}{5}$$

or, $$\mathcal{N}_1 = \frac{2}{3}W. \qquad \ldots \text{(iii)}$$

The normal force by the floor is

$$\mathcal{N}_2 = W = (10 \text{ kg})\,(9{\cdot}8 \text{ m/s}^2) = 98 \text{ N}.$$

The frictional force is

$$f = \mathcal{N}_1 = \frac{2}{3}W = 65 \text{ N}.$$

**9.** *The ladder shown in figure (10-W6) has negligible mass and rests on a frictionless floor. The crossbar connects the two legs of the ladder at the middle. The angle between the two legs is* 60°. *The fat person sitting on the ladder has a mass of* 80 kg. *Find the contact force exerted by the floor on each leg and the tension in the crossbar.*

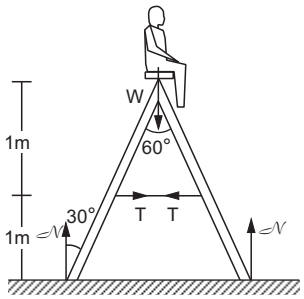



Figure 10-W6

***Solution*** **:** The forces acting on different parts are shown in figure (10-W6). Consider the vertical equilibrium of "the ladder plus the person" system. The forces acting on this system are its weight (80 kg)$g$ and the contact force $\mathcal{N} + \mathcal{N} = 2\mathcal{N}$ due to the floor. Thus,

$$2\mathcal{N} = (80 \text{ kg})\,g$$

or, $$\mathcal{N} = (40 \text{ kg})\,(9{\cdot}8 \text{ m/s}^2) = 392 \text{ N}.$$

Next consider the equilibrium of the left leg of the ladder. Taking torques of the forces acting on it about the upper end,

$$\mathcal{N}\,(2 \text{ m})\tan 30° = T(1 \text{ m})$$

or, $$T = \mathcal{N}\frac{2}{\sqrt{3}} = (392 \text{ N}) \times \frac{2}{\sqrt{3}} \approx 450 \text{ N}.$$

**10.** *Two small balls A and B, each of mass m, are attached rigidly to the ends of a light rod of length d. The structure rotates about the perpendicular bisector of the rod at an angular speed* ω. *Calculate the angular momentum of the individual balls and of the system about the axis of rotation.*

***Solution*** **:**

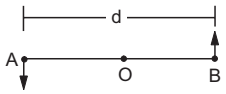



Figure 10-W7

Consider the situation shown in figure (10-W7). The velocity of the ball $A$ with respect to the centre $O$ is $v = \frac{\omega d}{2}$. The angular momentum of the ball with respect to the axis is $L_1 = mvr = m\left(\frac{\omega d}{2}\right)\left(\frac{d}{2}\right) = \frac{1}{4}m\omega d^2$.

The same is the angular momentum $L_2$ of the second ball. The angular momentum of the system is equal to sum of these two angular momenta i.e., $L = \frac{1}{2}m\omega d^2$.

**11.** *Two particles of mass m each are attached to a light rod of length d, one at its centre and the other at a free end.*

*The rod is fixed at the other end and is rotated in a plane at an angular speed ω. Calculate the angular momentum of the particle at the end with respect to the particle at the centre.*

***Solution*** :

Figure 10-W8

The situation is shown in figure (10-W8). The velocity of the particle $A$ with respect to the fixed end $O$ is $v_A = \omega\left(\frac{d}{2}\right)$ and that of $B$ with respect to $O$ is $v_B = \omega d$. Hence the velocity of $B$ with respect to $A$ is $v_B - v_A = \omega\left(\frac{d}{2}\right)$. The angular momentum of $B$ with respect to $A$ is, therefore,

$$L = mvr = m\omega\left(\frac{d}{2}\right)\frac{d}{2} = \frac{1}{4}m\omega d^2$$

along the direction perpendicular to the plane of rotation.

**12.** *A particle is projected at time $t$ = 0 from a point $P$ with a speed $v_0$ at an angle of* 45° *to the horizontal. Find the magnitude and the direction of the angular momentum of the particle about the point $P$ at time $t = v_0/g$.*

***Solution*** : Let us take the origin at $P$, $X$-axis along the horizontal and $Y$-axis along the vertically upward direction as shown in figure (10-W9). For horizontal motion during the time 0 to $t$,

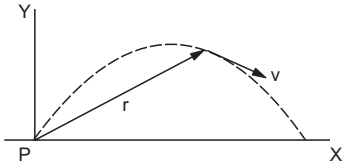



Figure 10-W9

$$v_x = v_0 \cos 45° = v_0/\sqrt{2}$$

and $$x = v_x t = \frac{v_0}{\sqrt{2}} \cdot \frac{v_0}{g} = \frac{v_0^2}{\sqrt{2}\,g}.$$

For vertical motion,

$$v_y = v_0 \sin 45° - gt = \frac{v_0}{\sqrt{2}} - v_0 = \frac{(1-\sqrt{2})}{\sqrt{2}}v_0$$

and $$y = (v_0 \sin 45°)\,t - \frac{1}{2}gt^2$$

$$= \frac{v_0^2}{\sqrt{2}\,g} - \frac{v_0^2}{2g} = \frac{v_0^2}{2g}(\sqrt{2}-1)$$

The angular momentum of the particle at time $t$ about the origin is

$$L = \vec{r} \times \vec{p} = m\,\vec{r} \times \vec{v}$$

$$= m\,(\vec{i}\,x + \vec{j}\,y) \times (\vec{i}\,v_x + \vec{j}\,v_y)$$

$$= m\,(\vec{k}\,x\,v_y - \vec{k}\,y\,v_x)$$

$$= m\,\vec{k}\left[\left(\frac{v_0^2}{\sqrt{2}\,g}\right)\frac{v_0}{\sqrt{2}}(1-\sqrt{2}) - \frac{v_0^2}{2g}(\sqrt{2}-1)\frac{v_0}{\sqrt{2}}\right]$$

$$= -\vec{k}\,\frac{mv_0^3}{2\sqrt{2}\,g}.$$

Thus, the angular momentum of the particle is $\frac{m v_0^3}{2\sqrt{2}\,g}$ in the negative $Z$-direction, i.e., perpendicular to the plane of motion, going into the plane.

**13.** *A uniform circular disc of mass* 200 g *and radius* 4·0 cm is rotated about one of its diameter at an angular speed of 10 rad/s. *Find the kinetic energy of the disc and its angular momentum about the axis of rotation.*

***Solution*** : The moment of inertia of the circular disc about its diameter is

$$I = \frac{1}{4}Mr^2 = \frac{1}{4}(0{\cdot}200\text{ kg})\,(0{\cdot}04\text{ m})^2$$

$$= 8{\cdot}0 \times 10^{-5}\text{ kg-m}^2.$$

The kinetic energy is

$$K = \frac{1}{2}I\omega^2 = \frac{1}{2}(8{\cdot}0 \times 10^{-5}\text{ kg-m}^2)\,(100\text{ rad}^2/\text{s}^2)$$

$$= 4{\cdot}0 \times 10^{-3}\text{ J}$$

and the angular momentum about the axis of rotation is

$$L = I\omega = (8{\cdot}0 \times 10^{-5}\text{ kg-m}^2)\,(10\text{ rad/s})$$

$$= 8{\cdot}0 \times 10^{-4}\text{ kg-m}^2/\text{s} = 8{\cdot}0 \times 10^{-4}\text{ J-s}.$$

**14.** *A wheel rotating at an angular speed of* 20 rad/s *is brought to rest by a constant torque in* 4·0 seconds. *If the moment of inertia of the wheel about the axis of rotation is* 0·20 kg-m$^2$, *find the work done by the torque in the first two seconds.*

***Solution*** : The angular deceleration of the wheel during the 4·0 seconds may be obtained by the equation

$$\omega = \omega_0 - \alpha t$$

or, $$\alpha = \frac{\omega_0 - \omega}{t} = \frac{20\text{ rad/s}}{4{\cdot}0\text{ s}} = 5{\cdot}0\text{ rad/s}^2.$$

The torque applied to produce this deceleration is

$$\Gamma = I\alpha = (0{\cdot}20\text{ kg-m}^2)\,(5{\cdot}0\text{ rad/s}^2) = 1{\cdot}0\text{ N-m}.$$

The angle rotated in the first two seconds is

$$\theta = \omega_0 t - \frac{1}{2}\alpha t^2$$

$$= (20\text{ rad/s})\,(2\text{ s}) - \frac{1}{2}(5{\cdot}0\text{ rad/s}^2)\,(4{\cdot}0\text{ s}^2)$$

$$= 40\text{ rad} - 10\text{ rad} = 30\text{ rad}.$$

The work done by the torque in the first 2 seconds is, therefore,

$$W = \Gamma\theta = (1{\cdot}0\text{ N-m})\,(30\text{ rad}) = 30\text{ J}.$$

**15.** *Two masses M and m are connected by a light string going over a pulley of radius r. The pulley is free to rotate about its axis which is kept horizontal. The moment of inertia of the pulley about the axis is I. The system is released from rest. Find the angular momentum of the system when the mass M has descended through a height h. The string does not slip over the pulley.*

***Solution*** **:**

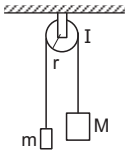



Figure 10-W10

The situation is shown in figure (10-W10). Let the speed of the masses be $v$ at time $t$. This will also be the speed of a point on the rim of the wheel and hence the angular velocity of the wheel at time $t$ will be $v/r$. If the height descended by the mass $M$ is $h$, the loss in the potential energy of the "masses plus the pulley" system is $Mgh - mgh$. The gain in kinetic energy is $\frac{1}{2}Mv^2 + \frac{1}{2}mv^2 + \frac{1}{2}I\left(\frac{v}{r}\right)^2$. As no energy is lost,

$$\frac{1}{2}\left(M + m + \frac{I}{r^2}\right)v^2 = (M - m)gh$$

or, $$v^2 = \frac{2(M-m)gh}{M + m + \frac{I}{r^2}}.$$

The angular momentum of the mass $M$ is $Mvr$ and that of the mass $m$ is $mvr$ in the same direction. The angular momentum of the pulley is $I\omega = Iv/r$. The total angular momentum is

$$\left[(M+m)r + \frac{I}{r}\right]v = \left[\left(M + m + \frac{I}{r^2}\right)r\right]\sqrt{\frac{2(M-m)gh}{M + m + \frac{I}{r^2}}}$$

$$= \sqrt{2(M-m)\left(M + m + \frac{I}{r^2}\right)r^2gh}\ .$$

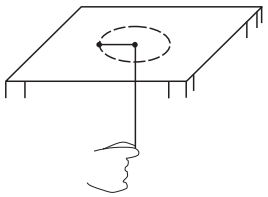


Figure 10-W11

**16.** *Figure (10-W11) shows a mass m placed on a frictionless horizontal table and attached to a string passing through a small hole in the surface. Initially, the mass moves in a circle of radius $r_0$ with a speed $v_0$ and the free end of the string is held by a person. The person pulls on the string slowly to decrease the radius of the circle to r. (a) Find the tension in the string when the mass moves in the circle of radius r. (b) Calculate the change in the kinetic energy of the mass.*

***Solution*** **:** The torque acting on the mass $m$ about the vertical axis through the hole is zero. The angular momentum about this axis, therefore, remains constant. If the speed of the mass is $v$ when it moves in the circle of radius $r$, we have

$$mv_0 r_0 = mvr$$

or, $$v = \frac{r_0}{r}v_0\ . \qquad \ldots \text{(i)}$$

(a) The tension $T = \frac{mv^2}{r} = \frac{mr_0^2 v_0^2}{r^3}$.

(b) The change in kinetic energy $= \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$.

By (i), it is $\frac{1}{2}mv_0^2\left[\frac{r_0^2}{r^2} - 1\right]$.

**17.** *A uniform rod of mass m and length l is kept vertical with the lower end clamped. It is slightly pushed to let it fall down under gravity. Find its angular speed when the rod is passing through its lowest position. Neglect any friction at the clamp. What will be the linear speed of the free end at this instant ?*

***Solution*** **:**

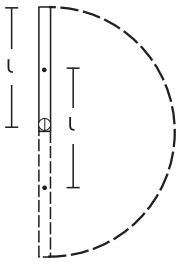



Figure 10-W12

As the rod reaches its lowest position, the centre of mass is lowered by a distance $l$. Its gravitational potential energy is decreased by $mgl$. As no energy is lost against friction, this should be equal to the increase in the kinetic energy. As the rotation occurs about the horizontal axis through the clamped end, the moment of inertia is $I = ml^2/3$. Thus,

$$\frac{1}{2}I\omega^2 = mgl$$

$$\frac{1}{2}\left(\frac{ml^2}{3}\right)\omega^2 = mgl$$

or, $$\omega = \sqrt{\frac{6g}{l}}\ .$$

The linear speed of the free end is

$$v = l\omega = \sqrt{6\,gl}.$$

**18.** *Four particles each of mass m are kept at the four corners of a square of edge a. Find the moment of inertia of the system about a line perpendicular to the plane of the square and passing through the centre of the square.*

***Solution*** **:**

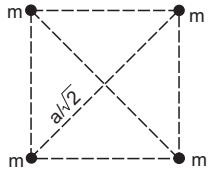



Figure 10-W13

The perpendicular distance of every particle from the given line is $a/\sqrt{2}$. The moment of inertia of one particle is, therefore, $m(a/\sqrt{2})^2 = \frac{1}{2}ma^2$. The moment of inertia of the system is, therefore, $4 \times \frac{1}{2}ma^2 = 2\,ma^2$.

**19.** *Two identical spheres each of mass* 1·20 kg *and radius* 10·0 cm *are fixed at the ends of a light rod so that the separation between the centres is* 50·0 cm. *Find the moment of inertia of the system about an axis perpendicular to the rod passing through its middle point.*

***Solution*** **:**

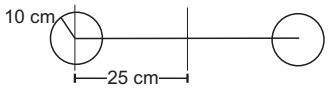



Figure 10-W14

Consider the diameter of one of the spheres parallel to the given axis. The moment of inertia of this sphere about the diameter is

$$I = \frac{2}{5}mR^2 = \frac{2}{5}(1{\cdot}20\text{ kg})(0{\cdot}1\text{ m})^2$$
$$= 4{\cdot}8 \times 10^{-3}\text{ kg-m}^2.$$

Its moment of inertia about the given axis is obtained by using the parallel axes theorem. Thus,

$$I = I_{cm} + md^2$$
$$= 4{\cdot}8 \times 10^{-3}\text{ kg-m}^2 + (1{\cdot}20\text{ kg})(0{\cdot}25\text{ m})^2$$
$$= 4{\cdot}8 \times 10^{-3}\text{ kg-m}^2 + 0{\cdot}075\text{ kg-m}^2$$
$$= 79{\cdot}8 \times 10^{-3}\text{ kg-m}^2.$$

The moment of inertia of the second sphere is also the same so that the moment of inertia of the system is

$$2 \times 79{\cdot}8 \times 10^{-3}\text{ kg-m}^2 \approx 0{\cdot}160\text{ kg-m}^2.$$

**20.** *Two uniform identical rods each of mass M and length l are joined to form a cross as shown in figure (10-W15). Find the moment of inertia of the cross about a bisector as shown dotted in the figure.*

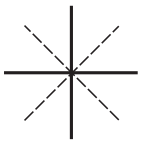


Figure 10-W15

***Solution*** **:** Consider the line perpendicular to the plane of the figure through the centre of the cross. The moment of inertia of each rod about this line is $\frac{Ml^2}{12}$ and hence the moment of inertia of the cross is $\frac{Ml^2}{6}$. The moment of inertia of the cross about the two bisectors are equal by symmetry and according to the theorem of perpendicular axes, the moment of inertia of the cross about the bisector is $\frac{Ml^2}{12}$.

**21.** *A uniform rod of mass M and length a lies on a smooth horizontal plane. A particle of mass m moving at a speed v perpendicular to the length of the rod strikes it at a distance a/4 from the centre and stops after the collision. Find (a) the velocity of the centre of the rod and (b) the angular velocity of the rod about its centre just after the collision.*

***Solution*** **:**

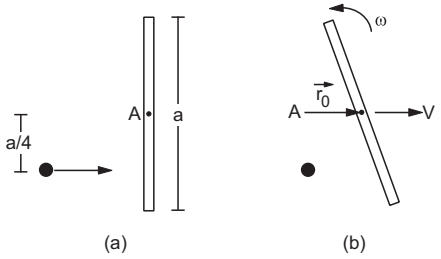



Figure 10-W16

The situation is shown in figure (10-W16a). Consider the rod and the particle together as the system. As there is no external resultant force, the linear momentum of the system will remain constant. Also there is no resultant external torque on the system and so the angular momentum of the system about any line will remain constant.

Suppose the velocity of the centre of the rod is $V$ and the angular velocity about the centre is $\omega$.

(a) The linear momentum before the collision is $mv$ and that after the collision is $MV$. Thus,

$$mv = MV, \text{ or } V = \frac{m}{M}v.$$

(b) Let $A$ be the centre of the rod when it is at rest. Let $AB$ be the line perpendicular to the plane of the figure. Consider the angular momentum of "the rod plus the particle" system about $AB$. Initially the rod is at rest. The angular momentum of the particle about $AB$ is

$$L = mv(a/4).$$

After the collision, the particle comes to rest. The angular momentum of the rod about $A$ is

$$\vec{L} = \vec{L}_{cm} + M\vec{r}_0 \times \vec{V}.$$

As $\vec{r}_0 \parallel \vec{V}$, $\vec{r}_0 \times \vec{V} = 0$.

Thus, $\vec{L} = \vec{L}_{cm}$.

Hence the angular momentum of the rod about $AB$ is

$$L = I\omega = \frac{Ma^2}{12}\omega.$$

Thus, $\frac{mva}{4} = \frac{Ma^2}{12}\omega$ or, $\omega = \frac{3mv}{Ma}$.

**22.** *A wheel of perimeter* 220 cm *rolls on a level road at a speed of* 9 km/h. *How many revolutions does the wheel make per second* ?

***Solution*** : As the wheel rolls on the road, its angular speed $\omega$ about the centre and the linear speed $v$ of the centre are related as $v = \omega r$.

$$\therefore\ \omega = \frac{v}{r} = \frac{9\text{ km/h}}{220\text{ cm}/2\pi} = \frac{2\pi \times 9 \times 10^5}{220 \times 3600}\text{ rad/s}.$$

$$= \frac{900}{22 \times 36}\text{ rev/s} = \frac{25}{22}\text{ rev/s}.$$

**23.** *A cylinder is released from rest from the top of an incline of inclination* $\theta$ *and length* $l$. *If the cylinder rolls without slipping, what will be its speed when it reaches the bottom* ?

***Solution*** : Let the mass of the cylinder be $m$ and its radius $r$. Suppose the linear speed of the cylinder when it reaches the bottom is $v$. As the cylinder rolls without slipping, its angular speed about its axis is $\omega = v/r$. The kinetic energy at the bottom will be

$$K = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2$$

$$= \frac{1}{2}\left(\frac{1}{2}mr^2\right)\omega^2 + \frac{1}{2}mv^2 = \frac{1}{4}mv^2 + \frac{1}{2}mv^2 = \frac{3}{4}mv^2.$$

This should be equal to the loss of potential energy $mgl\sin\theta$. Thus, $\frac{3}{4}mv^2 = mgl\sin\theta$

or, $v = \sqrt{\frac{4}{3}gl\sin\theta}$.

**24.** *A sphere of mass* $m$ *rolls without slipping on an inclined plane of inclination* $\theta$. *Find the linear acceleration of the sphere and the force of friction acting on it. What should be the minimum coefficient of static friction to support pure rolling* ?

***Solution*** : Suppose the radius of the sphere is $r$. The forces acting on the sphere are shown in figure (10-W17). They are (a) weight $mg$, (b) normal force $\mathcal{N}$ and (c) friction $f$.

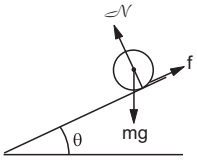



Figure 10-W17

Let the linear acceleration of the sphere down the plane be $a$. The equation for the linear motion of the centre of mass is

$$mg\sin\theta - f = ma. \qquad \ldots\ (i)$$

As the sphere rolls without slipping, its angular acceleration about the centre is $a/r$. The equation of rotational motion about the centre of mass is,

$$fr = \left(\frac{2}{5}mr^2\right)\left(\frac{a}{r}\right)$$

or, $f = \frac{2}{5}ma.$ ... (ii)

From (i) and (ii),

$$a = \frac{5}{7}g\sin\theta$$

and $f = \frac{2}{7}mg\sin\theta.$

The normal force is equal to $mg\cos\theta$ as there is no acceleration perpendicular to the incline. The maximum friction that can act is, therefore, $\mu mg\cos\theta$, where $\mu$ is the coefficient of static friction. Thus, for pure rolling

$$\mu mg\cos\theta > \frac{2}{7}mg\sin\theta$$

or, $\mu > \frac{2}{7}\tan\theta.$

**25.** *Figure (10-W18) shows two cylinders of radii* $r_1$ *and* $r_2$ *having moments of inertia* $I_1$ *and* $I_2$ *about their respective axes. Initially, the cylinders rotate about their axes with angular speeds* $\omega_1$ *and* $\omega_2$ *as shown in the figure. The cylinders are moved closer to touch each other keeping the axes parallel. The cylinders first slip over each other at the contact but the slipping finally ceases due to the friction between them. Find the angular speeds of the cylinders after the slipping ceases.*

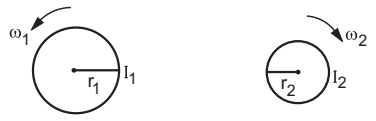



Figure 10-W18

***Solution*** : When slipping ceases, the linear speeds of the points of contact of the two cylinders will be equal. If $\omega'_1$ *and* $\omega'_2$ be the respective angular speeds, we have

$$\omega'_1 r_1 = \omega'_2 r_2. \quad \ldots \text{(i)}$$

The change in the angular speed is brought about by the frictional force which acts as long as the slipping exists. If this force $f$ acts for a time $t$, the torque on the first cylinder is $fr_1$ and that on the second is $fr_2$. Assuming $\omega_1 r_1 > \omega_2 r_2$, the corresponding angular impluses are $-fr_1 t$ and $fr_2 t$. We, therefore, have

$$-fr_1 t = I_1(\omega'_1 - \omega_1)$$

and

$$fr_2 t = I_2(\omega'_2 - \omega_2)$$

or,
$$-\frac{I_1}{r_1}(\omega'_1 - \omega_1) = \frac{I_2}{r_2}(\omega'_2 - \omega_2). \quad \ldots \text{(ii)}$$

Solving (i) and (ii),

$$\omega'_1 = \frac{I_1\omega_1 r_2 + I_2\omega_2 r_1}{I_2 r_1^2 + I_1 r_2^2} r_2 \text{ and } \omega'_2 = \frac{I_1\omega_1 r_2 + I_2\omega_2 r_1}{I_2 r_1^2 + I_1 r_2^2} r_1.$$

**26.** *A cylinder of mass m is suspended through two strings wrapped around it as shown in figure (10-W19). Find (a) the tension T in the string and (b) the speed of the cylinder as it falls through a distance h.*

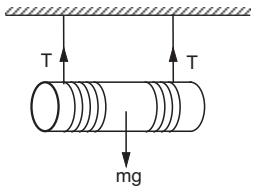



Figure 10-W19

***Solution*** : The portion of the strings between the ceiling and the cylinder is at rest. Hence the points of the cylinder where the strings leave it are at rest. The cylinder is thus rolling without slipping on the strings. Suppose the centre of the cylinder falls with an acceleration $a$. The angular acceleration of the cylinder about its axis is $\alpha = a/R$, as the cylinder does not slip over the strings.

The equation of motion for the centre of mass of the cylinder is

$$mg - 2T = ma \quad \ldots \text{(i)}$$

and for the motion about the centre of mass, it is

$$2Tr = \left(\frac{1}{2}mr^2\alpha\right) = \frac{1}{2}mra$$

or,
$$2T = \frac{1}{2}ma. \quad \ldots \text{(ii)}$$

From (i) and (ii),

$$a = \frac{2}{3}g \text{ and } T = \frac{mg}{6}.$$

As the centre of the cylinder starts moving from rest, the velocity after it has fallen through a distance $h$ is given by

$$v^2 = 2\left(\frac{2}{3}g\right)h$$

or,
$$v = \sqrt{\frac{4gh}{3}}.$$

**27.** *A force F acts tangentially at the highest point of a sphere of mass m kept on a rough horizontal plane. If the sphere rolls without slipping, find the acceleration of the centre of the sphere.*

***Solution*** :

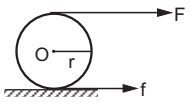



Figure 10-W20

The situation is shown in figure (10-W20). As the force $F$ rotates the sphere, the point of contact has a tendency to slip towards left so that the static friction on the sphere will act towards right. Let $r$ be the radius of the sphere and $a$ be the linear acceleration of the centre of the sphere. The angular acceleration about the centre of the sphere is $\alpha = a/r$, as there is no slipping.

For the linear motion of the centre

$$F + f = ma \quad \ldots \text{(i)}$$

and for the rotational motion about the centre,

$$Fr - fr = I\alpha = \left(\frac{2}{5}mr^2\right)\left(\frac{a}{r}\right)$$

or,
$$F - f = \frac{2}{5}ma. \quad \ldots \text{(ii)}$$

From (i) and (ii),

$$2F = \frac{7}{5}ma \text{ or, } a = \frac{10F}{7m}.$$

**28.** *A sphere of mass M and radius r shown in figure (10-W21) slips on a rough horizontal plane. At some instant it has translational velocity $v_0$ and rotational velocity about the centre $\frac{v_0}{2r}$. Find the translational velocity after the sphere starts pure rolling.*

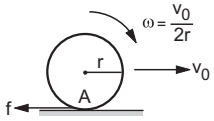



Figure 10-W21

***Solution*** : Velocity of the centre $= v_0$ and the angular velocity about the centre $= \omega_0 = \frac{v_0}{2r}$. Thus, $v_0 > \omega_0 r$. The sphere slips forward and thus the friction by the plane

on the sphere will act backward. As the friction is kinetic, its value is $\mu\mathcal{N} = \mu Mg$ and the sphere will be decelerated by $a_{cm} = f/M$. Hence,

$$v(t) = v_0 - \frac{f}{M}t \cdot \qquad \ldots \text{ (i)}$$

This friction will also have a torque $\Gamma = fr$ about the centre. This torque is clockwise and in the direction of $\omega_0$. Hence the angular acceleration about the centre will be

$$\alpha = f\frac{r}{(2/5)\,Mr^2} = \frac{5f}{2Mr}$$

and the clockwise angular velocity at time $t$ will be

$$\omega(t) = \omega_0 + \frac{5f}{2Mr}t = \frac{v_0}{2r} + \frac{5f}{2M}t$$

Pure rolling starts when $v(t) = r\omega(t)$

i.e., $$v(t) = \frac{v_0}{2} + \frac{5f}{2M}t \qquad \ldots \text{ (ii)}$$

Eliminating $t$ from (i) and (ii),

$$\frac{5}{2}v(t) + v(t) = \frac{5}{2}v_0 + \frac{v_0}{2}$$

or, $$v(t) = \frac{2}{7}\times 3\,v_0 = \frac{6}{7}v_0\,.$$

Thus, the sphere rolls with translational velocity $6v_0/7$ in the forward direction.

**Alternative :** Let us consider the torque about the initial point of contact $A$. The force of friction passes through this point and hence its torque is zero. The normal force and the weight balance each other. The net torque about $A$ is zero. Hence the angular momentum about $A$ is conserved.

Initial angular momentum is,

$$L = L_{cm} + Mrv_0 = I_{cm}\,\omega_0 + Mrv_0$$
$$= \left(\frac{2}{5}Mr^2\right)\left(\frac{v_0}{2r}\right) + Mrv_0 = \frac{6}{5}Mrv_0.$$

Suppose the translational velocity of the sphare, after it starts rolling, is $v$. The angular velocity is $v/r$. The angular momentum about $A$ is,

$$L = L_{cm} + Mrv$$
$$= \left(\frac{2}{5}Mr^2\right)\left(\frac{v}{r}\right) + Mrv = \frac{7}{5}Mrv.$$

Thus, $$\frac{6}{5}Mrv_0 = \frac{7}{5}Mrv$$

or, $$v = \frac{6}{7}v_0\,.$$

**29**. *The sphere shown in figure (10-W22) lies on a rough plane when a particle of mass m travelling at a speed $v_0$ collides and sticks with it. If the line of motion of the particle is at a distance h above the plane, find (a) the linear speed of the combined system just after the collision, (b) the angular speed of the system about the centre of the sphere just after the collision and (c) the value of h for which the sphere starts pure rolling on the plane. Assume that the mass M of the sphere is large compared to the mass of the particle so that the centre of mass of the combined system is not appreciably shifted from the centre of the sphere.*

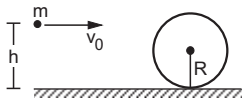



Figure 10-W22

***Solution*** **:** Take the particle plus the sphere as the system.

(a) Using conservation of linear momentum, the linear speed of the combined system $v$ is given by

$$mv_0 = (M+m)v \quad \text{or,} \quad v = \frac{mv_0}{M+m}\cdot \qquad \ldots \text{ (i)}$$

(b) Next, we shall use conservation of angular momentum about the centre of mass, which is to be taken at the centre of the sphere ($M >> m$). Angular momentum of the particle before collision is $mv_0(h-R)$. If the system rotates with angular speed $\omega$ after collision, the angular momentum of the system becomes

$$\left(\frac{2}{5}MR^2 + mR^2\right)\omega.$$

Hence,

$$mv_0(h-R) = \left(\frac{2}{5}M + m\right)R^2\omega$$

or, $$\omega = \frac{mv_0(h-R)}{\left(\frac{2}{5}M + m\right)R^2}\,.$$

(c) The sphere will start rolling just after the collision if

$$v = \omega R, \text{ i.e., } \frac{mv_0}{M+m} = \frac{mv_0(h-R)}{\left(\frac{2}{5}M + m\right)R}$$

giving, $$h = \left(\frac{\frac{7}{5}M + 2m}{M+m}\right)R \approx \frac{7}{5}R.$$

□

## QUESTIONS FOR SHORT ANSWER

**1.** Can an object be in pure translation as well as in pure rotation ?

**2.** A simple pendulum is a point mass suspended by a light thread from a fixed point. The particle is displaced towards one side and then released. It makes small oscillations. Is the motion of such a simple pendulum a pure rotation ? If yes, where is the axis of rotation ?

**3.** In a rotating body, $a = \alpha r$ and $v = \omega r$. Thus $\frac{a}{\alpha} = \frac{v}{\omega}$. Can you use the theorems of ratio and proportion studied in algebra so as to write

$$\frac{a+\alpha}{a-\alpha} = \frac{v+\omega}{v-\omega}$$

**4.** A ball is whirled in a circle by attaching it to a fixed point with a string. Is there an angular rotation of the ball about its centre ? If yes, is this angular velocity equal to the angular velocity of the ball about the fixed point ?

**5.** The moon rotates about the earth in such a way that only one hemisphere of the moon faces the earth (figure 10-Q1). Can we ever see the "other face" of the moon from the earth ? Can a person on the moon ever see all the faces of the earth ?

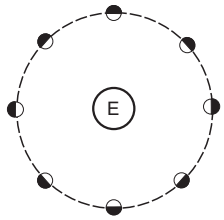



Figure 10-Q1

**6.** The torque of the weight of any body about any vertical axis is zero. Is it always correct ?

**7.** The torque of a force $\vec{F}$ about a point is defined as $\vec{\Gamma} = \vec{r} \times \vec{F}$. Suppose $\vec{r}, \vec{F}$ and $\vec{\Gamma}$ are all nonzero. Is $\vec{r} \times \vec{\Gamma} \parallel \vec{F}$ always true ? Is it ever true ?

**8.** A heavy particle of mass $m$ falls freely near the earth's surface. What is the torque acting on this particle about a point 50 cm east to the line of motion ? Does this torque produce any angular acceleration in the particle ?

**9.** If several forces act on a particle, the total torque on the particle may be obtained by first finding the resultant force and then taking torque of this resultant. Prove this. Is this result valid for the forces acting on different particles of a body in such a way that their lines of action intersect at a common point ?

**10.** If the sum of all the forces acting on a body is zero, is it necessarily in equilibrium ? If the sum of all the forces on a particle is zero, is it necessarily in equilibrium ?

**11.** If the angular momentum of a body is found to be zero about a point, is it necessary that it will also be zero about a different point ?

**12.** If the resultant torque of all the forces acting on a body is zero about a point, is it necessary that it will be zero about any other point ?

**13.** A body is in translational equilibrium under the action of coplanar forces. If the torque of these forces is zero about a point, is it necessary that it will also be zero about any other point ?

**14.** A rectangular brick is kept on a table with a part of its length projecting out. It remains at rest if the length projected is slightly less than half the total length but it falls down if the length projected is slightly more than half the total length. Give reason.

**15.** When a fat person tries to touch his toes, keeping the legs straight, he generally falls. Explain with reference to figure (10-Q2).

Figure 10-Q2

**16.** A ladder is resting with one end on a vertical wall and the other end on a horizontal floor. Is it more likely to slip when a man stands near the bottom or near the top ?

**17.** When a body is weighed on an ordinary balance we demand that the arm should be horizontal if the weights on the two pans are equal. Suppose equal weights are put on the two pans, the arm is kept at an angle with the horizontal and released. Is the torque of the two weights about the middle point (point of support) zero ? Is the total torque zero ? If so, why does the arm rotate and finally become horizontal ?

**18.** The density of a rod $AB$ continuously increases from $A$ to $B$. Is it easier to set it in rotation by clamping it at $A$ and applying a perpendicular force at $B$ or by clamping it at $B$ and applying the force at $A$ ?

**19.** When tall buildings are constructed on earth, the duration of day–night slightly increases. Is it true ?

**20.** If the ice at the poles melts and flows towards the equator, how will it affect the duration of day–night ?

**21.** A hollow sphere, a solid sphere, a disc and a ring all having same mass and radius are rolled down on an inclined plane. If no slipping takes place, which one will take the smallest time to cover a given length ?

**22.** A sphere rolls on a horizontal surface. Is there any point of the sphere which has a vertical velocity ?

## OBJECTIVE I

**1**. Let $\vec{A}$ be a unit vector along the axis of rotation of a purely rotating body and $\vec{B}$ be a unit vector along the velocity of a particle $P$ of the body away from the axis. The value of $\vec{A} \cdot \vec{B}$ is
(a) 1 (b) –1 (c) 0 (d) None of these.

**2.** A body is uniformly rotating about an axis fixed in an inertial frame of reference. Let $\vec{A}$ be a unit vector along the axis of rotation and $\vec{B}$ be the unit vector along the resultant force on a particle $P$ of the body away from the axis. The value of $\vec{A} \cdot \vec{B}$ is
(a) 1 (b) –1 (c) 0 (d) none of these.

**3.** A particle moves with a constant velocity parallel to the $X$-axis. Its angular momentum with respect to the origin
(a) is zero (b) remains constant
(c) goes on increasing (d) goes on decreasing.

**4.** A body is in pure rotation. The linear speed $v$ of a particle, the distance $r$ of the particle from the axis and the angular velocity ω of the body are related as $\omega = \frac{v}{r}$. Thus
(a) $\omega \propto \frac{1}{r}$ (b) $\omega \propto r$
(c) $\omega = 0$ (d) ω is independent of $r$.

**5.** Figure (10-Q3) shows a small wheel fixed coaxially on a bigger one of double the radius. The system rotates about the common axis. The strings supporting $A$ and $B$ do not slip on the wheels. If $x$ and $y$ be the distances travelled by $A$ and $B$ in the same time interval, then
(a) $x = 2y$ (b) $x = y$ (c) $y = 2x$ (d) none of these.

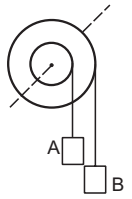



Figure 10-Q3

**6.** A body is rotating uniformly about a vertical axis fixed in an inertial frame. The resultant force on a particle of the body not on the axis is
(a) vertical (b) horizontal and skew with the axis
(c) horizontal and intersecting the axis
(d) none of these.

**7.** A body is rotating nonuniformly about a vertical axis fixed in an inertial frame. The resultant force on a particle of the body not on the axis is
(a) vertical (b) horizontal and skew with the axis
(c) horizontal and intersecting the axis
(d) none of these.

**8.** Let $\vec{F}$ be a force acting on a particle having position vector $\vec{r}$. Let $\vec{\Gamma}$ be the torque of this force about the origin, then
(a) $\vec{r} \cdot \vec{\Gamma} = 0$ and $\vec{F} \cdot \vec{\Gamma} = 0$ (b) $\vec{r} \cdot \vec{\Gamma} = 0$ but $\vec{F} \cdot \vec{\Gamma} \neq 0$
(c) $\vec{r} \cdot \vec{\Gamma} \neq 0$ but $\vec{F} \cdot \vec{\Gamma} = 0$ (d) $\vec{r} \cdot \vec{\Gamma} \neq 0$ and $\vec{F} \cdot \vec{\Gamma} \neq 0$.

**9.** One end of a uniform rod of mass $m$ and length $l$ is clamped. The rod lies on a smooth horizontal surface and rotates on it about the clamped end at a uniform angular velocity ω. The force exerted by the clamp on the rod has a horizontal component
(a) $m\omega^2 l$ (b) zero (c) $mg$ (d) $\frac{1}{2} m\omega^2 l$.

**10**. A uniform rod is kept vertically on a horizontal smooth surface at a point $O$. If it is rotated slightly and released, it falls down on the horizontal surface. The lower end will remain
(a) at $O$ (b) at a distance less than $l/2$ from $O$
(c) at a distance $l/2$ from $O$
(d) at a distance larger than $l/2$ from $O$.

**11.** A circular disc $A$ of radius $r$ is made from an iron plate of thickness $t$ and another circular disc $B$ of radius $4r$ is made from an iron plate of thickness $t/4$. The relation between the moments of inertia $I_A$ and $I_B$ is
(a) $I_A > I_B$ (b) $I_A = I_B$ (c) $I_A < I_B$
(d) depends on the actual values of $t$ and $r$.

**12.** Equal torques act on the discs $A$ and $B$ of the previous problem, initially both being at rest. At a later instant, the linear speeds of a point on the rim of $A$ and another point on the rim of $B$ are $v_A$ and $v_B$ respectively. We have
(a) $v_A > v_B$ (b) $v_A = v_B$ (c) $v_A < v_B$
(d) the relation depends on the actual magnitude of the torques.

**13.** A closed cylindrical tube containing some water (not filling the entire tube) lies in a horizontal plane. If the tube is rotated about a perpendicular bisector, the moment of inertia of water about the axis
(a) increases (b) decreases (c) remains constant
(d) increases if the rotation is clockwise and decreases if it is anticlockwise.

**14.** The moment of inertia of a uniform semicircular wire of mass $M$ and radius $r$ about a line perpendicular to the plane of the wire through the centre is
(a) $Mr^2$ (b) $\frac{1}{2} Mr^2$ (c) $\frac{1}{4} Mr^2$ (d) $\frac{2}{5} Mr^2$.

**15.** Let $I_1$ and $I_2$ be the moments of inertia of two bodies of identical geometrical shape, the first made of aluminium and the second of iron.
(a) $I_1 < I_2$ (b) $I_1 = I_2$ (c) $I_1 > I_2$
(d) relation between $I_1$ and $I_2$ depends on the actual shapes of the bodies.

**16.** A body having its centre of mass at the origin has three of its particles at $(a,0,0)$, $(0,a,0)$, $(0,0,a)$. The moments of inertia of the body about the $X$ and $Y$ axes are 0·20 kg–m$^2$ each. The moment of inertia about the $Z$-axis
(a) is 0·20 kg-m$^2$ (b) is 0·40 kg-m$^2$
(c) is $0{\cdot}20\sqrt{2}$ kg-m$^2$
(d) cannot be deduced with this information.

**17.** A cubical block of mass $M$ and edge $a$ slides down a rough inclined plane of inclination θ with a uniform

velocity. The torque of the normal force on the block about its centre has a magnitude

(a) zero (b) $Mga$ (c) $Mga \sin\theta$ (d) $\frac{1}{2} Mga \sin\theta$.

**18.** A thin circular ring of mass $M$ and radius $r$ is rotating about its axis with an angular speed $\omega$. Two particles having mass $m$ each are now attached at diametrically opposite points. The angular speed of the ring will become

(a) $\frac{\omega M}{M+m}$ (b) $\frac{\omega M}{M+2m}$

(c) $\frac{\omega(M-2m)}{M+2m}$ (d) $\frac{\omega(M+2m)}{M}$.

**19.** A person sitting firmly over a rotating stool has his arms stretched. If he folds his arms, his angular momentum about the axis of rotation

(a) increases (b) decreases

(c) remains unchanged (d) doubles.

**20.** The centre of a wheel rolling on a plane surface moves with a speed $v_0$. A particle on the rim of the wheel at the same level as the centre will be moving at speed

(a) zero (b) $v_0$ (c) $\sqrt{2}v_0$ (d) $2v_0$.

**21.** A wheel of radius 20 cm is pushed to move it on a rough horizontal surface. It is found to move through a distance of 60 cm on the road during the time it completes one revolution about the centre. Assume that the linear and the angular accelerations are uniform. The frictional force acting on the wheel by the surface is

(a) along the velocity of the wheel

(b) opposite to the velocity of the wheel

(c) perpendicular to the velocity of the wheel

(d) zero.

**22.** The angular velocity of the engine (and hence of the wheel) of a scooter is proportional to the petrol input per second. The scooter is moving on a frictionless road with uniform velocity. If the petrol input is increased by 10%, the linear velocity of the scooter is increased by

(a) 50% (b) 10% (c) 20% (d) 0%.

**23.** A solid sphere, a hollow sphere and a disc, all having same mass and radius, are placed at the top of a smooth incline and released. Least time will be taken in reaching the bottom by

(a) the solid sphere (b) the hollow sphere

(c) the disc (d) all will take same time.

**24.** A solid sphere, a hollow sphere and a disc, all having same mass and radius, are placed at the top of an incline and released. The friction coefficients between the objects and the incline are same and not sufficient to allow pure rolling. Least time will be taken in reaching the bottom by

(a) the solid sphere (b) the hollow sphere

(c) the disc (d) all will take same time.

**25.** In the previous question, the smallest kinetic energy at the bottom of the incline will be achieved by

(a) the solid sphere (b) the hollow sphere

(c) the disc (d) all will achieve same kinetic energy.

**26.** A string of negligible thickness is wrapped several times around a cylinder kept on a rough horizontal surface. A man standing at a distance $l$ from the cylinder holds one end of the string and pulls the cylinder towards him (figure 10-Q4). There is no slipping anywhere. The length of the string passed through the hand of the man while the cylinder reaches his hands is

(a) $l$ (b) $2l$ (c) $3l$ (d) $4l$.

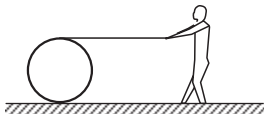


Figure 10-Q4

## OBJECTIVE II

**1.** The axis of rotation of a purely rotating body

(a) must pass through the centre of mass

(b) may pass through the centre of mass

(c) must pass through a particle of the body

(d) may pass through a particle of the body.

**2.** Consider the following two equations

(A) $L = I\omega$ (B) $\frac{dL}{dt} = \Gamma$

In noninertial frames

(a) both $A$ and $B$ are true (b) $A$ is true but $B$ is false

(c) $B$ is true but $A$ is false (d) both $A$ and $B$ are false.

**3.** A particle moves on a straight line with a uniform velocity. Its angular momentum

(a) is always zero

(b) is zero about a point on the straight line

(c) is not zero about a point away from the straight line

(d) about any given point remains constant.

**4.** If there is no external force acting on a nonrigid body, which of the following quantities must remain constant ?

(a) angular momentum (b) linear momentum

(c) kinetic energy (d) moment of inertia.

**5.** Let $I_A$ and $I_B$ be moments of inertia of a body about two axes $A$ and $B$ respectively. The axis $A$ passes through the centre of mass of the body but $B$ does not.

(a) $I_A < I_B$ (b) If $I_A < I_B$, the axes are parallel

(c) If the axes are parallel, $I_A < I_B$

(d) If the axes are not parallel, $I_A \geq I_B$.

**6.** A sphere is rotating about a diameter.

(a) The particles on the surface of the sphere do not have any linear acceleration.

(b) The particles on the diameter mentioned above do not have any linear acceleration.

(c) Different particles on the surface have different

angular speeds.
(d) All the particles on the surface have same linear speed.

**7.** The density of a rod gradually decreases from one end to the other. It is pivoted at an end so that it can move about a vertical axis through the pivot. A horizontal force $F$ is applied on the free end in a direction perpendicular to the rod. The quantities, that do not depend on which end of the rod is pivoted, are
(a) angular acceleration
(b) angular velocity when the rod completes one rotation
(c) angular momentum when the rod completes one rotation
(d) torque of the applied force.

**8.** Consider a wheel of a bicycle rolling on a level road at a linear speed $v_0$ (figure 10-Q5).
(a) the speed of the particle $A$ is zero
(b) the speed of $B$, $C$ and $D$ are all equal to $v_0$
(c) the speed of $C$ is $2\,v_0$
(d) the speed of $B$ is greater than the speed of $O$.

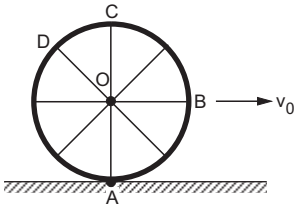



Figure 10-Q5

**9.** Two uniform solid spheres having unequal masses and unequal radii are released from rest from the same height on a rough incline. If the spheres roll without slipping,
(a) the heavier sphere reaches the bottom first
(b) the bigger sphere reaches the bottom first
(c) the two spheres reach the bottom together
(d) the information given is not sufficient to tell which sphere will reach the bottom first.

**10.** A hollow sphere and a solid sphere having same mass and same radii are rolled down a rough inclined plane.
(a) The hollow sphere reaches the bottom first.
(b) The solid sphere reaches the bottom with greater speed.
(c) The solid sphere reaches the bottom with greater kinetic energy.
(d) The two spheres will reach the bottom with same linear momentum.

**11.** A sphere cannot roll on
(a) a smooth horizontal surface
(b) a smooth inclined surface
(c) a rough horizontal surface
(d) a rough inclined surface.

**12.** In rear-wheel drive cars, the engine rotates the rear wheels and the front wheels rotate only because the car moves. If such a car accelerates on a horizontal road, the friction
(a) on the rear wheels is in the forward direction
(b) on the front wheels is in the backward direction
(c) on the rear wheels has larger magnitude than the friction on the front wheels
(d) on the car is in the backward direction.

**13.** A sphere can roll on a surface inclined at an angle $\theta$ if the friction coefficient is more than $\frac{2}{7}\tan\theta$. Suppose the friction coefficient is $\frac{1}{7}g\tan\theta$. If a sphere is released from rest on the incline,
(a) it will stay at rest
(b) it will make pure translational motion
(c) it will translate and rotate about the centre
(d) the angular momentum of the sphere about its centre will remain constant.

**14.** A sphere is rolled on a rough horizontal surface. It gradually slows down and stops. The force of friction tries to
(a) decrease the linear velocity
(b) increase the angular velocity
(c) increase the linear momentum
(d) decrease the angular velocity.

**15.** Figure (10-Q6) shows a smooth inclined plane fixed in a car accelerating on a horizontal road. The angle of incline $\theta$ is related to the acceleration $a$ of the car as $a = g\tan\theta$. If the sphere is set in pure rolling on the incline,
(a) it will continue pure rolling
(b) it will slip down the plane
(c) its linear velocity will increase
(d) its linear velocity will slowly decrease.

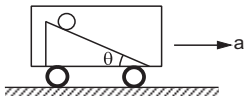



Figure 10-Q6

## EXERCISES

**1.** A wheel is making revolutions about its axis with uniform angular acceleration. Starting from rest, it reaches 100 rev/sec in 4 seconds. Find the angular acceleration. Find the angle rotated during these four seconds.

**2.** A wheel rotating with uniform angular acceleration covers 50 revolutions in the first five seconds after the start. Find the angular acceleration and the angular velocity at the end of five seconds.

**3.** A wheel starting from rest is uniformly accelerated at 4 rad/s$^2$ for 10 seconds. It is allowed to rotate uniformly

for the next 10 seconds and is finally brought to rest in the next 10 seconds. Find the total angle rotated by the wheel.

**4.** A body rotates about a fixed axis with an angular acceleration of one radian/second/second. Through what angle does it rotate during the time in which its angular velocity increases from 5 rad/s to 15 rad/s.

**5.** Find the angular velocity of a body rotating with an acceleration of 2 rev/s$^2$ as it completes the 5th revolution after the start.

**6.** A disc of radius 10 cm is rotating about its axis at an angular speed of 20 rad/s. Find the linear speed of
(a) a point on the rim,
(b) the middle point of a radius.

**7.** A disc rotates about its axis with a constant angular acceleration of 4 rad/s$^2$. Find the radial and tangential accelerations of a particle at a distance of 1 cm from the axis at the end of the first second after the disc starts rotating.

**8.** A block hangs from a string wrapped on a disc of radius 20 cm free to rotate about its axis which is fixed in a horizontal position. If the angular speed of the disc is 10 rad/s at some instant, with what speed is the block going down at that instant ?

**9.** Three particles, each of mass 200 g, are kept at the corners of an equilateral triangle of side 10 cm. Find the moment of inertia of the system about an axis
(a) joining two of the particles and
(b) passing through one of the particles and perpendicular to the plane of the particles.

**10.** Particles of masses 1 g, 2 g, 3 g, ..., 100 g are kept at the marks 1 cm, 2 cm, 3 cm, ..., 100 cm respectively on a metre scale. Find the moment of inertia of the system of particles about a perpendicular bisector of the metre scale.

**11.** Find the moment of inertia of a pair of spheres, each having a mass $m$ and radius $r$, kept in contact about the tangent passing through the point of contact.

**12.** The moment of inertia of a uniform rod of mass 0·50 kg and length 1 m is 0·10 kg-m$^2$ about a line perpendicular to the rod. Find the distance of this line from the middle point of the rod.

**13.** Find the radius of gyration of a circular ring of radius $r$ about a line perpendicular to the plane of the ring and passing through one of its particles.

**14.** The radius of gyration of a uniform disc about a line perpendicular to the disc equals its radius. Find the distance of the line from the centre.

**15.** Find the moment of inertia of a uniform square plate of mass $m$ and edge $a$ about one of its diagonals.

**16.** The surface density (mass/area) of a circular disc of radius $a$ depends on the distance from the centre as $\rho(r) = A + Br$. Find its moment of inertia about the line perpendicular to the plane of the disc through its centre.

**17.** A particle of mass $m$ is projected with a speed $u$ at an angle $\theta$ with the horizontal. Find the torque of the weight of the particle about the point of projection when the particle is at the highest point.

**18.** A simple pendulum of length $l$ is pulled aside to make an angle $\theta$ with the vertical. Find the magnitude of the torque of the weight $w$ of the bob about the point of suspension. When is the torque zero ?

**19.** When a force of 6·0 N is exerted at 30° to a wrench at a distance of 8 cm from the nut, it is just able to loosen the nut. What force $F$ would be sufficient to loosen it if it acts perpendicularly to the wrench at 16 cm from the nut ?

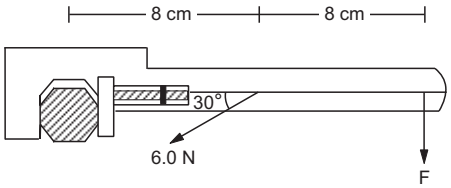



Figure 10-E1

**20.** Calculate the total torque acting on the body shown in figure (10-E2) about the point $O$.

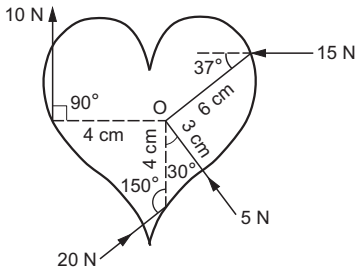



Figure 10-E2

**21.** A cubical block of mass $m$ and edge $a$ slides down a rough inclined plane of inclination $\theta$ with a uniform speed. Find the torque of the normal force acting on the block about its centre.

**22.** A rod of mass $m$ and length $L$, lying horizontally, is free to rotate about a vertical axis through its centre. A horizontal force of constant magnitude $F$ acts on the rod at a distance of $L/4$ from the centre. The force is always perpendicular to the rod. Find the angle rotated by the rod during the time $t$ after the motion starts.

**23.** A square plate of mass 120 g and edge 5·0 cm rotates about one of the edges. If it has a uniform angular acceleration of 0·2 rad/s$^2$, what torque acts on the plate ?

**24.** Calculate the torque on the square plate of the previous problem if it rotates about a diagonal with the same angular acceleration.

**25.** A flywheel of moment of inertia 5·0 kg-m$^2$ is rotated at a speed of 60 rad/s. Because of the friction at the axle, it comes to rest in 5·0 minutes. Find (a) the average torque of the friction, (b) the total work done by the friction and (c) the angular momentum of the wheel 1 minute before it stops rotating.

**26.** Because of the friction between the water in oceans with the earth's surface, the rotational kinetic energy of the earth is continuously decreasing. If the earth's angular speed decreases by 0·0016 rad/day in 100 years, find the

average torque of the friction on the earth. Radius of the earth is 6400 km and its mass is $6{\cdot}0 \times 10^{24}$ kg.

**27.** A wheel rotating at a speed of 600 rpm (revolutions per minute) about its axis is brought to rest by applying a constant torque for 10 seconds. Find the angular deceleration and the angular velocity 5 seconds after the application of the torque.

**28.** A wheel of mass 10 kg and radius 20 cm is rotating at an angular speed of 100 rev/min when the motor is turned off. Neglecting the friction at the axle, calculate the force that must be applied tangentially to the wheel to bring it to rest in 10 revolutions.

**29.** A cylinder rotating at an angular speed of 50 rev/s is brought in contact with an identical stationary cylinder. Because of the kinetic friction, torques act on the two cylinders, accelerating the stationary one and decelerating the moving one. If the common magnitude of the acceleration and deceleration be one revolution per second square, how long will it take before the two cylinders have equal angular speed ?

**30.** A body rotating at 20 rad/s is acted upon by a constant torque providing it a deceleration of 2 rad/s$^2$ . At what time will the body have kinetic energy same as the initial value if the torque continues to act ?

**31.** A light rod of length 1 m is pivoted at its centre and two masses of 5 kg and 2 kg are hung from the ends as shown in figure (10-E3). Find the initial angular acceleration of the rod assuming that it was horizontal in the beginning.

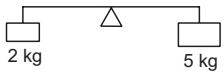



Figure 10-E3

**32.** Suppose the rod in the previous problem has a mass of 1 kg distributed uniformly over its length.
(a) Find the initial angular acceleration of the rod.
(b) Find the tension in the supports to the blocks of mass 2 kg and 5 kg.

**33.** Figure (10-E4) shows two blocks of masses $m$ and $M$ connected by a string passing over a pulley. The horizontal table over which the mass $m$ slides is smooth. The pulley has a radius $r$ and moment of inertia $I$ about its axis and it can freely rotate about this axis. Find the acceleration of the mass $M$ assuming that the string does not slip on the pulley.

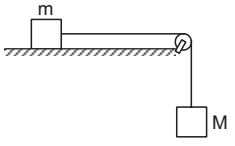



Figure 10-E4

**34.** A string is wrapped on a wheel of moment of inertia 0·20 kg-m$^2$ and radius 10 cm and goes through a light pulley to support a block of mass 2·0 kg as shown in figure (10-E5). Find the acceleration of the block.

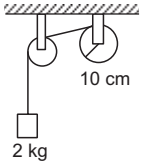



Figure 10-E5

**35.** Suppose the smaller pulley of the previous problem has its radius 5·0 cm and moment of inertia 0·10 kg-m$^2$. Find the tension in the part of the string joining the pulleys.

**36.** The pulleys in figure (10-E6) are identical, each having a radius $R$ and moment of inertia $I$. Find the acceleration of the block $M$.

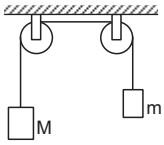



Figure 10-E6

**37.** The descending pulley shown in figure (10-E7) has a radius 20 cm and moment of inertia 0·20 kg-m$^2$. The fixed pulley is light and the horizontal plane frictionless. Find the acceleration of the block if its mass is 1·0 kg.

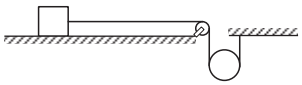


Figure 10-E7

**38.** The pulley shown in figure (10-E8) has a radius 10 cm and moment of inertia 0·5 kg-m$^2$ about its axis. Assuming the inclined planes to be frictionless, calculate the acceleration of the 4·0 kg block.

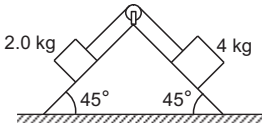



Figure 10-E8

**39.** Solve the previous problem if the friction coefficient between the 2·0 kg block and the plane below it is 0·5 and the plane below the 4·0 kg block is frictionless.

**40.** A uniform metre stick of mass 200 g is suspended from the ceiling through two vertical strings of equal lengths fixed at the ends. A small object of mass 20 g is placed on the stick at a distance of 70 cm from the left end. Find the tensions in the two strings.

**41.** A uniform ladder of length 10·0 m and mass 16·0 kg is resting against a vertical wall making an angle of 37° with it. The vertical wall is frictionless but the ground is rough. An electrician weighing 60·0 kg climbs up the ladder. If he stays on the ladder at a point 8·00 m from

the lower end, what will be the normal force and the force of friction on the ladder by the ground ? What should be the minimum coefficient of friction for the elctrician to work safely ?

**42.** Suppose the friction coefficient between the ground and the ladder of the previous problem is 0·540. Find the maximum weight of a mechanic who could go up and do the work from the same position of the ladder.

**43.** A 6·5 m long ladder rests against a vertical wall reaching a height of 6·0 m. A 60 kg man stands half way up the ladder. (a) Find the torque of the force exerted by the man on the ladder about the upper end of the ladder. (b) Assuming the weight of the ladder to be negligible as compared to the man and assuming the wall to be smooth, find the force exerted by the ground on the ladder.

**44.** The door of an almirah is 6 ft high, 1·5 ft wide and weighs 8 kg. The door is supported by two hinges situated at a distance of 1 ft from the ends. If the magnitudes of the forces exerted by the hinges on the door are equal, find this magnitude.

**45.** A uniform rod of length $L$ rests against a smooth roller as shown in figure (10-E9). Find the friction coefficient between the ground and the lower end if the minimum angle that the rod can make with the horizontal is $\theta$.

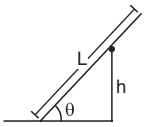



Figure 10-E9

**46.** A uniform rod of mass 300 g and length 50 cm rotates at a uniform angular speed of 2 rad/s about an axis perpendicular to the rod through an end. Calculate (a) the angular momentum of the rod about the axis of rotation, (b) the speed of the centre of the rod and (c) its kinetic energy.

**47.** A uniform square plate of mass 2·0 kg and edge 10 cm rotates about one of its diagonals under the action of a constant torque of 0·10 N-m. Calculate the angular momentum and the kinetic energy of the plate at the end of the fifth second after the start.

**48.** Calculate the ratio of the angular momentum of the earth about its axis due to its spinning motion to that about the sun due to its orbital motion. Radius of the earth = 6400 km and radius of the orbit of the earth about the sun = $1{\cdot}5 \times 10^8$ km.

**49.** Two particles of masses $m_1$ and $m_2$ are joined by a light rigid rod of length $r$. The system rotates at an angular speed $\omega$ about an axis through the centre of mass of the system and perpendicular to the rod. Show that the angular momentum of the system is $L = \mu r^2 \omega$ where $\mu$ is the reduced mass of the system defined as $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$.

**50.** A dumb-bell consists of two identical small balls of mass 1/2 kg each connected to the two ends of a 50 cm long light rod. The dumb-bell is rotating about a fixed axis through the centre of the rod and perpendicular to it at an angular speed of 10 rad/s. An impulsive force of average magnitude 5·0 N acts on one of the masses in the direction of its velocity for 0·10 s. Find the new angular velocity of the system.

**51.** A wheel of moment of inertia 0·500 kg-m$^2$ and radius 20·0 cm is rotating about its axis at an angular speed of 20·0 rad/s. It picks up a stationary particle of mass 200 g at its edge. Find the new angular speed of the wheel.

**52.** A diver having a moment of inertia of 6·0 kg-m$^2$ about an axis through its centre of mass rotates at an angular speed of 2 rad/s about this axis. If he folds his hands and feet to decrease the moment of inertia to 5·0 kg-m$^2$, what will be the new angular speed ?

**53.** A boy is seated in a revolving chair revolving at an angular speed of 120 revolutions per minute. Two heavy balls form part of the revolving system and the boy can pull the balls closer to himself or may push them apart. If by pulling the balls closer, the boy decreases the moment of inertia of the system from 6 kg-m$^2$ to 2 kg-m$^2$, what will be the new angular speed ?

**54.** A boy is standing on a platform which is free to rotate about its axis. The boy holds an open umbrella in his hand. The axis of the umbrella coincides with that of the platform. The moment of inertia of "the platform plus the boy system" is $3{\cdot}0 \times 10^{-3}$ kg-m$^2$ and that of the umbrella is $2{\cdot}0 \times 10^{-3}$ kg-m$^2$. The boy starts spinning the umbrella about the axis at an angular speed of 2·0 rev/s with respect to himself. Find the angular velocity imparted to the platform.

**55.** A wheel of moment of inertia 0·10 kg-m$^2$ is rotating about a shaft at an angular speed of 160 rev/minute. A second wheel is set into rotation at 300 rev/minute and is coupled to the same shaft so that both the wheels finally rotate with a common angular speed of 200 rev/minute. Find the moment of inertia of the second wheel.

**56.** A kid of mass $M$ stands at the edge of a platform of radius $R$ which can be freely rotated about its axis. The moment of inertia of the platform is $I$. The system is at rest when a friend throws a ball of mass $m$ and the kid catches it. If the velocity of the ball is $v$ horizontally along the tangent to the edge of the platform when it was caught by the kid, find the angular speed of the platform after the event.

**57.** Suppose the platform of the previous problem is brought to rest with the ball in the hand of the kid standing on the rim. The kid throws the ball horizontally to his friend in a direction tangential to the rim with a speed $v$ as seen by his friend. Find the angular velocity with which the platform will start rotating.

**58.** Suppose the platform with the kid in the previous problem is rotating in anticlockwise direction at an angular speed $\omega$. The kid starts walking along the rim with a speed $v$ relative to the platform also in the anticlockwise direction. Find the new angular speed of the platform.

**59.** A uniform rod of mass $m$ and length $l$ is struck at an end by a force $F$ perpendicular to the rod for a short time interval $t$. Calculate
(a) the speed of the centre of mass, (b) the angular speed of the rod about the centre of mass, (c) the kinetic energy of the rod and (d) the angular momentum of the rod about the centre of mass after the force has stopped to act. Assume that $t$ is so small that the rod does not appreciably change its direction while the force acts.

**60.** A uniform rod of length $L$ lies on a smooth horizontal table. A particle moving on the table strikes the rod perpendicularly at an end and stops. Find the distance travelled by the centre of the rod by the time it turns through a right angle. Show that if the mass of the rod is four times that of the particle, the collision is elastic.

**61.** Suppose the particle of the previous problem has a mass $m$ and a speed $v$ before the collision and it sticks to the rod after the collision. The rod has a mass $M$. (a) Find the velocity of the centre of mass $C$ of the system constituting "the rod plus the particle". (b) Find the velocity of the particle with respect to $C$ before the collision. (c) Find the velocity of the rod with respect to $C$ before the collision. (d) Find the angular momentum of the particle and of the rod about the centre of mass $C$ before the collision. (e) Find the moment of inertia of the system about the vertical axis through the centre of mass $C$ after the collision. (f) Find the velocity of the centre of mass $C$ and the angular velocity of the system about the centre of mass after the collision.

**62.** Two small balls $A$ and $B$, each of mass $m$, are joined rigidly by a light horizontal rod of length $L$. The rod is clamped at the centre in such a way that it can rotate freely about a vertical axis through its centre. The system is rotated with an angular speed $\omega$ about the axis. A particle $P$ of mass $m$ kept at rest sticks to the ball $A$ as the ball collides with it. Find the new angular speed of the rod.

**63.** Two small balls $A$ and $B$, each of mass $m$, are joined rigidly to the ends of a light rod of lengh $L$ (figure 10-E10). The system translates on a frictionless horizontal surface with a velocity $v_0$ in a direction perpendicular to the rod. A particle $P$ of mass $m$ kept at rest on the surface sticks to the ball $A$ as the ball collides with it. Find
(a) the linear speeds of the balls $A$ and $B$ after the collision, (b) the velocity of the centre of mass $C$ of the system $A + B + P$ and (c) the angular speed of the system about $C$ after the collision.

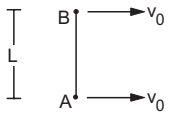



Figure 10-E10

[**Hint :** The light rod will exert a force on the ball $B$ only along its length.]

**64.** Suppose the rod with the balls $A$ and $B$ of the previous problem is clamped at the centre in such a way that it can rotate freely about a horizontal axis through the clamp. The system is kept at rest in the horizontal position. A particle $P$ of the same mass $m$ is dropped from a height $h$ on the ball $B$. The particle collides with $B$ and sticks to it. (a) Find the angular momentum and the angular speed of the system just after the collision. (b) What should be the minimum value of $h$ so that the system makes a full rotation after the collision.

**65.** Two blocks of masses 400 g and 200 g are connected through a light string going over a pulley which is free to rotate about its axis. The pulley has a moment of inertia $1{\cdot}6 \times 10^{-4}$ kg-m$^2$ and a radius 2·0 cm. Find (a) the kinetic energy of the system as the 400 g block falls through 50 cm, (b) the speed of the blocks at this instant.

**66.** The pulley shown in figure (10-E11) has a radius of 20 cm and moment of inertia 0·2 kg-m$^2$. The string going over it is attached at one end to a vertical spring of spring constant 50 N/m fixed from below, and supports a 1 kg mass at the other end. The system is released from rest with the spring at its natural length. Find the speed of the block when it has descended through 10 cm. Take $g = 10$ m/s$^2$.

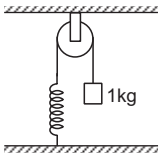



Figure 10-E11

**67.** A metre stick is held vertically with one end on a rough horizontal floor. It is gently allowed to fall on the floor. Assuming that the end at the floor does not slip, find the angular speed of the rod when it hits the floor.

**68.** A metre stick weighing 240 g is pivoted at its upper end in such a way that it can freely rotate in a vertical plane through this end (figure 10-E12). A particle of mass 100 g is attached to the upper end of the stick through a light string of length l m. Initially, the rod is kept vertical and the string horizontal when the system is released from rest. The particle collides with the lower end of the stick and sticks there. Find the maximum angle through which the stick will rise.

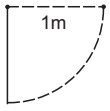



Figure 10-E12

**69.** A uniform rod pivoted at its upper end hangs vertically. It is displaced through an angle of 60° and then released. Find the magnitude of the force acting on a particle of mass $dm$ at the tip of the rod when the rod makes an angle of 37° with the vertical.

**70.** A cylinder rolls on a horizontal plane surface. If the speed of the centre is 25 m/s, what is the speed of the highest point ?

**71.** A sphere of mass $m$ rolls on a plane surface. Find its kinetic energy at an instant when its centre moves with speed $v$.

**72.** A string is wrapped over the edge of a uniform disc and the free end is fixed with the ceiling. The disc moves down, unwinding the string. Find the downward acceleration of the disc.

**73.** A small spherical ball is released from a point at a height $h$ on a rough track shown in figure (10-E13). Assuming that it does not slip anywhere, find its linear speed when it rolls on the horizontal part of the track.

Figure 10-E13

**74.** A small disc is set rolling with a speed $v$ on the horizontal part of the track of the previous problem from right to left. To what height will it climb up the curved part ?

**75.** A sphere starts rolling down an incline of inclination $\theta$. Find the speed of its centre when it has covered a distance $l$.

**76.** A hollow sphere is released from the top of an inclined plane of inclination $\theta$. (a) What should be the minimum coefficient of friction between the sphere and the plane to prevent sliding ? (b) Find the kinetic energy of the ball as it moves down a length $l$ on the incline if the friction coefficient is half the value calculated in part (a).

**77.** A solid sphere of mass $m$ is released from rest from the rim of a hemispherical cup so that it rolls along the surface. If the rim of the hemisphere is kept horizontal, find the normal force exerted by the cup on the ball when the ball reaches the bottom of the cup.

**78.** Figure (10-E14) shows a rough track, a portion of which is in the form of a cylinder of radius $R$. With what minimum linear speed should a sphere of radius $r$ be set rolling on the horizontal part so that it completely goes round the circle on the cylindrical part.

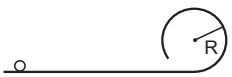



Figure 10-E14

**79.** Figure (10-E15) shows a small spherical ball of mass $m$ rolling down the loop track. The ball is released on the linear portion at a vertical height $H$ from the lowest point. The circular part shown has a radius $R$.
(a) Find the kinetic energy of the ball when it is at a point $A$ where the radius makes an angle $\theta$ with the horizontal.
(b) Find the radial and the tangential accelerations of the centre when the ball is at $A$.
(c) Find the normal force and the frictional force acting on the ball if $H = 60$ cm, $R = 10$ cm, $\theta = 0$ and $m = 70$ g.

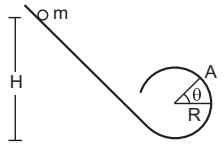



Figure 10-E15

**80.** A thin spherical shell of radius $R$ lying on a rough horizontal surface is hit sharply and horizontally by a cue. Where should it be hit so that the shell does not slip on the surface ?

**81.** A uniform wheel of radius $R$ is set into rotation about its axis at an angular speed $\omega$. This rotating wheel is now placed on a rough horizontal surface with its axis horizontal. Because of friction at the contact, the wheel accelerates forward and its rotation decelerates till the wheel starts pure rolling on the surface. Find the linear speed of the wheel after it starts pure rolling.

**82.** A thin spherical shell lying on a rough horizontal surface is hit by a cue in such a way that the line of action passes through the centre of the shell. As a result, the shell starts moving with a linear speed $v$ without any initial angular velocity. Find the linear speed of the shell after it starts pure rolling on the surface.

**83.** A hollow sphere of radius $R$ lies on a smooth horizontal surface. It is pulled by a horizontal force acting tangentially from the highest point. Find the distance travelled by the sphere during the time it makes one full rotation.

**84.** A solid sphere of mass 0·50 kg is kept on a horizontal surface. The coefficient of static friction between the surfaces in contact is 2/7. What maximum force can be applied at the highest point in the horizontal direction so that the sphere does not slip on the surface ?

**85.** A solid sphere is set into motion on a rough horizontal surface with a linear speed $v$ in the forward direction and an angular speed $v/R$ in the anticlockwise direction as shown in figure (10-E16). Find the linear speed of the sphere (a) when it stops rotating and (b) when slipping finally ceases and pure rolling starts.

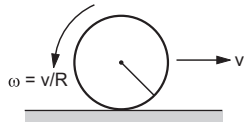



Figure 10-E16

**86.** A solid sphere rolling on a rough horizontal surface with a linear speed $v$ collides elastically with a fixed, smooth, vertical wall. Find the speed of the sphere after it has started pure rolling in the backward direction.

□

## ANSWERS

### OBJECTIVE I

| | | | | | |
|---|---|---|---|---|---|
| 1. (c) | 2. (c) | 3. (b) | 4. (d) | 5. (c) | 6. (c) |
| 7. (b) | 8. (a) | 9. (d) | 10. (c) | 11. (c) | 12. (a) |
| 13. (a) | 14. (a) | 15. (a) | 16. (d) | 17. (d) | 18. (b) |
| 19. (b) | 20. (c) | 21. (a) | 22. (d) | 23. (d) | 24. (d) |
| 25. (b) | 26. (b) | | | | |

### OBJECTIVE II

| | | |
|---|---|---|
| 1. (b), (d) | 2. (b) | 3. (b), (c), (d) |
| 4. (a), (b) | 5. (c) | 6. (b) |
| 7. (d) | 8. (a), (c), (d) | 9. (c) |
| 10. (b) | 11. (b) | 12. (a), (b), (c) |
| 13. (c) | 14. (a), (b) | 15. (a) |

### EXERCISES

1. 25 rev/s$^2$, 400 π rad
2. 4 rev/s$^2$, 20 rev/s
3. 800 rad
4. 100 rad
5. 2 √5 rev/s
6. 2 m/s, 1 m/s
7. 16 cm/s$^2$, 4 cm/s$^2$
8. 2 m/s
9. $1{\cdot}5 \times 10^{-3}$ kg–m$^2$, $4{\cdot}0 \times 10^{-3}$ kg–m$^2$
10. 0·43 kg–m$^2$
11. $\frac{14\,mr^2}{5}$
12. 0·34 m
13. $\sqrt{2}\,r$
14. $r/\sqrt{2}$
15. $ma^2/12$
16. $2\pi\left(\frac{Aa^4}{4}+\frac{Ba^5}{5}\right)$
17. $mu^2 \sin\theta \cos\theta$ perpendicular to the plane of motion
18. $wl \sin\theta$, when the bob is at the lowest point
19. 1·5 N
20. 0·54 N-m
21. $\frac{1}{2} mg\,a \sin\theta$
22. $\frac{3\,Ft^2}{2\,ml}$
23. $2{\cdot}0 \times 10^{-5}$ N-m
24. $0{\cdot}5 \times 10^{-5}$ N-m
25. (a) 1·0 N-m (b) 9·0 kJ (c) 60 kg-m$^2$/s
26. $5{\cdot}8 \times 10^{20}$ N-m
27. 1 rev/s$^2$, 5 rev/s
28. 0·87 N
29. 25 s
30. 20 s
31. 8·4 rad/s$^2$
32. 8·0 rad/s$^2$, 27·6 N, 29 N
33. $\frac{Mg}{M+m+I/r^2}$
34. 0·89 m/s$^2$
35. 6·3 N
36. $\frac{(M-m)g}{M+m+2\,I/R^2}$
37. 10 m/s$^2$
38. 0·25 m/s$^2$
39. 0·125 m/s$^2$
40. 1·04 N in the left string and 1·12 N in the right
41. 745 N, 412 N, 0·553
42. 44·0 kg
43. (a) 740 N-m
    (b) 590 N vertical and 120 N horizontal
44. 43 N
45. $\frac{L\cos\theta \sin^2\theta}{2\,h - L\cos^2\theta \sin\theta}$
46. (a) 0·05 kg-m$^2$/s (b) 50 cm/s (c) 0·05 J
47. 0·5 kg-m$^2$/s, 75 J
48. $2{\cdot}66 \times 10^{-7}$
50. 12 rad/s
51. 19·7 rad/s
52. 2·4 rad/s
53. 360 rev/minute
54. 0·8 rev/s
55. 0·04 kg-m$^2$
56. $\frac{mvR}{I+(M+m)R^2}$
57. $\frac{mvR}{I+MR^2}$
58. $\omega - \frac{MvR}{I+MR^2}$
59. (a) $\frac{Ft}{m}$ (b) $\frac{6\,Ft}{ml}$ (c) $\frac{2\,F^2t^2}{m}$ (d) $\frac{Flt}{2}$
60. $\pi L/12$
61. (a) $\frac{mv}{M+m}$ (b) $\frac{Mv}{M+m}$ (c) $-\frac{mv}{M+m}$
    (d) $\frac{M^2mvl}{2(M+m)^2}$, $\frac{Mm^2vl}{2(M+m)^2}$ (e) $\frac{M(M+4\,m)L^2}{12(M+m)}$
    (f) $\frac{mv}{M+m}$, $\frac{6\,mv}{(M+4\,m)L}$
62. $2\omega/3$

63. (a) $\frac{v_0}{2}, v_0$ (b) $\frac{2}{3}v_0$ along the initial motion of the rod
(c) $\frac{v_0}{2L}$

64. (a) $\frac{mL\sqrt{gh}}{\sqrt{2}}, \frac{\sqrt{8gh}}{3L}$ (b) $\frac{3}{2}L$

65. (a) 0·98 J (b) 1·4 m/s

66. 0·5 m/s

67. 5·4 rad/s

68. 41°

69. $0{\cdot}9\sqrt{2}\,dm\,g$

70. 50 m/s

71. $\frac{7}{10}mv^2$

72. $\frac{2}{3}g$

73. $\sqrt{10\,gh/7}$

74. $\frac{3v^2}{4g}$

75. $\sqrt{\frac{10}{7}g\,l\sin\theta}$

76. (a) $\frac{2}{5}\tan\theta$ (b) $\frac{7}{8}mgl\sin\theta$

77. 17 $mg$/7

78. $\sqrt{\frac{27}{7}g(R-r)}$

79. (a) $mg(H-R-R\sin\theta)$,
(b) $\frac{10}{7}g\left(\frac{H}{R}-1-\sin\theta\right), -\frac{5}{7}g\cos\theta$
(c) 4·9 N, 0·196 N upward

80. 2 $R$/3 above the centre

81. $\omega R/3$

82. 3 $v$/5

83. $4\pi R/3$

84. 3·3 N

85. (a) 3 $v$/5 (b) 3 $v$/7

86. 3 $v$/7

□

## CHAPTER 11

# GRAVITATION

### 11.1 HISTORICAL INTRODUCTION

The motion of celestial bodies such as the moon, the earth, the planets, etc., has been a subject of great interest for a long time. Famous Indian astronomer and mathematician, Aryabhat, studied these motions in great detail, most likely in the 5th century A.D., and wrote his conclusions in his book *Aryabhatiya*. He established that the earth revolves about its own axis. He also gave description of motion of other celestial bodies as seen from the earth.

About a thousand years after Aryabhat, the brilliant combination of Tycho Brahe (1546–1601) and Johannes Kepler (1571–1630) studied the planetary motion in great detail. Kepler formulated his important findings in his three laws of planetary motion :

1. *All planets move in elliptical orbits with the sun at a focus.*
2. *The radius vector from the sun to the planet sweeps equal area in equal time.*
3. *The square of the time period of a planet is proportional to the cube of the semimajor axis of the ellipse.*

The year 1665 was very fruitful for Isaac Newton aged 23. He was forced to take rest at his home in Lincolnshire after his college at Cambridge was closed for an indefinite period due to plague. In this year, he performed brilliant theoretical and experimental tasks mainly in the field of mechanics and optics. In this same year he focussed his attention on the motion of the moon about the earth.

The moon makes a revolution about the earth in $T = 27{\cdot}3$ days. The distance of the moon from the earth is $R = 3{\cdot}85 \times 10^5$ km. The acceleration of the moon is, therefore,

$$a = \omega^2 R$$

$$= \frac{4\pi^2 \times (3{\cdot}85 \times 10^5 \text{ km})}{(27{\cdot}3 \text{ days})^2} = 0{\cdot}0027 \text{ m s}^{-2} .$$

The first question before Newton was that what is the force that produces this acceleration. The acceleration is towards the centre of the orbit, that is towards the centre of the earth. Hence the force must act towards the centre of the earth. A natural guess was that the earth is attracting the moon. The saying goes that Newton was sitting under an apple tree when an apple fell down from the tree on the earth. This sparked the idea that the earth attracts all bodies towards its centre. The next question was what is the law governing this force.

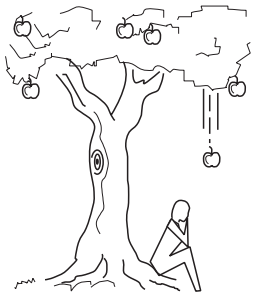


Figure 11.1

Newton had to make several daring assumptions which proved to be turning points in science and philosophy. He declared that the laws of nature are the same for earthly and celestial bodies. The force operating between the earth and an apple and that operating between the earth and the moon, must be governed by the same laws. This statement may look very obvious today but in the era before Newton, there was a general belief in the western countries that the earthly bodies are governed by certain rules and the heavenly bodies are governed by different rules. In particular, this heavenly structure was supposed to be so perfect that there could not be any change in the sky. This distinction was so sharp that when Tycho Brahe saw a new star in the sky, he did not believe

his eyes as there could be no change in the sky. So the Newton's declaration was indeed revolutionary.

The acceleration of a body falling near the earth's surface is about 9·8 $ms^{-2}$. Thus,

$$\frac{a_{apple}}{a_{moon}} = \frac{9{\cdot}8 \text{ m s}^{-2}}{0{\cdot}0027 \text{ m s}^{-2}} = 3600.$$

Also,

$$\frac{\text{distance of the moon from the earth}}{\text{distance of the apple from the earth}}$$

$$= \frac{d_{moon}}{d_{apple}} = \frac{3{\cdot}85 \times 10^5 \text{ km}}{6400 \text{ km}}$$

$$= 60$$

Thus, $$\frac{a_{apple}}{a_{moon}} = \left(\frac{d_{moon}}{d_{apple}}\right)^2.$$

Newton guessed that the acceleration of a body towards the earth is inversely proportional to the square of the distance of the body from the centre of the earth.

Thus, $a \propto \frac{1}{r^2}$.

Also, the force is mass times acceleration and so it is proportional to the mass of the body.

Hence,

$$F \propto \frac{m}{r^2}.$$

By the third law of motion, the force on a body due to the earth must be equal to the force on the earth due to the body. Therefore, this force should also be proportional to the mass of the earth. Thus, the force between the earth and a body is

$$F \propto \frac{Mm}{r^2}$$

or, $$F = \frac{GMm}{r^2}. \qquad \ldots (11.1)$$

Newton further generalised the law by saying that not only the earth but all material bodies in the universe attract each other according to equation (11.1) with same value of $G$. The constant $G$ is called *universal constant of gravitation* and its value is found to be $6{\cdot}67 \times 10^{-11}$ N–$m^2$/$kg^2$. Equation (11.1) is known as the *universal law of gravitation.*

In this argument, the distance of the apple from the earth is taken to be equal to the radius of the earth. This means we have assumed that earth can be treated as a single particle placed at its centre. This is of course not obvious. Newton had spent several years to prove that indeed this can be done. A spherically symmetric body can be replaced by a point particle of equal mass placed at its centre for the purpose of calculating gravitational force. In the process he discovered the methods of calculus that we have already learnt in Chapter 2. There is evidence that quite a bit of differential calculus was known to the ancient Indian mathematicians but this literature was almost certainly not known to Newton or other scientists of those days.

***Example 11.1***

*Two particles of masses* 1·0 kg *and* 2·0 kg *are placed at a separation of* 50 cm. *Assuming that the only forces acting on the particles are their mutual gravitation, find the initial accelerations of the two particles.*

***Solution*** : The force of gravitation exerted by one particle on another is

$$F = \frac{Gm_1m_2}{r^2}$$

$$= \frac{6{\cdot}67 \times 10^{-11} \frac{\text{N–m}^2}{\text{kg}^2} \times (1{\cdot}0 \text{ kg}) \times (2{\cdot}0 \text{ kg})}{(0{\cdot}5 \text{ m})^2}$$

$$= 5{\cdot}3 \times 10^{-10} \text{ N}.$$

The acceleration of 1·0 kg particle is

$$a_1 = \frac{F}{m_1} = \frac{5{\cdot}3 \times 10^{-10} \text{ N}}{1{\cdot}0 \text{ kg}}$$

$$= 5{\cdot}3 \times 10^{-10} \text{ m s}^{-2}.$$

This acceleration is towards the 2·0 kg particle. The acceleration of the 2·0 kg particle is

$$a_2 = \frac{F}{m_2} = \frac{5{\cdot}3 \times 10^{-10} \text{ N}}{2{\cdot}0 \text{ kg}}$$

$$= 2{\cdot}65 \times 10^{-10} \text{ m s}^{-2}.$$

This acceleration is towards the 1·0 kg particle.

## 11.2 MEASUREMENT OF GRAVITATIONAL CONSTANT *G*

The gravitational constant $G$ is a small quantity and its measurement needs very sensitive arrangement. The first important successful measurement of this quantity was made by Cavendish in 1736 about 71 years after the law was formulated.

In this method, two small balls of equal mass are attached at the two ends of a light rod to form a dumb-bell. The rod is suspended vertically by a fine quartz wire. Two large spheres of equal mass are placed near the smaller spheres in such a way that all the four spheres are on a horizontal circle. The centre of the circle is at the middle point of the rod (figure 11.2).

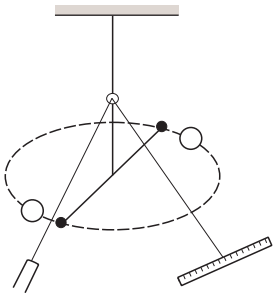


Figure 11.2

Two larger spheres lie on the opposite sides of the smaller balls at equal distance. A small plane mirror is attached to the vertical wire. A light beam, incident on the mirror, falls on a scale after reflection. If the wire rotates by an angle θ, the reflected beam rotates by 2θ and the spot on the scale moves. By measuring this movement of the spot on the scale and the distance between the mirror and the scale, the angle of deviation can be calculated. When the heavy balls are placed close to the small balls, a torque acts on the dumb-bell to rotate it. As the dumb-bell rotates, the suspension wire gets twisted and produces a torque on the dumb-bell in opposite direction. This torque is proportional to the angle rotated. The dumb-bell stays in equilibrium where the two torques have equal magnitude.

Let the mass of a heavy ball = $M$,

the mass of a small ball = $m$,

the distance between the centres of a heavy ball and the small ball placed close to it = $r$,

the deflection of the dumb-bell as it comes to equilibrium = θ,

the torsional constant of the suspension wire = $k$,

the length of the rod = $l$ and

the distance between the scale and the mirror = $D$.

The force acting on each of the small balls is

$$F = G\frac{Mm}{r^2}.$$

Here we have used the fact that the gravitational force due to a uniform sphere is same as that due to a single particle of equal mass placed at the centre of the sphere. As the four balls are on the same horizontal circle and the heavy balls are placed close to the smaller balls, this force acts in a horizontal direction perpendicular to the length of the dumb-bell. The torque due to each of these gravitational forces about the suspension wire is $F(l/2)$.

The total gravitational torque on the dumb-bell is, therefore,

$$\Gamma = 2F(l/2)$$
$$= Fl\ .$$

The opposing torque produced by the suspension wire is $k\theta$. For rotational equilibrium,

$$Fl = k\theta$$

or,
$$\frac{GMml}{r^2} = k\theta$$

$$G = \frac{k\theta r^2}{Mml}. \qquad \ldots \text{ (i)}$$

In an experiment, the heavy balls are placed close to the smaller balls as shown in the figure and the dumb-bell is allowed to settle down. The light beam is adjusted so that the beam reflected by the plane mirror falls on the scale. Now the heavy balls are shifted in such a way that they are placed on the same horizontal circle at same distance from the smaller balls but on the opposite side. In figure (11.3), the original positions of the heavy balls are shown by $A$, $B$ and the shifted positions by $A'$, $B'$.

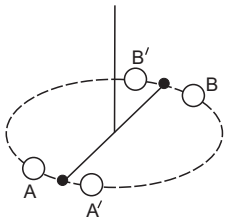



Figure 11.3

As the heavy balls are shifted to the new position, the dumb-bell rotates. If it was settled previously at an angle θ deviated from the mean position, it will now settle at the same angle θ on the other side. Thus, the total deflection of the dumb-bell due to the change in the positions of the heavy balls is 2θ. The reflected light beam deviates by an angle of 4θ.

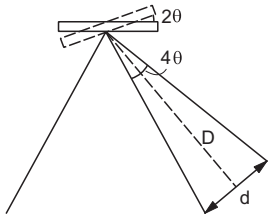



Figure 11.4

If the linear displacement of the light spot is $d$, we have (figure 11.4)

$$4\theta = \frac{d}{D}$$

or,
$$\theta = \frac{d}{4D}.$$

Substituting in (i),

$$G = \frac{kdr^2}{4MmlD}.$$

All the quantities on the right hand side are experimentally known and hence the value of $G$ may be calculated.

## 11.3 GRAVITATIONAL POTENTIAL ENERGY

The concept of potential energy of a system was introduced in Chapter-8. The potential energy of a system corresponding to a conservative force was defined as

$$U_f - U_i = -\int_i^f \vec{F} \cdot d\vec{r}.$$

The change in potential energy is equal to the negative of the work done by the internal forces. We also calculated the change in gravitational potential energy of the earth–particle system when the particle was raised through a small height over earth's surface. In this case the force $mg$ may be treated as constant and the change in potential energy is

$$U_f - U_i = mgh$$

where the symbols have their usual meanings. We now derive the general expression for the change in gravitational potential energy of a two-particle system.

Let a particle of mass $m_1$ be kept fixed at a point $A$ (figure 11.5) and another particle of mass $m_2$ is taken from a point $B$ to a point $C$. Initially, the distance between the particles is $AB = r_1$ and finally it becomes $AC = r_2$. We have to calculate the change in potential energy of the system of the two particles as the distance changes from $r_1$ to $r_2$.

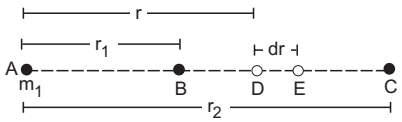



Figure 11.5

Consider a small displacement when the distance between the particles changes from $r$ to $r + dr$. In the figure, this corresponds to the second particle going from $D$ to $E$.

The force on the second particle is

$$F = \frac{Gm_1m_2}{r^2} \text{ along } \vec{DA}.$$

The work done by the gravitational force in the displacement is

$$dW = -\frac{Gm_1m_2}{r^2}\,dr.$$

The increase in potential energy of the two-particle system during this displacement is

$$dU = -dW = \frac{Gm_1m_2}{r^2}\,dr.$$

The increase in potential energy as the distance between the particles changes from $r_1$ to $r_2$ is

$$\begin{aligned} U(r_2) - U(r_1) &= \int dU \\ &= \int_{r_1}^{r_2} \frac{Gm_1m_2}{r^2}\,dr = Gm_1m_2 \int_{r_1}^{r_2} \frac{1}{r^2}\,dr \\ &= Gm_1m_2\left[-\frac{1}{r}\right]_{r_1}^{r_2} \\ &= Gm_1m_2\left(\frac{1}{r_1} - \frac{1}{r_2}\right). \end{aligned} \quad \ldots (11.2)$$

We choose the potential energy of the two-particle system to be zero when the distance between them is infinity. This means that we choose $U(\infty) = 0$. By (11.2) the potential energy $U(r)$, when the separation between the particles is $r$, is

$$\begin{aligned} U(r) &= U(r) - U(\infty) \\ &= Gm_1m_2\left[\frac{1}{\infty} - \frac{1}{r}\right] = -\frac{Gm_1m_2}{r}. \end{aligned}$$

*The gravitational potential energy of a two-particle system is*

$$U(r) = -\frac{Gm_1m_2}{r} \quad \ldots (11.3)$$

*where $m_1$ and $m_2$ are the masses of the particles, $r$ is the separation between the particles and the potential energy is chosen to be zero when the separation is infinite.*

We have proved this result by assuming that one of the particles is kept at rest and the other is displaced. However, as the potential energy depends only on the separation and not on the location of the particles, equation (11.3) is general.

Equation (11.3) gives the potential energy of a pair of particles. If there are three particles $A$, $B$ and $C$, there are three pairs $AB$, $AC$ and $BC$. The potential energy of the three-particle system is equal to the sum of the potential energies of the three pairs. For an $N$-particle system there are $N(N-1)/2$ pairs and the potential energy is calculated for each pair and added to get the total potential energy of the system.

***Example 11.2***

*Find the work done in bringing three particles, each having a mass of* 100 g, *from large distances to the vertices of an equilateral triangle of side* 20 cm.

***Solution*** **:** When the separations are large, the gravitational potential energy is zero. When the particles are brought at the vertices of the triangle *ABC,* three pairs *AB, BC* and *CA* are formed. The potential energy of each pair is $-Gm_1m_2/r$ and hence the total potential energy becomes

$$U = 3\times\left[-\frac{Gm_1m_2}{r}\right]$$

$$= 3\times\left[-\frac{6{\cdot}67\times10^{-11}\ \text{N–m}^2/\text{kg}^2\times(0{\cdot}1\ \text{kg})\times(0{\cdot}1\ \text{kg})}{0{\cdot}20\ \text{m}}\right]$$

$$= -1{\cdot}0\times10^{-11}\ \text{J}.$$

The work done by the gravitational forces is $W = -U = 1{\cdot}0\times10^{-11}$ J. If the particles are brought by some external agency without changing the kinetic energy, the work done by the external agency is equal to the change in potential energy $= -1{\cdot}0\times10^{-11}$ J.

## 11.4 GRAVITATIONAL POTENTIAL

Suppose a particle of mass $m$ is taken from a point $A$ to a point $B$ while keeping all other masses fixed. Let $U_A$ and $U_B$ denote the gravitational potential energy when the mass $m$ is at point $A$ and point $B$ respectively.

We define the "change in potential" $V_B - V_A$ between the two points as

$$V_B - V_A = \frac{U_B - U_A}{m}. \qquad \ldots\ (11.4)$$

The equation defines only the change in potential. We can choose any point to have zero potential. Such a point is called a *reference point*. If $A$ be the reference point, $V_A = 0$ and

$$V_B = \frac{U_B - U_A}{m}. \qquad \ldots\ (11.5)$$

Thus, *gravitational potential at a point is equal to the change in potential energy per unit mass, as the mass is brought from the reference point to the given point.* If the particle is slowly brought without increasing the kinetic energy, the work done by the external agent equals the change in potential energy. Thus, the potential at a point may also be defined as *the work done per unit mass by an external agent in bringing a particle slowly from the reference point to the given point.* Generally the reference point is chosen at infinity so that the potential at infinity is zero.

The *SI* unit of gravitational potential is J $kg^{-1}$.

## 11.5 CALCULATION OF GRAVITATIONAL POTENTIAL

### (A) Potential due to a Point Mass

Suppose a particle of mass $M$ is kept at a point $A$ (figure 11.6) and we have to calculate the potential at a point $P$ at a distance $r$ away from $A$. The reference point is at infinity.

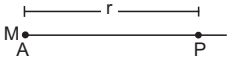



Figure 11.6

From equation (11.5), the potential at the point $P$ is

$$V(r) = \frac{U(r) - U(\infty)}{m}.$$

But $\quad U(r) - U(\infty) = -\dfrac{GMm}{r}.$

so that,

$$V = -\frac{GM}{r}. \qquad \ldots\ (11.6)$$

The gravitational potential due to a point mass $M$ at a distance $r$ is $-\dfrac{GM}{r}$.

### (B) Potential due to a Uniform Ring at A Point on its Axis

Let the mass of the ring be $M$ and its radius be $a$. We have to calculate the gravitational potential at a point $P$ on the axis of the ring (figure 11.7). The centre is at $O$ and $OP = r$.

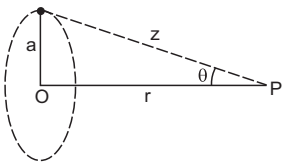



Figure 11.7

Consider any small part of the ring of mass $dm$. The point $P$ is at a distance $z = \sqrt{a^2 + r^2}$ from $dm$.

The potential at $P$ due to $dm$ is

$$dV = -\frac{G\,dm}{z} = -\frac{G\,dm}{\sqrt{a^2 + r^2}}.$$

The potential $V$ due to the whole ring is obtained by summing the contributions from all the parts. As the potential is a scalar quantity, we have

$$V = \int dV$$
$$= \int -\frac{G\,dm}{\sqrt{a^2 + r^2}}$$
$$= -\frac{G}{\sqrt{a^2 + r^2}} \int dm$$
$$= -\frac{GM}{\sqrt{a^2 + r^2}}. \quad \ldots (11.7)$$

In terms of the distance $z$ between the point $P$ and any point of the ring, the expression for the potential is given by

$$V = -\frac{GM}{z}. \quad \ldots (11.8)$$

**(C) Potential due to a Uniform Thin Spherical Shell**

Let the mass of the given spherical shell be $M$ and the radius $a$. We have to calculate the potential due to this shell at a point $P$. The centre of the shell is at $O$ and $OP = r$ (figure 11.8).

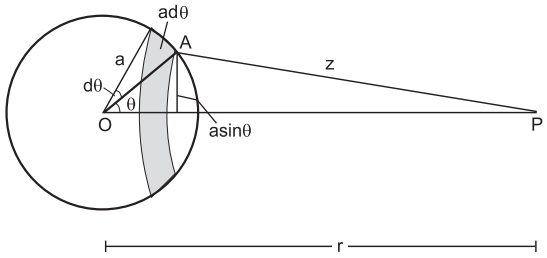



Figure 11.8

Let us draw a radius $OA$ making an angle θ with $OP$. Let us rotate this radius about $OP$ keeping the angle $AOP$ fixed at value θ. The point $A$ traces a circle on the surface of the shell. Let us now consider another radius at an angle $\theta + d\theta$ and likewise rotate it about $OP$. Another circle is traced on the surface of the shell. The part of the shell included between these two circles (shown shaded in the figure) may be treated as a ring.

The radius of this ring is $a \sin\theta$ and hence the perimeter is $2\pi a \sin\theta$. The width of the ring is $a d\theta$. The area of the ring is

$$(2\pi a \sin\theta)\,(a d\theta)$$
$$= 2\pi a^2 \sin\theta\, d\theta.$$

The total area of the shell is $4\pi a^2$. As the shell is uniform, the mass of the ring enclosed is

$$dm = \frac{M}{4\pi a^2}(2\pi a^2 \sin\theta\, d\theta)$$
$$= \frac{M}{2}\sin\theta\, d\theta.$$

Let the distance of any point of the ring from $P$ be $AP = z$. From the triangle $OAP$

$$z^2 = a^2 + r^2 - 2ar\cos\theta$$

or, $$2z\,dz = 2ar\sin\theta\, d\theta$$

or, $$\sin\theta\, d\theta = \frac{z\,dz}{ar}.$$

Thus, the mass of the ring is

$$dm = \frac{M}{2}\sin\theta\, d\theta = \frac{M}{2ar} z\, dz.$$

As the distance of any point of the ring from $P$ is $z$, the potential at $P$ due to the ring is

$$dV = -\frac{G\,dm}{z}$$
$$= -\frac{GM}{2ar} dz.$$

As we vary θ from 0 to π, the rings formed on the shell cover up the whole shell. The potential due to the whole shell is obtained by integrating $dV$ within the limits $\theta = 0$ to $\theta = \pi$.

**Case I : *P* is outside the shell (*r* > *a*)**

As figure (11.8) shows, when $\theta = 0$, the distance $z = AP = r - a$. When $\theta = \pi$, it is $z = r + a$. Thus, as θ varies from 0 to π, the distance $z$ varies from $r - a$ to $r + a$. Thus,

$$V = \int dV = -\frac{GM}{2ar}\int_{r-a}^{r+a} dz$$
$$= -\frac{GM}{2ar}[z]_{r-a}^{r+a}$$
$$= -\frac{GM}{2ar}[(r + a) - (r - a)]$$
$$= -\frac{GM}{r}. \quad \ldots (11.9)$$

*To calculate the potential at an external point, a uniform spherical shell may be treated as a point particle of equal mass placed at its centre.*

**Case II : *P* is inside the shell (*r* < *a*)**

In this case when $\theta = 0$, the distance $z = AP = a - r$ and when $\theta = \pi$ it is $z = a + r$ (figure 11.9). Thus, as θ varies from 0 to π, the distance $z$ varies from $a - r$ to $a + r$. Thus, the potential due to the shell is

$$V = \int dV$$
$$= -\frac{GM}{2ar}[z]_{a-r}^{a+r}$$
$$= -\frac{GM}{2ar}[(a + r) - (a - r)]$$
$$= -\frac{GM}{a}. \quad \ldots (11.10)$$

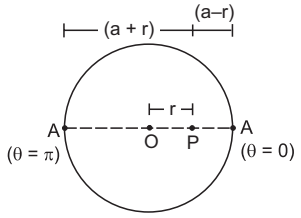



Figure 11.9

This does not depend on $r$. Thus, *the potential due to a uniform spherical shell is constant throughout the cavity of the shell.*

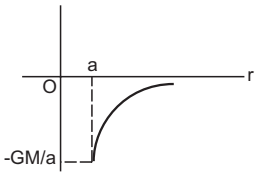



Figure 11.10

Figure (11.10) shows graphically the variation of potential with the distance from the centre of the shell.

***Example 11.3***

*A particle of mass M is placed at the centre of a uniform spherical shell of equal mass and radius a. Find the gravitational potential at a point P at a distance a/2 from the centre.*

***Solution*** **:** The gravitational potential at the point $P$ due to the particle at the centre is

$$V_1 = -\frac{GM}{a/2} = -\frac{2GM}{a}.$$

The potential at $P$ due to the shell is

$$V_2 = -\frac{GM}{a}.$$

The net potential at $P$ is $V_1 + V_2 = -\frac{3GM}{a}$.

**(D) Potential due to a Uniform Solid Sphere**

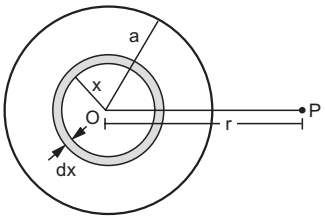



Figure 11.11

The situation is shown in figure (11.11). Let the mass of the sphere be $M$ and its radius $a$. We have to calculate the gravitational potential at a point $P$. Let $OP = r$.

Let us draw two spheres of radii $x$ and $x + dx$ concentric with the given sphere. These two spheres enclose a thin spherical shell of volume $4\pi x^2 dx$. The volume of the given sphere is $\frac{4}{3}\pi a^3$. As the sphere is uniform, the mass of the shell is

$$dm = \frac{M}{\frac{4}{3}\pi a^3} 4\pi x^2 dx = \frac{3M}{a^3} x^2 dx.$$

The potential due to this shell at the point $P$ is

$$dV = -\frac{G\,dm}{r} \text{ if } x < r \text{ and } dV = -\frac{G\,dm}{x} \text{ if } x > r.$$

**Case I : Potential at an external point**

Suppose the point $P$ is outside the sphere (figure 11.11). The potential at $P$ due to the shell considered is

$$dV = -\frac{G\,dm}{r}.$$

Thus, the potential due to the whole sphere is

$$V = \int dV = -\frac{G}{r}\int dm$$

$$= -\frac{GM}{r}. \qquad \ldots (11.11)$$

*The gravitational potential due to a uniform sphere at an external point is same as that due to a single particle of equal mass placed at its centre.*

**Case II : Potential at an internal point**

Let us divide the sphere in two parts by imagining a concentric spherical surface passing through $P$. The inner part has a mass

$$M' = \frac{M}{\frac{4}{3}\pi a^3} \times \frac{4}{3}\pi r^3 = \frac{Mr^3}{a^3}.$$

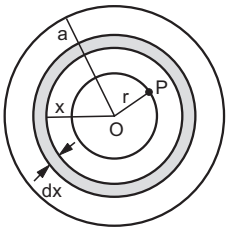



Figure 11.12

The potential at $P$ due to this inner part is by equation (11.11)

$$V_1 = -\frac{GM'}{r}$$

$$= -\frac{GMr^2}{a^3}. \qquad \ldots (i)$$

To get the potential at $P$ due to the outer part of the sphere, we divide this part in concentric shells. The mass of the shell between radii $x$ and $x + dx$ is

$$dm = \frac{M}{\frac{4}{3}\pi a^3} 4\pi x^2 dx = \frac{3Mx^2 dx}{a^3}.$$

The potential at $P$ due to this shell is,

$$\frac{-G\,dm}{x} = -3\frac{GM}{a^3}x\,dx.$$

The potential due to the outer part is

$$V_2 = \int_r^a -\frac{3GM}{a^3}x\,dx$$

$$= -\frac{3GM}{a^3}\left[\frac{x^2}{2}\right]_r^a$$

$$= \frac{-3GM}{2a^3}(a^2 - r^2). \quad \ldots \text{(ii)}$$

By (i) and (ii) the total potential at $P$ is

$$V = V_1 + V_2$$

$$= -\frac{GMr^2}{a^3} - \frac{3GM}{2a^3}(a^2 - r^2)$$

$$= -\frac{GM}{2a^3}(3a^2 - r^2). \quad \ldots \text{(11.12)}$$

At the centre of the sphere the potential is

$$V = -\frac{3GM}{2a}.$$

## 11.6 GRAVITATIONAL FIELD

We have been saying all through that a body $A$ exerts a force of gravitation on another body $B$ kept at a distance. This is called *action at a distance* viewpoint. However, this viewpoint creates certain problems when one deals with objects separated by large distances. It is now assumed that a body can not directly interact with another body kept at a distance. The force between two objects is seen to be a two-step process.

In the first step, it is assumed that the body $A$ creates a *gravitational field* in the space around it. The field has its own existence and has energy and momentum. This field has a definite direction at each point of the space and its intensity varies from point to point.

In the second step, it is assumed that when a body $B$ is placed in a gravitational field, this field exerts a force on it. The direction and the intensity of the field is defined in terms of the force it exerts on a body placed in it. We define the *intensity of gravitational field* $\vec{E}$ at a point by the equation

$$\vec{E} = \frac{\vec{F}}{m} \quad \ldots \text{(11.13)}$$

where $\vec{F}$ is the force exerted by the field on a body of mass $m$ placed in the field. Quite often the intensity of gravitational field is abbreviated as *gravitational field*. Its SI unit is N kg$^{-1}$.

Gravitational field adds according to the rules of vector addition. If $\vec{E}_1$ is the field due to a source $S_1$ and $\vec{E}_2$ is the field at the same point due to another source $S_2$, the resultant field when both the sources are present is $\vec{E}_1 + \vec{E}_2$.

If a mass $m$ is placed close to the surface of the earth, the force on it is $mg$. We say that the earth has set up a gravitational field and this field exerts a force on the mass. The intensity of the field is

$$\vec{E} = \frac{\vec{F}}{m} = \frac{m\vec{g}}{m} = \vec{g}.$$

Thus, the intensity of the gravitational field near the surface of the earth is equal to the acceleration due to gravity. It should be clearly understood that the intensity of the gravitational field and the acceleration due to gravity are two separate physical quantities having equal magnitudes and directions.

***Example 11.4***

*A particle of mass* 50 g *experiences a gravitational force of* 2·0 N *when placed at a particular point. Find the gravitational field at that point.*

***Solution*** : The gravitational field has a magnitude

$$E = \frac{F}{m} = \frac{2{\cdot}0\text{ N}}{(50 \times 10^{-3}\text{ kg})} = 40\text{ N kg}^{-1}.$$

This field is along the direction of the force.

## 11.7 RELATION BETWEEN GRAVITATIONAL FIELD AND POTENTIAL

Suppose the gravitational field at a point $\vec{r}$ due to a given mass distribution is $\vec{E}$. By definition (equation 11.13), the force on a particle of mass $m$ when it is at $\vec{r}$ is

$$\vec{F} = m\vec{E}.$$

As the particle is displaced from $\vec{r}$ to $\vec{r} + d\vec{r}$ the work done by the gravitational force on it is

$$dW = \vec{F} \cdot d\vec{r}$$

$$= m\vec{E} \cdot d\vec{r}.$$

The change in potential energy during this displacement is

$$dU = -dW = -m\vec{E} \cdot d\vec{r}.$$

The change in potential is, by equation (11.4),

$$dV = \frac{dU}{m} = -\vec{E} \,.\, d\vec{r}. \qquad \ldots \quad (11.14)$$

Integrating between $\vec{r}_1$ and $\vec{r}_2$

$$V(\vec{r}_2) - V(\vec{r}_1) = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{E} \,.\, d\vec{r}. \qquad \ldots \quad (11.15)$$

If $\vec{r}_1$ is taken at the reference point, $V(\vec{r}_1) = 0$. The potential $V(\vec{r})$ at any point $\vec{r}$ is, therefore,

$$V(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{E} \,.\, d\vec{r} \qquad \ldots \quad (11.16)$$

where $\vec{r}_0$ denotes the reference point.

If we work in Cartesian coordinates, we can write

$$\vec{E} = \vec{i}E_x + \vec{j}E_y + \vec{k}E_z$$

and $$d\vec{r} = \vec{i}\,dx + \vec{j}\,dy + \vec{k}\,dz$$

so that

$$\vec{E} \,.\, d\vec{r} = E_x\,dx + E_y\,dy + E_z\,dz \,.$$

Equation (11.14) may be written as

$$dV = -E_x\,dx - E_y\,dy - E_z\,dz.$$

If $y$ and $z$ remain constant, $dy = dz = 0$ .

Thus, $$E_x = -\frac{\partial V}{\partial x} \qquad \ldots \quad (11.17)$$

Similarly, $$E_y = -\frac{\partial V}{\partial y} \text{ and } E_z = -\frac{\partial V}{\partial z}.$$

The symbol $\frac{\partial}{\partial x}$ means partial differentiation with respect to $x$ treating $y$ and $z$ to be constants. Similarly for $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$.

If the field is known, the potential may be obtained by integrating the field according to equation (11.16) and if the potential is known, the field may be obtained by differentiating the potential according to equation (11.17).

***Example 11.5***

*The gravitational field due to a mass distribution is given by $E = K/x^3$ in X-direction. Taking the gravitational potential to be zero at infinity, find its value at a distance x.*

***Solution*** : The potential at a distance $x$ is

$$V(x) = -\int_{\infty}^{x} E\,dx = -\int_{\infty}^{x} \frac{K}{x^3}\,dx$$

$$= \left[\frac{K}{2x^2}\right]_{\infty}^{x} = \frac{K}{2x^2}.$$

***Example 11.6***

*The gravitational potential due to a mass distribution is $V = \frac{A}{\sqrt{x^2 + a^2}}$. Find the gravitational field.*

***Solution*** : $$V = \frac{A}{\sqrt{x^2 + a^2}} = A(x^2 + a^2)^{-1/2}.$$

If the gravitational field is $E$,

$$E_x = -\frac{\partial V}{\partial x} = -A\left(-\frac{1}{2}\right)(x^2 + a^2)^{-3/2}(2x)$$

$$= \frac{Ax}{(x^2 + a^2)^{3/2}}$$

$$E_y = -\frac{\partial V}{\partial y} = 0 \text{ and } E_z = -\frac{\partial V}{\partial z} = 0.$$

The gravitational field is $\frac{Ax}{(x^2 + a^2)^{3/2}}$ in the $x$-direction.

## 11.8 CALCULATION OF GRAVITATIONAL FIELD

### (A) Field due to a Point Mass

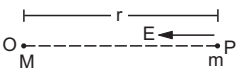



Figure 11.13

Suppose a particle of mass $M$ is placed at a point $O$ (figure 11.13) and a second particle of mass $m$ is placed at a point $P$. Let $OP = r$. The mass $M$ creates a field $\vec{E}$ at the site of mass $m$ and this field exerts a force

$$\vec{F} = m\vec{E}$$

on the mass $m$. But the force $\vec{F}$ on the mass $m$ due to the mass $M$ is

$$F = \frac{GMm}{r^2}$$

acting along $\vec{PO}$ . Thus, the gravitational field at $P$ is

$$E = \frac{GM}{r^2} \qquad \ldots \quad (11.18)$$

along $\vec{PO}$ . If $O$ is taken as the origin, the position vector of mass $m$ is $\vec{r} = \vec{OP}$ . Equation (11.18) may be rewritten in vector form as

$$= -\frac{GM}{r^2}\vec{e}_r \qquad \ldots \quad (11.19)$$

where $\vec{e}_r$ is the unit vector along $\vec{r}$.

### (B) Field due to a Uniform Circular Ring at a Point on its Axis

Figure (11.14) shows a uniform circular ring of radius $a$ and mass $M$. Let $P$ be a point on its axis at

a distance $r$ from the centre. We have to obtain the gravitational field at $P$ due to the ring. By symmetry the field must be towards the centre that is along $\overrightarrow{PO}$.

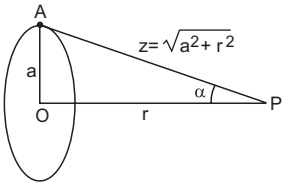



Figure 11.14

Consider any particle of mass $dm$ on the ring, say at point $A$. The distance of this particle from $P$ is $AP = z = \sqrt{a^2 + r^2}$. The gravitational field at $P$ due to $dm$ is along $\overrightarrow{PA}$ and its magnitude is

$$dE = \frac{G\,dm}{z^2}.$$

The component along $PO$ is

$$dE\cos\alpha = \frac{G\,dm}{z^2}\cos\alpha.$$

The net gravitational field at $P$ due to the ring is

$$E = \int \frac{G\,dm}{z^2}\cos\alpha = \frac{G\cos\alpha}{z^2}\int dm = \frac{GM\cos\alpha}{z^2}$$

$$= \frac{GMr}{(a^2 + r^2)^{3/2}}. \qquad \ldots (11.20)$$

The field is directed towards the centre of the ring.

**(C) Field due to a Uniform Disc at a Point on its Axis**

The situation is shown in figure (11.15). Let the mass of the disc be $M$ and its radius be $a$. Let $O$ be the centre of the disc and $P$ be a point on its axis at a distance $r$ from the centre. We have to find the gravitational field at $P$ due to the disc.

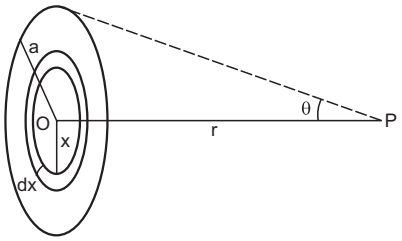



Figure 11.15

Let us draw a circle of radius $x$ with the centre at $O$. We draw another concentric circle of radius $x + dx$. The part of the disc enclosed between these two circles can be treated as a uniform ring of radius $x$. The point $P$ is on its axis at a distance $r$ from the centre. The area of this ring is $2\pi\,x\,dx$. The area of the whole disc is $\pi a^2$. As the disc is uniform, the mass of this ring is

$$dm = \frac{M}{\pi a^2}2\pi\,x\,dx$$

$$= \frac{2M\,x\,dx}{a^2}.$$

The gravitational field at $P$ due to the ring is, by equation (11.20),

$$dE = \frac{G\left(\frac{2M\,x\,dx}{a^2}\right)r}{(r^2 + x^2)^{3/2}}$$

$$= \frac{2GMr}{a^2}\frac{x\,dx}{(r^2 + x^2)^{3/2}}.$$

As $x$ varies from 0 to $a$, the rings cover up the whole disc. The field due to each of these rings is in the same direction $PO$. Thus, the net field due to the whole disc is along $PO$ and its magnitude is

$$E = \int_0^a \frac{2GMr}{a^2}\frac{x\,dx}{(r^2 + x^2)^{3/2}}$$

$$= \frac{2GMr}{a^2}\int_0^a \frac{x\,dx}{(r^2 + x^2)^{3/2}}. \qquad \ldots (i)$$

Let $r^2 + x^2 = z^2$.

Then $2x\,dx = 2z\,dz$ and

$$\int \frac{x\,dx}{(r^2 + x^2)^{3/2}} = \int \frac{z\,dz}{z^3}$$

$$= \int \frac{1}{z^2}dz = -\frac{1}{z} = -\frac{1}{\sqrt{r^2 + x^2}}.$$

From (i), $$E = \frac{2GMr}{a^2}\left[-\frac{1}{\sqrt{r^2 + x^2}}\right]_0^a$$

$$= \frac{2GMr}{a^2}\left[\frac{1}{r} - \frac{1}{\sqrt{r^2 + a^2}}\right] \ldots (11.21)$$

Equation (11.21) may be expressed in terms of the angle $\theta$ subtended by a radius of the disc at $P$ as,

$$E = \frac{2GM}{a^2}(1 - \cos\theta).$$

**(D) Field due to a Uniform Thin Spherical Shell**

We can use the construction of figure (11.8) to find the gravitational field at a point due to a uniform thin spherical shell. The figure is reproduced here (figure 11.16) with symbols having same meanings. The shaded ring has mass $dm = \frac{M}{2}\sin\theta\,d\theta$. The field at $P$ due to this ring is

$$dE = \frac{G\,dm}{z^2}\cos\alpha = \frac{GM}{2}\frac{\sin\theta\,d\theta\cos\alpha}{z^2}. \qquad \ldots (i)$$

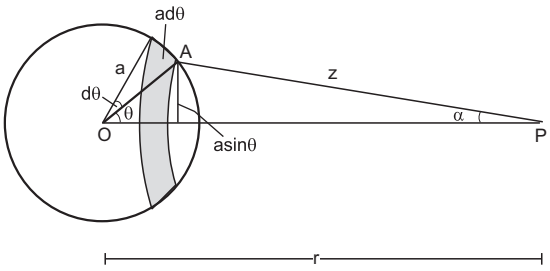



Figure 11.16

From the triangle *OAP*,

$$z^2 = a^2 + r^2 - 2ar\cos\theta$$

or, $$2z\,dz = 2ar\sin\theta\,d\theta$$

or, $$\sin\theta\,d\theta = \frac{z\,dz}{ar}. \qquad \ldots \text{(ii)}$$

Also from the triangle *OAP*,

$$a^2 = z^2 + r^2 - 2zr\cos\alpha$$

or, $$\cos\alpha = \frac{z^2 + r^2 - a^2}{2zr}. \qquad \ldots \text{(iii)}$$

Putting from (ii) and (iii) in (i),

$$dE = \frac{GM}{4ar^2}\left(1 - \frac{a^2 - r^2}{z^2}\right)dz$$

or, $$\int dE = \frac{GM}{4ar^2}\left[z + \frac{a^2 - r^2}{z}\right]$$

**Case I : *P* is outside the shell ($r > a$)**

In this case $z$ varies from $r - a$ to $r + a$. The field due to the whole shell is

$$E = \frac{GM}{4ar^2}\left[z + \frac{a^2 - r^2}{z}\right]_{r-a}^{r+a} = \frac{GM}{r^2}.$$

We see that *the shell may be treated as a point particle of the same mass placed at its centre to calculate the gravitational field at an external point.*

**Case II : *P* is inside the shell ($r < a$)**

In this case $z$ varies from $a - r$ to $a + r$ (figure 11.9). The field at $P$ due to the whole shell is

$$E = \frac{GM}{4ar^2}\left[z + \frac{a^2 - r^2}{z}\right]_{a-r}^{a+r} = 0.$$

Hence *the field inside a uniform spherical shell is zero.*

**(E) Gravitational Field due to a Uniform Solid Sphere**

**Case I : Field at an external point**

Let the mass of the sphere be $M$ and its radius be $a$. We have to calculate the gravitational field due to the sphere at a point outside the sphere at a distance $r$ from the centre. Figure (11.17) shows the situation. The centre of the sphere is at $O$ and the field is to be calculated at $P$.

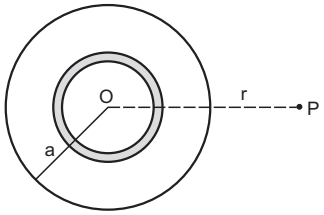



Figure 11.17

Let us divide the sphere into thin spherical shells each centred at $O$. Let the mass of one such shell be $dm$. To calculate the gravitational field at $P$, we can replace the shell by a single particle of mass $dm$ placed at the centre of the shell that is at $O$. The field at $P$ due to this shell is then

$$dE = \frac{G\,dm}{r^2}$$

towards $PO$. The field due to the whole sphere may be obtained by summing the fields of all the shells making the solid sphere.

Thus, $$E = \int dE$$

$$= \int \frac{G\,dm}{r^2} = \frac{G}{r^2}\int dm$$

$$= \frac{GM}{r^2}. \qquad \ldots \text{(11.23)}$$

Thus, *a uniform sphere may be treated as a single particle of equal mass placed at its centre for calculating the gravitational field at an external point.*

This allows us to treat the earth as a point particle placed at its centre while calculating the force between the earth and an apple.

**Case II : Field at an internal point**

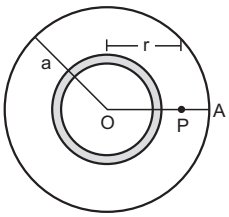



Figure 11.18

Suppose the point $P$ is inside the solid sphere (figure 11.18). In this case $r < a$. The sphere may be divided into thin spherical shells all centered at $O$. Suppose the mass of such a shell is $dm$. If the radius

of the shell is less than $r$, the point $P$ is outside the shell and the field due to the shell is

$$dE = \frac{G\,dm}{r^2} \text{ along } PO .$$

If the radius of the shell considered is greater than $r$, the point $P$ is internal and the field due to such a shell is zero. The total field due to the whole sphere is obtained by summing the fields due to all the shells. As all these fields are along the same direction, the net field is

$$E = \int dE$$

$$= \int \frac{G\,dm}{r^2} = \frac{G}{r^2} \int dm . \qquad \ldots \text{ (i)}$$

Only the masses of the shells with radii less than $r$ should be added to get $\int dm$. These shells form a solid sphere of radius $r$. The volume of this sphere is $\frac{4}{3}\pi r^3$. The volume of the whole sphere is $\frac{4}{3}\pi a^3$. As the given sphere is uniform, the mass of the sphere of radius $r$ is

$$\frac{M}{\frac{4}{3}\pi a^3} \cdot \left(\frac{4}{3}\pi r^3\right) = \frac{Mr^3}{a^3} .$$

Thus, $$\int dm = \frac{Mr^3}{a^3}$$

and by (i) $$E = \frac{G}{r^2}\frac{Mr^3}{a^3}$$

$$= \frac{GM}{a^3} r. \qquad \ldots \text{ (11.24)}$$

*The gravitational field due to a uniform sphere at an internal point is proportional to the distance of the point from the centre of the sphere.* At the centre itself, $r = 0$ and the field is zero. This is also expected from symmetry because any particle at the centre is equally pulled from all sides and the resultant must be zero. At the surface of the sphere, $r = a$ and

$$E = \frac{GM}{a^2} .$$

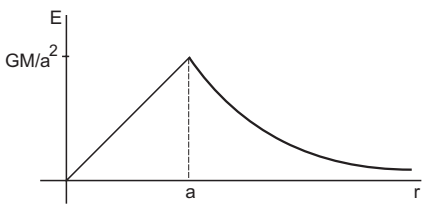



Figure 11.19

The formula (11.23) for the field at an external point also gives $E = \frac{GM}{a^2}$ at the surface of the sphere. The two formulae agree at $r = a$. Figure (11.19) shows graphically the variation of gravitational field due to a solid sphere with the distance from its centre.

***Example 11.7***

*Find the gravitational field due to the moon at its surface. The mass of the moon is* $7{\cdot}36 \times 10^{22}$ kg *and the radius of the moon is* $1{\cdot}74 \times 10^6$ m. *Assume the moon to be a spherically symmetric body.*

***Solution*** **:** To calculate the gravitational field at an external point, the moon may be replaced by a single particle of equal mass placed at its centre. Then the field at the surface is

$$E = \frac{GM}{a^2}$$

$$= \frac{6{\cdot}67 \times 10^{-11}\text{N–m}^2/\text{kg}^2 \times 7{\cdot}36 \times 10^{22}\text{ kg}}{(1{\cdot}74 \times 10^6\text{ m})^2}$$

$$= 1{\cdot}62\text{ N kg}^{-1} .$$

This is about one sixth of the gravitational field due to the earth at its surface.

## 11.9 VARIATION IN THE VALUE OF *g*

The acceleration due to gravity is given by

$$g = \frac{F}{m}$$

where $F$ is the force exerted by the earth on an object of mass $m$. This force is affected by a number of factors and hence $g$ also depends on these factors.

### (a) Height from the Surface of the Earth

If the object is placed at a distance $h$ above the surface of the earth, the force of gravitation on it due to the earth is

$$F = \frac{GMm}{(R+h)^2}$$

where $M$ is the mass of the earth and $R$ is its radius.

Thus, $$g = \frac{F}{m} = \frac{GM}{(R+h)^2} .$$

We see that the value of $g$ decreases as one goes up. We can write,

$$g = \frac{GM}{R^2\left(1+\frac{h}{R}\right)^2} = \frac{g_0}{\left(1+\frac{h}{R}\right)^2}$$

where $g_0 = \frac{GM}{R^2}$ is the value of $g$ at the surface of the earth. If $h << R$,

$$g = g_0\left(1+\frac{h}{R}\right)^{-2} \approx g_0\left(1-\frac{2h}{R}\right).$$

If one goes a distance $h$ inside the earth such as in mines, the value of $g$ again decreases. The force by the

earth is, by equation (11.24),

$$F = \frac{GMm}{R^3}(R - h)$$

or, $$g = \frac{F}{m} = \frac{GM}{R^2}\left(\frac{R-h}{R}\right)$$

$$= g_0\left(1 - \frac{h}{R}\right).$$

The value of $g$ is maximum at the surface of the earth and decreases with the increase in height as well as with depth similar to that shown in figure (11.19).

***Example 11.8***

*Calculate the value of acceleration due to gravity at a point (a)* 5·0 km *above the earth's surface and (b)* 5·0 km *below the earth's surface. Radius of earth* = 6400 km *and the value of g at the surface of the earth is* 9·80 m s$^{-2}$.

***Solution*** :

(a) The value of $g$ at a height $h$ is (for $h << R$)

$$g = g_0\left(1 - \frac{2h}{R}\right)$$

$$= (9{\cdot}80 \text{ m s}^{-2})\left(1 - \frac{2 \times 5{\cdot}0 \text{ km}}{6400 \text{ km}}\right)$$

$$= 9{\cdot}78 \text{ m s}^{-2}.$$

(b) The value at a depth $h$ is

$$g = g_0\left(1 - \frac{h}{R}\right)$$

$$= (9{\cdot}8 \text{ m s}^{-2})\left(1 - \frac{5{\cdot}0 \text{ km}}{6400 \text{ km}}\right)$$

$$= 9{\cdot}79 \text{ m s}^{-2}.$$

**(b) Rotation of the Earth**

As the earth rotates about its own axis the frame attached to the earth is noninertial. If we wish to use the familiar Newton's laws, we have to include pseudo forces. For an object at rest with respect to the earth, a centrifugal force $m\omega^2 r$ is to be added where $m$ is the mass of the object, $\omega$ is the angular velocity of the earth and $r$ is the radius of the circle in which the particle rotates.

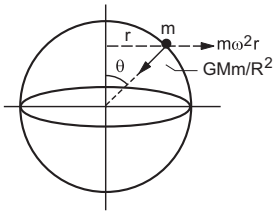



Figure 11.20

If the colatitude of the location of the particle is $\theta$ (figure 11.20), $r = R \sin\theta$ where $R$ is the radius of the earth. Acceleration of an object falling near the earth's surface, as measured from the earth frame, is $F/m$ where $F$ is the vector sum of the gravitational force $\frac{GMm}{R^2} = mg$ and the centrifugal force $m\omega^2 r = m\omega^2 R \sin\theta$. The acceleration $F/m = g'$ is the apparent value of the acceleration due to gravity.

At the equator, $\theta = \pi/2$ and the centrifugal force is just opposite to the force of gravity. The resultant of these two is

$$F = mg - m\omega^2 R$$

or, $$g' = g - \omega^2 R.$$

At the poles, $\theta = 0$ and the centrifugal force $m\omega^2 R \sin\theta = 0$. Thus, $F = mg$ and $g' = g$. Thus, the observed value of the acceleration due to gravity is minimum at the equator and is maximum at the poles. This effect had been discussed in the chapter on circular motion.

**(c) Nonsphericity of the Earth**

All formulae and equations have been derived by assuming that the earth is a uniform solid sphere. The shape of the earth slightly deviates from the perfect sphere. The radius in the equatorial plane is about 21 km larger than the radius along the poles. Due to this the force of gravity is more at the poles and less at the equator. The value of $g$ is accordingly larger at the poles and less at the equator. Note that due to rotation of earth also, the value of $g$ is smaller at the equator than that at the poles.

**(d) Nonuniformity of the Earth**

The earth is not a uniformly dense object. There are a variety of minerals, metals, water, oil, etc., inside the earth. Then at the surface there are mountains, seas, etc. Due to these nonuniformities in the mass distribution, the value of $g$ is locally affected.

**"Weighing" the Earth**

The force exerted by the earth on a body is called the *weight* of the body. In this sense "weight of the earth" is a meaningless concept. However, the mass of the earth can be determined by noting the acceleration due to gravity near the surface of the earth. We have,

$$g = \frac{GM}{R^2}$$

or, $$M = gR^2/G$$

Putting $g = 9{\cdot}8 \text{ m s}^{-2}$, $R = 6400$ km

and $$G = 6{\cdot}67 \times 10^{-11} \frac{\text{N–m}^2}{\text{kg}^2}$$

the mass of the earth comes out to be $5{\cdot}98 \times 10^{24}$ kg.

## 11.10 PLANETS AND SATELLITES

### Planets

Planets move round the sun due to the gravitational attraction of the sun. The path of these planets are elliptical with the sun at a focus. However, the difference in major and minor axes is not large. The orbits can be treated as nearly circular for not too sophisticated calculations. Let us derive certain characteristics of the planetary motion in terms of the radius of the orbit assuming it to be perfectly circular.

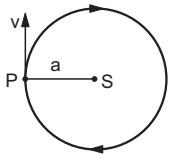



Figure 11.21

Let the mass of the sun be $M$ and that of the planet under study be $m$. The mass of the sun is many times larger than the mass of the planet. The sun may, therefore, be treated as an inertial frame of reference.

#### Speed

Let the radius of the orbit be $a$ and the speed of the planet in the orbit be $v$. By Newton's second law, the force on the planet equals its mass times the acceleration. Thus,

$$\frac{GMm}{a^2} = m\left(\frac{v^2}{a}\right)$$

or, $$v = \sqrt{\frac{GM}{a}} \; . \qquad \ldots \; (11.25)$$

The speed of a planet is inversely proportional to the square root of the radius of its orbit.

#### Time period

The time taken by a planet in completing one revolution is its time period $T$. In one revolution it covers a linear distance of $2\pi a$ at speed $v$. Thus,

$$T = \frac{2\pi a}{v}$$

$$= \frac{2\pi a}{\sqrt{\frac{GM}{a}}} = \frac{2\pi}{\sqrt{GM}} a^{3/2}$$

or, $$T^2 = \frac{4\pi^2}{GM} a^3 . \qquad \ldots \; (11.26)$$

#### Energy

The kinetic energy of the planet is

$$K = \frac{1}{2} mv^2 .$$

Using (11.25),

$$K = \frac{1}{2} m \; \frac{GM}{a} = \frac{GMm}{2a} .$$

The gravitational potential energy of the sun–planet system is

$$U = -\frac{GMm}{a} .$$

The total mechanical energy of the sun–planet system is

$$E = K + U = \frac{GMm}{2a} - \frac{GMm}{a} = -\frac{GMm}{2a} .$$

The total energy is negative. This is true for any bound system if the potential energy is taken to be zero at infinite separation.

### Satellite

Satellites are launched from the earth so as to move round it. A number of rockets are fired from the satellite at proper time to establish the satellite in the desired orbit. Once the satellite is placed in the desired orbit with the correct speed for that orbit, it will continue to move in that orbit under gravitational attraction of the earth. All the equations derived above for planets are also true for satellites with $M$ representing the mass of the earth and $m$ representing the mass of the satellite.

***Example 11.9***

*A satellite is revolving round the earth at a height of* 600 km. *Find (a) the speed of the satellite and (b) the time period of the satellite. Radius of the earth* = 6400 km *and mass of the earth* = $6 \times 10^{24}$ kg.

***Solution*** : The distance of the satellite from the centre of the earth is 6400 km + 600 km = 7000 km.

The speed of the satellite is

$$v = \sqrt{\frac{GM}{a}}$$

$$= \sqrt{\frac{6{\cdot}67 \times 10^{-11} \text{ N–m}^2/\text{kg}^2 \times 6 \times 10^{24} \text{ kg}}{7000 \times 10^3 \text{ m}}}$$

$$= 7{\cdot}6 \times 10^3 \text{ m s}^{-1} = 7{\cdot}6 \text{ km s}^{-1} .$$

The time period is

$$T = \frac{2\pi a}{v}$$

$$= \frac{2\pi \times 7000 \times 10^3 \text{ m}}{7{\cdot}6 \times 10^3 \text{ m s}^{-1}} = 5{\cdot}8 \times 10^3 \text{ s}.$$

**Geostationary Satellite**

The earth rotates about its own axis (the line joining the north pole and the south pole) once in 24 hours. Suppose a satellite is established in an orbit in the plane of the equator. Suppose the height is such that the time period of the satellite is 24 hours and it moves in the same sense as the earth. The satellite will always be overhead a particular place on the equator. As seen from the earth, this satellite will appear to be stationary. Such a satellite is called *a geostationary satellite*. Such satellites are used for telecommunication, weather forecasting and other applications.

According to equation (11.26),

$$T^2 = \frac{4\pi^2}{GM} a^3$$

or,
$$a = \left(\frac{GMT^2}{4\pi^2}\right)^{1/3}.$$

Putting the values of $G$, $M = (6 \times 10^{24}$ kg) and $T = (24$ hours); the radius of the geostationary orbit comes out to be $a = 4{\cdot}2 \times 10^4$ km. The height above the surface of the earth is about $3{\cdot}6 \times 10^4$ km.

## 11.11 KEPLER'S LAWS

From the observations of Tycho Brahe, Kepler formulated the laws of planetary motion which we have listed in the first section of this chapter. The first law states that the path of a planet is elliptical with the sun at a focus. Circular path is a special case of an ellipse when the major and minor axes are equal. For a circular path, the planet should have velocity perpendicular to the line joining it with the sun and the magnitude should satisfy equation (11.25), that is $v = \sqrt{\frac{GM}{a}}$. If these conditions are not satisfied, the planet moves in an ellipse.

The second law states that the radius vector from the sun to the planet sweeps out equal area in equal time. For a circular orbit, this is obvious because the speed of the particle remains constant.

The third law of Kepler states that the square of the time period of a planet is proportional to the cube of the semimajor axis. For a circular orbit semimajor axis is same as the radius. We have already proved this law for circular orbits in equation (11.26). As $M$ denotes mass of the sun, $\frac{4\pi^2}{GM}$ is fixed for all planets and $T^2 \propto a^3$.

## 11.12 WEIGHTLESSNESS IN A SATELLITE

A satellite moves round the earth in a circular orbit under the action of gravity. The acceleration of the satellite is $\frac{GM}{R^2}$ towards the centre of the earth, where $M$ is the mass of the earth and $R$ is the radius of the orbit of the satellite. Consider a body of mass $m$ placed on a surface inside a satellite moving round the earth. The forces on the body are

(a) the gravitational pull of the earth $= \frac{GMm}{R^2}$,

(b) the contact force $\mathcal{N}$ by the surface.

By Newton's law,

$$G\frac{Mm}{R^2} - \mathcal{N} = m\left(\frac{GM}{R^2}\right) \text{ or, } \mathcal{N} = 0.$$

Thus, the surface does not exert any force on the body and hence its apparent weight is zero. No support is needed to hold a body in the satellite. All positions shown in figure (11.22) are equally comfortable.

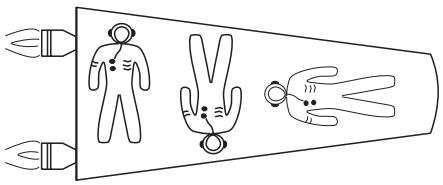


Figure 11.22

One can analyse the situation from the frame of the satellite. Working in the satellite frame we have to add a centrifugal force on all bodies. If the mass of a body is $m$, the centrifugal force is $m\left(\frac{GM}{R^2}\right)$ away from the centre of the earth. This pseudo force exactly balances the weight of the body which is $\frac{GMm}{R^2}$ towards the centre of the earth. A body needs no support to stay at rest in the satellite and hence all positions are equally comfortable. Water will not fall down from the glass even if it is inverted. It will act like a "gravity-free hall". Such a state is called *weightlessness.*

It should be clear that the earth still attracts a body with the same force $\frac{GMm}{R^2}$. The feeling of weightlessness arises because one stays in a rotating frame.

## 11.13 ESCAPE VELOCITY

When a stone is thrown up it goes up to a maximum height and then returns. As the particle

goes up, the gravitational potential energy increases and the kinetic energy of the particle decreases. The particle will continue to go up till its kinetic energy becomes zero and will return from there.

Let the initial velocity of the particle be $u$. The kinetic energy of the particle is $K = \frac{1}{2}mu^2$ and the gravitational potential energy of the earth–particle system is $U = -\frac{GMm}{R}$, where $M$ is the mass of the earth, $m$ is the mass of the particle and $R$ is the radius of the earth. When it reaches a height $h$ above the earth's surface, its speed becomes $v$. The kinetic energy there is $\frac{1}{2}mv^2$ and the gravitational potential energy is $-\frac{GMm}{R+h}$.

By conservation of energy

$$\frac{1}{2}mu^2 - \frac{GMm}{R} = \frac{1}{2}mv^2 - \frac{GMm}{R+h}$$

or, $$\frac{1}{2}mv^2 = \left[\frac{1}{2}mu^2 - \frac{GMm}{R}\right] + \frac{GMm}{R+h}. \quad \ldots \text{(i)}$$

The particle will reach the maximum height when $v$ becomes zero.

If $\frac{1}{2}mu^2 - \frac{GMm}{R} \geq 0$, the right-hand side of (i) is greater than zero for all values of $h$. Thus, $\frac{1}{2}mv^2$ never becomes zero. The particle's velocity never reaches zero and so the particle will continue to go farther and farther away from the earth. Thus, the particle will never return to the earth if

$$\frac{1}{2}mu^2 - \frac{GMm}{R} \geq 0$$

or, $$u \geq \sqrt{\frac{2GM}{R}}. \quad \ldots \text{(11.27)}$$

This critical initial velocity is called the *escape velocity*. Putting the values of $G, M$ and $R$, the escape velocity from the earth comes out to be 11·6 km s$^{-1}$. In this we have neglected the effect of other planets, stars and other objects in space. In fact, even if the initial velocity is somewhat less than the escape velocity, the particle may get attracted by some other celestial object and land up there.

Equation (11.27) is valid for any celestial object. For example, if something is thrown up from the surface of the moon, it will never return to the moon if the initial velocity is greater than $\sqrt{\frac{2GM}{R}}$, where $M$ is the mass of the moon and $R$ is the radius of the moon.

***Example 11.10***

*Calculate the escape velocity from the moon. The mass of the moon* $= 7{\cdot}4 \times 10^{22}$ kg *and radius of the moon* = 1740 km.

***Solution*** : The escape velocity is

$$v = \sqrt{\frac{2GM}{R}}$$

$$= \sqrt{\frac{2 \times 6{\cdot}67 \times 10^{-11}\ \text{N–m}^2/\text{kg}^2 \times 7{\cdot}4 \times 10^{22}\ \text{kg}}{1740 \times 10^3\ \text{m}}}$$

$$= 2{\cdot}4\ \text{km s}^{-1}.$$

## 11.14 GRAVITATIONAL BINDING ENERGY

We have seen that if a particle of mass $m$ placed on the earth is given an energy $\frac{1}{2}mu^2 = \frac{GMm}{R}$ or more, it finally escapes from earth. The minimum energy needed to take the particle infinitely away from the earth is called the *binding energy* of the earth–particle system. Thus, the binding energy of the earth–particle system is $\frac{GMm}{R}$.

## 11.15 BLACK HOLES

Consider a spherical body of mass $M$ and radius $R$. Suppose, due to some reason the volume goes on decreasing while the mass remains the same. The escape velocity $\sqrt{\frac{2GM}{R}}$ from such a dense material will be very high. Suppose the radius is so small that

$$\sqrt{\frac{2GM}{R}} \geq c$$

where $c = 3 \times 10^8$ m s$^{-1}$ is the speed of light. The escape velocity for such an object is equal to or greater than the speed of light. This means, anything starting from the object with a speed less than the speed of light will return to the object (neglecting the effect of other objects in space). According to the theory of relativity it is not possible to achieve a velocity greater than $c$ for any material object. Thus, nothing can escape from such a dense material. Such objects are known as *black holes*. A number of such black holes exist in space. Even light cannot escape from a black hole.

## 11.16 INERTIAL AND GRAVITATIONAL MASS

Given two objects $A$ and $B$, how can we determine the ratio of the mass of $A$ to the mass of $B$. One way is to use Newton's second law of motion. If we apply equal forces $F$ on each of the two objects,

$$F = m_A a_A \quad \text{and also} \quad F = m_B a_B.$$

Thus, $$\frac{m_A}{m_B} = \frac{a_B}{a_A}$$

or, $$m_A = \frac{a_B}{a_A} m_B . \quad \ldots \text{(i)}$$

This equation may be used to "define the mass" of an object. Taking the object $B$ to be the standard kilogram ($m_B = 1$ kg), mass of any object may be obtained by measuring their accelerations under equal force and using (i). The mass so defined is called *inertial mass.*

Another way to compare masses of two objects is based on the law of gravitation. The gravitational force exerted by a massive body on an object is proportional to the mass of the object. If $F_A$ and $F_B$ be the forces of attraction on the two objects due to the earth,

$$F_A = \frac{Gm_A M}{R^2} \quad \text{and} \quad F_B = \frac{Gm_B M}{R^2} .$$

Thus, $$\frac{m_A}{m_B} = \frac{F_A}{F_B}$$

or, $$m_A = \frac{F_A}{F_B} m_B . \quad \ldots \text{(ii)}$$

We can use this equation to "define the mass" of an object. If $B$ is a standard unit mass, by measuring the gravitational forces $F_A$ and $F_B$ we can obtain the mass of the object $A$. The mass so defined is called *gravitational mass.* When we measure the mass using a spring balance, we actually measure the gravitational mass.

**Equivalence of Inertial and Gravitational Mass**

The two definitions of mass, described above, are quite independent of each other. There is no obvious reason why the two should be identical. However, they happen to be identical. Several sophisticated experiments have been performed to test this equivalence and none of them has supplied any evidence against it. The general theory of relativity is based on the principle of equivalence of inertial and gravitational mass.

## 11.17 POSSIBLE CHANGES IN THE LAW OF GRAVITATION

There is some indication that the force between two masses is not as described in this chapter. The deviation from the simple law $F = \frac{GMm}{R^2}$ is being taken as an indication of the existence of a fifth interaction besides gravitational, electromagnetic, nuclear and weak. It has been reported (Phys. Rev. Lett. Jan 6, 1986) that the force between two masses may be better represented by

$$F = \frac{G_\infty m_1 m_2}{r^2}\left[1 + \left(1 + \frac{r}{\lambda}\right)\alpha\, e^{-\frac{r}{\lambda}}\right]$$

with $\alpha \approx -0{\cdot}007$ and $\lambda \approx 200$ m. As $\alpha$ is negative, the second term in the square bracket represents a repulsive force. For $r >> 200$ m

$$F = \frac{G_\infty m_1 m_2}{r^2}$$

which is the force operative between the earth and other objects. For $r << 200$ m

$$F = \frac{G_\infty m_1 m_2(1+\alpha)}{r^2} = \frac{G' m_1 m_2}{r^2}$$

where $G' = G_\infty(1 + \alpha)$.

This is the force we measure in a Cavendish-experiment. The value of $G$ for small distances is about 1% less than the value of $G$ for large distances.

### *Worked Out Examples*

**1.** *Three particles A, B and C, each of mass m, are placed in a line with AB = BC = d. Find the gravitational force on a fourth particle P of same mass, placed at a distance d from the particle B on the perpendicular bisector of the line AC.*

***Solution :***

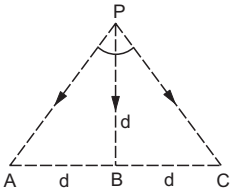



Figure 11-W1

The force at $P$ due to $A$ is

$$F_A = \frac{G m^2}{(AP)^2} = \frac{G m^2}{2 d^2}$$

along $PA$. The force at $P$ due to $C$ is

$$F_C = \frac{G m^2}{(CP)^2} = \frac{G m^2}{2 d^2}$$

along $PC$. The force at $P$ due to $B$ is

$$F_B = \frac{G m^2}{d^2} \quad \text{along } PB.$$

The resultant of $F_A$, $F_B$ and $F_C$ will be along $PB$.

Clearly $\angle APB = \angle BPC = 45°$.

Component of $F_A$ along $PB = F_A \cos 45° = \frac{G m^2}{2\sqrt{2}\, d^2}$.

Component of $F_C$ along $PB = F_C \cos 45° = \frac{G m^2}{2\sqrt{2}\, d^2}$.

Component of $F_B$ along $PB = \frac{G m^2}{d^2}$.

Hence, the resultant of the three forces is

$$\frac{G m^2}{d^2}\left(\frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} + 1\right) = \frac{G m^2}{d^2}\left(1 + \frac{1}{\sqrt{2}}\right) \text{ along } PB.$$

**2.** *Find the distance of a point from the earth's centre where the resultant gravitational field due to the earth and the moon is zero. The mass of the earth is* $6{\cdot}0 \times 10^{24}$ kg *and that of the moon is* $7{\cdot}4 \times 10^{22}$ kg. *The distance between the earth and the moon is* $4{\cdot}0 \times 10^5$ km.

***Solution*** **:** The point must be on the line joining the centres of the earth and the moon and in between them. If the distance of the point from the earth is $x$, the distance from the moon is $(4{\cdot}0 \times 10^5 \text{ km} - x)$. The magnitude of the gravitational field due to the earth is

$$E_1 = \frac{GM_e}{x^2} = \frac{G \times 6 \times 10^{24} \text{ kg}}{x^2}$$

and magnitude of the gravitational field due to the moon is

$$E_2 = \frac{GM_m}{(4{\cdot}0 \times 10^5 \text{ km} - x)^2} = \frac{G \times 7{\cdot}4 \times 10^{22} \text{ kg}}{(4{\cdot}0 \times 10^5 \text{ km} - x)^2}.$$

These fields are in opposite directions. For the resultant field to be zero $E_1 = E_2$,

or, $$\frac{6 \times 10^{24} \text{ kg}}{x^2} = \frac{7{\cdot}4 \times 10^{22} \text{ kg}}{(4{\cdot}0 \times 10^5 \text{ km} - x)^2}$$

or, $$\frac{x}{4{\cdot}0 \times 10^5 \text{ km} - x} = \sqrt{\frac{6 \times 10^{24}}{7{\cdot}4 \times 10^{22}}} = 9$$

or, $$x = 3{\cdot}6 \times 10^5 \text{ km}.$$

**3.** *Two particles of equal mass go round a circle of radius* $R$ *under the action of their mutual gravitational attraction. Find the speed of each particle.*

***Solution*** **:** The particles will always remain diametrically opposite so that the force on each particle will be directed along the radius. Consider the motion of one of the particles. The force on the particle is $F = \frac{G m^2}{4R^2}$. If the speed is $v$, its acceleration is $v^2/R$.

Thus, by Newton's law,

$$\frac{G m^2}{4R^2} = \frac{m v^2}{R}$$

or, $$v = \sqrt{\frac{G m}{4R}}.$$

**4.** *Two particles* $A$ *and* $B$ *of masses* 1 kg *and* 2 kg *respectively are kept* 1 m *apart and are released to move under mutual attraction. Find the speed of* $A$ *when that of* $B$ *is* 3·6 cm/hour. *What is the separation between the particles at this instant* ?

***Solution*** **:** The linear momentum of the pair $A + B$ is zero initially. As only mutual attraction is taken into account, which is internal when $A + B$ is taken as the system, the linear momentum will remain zero. The particles move in opposite directions. If the speed of $A$ is $v$ when the speed of $B$ is 3·6 cm/hour $= 10^{-5}$ m s$^{-1}$,

$$(1 \text{ kg})\, v = (2 \text{ kg})\, (10^{-5} \text{ m s}^{-1})$$

or, $$v = 2 \times 10^{-5} \text{ m s}^{-1}.$$

The potential energy of the pair is $-\frac{G m_A m_B}{R}$ with usual symbols. Initial potential energy

$$= -\frac{6{\cdot}67 \times 10^{-11} \text{ N-m}^2/\text{kg}^2 \times 2 \text{ kg} \times 1 \text{ kg}}{1 \text{ m}}$$

$$= -13{\cdot}34 \times 10^{-11} \text{ J}.$$

If the separation at the given instant is $d$, using conservation of energy,

$$-13{\cdot}34 \times 10^{-11} \text{ J} + 0$$

$$= -\frac{13{\cdot}34 \times 10^{-11} \text{ J-m}}{d} + \frac{1}{2}(2 \text{ kg})\, (10^{-5} \text{ m s}^{-1})^2$$

$$+ \frac{1}{2}(1 \text{ kg})\, (2 \times 10^{-5} \text{ m s}^{-1})^2$$

Solving this, $d$ = 0·31 m.

**5.** *The gravitational field in a region is given by* $\vec{E} = (10 \text{ N kg}^{-1})\, (\vec{i} + \vec{j})$. *Find the work done by an external agent to slowly shift a particle of mass* 2 kg *from the point* (0,0) *to a point* (5 m, 4 m).

***Solution*** **:** As the particle is slowly shifted, its kinetic energy remains zero. The total work done on the particle is thus zero. The work done by the external agent should be negative of the work done by the gravitational field. The work done by the field is

$$\int_i^f \vec{F} \cdot d\vec{r}$$

Consider figure (11-W2). Suppose the particle is taken from $O$ to $A$ and then from $A$ to $B$. The force on the particle is

$$\vec{F} = m\vec{E} = (2 \text{ kg})\, (10 \text{ N kg}^{-1})\, (\vec{i} + \vec{j}) = (20 \text{ N})\, (\vec{i} + \vec{j}).$$

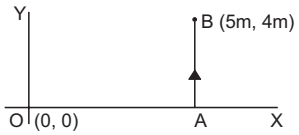



Figure 11-W2

The work done by the field during the displacement $OA$ is

$$W_1 = \int_0^{5m} F_x\, dx$$

$$= \int_0^{5m} (20\text{ N})\, dx = 20\text{ N} \times 5\text{ m} = 100\text{ J}.$$

Similarly, the work done in displacement $AB$ is

$$W_2 = \int_0^{4m} F_y\, dy = \int_0^{4m} (20\text{ N})\, dy$$

$$= (20\text{ N})\,(4\text{ m}) = 80\text{ J}.$$

Thus, the total work done by the field, as the particle is shifted from $O$ to $B$, is 180 J.

The work done by the external agent is –180 J.

Note that the work is independent of the path so that we can choose any path convenient to us from $O$ to $B$.

**6.** *A uniform solid sphere of mass $M$ and radius $a$ is surrounded symmetrically by a uniform thin spherical shell of equal mass and radius $2a$. Find the gravitational field at a distance* (a) $\frac{3}{2}a$ *from the centre,* (b) $\frac{5}{2}a$ *from the centre.*

***Solution*** **:**

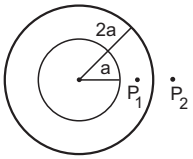



Figure 11-W3

Figure (11-W3) shows the situation. The point $P_1$ is at a distance $\frac{3}{2}a$ from the centre and $P_2$ is at a distance $\frac{5}{2}a$ from the centre. As $P_1$ is inside the cavity of the thin spherical shell, the field here due to the shell is zero. The field due to the solid sphere is

$$E = \frac{GM}{\left(\frac{3}{2}a\right)^2} = \frac{4\,GM}{9\,a^2}.$$

This is also the resultant field. The direction is towards the centre. The point $P_2$ is outside the sphere as well as the shell. Both may be replaced by single particles of the same mass at the centre. The field due to each of them is

$$E' = \frac{GM}{\left(\frac{5}{2}a\right)^2} = \frac{4\,GM}{25\,a^2}.$$

The resultant field is $E = 2\,E' = \frac{8\,GM}{25\,a^2}$ towards the centre.

**7.** *The density inside a solid sphere of radius $a$ is given by $\rho = \rho_0\, a/r$, where $\rho_0$ is the density at the surface and $r$ denotes the distance from the centre. Find the gravitational field due to this sphere at a distance $2a$ from its centre.*

***Solution*** **:** The field is required at a point outside the sphere. Dividing the sphere in concentric shells, each shell can be replaced by a point particle at its centre having mass equal to the mass of the shell. Thus, the whole sphere can be replaced by a point particle at its centre having mass equal to the mass of the given sphere. If the mass of the sphere is $M$, the gravitational field at the given point is

$$E = \frac{GM}{(2a)^2} = \frac{GM}{4a^2}. \qquad \ldots \text{ (i)}$$

The mass $M$ may be calculated as follows. Consider a concentric shell of radius $r$ and thickness $dr$. Its volume is

$$dV = (4\pi r^2)\, dr$$

and its mass is

$$dM = \rho dV = \left(\rho_0 \frac{a}{r}\right)(4\pi r^2 dr)$$

$$= 4\pi\rho_0\, ar\, dr.$$

The mass of the whole sphere is

$$M = \int_0^a 4\pi\rho_0\, ar\, dr$$

$$= 2\pi\rho_0 a^3.$$

Thus, by (i) the gravitational field is

$$E = \frac{2\pi G\rho_0 a^3}{4a^2} = \frac{1}{2}\pi G\rho_0 a.$$

**8.** *A uniform ring of mass $m$ and radius $a$ is placed directly above a uniform sphere of mass $M$ and of equal radius. The centre of the ring is at a distance $\sqrt{3}\,a$ from the centre of the sphere. Find the gravitational force exerted by the sphere on the ring.*

***Solution*** **:** The gravitational field at any point on the ring due to the sphere is equal to the field due to a single particle of mass $M$ placed at the centre of the sphere. Thus, the force on the ring due to the sphere is also equal to the force on it by a particle of mass $M$ placed at this point. By Newton's third law it is equal to the force on the particle by the ring. Now the gravitational field due to the ring at a distance $d = \sqrt{3}\,a$ on its axis is

$$E = \frac{Gmd}{(a^2 + d^2)^{3/2}} = \frac{\sqrt{3}\,Gm}{8a^2}.$$

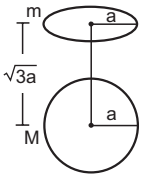



Figure 11-W4

The force on a particle of mass $M$ placed here is

$$F = ME = \frac{\sqrt{3}GMm}{8a^2}.$$

This is also the force due to the sphere on the ring.

**9.** *A particle is fired vertically upward with a speed of* $9{\cdot}8\ \text{km s}^{-1}$. *Find the maximum height attained by the particle. Radius of earth* = 6400 km *and g at the surface* $= 9{\cdot}8\ \text{m s}^{-2}$. *Consider only earth's gravitation.*

***Solution*** : At the surface of the earth, the potential energy of the earth–particle system is $-\frac{GMm}{R}$ with usual symbols. The kinetic energy is $\frac{1}{2}mv_0^2$ where $v_0 = 9{\cdot}8\ \text{km s}^{-1}$. At the maximum height the kinetic energy is zero. If the maximum height reached is $H$, the potential energy of the earth–particle system at this instant is $-\frac{GMm}{R+H}$. Using conservation of energy,

$$-\frac{GMm}{R} + \frac{1}{2}mv_0^2 = -\frac{GMm}{R+H}.$$

Writing $GM = gR^2$ and dividing by m,

$$-gR + \frac{v_0^2}{2} = \frac{-gR^2}{R+H}$$

or, $$\frac{R^2}{R+H} = R - \frac{v_0^2}{2g}$$

or, $$R + H = \frac{R^2}{R - \frac{v_0^2}{2g}}.$$

Putting the values of $R$, $v_0$ and $g$ on the right side,

$$R + H = \frac{(6400\ \text{km})^2}{6400\ \text{km} - \frac{(9{\cdot}8\ \text{km s}^{-1})^2}{2 \times 9{\cdot}8\ \text{m s}^{-2}}} = \frac{(6400\ \text{km})^2}{1500\ \text{km}} = 27300\ \text{km}$$

or, $$H = (27300 - 6400)\ \text{km} = 20900\ \text{km}.$$

**10.** *A particle hanging from a spring stretches it by* 1 cm *at earth's surface. How much will the same particle stretch the spring at a place* 800 km *above the earth's surface* ? *Radius of the earth* = 6400 km.

***Solution*** : Suppose the mass of the particle is $m$ and the spring constant of the spring is $k$. The acceleration due to gravity at earth's surface is $g = \frac{GM}{R^2}$ with usual symbols. The extension in the spring is $mg/k$.

Hence, $$1\ \text{cm} = \frac{GMm}{kR^2}. \quad \ldots \text{(i)}$$

At a height $h = 800$ km, the extension is given by

$$x = \frac{GMm}{k(R+h)^2}. \quad \ldots \text{(ii)}$$

By (i) and (ii), $$\frac{x}{1\ \text{cm}} = \frac{R^2}{(R+h)^2}$$

$$\frac{(6400\ \text{km})^2}{(7200\ \text{km})^2} = 0{\cdot}79.$$

Hence, $$x = 0{\cdot}79\ \text{cm}.$$

**11.** *A simple pendulum has a time period exactly* 2 s *when used in a laboratory at north pole. What will be the time period if the same pendulum is used in a laboratory at equator* ? *Account for the earth's rotation only. Take* $g = \frac{GM}{R^2} = 9{\cdot}8\ \text{m s}^{-2}$ *and radius of earth* = 6400 km.

***Solution*** : Consider the pendulum in its mean position at the north pole. As the pole is on the axis of rotation, the bob is in equilibrium. Hence in the mean position, the tension $T$ is balanced by earth's attraction. Thus, $T = \frac{GMm}{R^2} = mg$. The time period $t$ is

$$t = 2\pi\sqrt{\frac{l}{T/m}} = 2\pi\sqrt{\frac{l}{g}}. \quad \ldots \text{(1)}$$

At equator, the lab and the pendulum rotate with the earth at angular velocity $\omega = \frac{2\pi\ \text{radian}}{24\ \text{hour}}$ in a circle of radius equal to 6400 km. Using Newton's second law,

$$\frac{GMm}{R^2} - T' = m\omega^2 R \quad \text{or, } T' = m(g - \omega^2 R)$$

where $T'$ is the tension in the string.

The time period will be

$$t' = 2\pi\sqrt{\frac{l}{(T'/m)}} = 2\pi\sqrt{\frac{l}{g - \omega^2 R}}. \quad \ldots \text{(ii)}$$

By (i) and (ii)

$$\frac{t'}{t} = \sqrt{\frac{g}{g - \omega^2 R}} = \left(1 - \frac{\omega^2 R}{g}\right)^{-1/2}$$

or, $$t' \approx t\left(1 + \frac{\omega^2 R}{2g}\right).$$

Putting the values, $t' = 2{\cdot}004$ seconds.

**12.** *A satellite is to revolve round the earth in a circle of radius* 8000 km. *With what speed should this satellite be projected into orbit* ? *What will be the time period* ? *Take g at the surface* $= 9{\cdot}8\ \text{m s}^{-2}$ *and radius of the earth* = 6400 km.

***Solution*** : Suppose, the speed of the satellite is $v$. The acceleration of the satellite is $v^2/r$, where $r$ is the radius of the orbit. The force on the satellite is $\frac{GMm}{r^2}$ with usual symbols. Using Newton's second law,

$$\frac{GMm}{r^2} = m\frac{v^2}{r}$$

or, $$v^2 = \frac{GM}{r} = \frac{gR^2}{r} = \frac{(9{\cdot}8 \text{ m s}^{-2})\,(6400 \text{ km})^2}{(8000 \text{ km})}$$

giving $v = 7{\cdot}08 \text{ km s}^{-1}$.

The time period is $\frac{2\pi r}{v} = \frac{2\pi\,(8000 \text{ km})}{(7{\cdot}08 \text{ km s}^{-1})} \approx 118$ minutes.

**13.** *Two satellites $S_1$ and $S_2$ revolve round a planet in coplanar circular orbits in the same sense. Their periods of revolution are* 1 h *and* 8 h *respectively. The radius of the orbit of $S_1$ is* $10^4$ km. *When $S_2$ is closest to $S_1$, find (a) the speed of $S_2$ relative to $S_1$ and (b) the angular speed of $S_2$ as observed by an astronaut in $S_1$.*

***Solution*** **:** Let the mass of the planet be $M$, that of $S_1$ be $m_1$ and of $S_2$ be $m_2$. Let the radius of the orbit of $S_1$ be $R_1$ ( = $10^4$ km) and of $S_2$ be $R_2$.

Let $v_1$ and $v_2$ be the linear speeds of $S_1$ and $S_2$ with respect to the planet. Figure (11-W5) shows the situation.

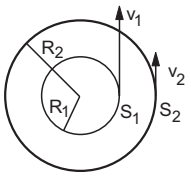



Figure 11-W5

As the square of the time period is proportional to the cube of the radius,

$$\left(\frac{R_2}{R_1}\right)^3 = \left(\frac{T_2}{T_1}\right)^2 = \left(\frac{8 \text{ h}}{1 \text{ h}}\right)^2 = 64$$

or, $$\frac{R_2}{R_1} = 4$$

or, $$R_2 = 4R_1 = 4 \times 10^4 \text{ km}.$$

Now the time period of $S_1$ is 1 h. So,

$$\frac{2\pi R_1}{v_1} = 1 \text{ h}$$

or, $$v_1 = \frac{2\pi R_1}{1 \text{ h}} = 2\pi \times 10^4 \text{ km h}^{-1}$$

similarly, $$v_2 = \frac{2\pi R_2}{8\,h} = \pi \times 10^4 \text{ km h}^{-1}.$$

(a) At the closest separation, they are moving in the same direction. Hence the speed of $S_2$ with respect to $S_1$ is $|v_2 - v_1| = \pi \times 10^4 \text{ km h}^{-1}$.

(b) As seen from $S_1$, the satellite $S_2$ is at a distance $R_2 - R_1 = 3 \times 10^4$ km at the closest separation. Also, it is moving at $\pi \times 10^4 \text{ km h}^{-1}$ in a direction perpendicular to the line joining them. Thus, the angular speed of $S_2$ as observed by $S_1$ is

$$\omega = \frac{\pi \times 10^4 \text{ km h}^{-1}}{3 \times 10^4 \text{ km}} = \frac{\pi}{3} \text{ rad h}^{-1}.$$

□

## QUESTIONS FOR SHORT ANSWERS

**1.** Can two particles be in equilibrium under the action of their mutual gravitational force ? Can three particles be ? Can one of the three particles be ?

**2.** Is there any meaning of "Weight of the earth" ?

**3.** If heavier bodies are attracted more strongly by the earth, why don't they fall faster than the lighter bodies ?

**4.** Can you think of two particles which do not exert gravitational force on each other ?

**5.** The earth revolves round the sun because the sun attracts the earth. The sun also attracts the moon and this force is about twice as large as the attraction of the earth on the moon. Why does the moon not revolve round the sun ? Or does it ?

**6.** At noon, the sun and the earth pull the objects on the earth's surface in opposite directions. At midnight, the sun and the earth pull these objects in same direction. Is the weight of an object, as measured by a spring balance on the earth's surface, more at midnight as compared to its weight at noon ?

**7.** An apple falls from a tree. An insect in the apple finds that the earth is falling towards it with an acceleration $g$. Who exerts the force needed to accelerate the earth with this acceleration $g$ ?

**8.** Suppose the gravitational potential due to a small system is $k/r^2$ at a distance $r$ from it. What will be the gravitational field ? Can you think of any such system ? What happens if there were negative masses ?

**9.** The gravitational potential energy of a two-particle system is derived in this chapter as $U = -\frac{Gm_1m_2}{r}$. Does it follow from this equation that the potential energy for $r = \infty$ must be zero ? Can we choose the potential energy for $r = \infty$ to be 20 J and still use this formula ? If no, what formula should be used to calculate the gravitational potential energy at separation $r$ ?

**10.** The weight of an object is more at the poles than at the equator. Is it beneficial to purchase goods at equator and sell them at the pole ? Does it matter whether a spring balance is used or an equal-beam balance is used ?

**11.** The weight of a body at the poles is greater than the weight at the equator. Is it the actual weight or the apparent weight we are talking about ? Does your

answer depend on whether only the earth's rotation is taken into account or the flattening of the earth at the poles is also taken into account ?

**12.** If the radius of the earth decreases by 1% without changing its mass, will the acceleration due to gravity at the surface of the earth increase or decrease ? If so, by what per cent ?

**13.** A nut becomes loose and gets detached from a satellite revolving around the earth. Will it land on the earth ? If yes, where will it land ? If no, how can an astronaut make it land on the earth ?

**14.** Is it necessary for the plane of the orbit of a satellite to pass through the centre of the earth ?

**15.** Consider earth satellites in circular orbits. A geostationary satellite must be at a height of about 36000 km from the earth's surface. Will any satellite moving at this height be a geostationary satellite ? Will any satellite moving at this height have a time period of 24 hours ?

**16.** No part of India is situated on the equator. Is it possible to have a geostationary satellite which always remains over New Delhi ?

**17.** As the earth rotates about its axis, a person living in his house at the equator goes in a circular orbit of radius equal to the radius of the earth. Why does he/she not feel weightless as a satellite passenger does ?

**18.** Two satellites going in equatorial plane have almost same radii. As seen from the earth one moves from east to west and the other from west to east. Will they have the same time period as seen from the earth ? If not, which one will have less time period ?

**19.** A spacecraft consumes more fuel in going from the earth to the moon than it takes for a return trip. Comment on this statement.

## OBJECTIVE I

**1.** The acceleration of moon with respect to earth is $0{\cdot}0027\ \text{m s}^{-2}$ and the acceleration of an apple falling on earth's surface is about $10\ \text{m s}^{-2}$. Assume that the radius of the moon is one fourth of the earth's radius. If the moon is stopped for an instant and then released, it will fall towards the earth. The initial acceleration of the moon towards the earth will be
(a) $10\ \text{m s}^{-2}$ (b) $0{\cdot}0027\ \text{m s}^{-2}$ (c) $6{\cdot}4\ \text{m s}^{-2}$ (d) $5{\cdot}0\ \text{m s}^{-2}$.

**2.** The acceleration of the moon just before it strikes the earth in the previous question is
(a) $10\ \text{m s}^{-2}$ (b) $0{\cdot}0027\ \text{m s}^{-2}$ (c) $6{\cdot}4\ \text{m s}^{-2}$ (d) $5{\cdot}0\ \text{m s}^{-2}$

**3.** Suppose, the acceleration due to gravity at the earth's surface is $10\ \text{m s}^{-2}$ and at the surface of Mars it is $4{\cdot}0\ \text{m s}^{-2}$. A 60 kg passenger goes from the earth to the Mars in a spaceship moving with a constant velocity. Neglect all other objects in the sky. Which part of figure (11-Q1) best represents the weight (net gravitational force) of the passenger as a function of time ?
(a) *A* (b) *B* (c) *C* (d) *D*.

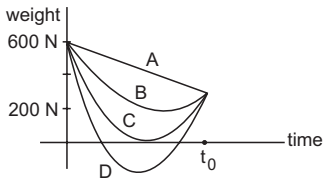



Figure 11-Q2

**4.** Consider a planet in some solar system which has a mass double the mass of the earth and density equal to the average density of the earth. An object weighing $W$ on the earth will weigh
(a) $W$ (b) $2W$ (c) $W/2$ (d) $2^{1/3}W$ at the planet.

**5.** If the acceleration due to gravity at the surface of the earth is $g$, the work done in slowly lifting a body of mass $m$ from the earth's surface to a height $R$ equal to the radius of the earth is
(a) $\frac{1}{2}mgR$ (b) $2mgR$ (c) $mgR$ (d) $\frac{1}{4}mgR$.

**6.** A person brings a mass of 1 kg from infinity to a point *A*. Initially the mass was at rest but it moves at a speed of $2\ \text{m s}^{-1}$ as it reaches *A*. The work done by the person on the mass is –3 J. The potential at *A* is
(a) $-3\ \text{J kg}^{-1}$ (b) $-2\ \text{J kg}^{-1}$ (c) $-5\ \text{J kg}^{-1}$ (d) none of these.

**7.** Let $V$ and $E$ be the gravitational potential and gravitational field at a distance $r$ from the centre of a uniform spherical shell. Consider the following two statements :
(A) The plot of $V$ against $r$ is discontinuous.
(B) The plot of $E$ against $r$ is discontinuous.
(a) Both *A* and *B* are correct.
(b) *A* is correct but *B* is wrong.
(c) *B* is correct but *A* is wrong.
(d) Both *A* and *B* are wrong.

**8.** Let $V$ and $E$ represent the gravitational potential and field at a distance $r$ from the centre of a uniform solid sphere. Consider the two statements:
(A) the plot of $V$ against $r$ is discontinuous.
(B) The plot of $E$ against $r$ is discontinuous.
(a) Both *A* and *B* are correct.
(b) *A* is correct but *B* is wrong.
(c) *B* is correct but *A* is wrong.
(d) Both *A* and *B* are wrong.

**9.** Take the effect of bulging of earth and its rotation in account. Consider the following statements:
(A) There are points outside the earth where the value of $g$ is equal to its value at the equator.
(B) There are points outside the earth where the value of $g$ is equal to its value at the poles.

(a) Both $A$ and $B$ are correct.
(b) $A$ is correct but $B$ is wrong.
(c) $B$ is correct but $A$ is wrong.
(d) Both $A$ and $B$ are wrong.

**10.** The time period of an earth-satellite in circular orbit is independent of
(a) the mass of the satellite (b) radius of the orbit
(c) none of them (d) both of them.

**11.** The magnitude of gravitational potential energy of the moon–earth system is $U$ with zero potential energy at infinite separation. The kinetic energy of the moon with respect to the earth is $K$.
(a) $U < K$ (b) $U > K$ (c) $U = K$.

**12.** Figure (11-Q2) shows the elliptical path of a planet about the sun. The two shaded parts have equal area. If $t_1$ and $t_2$ be the time taken by the planet to go from $a$ to $b$ and from $c$ to $d$ respectively,
(a) $t_1 < t_2$ (b) $t_1 = t_2$ (c) $t_1 > t_2$
(d) insufficient information to deduce the relation between $t_1$ and $t_2$.

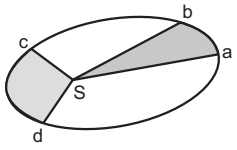



Figure 11-Q2

**13.** A person sitting in a chair in a satellite feels weightless because
(a) the earth does not attract the objects in a satellite
(b) the normal force by the chair on the person balances the earth's attraction
(c) the normal force is zero
(d) the person in satellite is not accelerated.

**14.** A body is suspended from a spring balance kept in a satellite. The reading of the balance is $W_1$ when the satellite goes in an orbit of radius $R$ and is $W_2$ when it goes in an orbit of radius $2R$.
(a) $W_1 = W_2$ (b) $W_1 < W_2$ (c) $W_1 > W_2$ (d) $W_1 \neq W_2$.

**15.** The kinetic energy needed to project a body of mass $m$ from the earth's surface to infinity is
(a) $\frac{1}{4}mgR$ (b) $\frac{1}{2}mgR$ (c) $mgR$ (d) $2\,mgR$.

**16.** A particle is kept at rest at a distance $R$ (earth's radius) above the earth's surface. The minimum speed with which it should be projected so that it does not return is
(a) $\sqrt{\frac{GM}{4R}}$ (b) $\sqrt{\frac{GM}{2R}}$ (c) $\sqrt{\frac{GM}{R}}$ (d) $\sqrt{\frac{2GM}{R}}$.

**17.** A satellite is orbiting the earth close to its surface. A particle is to be projected from the satellite to just escape from the earth. The escape speed from the earth is $v_e$. Its speed with respect to the satellite
(a) will be less than $v_e$
(b) will be more than $v_e$
(c) will be equal to $v_e$
(d) will depend on direction of projection.

## OBJECTIVE II

**1.** Let $V$ and $E$ denote the gravitational potential and gravitational field at a point. It is possible to have
(a) $V = 0$ and $E = 0$ (b) $V = 0$ and $E \neq 0$
(c) $V \neq 0$ and $E = 0$ (d) $V \neq 0$ and $E \neq 0$.

**2.** Inside a uniform spherical shell
(a) the gravitational potential is zero
(b) the gravitational field is zero
(c) the gravitational potential is same everywhere
(d) the gravitational field is same everywhere.

**3.** A uniform spherical shell gradually shrinks maintaining its shape. The gravitational potential at the centre
(a) increases (b) decreases
(c) remains constant (d) oscillates.

**4.** Consider a planet moving in an elliptical orbit round the sun. The work done on the planet by the gravitational force of the sun
(a) is zero in any small part of the orbit
(b) is zero in some parts of the orbit
(c) is zero in one complete revolution
(d) is zero in no part of the motion.

**5.** Two satellites $A$ and $B$ move round the earth in the same orbit. The mass of $B$ is twice the mass of $A$.
(a) Speeds of $A$ and $B$ are equal.
(b) The potential energy of earth+$A$ is same as that of earth+$B$.
(c) The kinetic energy of $A$ and $B$ are equal.
(d) The total energy of earth+$A$ is same as that of earth+$B$.

**6.** Which of the following quantities remain constant in a planetary motion (consider elliptical orbits) as seen from the sun ?
(a) Speed (b) Angular speed
(c) Kinetic Energy (d) Angular momentum.

## EXERCISES

**1.** Two spherical balls of mass 10 kg each are placed 10 cm apart. Find the gravitational force of attraction between them.

**2.** Four particles having masses $m$, $2m$, $3m$ and $4m$ are placed at the four corners of a square of edge $a$. Find

the gravitational force acting on a particle of mass $m$ placed at the centre.

**3.** Three equal masses $m$ are placed at the three corners of an equilateral triangle of side $a$. Find the force exerted by this system on another particle of mass $m$ placed at (a) the mid-point of a side, (b) at the centre of the triangle.

**4.** Three uniform spheres each having a mass $M$ and radius $a$ are kept in such a way that each touches the other two. Find the magnitude of the gravitational force on any of the spheres due to the other two.

**5.** Four particles of equal masses $M$ move along a circle of radius $R$ under the action of their mutual gravitational attraction. Find the speed of each particle.

**6.** Find the acceleration due to gravity of the moon at a point 1000 km above the moon's surface. The mass of the moon is $7{\cdot}4 \times 10^{22}$ kg and its radius is 1740 km.

**7.** Two small bodies of masses 10 kg and 20 kg are kept a distance 1·0 m apart and released. Assuming that only mutual gravitational forces are acting, find the speeds of the particles when the separation decreases to 0·5 m.

**8.** A semicircular wire has a length $L$ and mass $M$. A particle of mass $m$ is placed at the centre of the circle. Find the gravitational attraction on the particle due to the wire.

**9.** Derive an expression for the gravitational field due to a uniform rod of length $L$ and mass $M$ at a point on its perpendicular bisector at a distance $d$ from the centre.

**10.** Two concentric spherical shells have masses $M_1, M_2$ and radii $R_1, R_2$ $(R_1 < R_2)$. What is the force exerted by this system on a particle of mass $m$ if it is placed at a distance $(R_1 + R_2)/2$ from the centre ?

**11.** A tunnel is dug along a diameter of the earth. Find the force on a particle of mass $m$ placed in the tunnel at a distance $x$ from the centre.

**12.** A tunnel is dug along a chord of the earth at a perpendicular distance $R/2$ from the earth's centre. The wall of the tunnel may be assumed to be frictionless. Find the force exerted by the wall on a particle of mass $m$ when it is at a distance $x$ from the centre of the tunnel.

**13.** A solid sphere of mass $m$ and radius $r$ is placed inside a hollow thin spherical shell of mass $M$ and radius $R$ as shown in figure (11-E1). A particle of mass $m'$ is placed on the line joining the two centres at a distance $x$ from the point of contact of the sphere and the shell. Find the magnitude of the resultant gravitational force on this particle due to the sphere and the shell if (a) $r < x < 2r$, (b) $2r < x < 2R$ and (c) $x > 2R$.

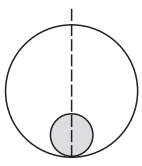


Figure 11-E1

**14.** A uniform metal sphere of radius $a$ and mass $M$ is surrounded by a thin uniform spherical shell of equal mass and radius $4a$ (figure 11-E2). The centre of the shell falls on the surface of the inner sphere. Find the gravitational field at the points $P_1$ and $P_2$ shown in the figure.

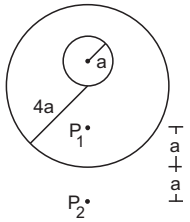



Figure 11-E2

**15.** A thin spherical shell having uniform density is cut in two parts by a plane and kept separated as shown in figure (11-E3). The point $A$ is the centre of the plane section of the first part and $B$ is the centre of the plane section of the second part. Show that the gravitational field at $A$ due to the first part is equal in magnitude to the gravitational field at $B$ due to the second part.

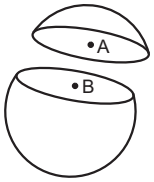



Figure 11-E3

**16.** Two small bodies of masses 2.00 kg and 4.00 kg are kept at rest at a separation of 2·0 m. Where should a particle of mass 0.10 kg be placed to experience no net gravitational force from these bodies ? The particle is placed at this point. What is the gravitational potential energy of the system of three particles with usual reference level ?

**17.** Three particles of mass $m$ each are placed at the three corners of an equilateral triangle of side $a$. Find the work which should be done on this system to increase the sides of the triangle to $2a$.

**18.** A particle of mass 100 g is kept on the surface of a uniform sphere of mass 10 kg and radius 10 cm. Find the work to be done against the gravitational force between them to take the particle away from the sphere.

**19.** The gravitational field in a region is given by $\vec{E} = (5 \text{ N kg}^{-1})\,\vec{i} + (12 \text{ N kg}^{-1})\,\vec{j}$. (a) Find the magnitude of the gravitational force acting on a particle of mass 2 kg placed at the origin. (b) Find the potential at the points (12 m, 0) and (0, 5 m) if the potential at the origin is taken to be zero. (c) Find the change in gravitational potential energy if a particle of mass 2 kg is taken from the origin to the point (12 m, 5 m). (d) Find the change in potential energy if the particle is taken from (12 m, 0) to (0, 5 m).

**20.** The gravitational potential in a region is given by $V = 20\ \text{N kg}^{-1}\ (x + y)$. (a) Show that the equation is dimensionally correct. (b) Find the gravitational field at the point $(x, y)$. Leave your answer in terms of the unit vectors $\vec{i}, \vec{j}, \vec{k}$. (c) Calculate the magnitude of the gravitational force on a particle of mass 500 g placed at the origin.

**21.** The gravitational field in a region is given by $\vec{E} = (2\vec{i} + 3\vec{j})\ \text{N kg}^{-1}$. Show that no work is done by the gravitational field when a particle is moved on the line $3y + 2x = 5$.

**[Hint :** If a line $y = mx + c$ makes angle $\theta$ with the $X$-axis, $m = \tan\theta$.]

**22.** Find the height over the earth's surface at which the weight of a body becomes half of its value at the surface.

**23.** What is the acceleration due to gravity on the top of Mount Everest ? Mount Everest is the highest mountain peak of the world at the height of 8848 m. The value at sea level is $9{\cdot}80\ \text{m s}^{-2}$.

**24.** Find the acceleration due to gravity in a mine of depth 640 m if the value at the surface is $9{\cdot}800\ \text{m s}^{-2}$. The radius of the earth is 6400 km.

**25.** A body is weighed by a spring balance to be 1·000 kg at the north pole. How much will it weigh at the equator ? Account for the earth's rotation only.

**26.** A body stretches a spring by a particular length at the earth's surface at equator. At what height above the south pole will it stretch the same spring by the same length ? Assume the earth to be spherical.

**27.** At what rate should the earth rotate so that the apparent $g$ at the equator becomes zero ? What will be the length of the day in this situation ?

**28.** A pendulum having a bob of mass $m$ is hanging in a ship sailing along the equator from east to west. When the ship is stationary with respect to water the tension in the string is $T_0$. (a) Find the speed of the ship due to rotation of the earth about its axis. (b) Find the difference between $T_0$ and the earth's attraction on the bob. (c) If the ship sails at speed $v$, what is the tension in the string ? Angular speed of earth's rotation is $\omega$ and radius of the earth is $R$.

**29.** The time taken by Mars to revolve round the sun is 1·88 years. Find the ratio of average distance between Mars and the sun to that between the earth and the sun.

**30.** The moon takes about 27·3 days to revolve round the earth in a nearly circular orbit of radius $3{\cdot}84 \times 10^5$ km. Calculate the mass of the earth from these data.

**31.** A Mars satellite moving in an orbit of radius $9{\cdot}4 \times 10^3$ km takes 27540 s to complete one revolution. Calculate the mass of Mars.

**32.** A satellite of mass 1000 kg is supposed to orbit the earth at a height of 2000 km above the earth's surface. Find (a) its speed in the orbit, (b) its kinetic energy, (c) the potential energy of the earth–satellite system and (d) its time period. Mass of the earth $= 6 \times 10^{24}$ kg.

**33.** (a) Find the radius of the circular orbit of a satellite moving with an angular speed equal to the angular speed of earth's rotation. (b) If the satellite is directly above the north pole at some instant, find the time it takes to come over the equatorial plane. Mass of the earth $= 6 \times 10^{24}$ kg.

**34.** What is the true weight of an object in a geostationary satellite that weighed exactly 10·0 N at the north pole ?

**35.** The radius of a planet is $R_1$ and a satellite revolves round it in a circle of radius $R_2$. The time period of revolution is $T$. Find the acceleration due to the gravitation of the planet at its surface.

**36.** Find the minimum colatitude which can directly receive a signal from a geostationary satellite.

**37.** A particle is fired vertically upward from earth's surface and it goes up to a maximum height of 6400 km. Find the initial speed of particle.

**38.** A particle is fired vertically upward with a speed of $15\ \text{km s}^{-1}$. With what speed will it move in intersteller space. Assume only earth's gravitational field.

**39.** A mass of $6 \times 10^{24}$ kg (equal to the mass of the earth) is to be compressed in a sphere in such a way that the escape velocity from its surface is $3 \times 10^8\ \text{m s}^{-1}$. What should be the radius of the sphere ?

□

## ANSWERS

### OBJECTIVE I

| | | | | | |
|---|---|---|---|---|---|
| 1. (b) | 2. (c) | 3. (c) | 4. (d) | 5. (a) | 6. (c) |
| 7. (c) | 8. (d) | 9. (b) | 10. (a) | 11. (b) | 12. (b) |
| 13. (c) | 14. (a) | 15. (c) | 16. (c) | 17. (d) | |

### OBJECTIVE II

1. all 2. (b), (c), (d) 3. (b) 4. (b), (c) 5. (a) 6. (d)

### EXERCISES

1. $6{\cdot}67 \times 10^{-7}$ N
2. $\dfrac{4\sqrt{2}\,Gm^2}{a^2}$
3. (a) $\dfrac{4Gm^2}{3a^2}$, (b) zero
4. $\dfrac{\sqrt{3}\,GM^2}{4a^2}$

5. $\sqrt{\frac{GM}{R}\left(\frac{2\sqrt{2}+1}{4}\right)}$
6. $0{\cdot}65\ \text{m s}^{-2}$
7. $4{\cdot}2\times10^{-5}\ \text{m s}^{-1}$ and $2{\cdot}1\times10^{-5}\ \text{m s}^{-1}$
8. $\frac{2\pi GMm}{L^2}$
9. $\frac{2Gm}{d\sqrt{L^2+4d^2}}$
10. $\frac{4GM_1m}{(R_1+R_2)^2}$
11. $\frac{GM_em}{R^3}x$
12. $\frac{GM_em}{2R^2}$
13. (a) $\frac{Gmm'(x-r)}{r^3}$ (b) $\frac{Gmm'}{(x-r)^2}$ (c) $\frac{GMm'}{(x-R)^2}+\frac{Gmm'}{(x-r)^2}$
14. $\frac{GM}{16a^2}, \frac{61GM}{900a^2}$
16. 0·83 m from the 2.00 kg body towards the other body, $-3{\cdot}06\times10^{-10}$ J
17. $\frac{3Gm^2}{2a}$
18. $6{\cdot}67\times10^{-10}$ J
19. (a) 26 N (b) $-60\ \text{J kg}^{-1}$, $-60\ \text{J kg}^{-1}$ (c) $-240$ J (d) zero
20. (b) $-20(\vec{i}+\vec{j})\ \text{Nkg}^{-1}$ (c) $10\sqrt{2}$ N
22. $(\sqrt{2}-1)$ times the radius of the earth
23. $9{\cdot}77\ \text{m s}^{-2}$
24. $9{\cdot}799\ \text{m s}^{-2}$
25. 0·997 kg
26. 10 km approx.
27. $1{\cdot}237\times10^{-3}\ \text{rads}^{-1}$, 1·41 h
28. (a) $\omega R$ (b) $m\omega^2R$ (c) $T_0+2m\omega v$ approx.
29. 1·52
30. $6{\cdot}02\times10^{24}$ kg
31. $6{\cdot}5\times10^{23}$ kg
32. (a) $6{\cdot}90\ \text{kms}^{-1}$ (b) $2{\cdot}38\times10^{10}$ J (c) $-4{\cdot}76\times10^{10}$ J with usual reference (d) 2·12 hours
33. (a) 42300 km (b) 6 hours
34. 0·23 N
35. $\frac{4\pi^2R_2^3}{T^2R_1^2}$
36. $\sin^{-1}(0{\cdot}15)$
37. $7{\cdot}9\ \text{km s}^{-1}$
38. $10{\cdot}0\ \text{km s}^{-1}$
39. $\approx 9$ mm

□

## CHAPTER 12

# SIMPLE HARMONIC MOTION

### 12.1 SIMPLE HARMONIC MOTION

When a body repeats its motion after regular time intervals we say that it is in *harmonic motion* or *periodic motion.* The time interval after which the motion is repeated is called the *time period*. If a body moves to and fro on the same path, it is said to perform *oscillations. Simple harmonic motion* (SHM) is a special type of oscillation in which the particle oscillates on a straight line, the acceleration of the particle is always directed towards a fixed point on the line and its magnitude is proportional to the displacement of the particle from this point. This fixed point is called the *centre of oscillation.* Taking this point as the origin and the line of motion as the $X$-axis, we can write the defining equation of a simple harmonic motion as

$$a = -\omega^2 x \qquad \ldots \quad (12.1)$$

where $\omega^2$ is a positive constant. If $x$ is positive, $a$ is negative and if $x$ is negative, $a$ is positive. This means that the acceleration is always directed towards the centre of oscillation.

If we are looking at the motion from an inertial frame,

$$a = F/m.$$

The defining equation (12.1) may thus be written as

$$F/m = -\omega^2 x$$

or, $$F = -m\omega^2 x$$

or, $$F = -kx. \qquad \ldots \quad (12.2)$$

We can use equation (12.2) as the definition of SHM. A particle moving on a straight line executes simple harmonic motion if the resultant force acting on it is directed towards a fixed point on the line and is proportional to the displacement of the particle from this fixed point. The constant $k = m\omega^2$ is called the *force constant* or *spring constant*. The resultant force on the particle is zero when it is at the centre of oscillation. The centre of oscillation is, therefore, the equilibrium position. A force which takes the particle back towards the equilibrium position is called a *restoring force*. Equation (12.2) represents a restoring force which is linear. Figure (12.1) shows the linear restoring force graphically.

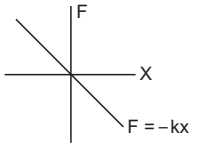



Figure 12.1

***Example 12.1***

*The resultant force acting on a particle executing simple harmonic motion is* 4 N *when it is* 5 cm *away from the centre of oscillation. Find the spring constant.*

***Solution*** **:** The simple harmonic motion is defined as

$$F = -k\,x.$$

The spring constant is $k = \left| \frac{F}{x} \right|$

$$= \frac{4\text{ N}}{5\text{ cm}} = \frac{4\text{ N}}{5 \times 10^{-2}\text{ m}} = 80\text{ N m}^{-1}.$$

### 12.2 QUALITATIVE NATURE OF SIMPLE HARMONIC MOTION

Let us consider a small block of mass $m$ placed on a smooth horizontal surface and attached to a fixed wall through a spring as shown in figure (12.2). Let the spring constant of the spring be $k$.

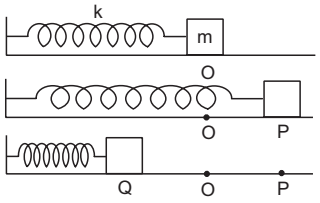



Figure 12.2

The block is at a position $O$ when the spring is at its natural length. Suppose the block is taken to a point $P$ stretching the spring by the distance $OP = A$ and is released from there.

At any point on its path the displacement $x$ of the particle is equal to the extension of the spring from its natural length. The resultant force on the particle is given by $F = -kx$ and hence by definition the motion of the block is simple harmonic.

When the block is released from $P$, the force acts towards the centre $O$. The block is accelerated in that direction. The force continues to act towards $O$ until the block reaches $O$. The speed thus increases all the time from $P$ to $O$. When the block reaches $O$, its speed is maximum and it is going towards left. As it moves towards left from $O$, the spring becomes compressed. The spring pushes the block towards right and hence its speed decreases. The block moves to a point $Q$ when its speed becomes zero. The potential energy of the system (block + spring), when the block is at $P$, is $\frac{1}{2} k (OP)^2$ and when the block is at $Q$ it is $\frac{1}{2} k (OQ)^2$. Since the block is at rest at $P$ as well as at $Q$, the kinetic energy is zero at both these positions. As we have assumed frictionless surface, principle of conservation of energy gives

$$\frac{1}{2} k (OP)^2 = \frac{1}{2} k (OQ)^2$$

or, $$OP = OQ.$$

The spring is now compressed and hence it pushes the block towards right. The block starts moving towards right, its speed increases upto $O$ and then decreases to zero when it reaches $P$. Thus the particle oscillates between $P$ and $Q$. As $OP = OQ$, it moves through equal distances on both sides of the centre of oscillation. The maximum displacement on either side from the centre of oscillation is called the *amplitude*.

***Example 12.2***

*A particle of mass* 0·50 kg *executes a simple harmonic motion under a force* $F = -(50 \text{ N m}^{-1})x$. *If it crosses the centre of oscillation with a speed of* 10 m s$^{-1}$, *find the amplitude of the motion.*

***Solution*** : The kinetic energy of the particle when it is at the centre of oscillation is

$$E = \frac{1}{2} m v^2$$

$$= \frac{1}{2} (0{\cdot}50 \text{ kg}) (10 \text{ m s}^{-1})^2$$

$$= 25 \text{ J}.$$

The potential energy is zero here. At the maximum displacement $x = A$, the speed is zero and hence the kinetic energy is zero. The potential energy here is $\frac{1}{2} k A^2$. As there is no loss of energy,

$$\frac{1}{2} k A^2 = 25 \text{ J}. \quad \ldots \text{ (i)}$$

The force on the particle is given by

$$F = -(50 \text{ N m}^{-1})x.$$

Thus, the spring constant is $k = 50 \text{ N m}^{-1}$.

Equation (i) gives

$$\frac{1}{2} (50 \text{ N m}^{-1}) A^2 = 25 \text{ J}$$

or, $$A = 1 \text{ m}.$$

## 12.3 EQUATION OF MOTION OF A SIMPLE HARMONIC MOTION

Consider a particle of mass $m$ moving along the $X$-axis. Suppose, a force $F = -kx$ acts on the particle where $k$ is a positive constant and $x$ is the displacement of the particle from the assumed origin. The particle then executes a simple harmonic motion with the centre of oscillation at the origin. We shall calculate the displacement $x$ and the velocity $v$ as a function of time.

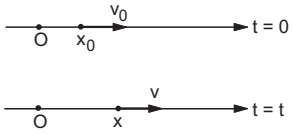



Figure 12.3

Suppose the position of the particle at $t = 0$ is $x_0$ and its velocity is $v_0$. Thus,

at $t = 0, x = x_0$ and $v = v_0$.

The acceleration of the particle at any instant is

$$a = \frac{F}{m} = -\frac{k}{m} x = -\omega^2 x$$

where $\omega = \sqrt{km^{-1}}$.

Thus, $$\frac{dv}{dt} = -\omega^2 x \quad \ldots \text{ (12.3)}$$

or, $$\frac{dv}{dx} \frac{dx}{dt} = -\omega^2 x$$

or, $$v \frac{dv}{dx} = -\omega^2 x$$

or, $$v dv = -\omega^2 x \, dx.$$

The velocity of the particle is $v_0$ when the particle is at $x = x_0$. It becomes $v$ when the displacement becomes $x$. We can integrate the above equation and write

$$\int_{v_0}^{v} v \, dv = \int_{x_0}^{x} -\omega^2 x \, dx$$

or, $$\left[\frac{v^2}{2}\right]_{v_0}^{v} = -\omega^2\left[\frac{x^2}{2}\right]_{x_0}^{x}$$

or, $$v^2 - v_0^2 = -\omega^2(x^2 - x_0^2)$$

or, $$v^2 = (v_0^2 + \omega^2 x_0^2 - \omega^2 x^2)$$

or, $$v = \sqrt{(v_0^2 + \omega^2 x_0^2) - \omega^2 x^2}$$

or, $$v = \omega\sqrt{\left(\frac{v_0^2}{\omega^2} + x_0^2\right) - x^2}\,.$$

Writing $$\left(\frac{v_0}{\omega}\right)^2 + x_0^2 = A^2 \qquad \ldots (12.4)$$

the above equation becomes

$$v = \omega\sqrt{A^2 - x^2}. \qquad \ldots (12.5)$$

We can write this equation as

$$\frac{dx}{dt} = \omega\sqrt{A^2 - x^2}$$

or, $$\frac{dx}{\sqrt{A^2 - x^2}} = \omega\, dt.$$

At time $t = 0$ the displacement is $x = x_0$ and at time $t$ the displacement becomes $x$. The above equation can be integrated as

$$\int_{x_0}^{x} \frac{dx}{\sqrt{A^2 - x^2}} = \int_0^t \omega\, dt$$

or, $$\left[\sin^{-1}\frac{x}{A}\right]_{x_0}^{x} = [\omega t]_0^t$$

or, $$\sin^{-1}\frac{x}{A} - \sin^{-1}\frac{x_0}{A} = \omega t.$$

Writing $\sin^{-1}\frac{x_0}{A} = \delta$, this becomes

$$\sin^{-1}\frac{x}{A} = \omega t + \delta$$

or, $$x = A\sin(\omega t + \delta). \qquad \ldots (12.6)$$

The velocity at time $t$ is

$$v = \frac{dx}{dt} = A\,\omega\cos(\omega t + \delta). \qquad \ldots (12.7)$$

## 12.4 TERMS ASSOCIATED WITH SIMPLE HARMONIC MOTION

### (a) Amplitude

Equation (12.6) gives the displacement of a particle in simple harmonic motion. As $\sin(\omega t + \delta)$ can take values between $-1$ and $+1$, the displacement $x$ can take values between $-A$ and $+A$. This gives the physical significance of the constant $A$. It is the maximum displacement of the particle from the centre of oscillation, i.e, the amplitude of oscillation.

### (b) Time Period

A particle in simple harmonic motion repeats its motion after a regular time interval. Suppose the particle is at a position $x$ and its velocity is $v$ at a certain time $t$. After some time the position of the particle will again be $x$ and its velocity will again be $v$ in the same direction. This part of the motion is called *one complete oscillation* and the time taken in one complete oscillation is called the *time period* $T$. Thus, in figure (12.4) $Q$ to $P$ and then back to $Q$ is a complete oscillation, $R$ to $P$ to $Q$ to $R$ is a complete oscillation, $O$ to $P$ to $Q$ to $O$ is a complete oscillation, etc. Both the position and the velocity (magnitude as well as direction) repeat after each complete oscillation.

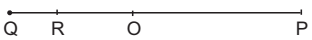



Figure 12.4

We have,

$$x = A\sin(\omega t + \delta).$$

If $T$ be the time period, $x$ should have same value at $t$ and $t + T$.

Thus, $\sin(\omega t + \delta) = \sin[\omega(t + T) + \delta]$.

Now the velocity is (equation 12.7)

$$v = A\,\omega\cos(\omega t + \delta).$$

As the velocity also repeats its value after a time period, $\cos(\omega t + \delta) = \cos[\omega(t + T) + \delta]$.

Both $\sin(\omega t + \delta)$ and $\cos(\omega t + \delta)$ will repeat their values if the angle $(\omega t + \delta)$ increases by $2\pi$ or its multiple. As $T$ is the smallest time for repetition,

$$\omega(t + T) + \delta = (\omega t + \delta) + 2\pi$$

or, $$\omega T = 2\pi$$

or, $$T = \frac{2\pi}{\omega}\,.$$

Remembering that $\omega = \sqrt{km^{-1}}$, we can write for the time period,

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \qquad \ldots (12.8)$$

where $k$ is the force constant and $m$ is the mass of the particle.

***Example 12.3***

*A particle of mass* 200 g *executes a simple harmonic motion. The restoring force is provided by a spring of spring constant* 80 N m$^{-1}$. *Find the time period.*

***Solution*** : The time period is

$$T = 2\pi\sqrt{\frac{m}{k}}$$

$$= 2\pi \sqrt{\frac{200 \times 10^{-3}\ \text{kg}}{80\ \text{N m}^{-1}}}$$

$$= 2\pi \times 0{\cdot}05\ \text{s} = 0{\cdot}31\ \text{s}.$$

**(c) Frequency and Angular Frequency**

The reciprocal of time period is called the *frequency*. Physically, the frequency represents the number of oscillations per unit time. It is measured in cycles per second also known as *hertz* and written in symbols as Hz. Equation (12.8) shows that the frequency is

$$\nu = \frac{1}{T} = \frac{\omega}{2\pi} \qquad \ldots (12.9)$$

$$= \frac{1}{2\pi}\sqrt{\frac{k}{m}}\ . \qquad \ldots (12.10)$$

The constant $\omega$ is called the *angular frequency*.

**(d) Phase**

The quantity $\phi = \omega t + \delta$ is called the phase. It determines the status of the particle in simple harmonic motion. If the phase is zero at a certain instant, $x = A\sin(\omega t + \delta) = 0$ and $v = A\,\omega\cos(\omega t + \delta) = A\,\omega$. This means that the particle is crossing the mean position and is going towards the positive direction. If the phase is $\pi/2$, we get $x = A$, $v = 0$ so that the particle is at the positive extreme position. Figure (12.5) shows the status of the particle at different phases.

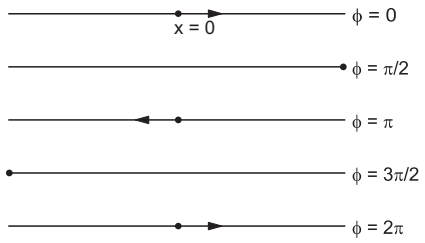



Figure 12.5

We see that as time increases the phase increases. An increase of $2\pi$ brings the particle to the same status in the motion. Thus, a phase $\omega t + \delta$ is equivalent to a phase $\omega t + \delta + 2\pi$. Similarly, a phase change of $4\pi$, $6\pi$, $8\pi$, …, etc., are equivalent to no phase change.

Figure (12.6) shows graphically the variation of position and velocity as a function of the phase.

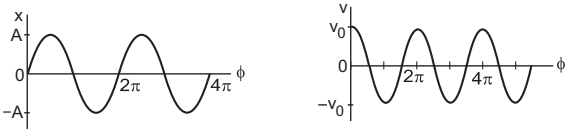



Figure 12.6

**(e) Phase constant**

The constant $\delta$ appearing in equation (12.6) is called the *phase constant*. This constant depends on the choice of the instant $t = 0$. To describe the motion quantitatively, a particular instant should be called $t = 0$ and measurement of time should be made from this instant. This instant may be chosen according to the convenience of the problem. Suppose we choose $t = 0$ at an instant when the particle is passing through its mean position and is going towards the positive direction. The phase $\omega t + \delta$ should then be zero. As $t = 0$ this means $\delta$ will be zero. The equation for displacement can then be written as

$$x = A\sin\omega t.$$

If we choose $t = 0$ at an instant when the particle is at its positive extreme position, the phase is $\pi/2$ at this instant. Thus $\omega t + \delta = \pi/2$ and hence $\delta = \pi/2$. The equation for the displacement is $x = A\sin(\omega t + \pi/2)$

or, $$x = A\cos\omega t.$$

Any instant can be chosen as $t = 0$ and hence the phase constant can be chosen arbitrarily. Quite often we shall choose $\delta = 0$ and write the equation for displacement as $x = A\sin\omega t$. Sometimes we may have to consider two or more simple harmonic motions together. The phase constant of any one can be chosen as $\delta = 0$. The phase constants of the rest will be determined by the actual situation. The general equation for displacement may be written as

$$x = A\sin(\omega t + \delta)$$

$$= A\sin\left(\omega t + \frac{\pi}{2} + \delta'\right)$$

$$= A\cos(\omega t + \delta')$$

where $\delta'$ is another arbitrary constant. The sine form and the cosine form are, therefore, equivalent. The value of phase constant, however, depends on the form chosen.

***Example 12.4***

*A particle executes simple harmonic motion of amplitude A along the X-axis. At $t = 0$, the position of the particle is $x = A/2$ and it moves along the positive x-direction. Find the phase constant $\delta$ if the equation is written as $x = A\sin(\omega t + \delta)$.*

***Solution*** : We have $x = A\sin(\omega t + \delta)$. At $t = 0$, $x = A/2$.

Thus, $$A/2 = A\sin\delta$$

or, $$\sin\delta = 1/2$$

or, $$\delta = \pi/6 \quad \text{or} \quad 5\pi/6.$$

The velocity is $v = \dfrac{dx}{dt} = A\,\omega\cos(\omega t + \delta)$.

At $$t = 0,\ v = A\,\omega\cos\delta.$$

Now, $$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \text{ and } \cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}.$$

As $v$ is positive at $t = 0$, $\delta$ must be equal to $\pi/6$.

## 12.5 SIMPLE HARMONIC MOTION AS A PROJECTION OF CIRCULAR MOTION

Consider a particle $P$ moving on a circle of radius $A$ with a constant angular speed $\omega$ (figure 12.7). Let us take the centre of the circle as the origin and two perpendicular diameters as the $X$ and $Y$-axes. Suppose the particle $P$ is on the $X$-axis at $t = 0$. The radius $OP$ will make an angle $\theta = \omega t$ with the $X$-axis at time $t$. Drop perpendicular $PQ$ on $X$-axis and $PR$ on $Y$-axis. The $x$ and $y$-coordinates of the particle at time $t$ are

$$x = OQ = OP \cos \omega t$$

or, $$x = A \cos \omega t \quad \ldots (12.11)$$

and $$y = OR = OP \sin \omega t$$

or, $$y = A \sin \omega t. \quad \ldots (12.12)$$

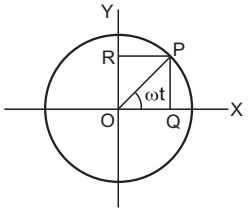



Figure 12.7

Equation (12.11) shows that the foot of perpendicular $Q$ executes a simple harmonic motion on the $X$-axis. The amplitude is $A$ and the angular frequency is $\omega$. Similarly, equation (12.12) shows that the foot of perpendicular $R$ executes a simple harmonic motion on the $Y$-axis. The amplitude is $A$ and the angular frequency is $\omega$. The phases of the two simple harmonic motions differ by $\pi/2$ [remember $\cos\omega t = \sin(\omega t + \pi/2)$].

Thus, the projection of a uniform circular motion on a diameter of the circle is a simple harmonic motion.

## 12.6 ENERGY CONSERVATION IN SIMPLE HARMONIC MOTION

Simple harmonic motion is defined by the equation

$$F = -kx.$$

The work done by the force $F$ during a displacement from $x$ to $x + dx$ is

$$dW = F\,dx$$
$$= -kx\,dx.$$

The work done in a displacement from $x = 0$ to $x$ is

$$W = \int_0^x (-kx)dx = -\frac{1}{2}kx^2.$$

Let $U(x)$ be the potential energy of the system when the displacement is $x$. As the change in potential energy corresponding to a force is negative of the work done by this force,

$$U(x) - U(0) = -W = \frac{1}{2}kx^2.$$

Let us choose the potential energy to be zero when the particle is at the centre of oscillation $x = 0$.

Then $$U(0) = 0 \text{ and } U(x) = \frac{1}{2}kx^2.$$

This expression for potential energy is same as that for a spring and has been used so far in this chapter.

As $$\omega = \sqrt{\frac{k}{m}}, \; k = m\omega^2$$

we can write $$U(x) = \frac{1}{2}m\omega^2 x^2. \quad \ldots (12.13)$$

The displacement and the velocity of a particle executing a simple harmonic motion are given by

$$x = A \sin(\omega t + \delta)$$

and $$v = A\omega \cos(\omega t + \delta).$$

The potential energy at time $t$ is, therefore,

$$U = \frac{1}{2}m\omega^2 x^2$$
$$= \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \delta),$$

and the kinetic energy at time $t$ is

$$K = \frac{1}{2}mv^2$$
$$= \frac{1}{2}mA^2\omega^2 \cos^2(\omega t + \delta).$$

The total mechanical energy at time $t$ is

$$E = U + K$$
$$= \frac{1}{2}m\omega^2 A^2 [\sin^2(\omega t + \delta) + (\cos^2(\omega t + \delta)]$$
$$= \frac{1}{2}m\omega^2 A^2. \quad \ldots (12.14)$$

We see that the total mechanical energy at time $t$ is independent of $t$. Thus, the mechanical energy remains constant as expected.

As an example, consider a small block of mass $m$ placed on a smooth horizontal surface and attached to a fixed wall through a spring of spring constant $k$ (figure 12.8).

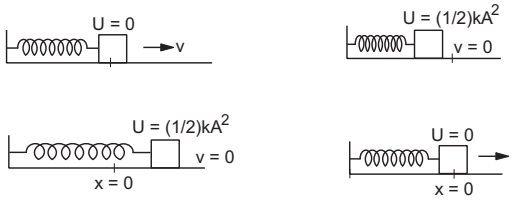



Figure 12.8

When displaced from the mean position (where the spring has its natural length), the block executes a simple harmonic motion. The spring is the agency exerting a force $F = -kx$ on the block. The potential energy of the system is the elastic potential energy stored in the spring.

At the mean position $x = 0$, the potential energy is zero. The kinetic energy is $\frac{1}{2} m v_0^2 = \frac{1}{2} m \omega^2 A^2$. All the mechanical energy is in the form of kinetic energy here. As the particle is displaced away from the mean position, the kinetic energy decreases and the potential energy increases. At the extreme positions $x = \pm A$, the speed $v$ is zero and the kinetic energy decreases to zero. The potential energy is increased to its maximum value $\frac{1}{2} kA^2 = \frac{1}{2} m \omega^2 A^2$. All the mechanical energy is in the form of potential energy here.

***Example 12.5***

*A particle of mass* 40 g *executes a simple harmonic motion of amplitude* 2·0 cm. *If the time period is* 0·20 s, *find the total mechanical energy of the system.*

***Solution*** : The total mechanical energy of the system is

$$E = \frac{1}{2} m \omega^2 A^2$$

$$= \frac{1}{2} m \left(\frac{2\pi}{T}\right)^2 A^2 = \frac{2\pi^2 m A^2}{T^2}$$

$$= \frac{2\pi^2 (40 \times 10^{-3} \text{ kg}) (2{\cdot}0 \times 10^{-2} \text{ m})^2}{(0{\cdot}20 \text{ s})^2}$$

$$= 7{\cdot}9 \times 10^{-3} \text{ J}.$$

## 12.7 ANGULAR SIMPLE HARMONIC MOTION

A body free to rotate about a given axis can make angular oscillations. For example, a hanging umbrella makes angular oscillations when it is slightly pushed aside and released. The angular oscillations are called angular simple harmonic motion if

(a) there is a position of the body where the resultant torque on the body is zero, this position is the mean position $\theta = 0$,

(b) when the body is displaced through an angle from the mean position, a resultant torque acts which is proportional to the angle displaced, and

(c) this torque has a sense (clockwise or anticlockwise) so as to bring the body towards the mean position.

If the angular displacement of the body at an instant is $\theta$, the resultant torque acting on the body in angular simple harmonic motion should be

$$\Gamma = -k\theta.$$

If the moment of inertia is $I$, the angular acceleration is

$$\alpha = \frac{\Gamma}{I} = -\frac{k}{I}\theta$$

or, $$\frac{d^2\theta}{dt^2} = -\omega^2 \theta \qquad \ldots \quad (12.15)$$

where $$\omega = \sqrt{kI^{-1}}.$$

Equation (12.15) is identical to equation (12.3) except for the symbols. The linear displacement $x$ in (12.3) is replaced here by the angular displacement $\theta$. Thus, equation (12.15) may be integrated in the similar manner and we shall get an equation similar to (12.6), i.e.,

$$\theta = \theta_0 \sin(\omega t + \delta) \qquad \ldots \quad (12.16)$$

where $\theta_0$ is the maximum angular displacement on either side. The angular velocity at time $t$ is given by,

$$\Omega = \frac{d\theta}{dt} = \theta_0 \omega \cos(\omega t + \delta). \qquad \ldots \quad (12.17)$$

The time period of oscillation is

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{I}{k}} \qquad \ldots \quad (12.18)$$

and the frequency of oscillation is

$$\nu = \frac{1}{T} = \frac{1}{2\pi} \sqrt{\frac{k}{I}}. \qquad \ldots \quad (12.19)$$

The quantity $\omega = \sqrt{kI^{-1}}$ is the angular frequency.

***Example 12.6***

*A body makes angular simple harmonic motion of amplitude* $\pi/10$ rad *and time period* 0·05 s. *If the body is at a displacement* $\theta = \pi/10$ rad *at* $t = 0$, *write the equation giving the angular displacement as a function of time.*

***Solution*** : Let the required equation be

$$\theta = \theta_0 \sin(\omega t + \delta).$$

Here $\theta_0 =$ amplitude $= \frac{\pi}{10}$ rad

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0{\cdot}05 \text{ s}} = 40\pi \text{ s}^{-1}$$

so that $$\theta = \left(\frac{\pi}{10} \text{ rad}\right) \sin\left[\left(40\pi \text{ s}^{-1}\right) t + \delta\right]. \qquad \ldots \text{ (i)}$$

At $t = 0$, $\theta = \pi/10$ rad. Putting in (i),

$$\frac{\pi}{10} = \left(\frac{\pi}{10}\right) \sin\delta$$

or, $$\sin\delta = 1$$

or, $$\delta = \pi/2.$$

Thus by (i),

$$\theta = \left(\frac{\pi}{10} \text{ rad}\right) \sin\left[(40\pi \text{ s}^{-1}) t + \frac{\pi}{2}\right]$$

$$= \left(\frac{\pi}{10} \text{ rad}\right) \cos[(40\pi \text{ s}^{-1}) t\ ].$$

**Energy**

The potential energy is

$$U = \frac{1}{2} k \theta^2 = \frac{1}{2} I \omega^2 \theta^2$$

and the kinetic energy is

$$K = \frac{1}{2} I \Omega^2.$$

The total energy is

$$E = U + K$$

$$= \frac{1}{2} I \omega^2 \theta^2 + \frac{1}{2} I \Omega^2.$$

Using $\theta = \theta_0 \sin(\omega t + \delta)$

$$E = \frac{1}{2} I \omega^2 \theta_0^2 \sin^2(\omega t + \delta) + \frac{1}{2} I \theta_0^2 \omega^2 \cos^2(\omega t + \delta)$$

$$= \frac{1}{2} I \omega^2 \theta_0^2. \quad \ldots (12.20)$$

## 12.8 SIMPLE PENDULUM

A simple pendulum consists of a heavy particle suspended from a fixed support through a light inextensible string. Simple pendulum is an idealised model. In practice, one takes a small metallic sphere and suspends it through a string.

Figure (12.9) shows a simple pendulum in which a particle of mass $m$ is suspended from the fixed support $O$ through a light string of length $l$. The system can stay in equilibrium if the string is vertical. This is the mean or equilibrium position. If the particle is pulled aside and released, it oscillates in a circular arc with the centre at the point of suspension $O$.

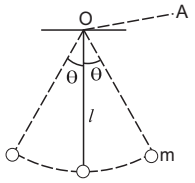



Figure 12.9

The position of the particle at any time can be described by the angle θ between the string and the vertical. The mean position or the equilibrium position corresponds to $\theta = 0$. The particle makes pure rotation about the horizontal line $OA$ (figure 12.9) which is perpendicular to the plane of motion.

Let us see whether the motion of the particle is simple harmonic or not and find out its time period of oscillation.

Let the particle be at $P$ at a time $t$ when the string $OP$ makes an angle θ with the vertical (figure 12.10). Let $OQ$ be the horizontal line in the plane of motion. Let $PQ$ be the perpendicular to $OQ$.

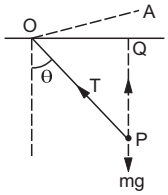



Figure 12.10

Forces acting on the particle are (a) the weight $mg$ and (b) the tension $T$.

The torque of $T$ about $OA$ is zero as it intersects $OA$. The magnitude of the torque of $mg$ about $OA$ is

$$|\Gamma| = (mg)(OQ)$$

$$= mg\,(OP) \sin\theta$$

$$= mgl \sin\theta.$$

Also, the torque tries to bring the particle back towards $\theta = 0$. Thus, we can write

$$\Gamma = -mgl \sin\theta. \quad \ldots (12.21)$$

We see that the resultant torque is not proportional to the angular displacement and hence the motion is not angular simple harmonic. However, if the angular displacement is small, sinθ is approximately equal to θ (expressed in radians) and equation (12.21) may be written as

$$\Gamma = -mgl\,\theta. \quad \ldots (12.22)$$

Thus, if the amplitude of oscillation is small, the motion of the particle is approximately angular simple harmonic. The moment of inertia of the particle about the axis of rotation $OA$ is

$$I = m(OP)^2 = ml^2.$$

The angular acceleration is

$$\alpha = \frac{\Gamma}{I} = -\frac{mgl\,\theta}{ml^2} = -\frac{g}{l}\theta$$

or, $\alpha = -\omega^2 \theta$

where $\omega = \sqrt{gl^{-1}}$.

This is the equation of an angular simple harmonic motion. The constant $\omega = \sqrt{gl^{-1}}$ represents the angular frequency. The time period is

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{l/g}. \quad \ldots (12.23)$$

***Example 12.7***

*Calculate the time period of a simple pendulum of length one meter. The acceleration due to gravity at the place is* $\pi^2$ m $s^{-2}$.

***Solution*** : The time period is

$$T = 2\pi \sqrt{lg^{-1}}$$

$$= 2\pi \sqrt{\frac{1{\cdot}00 \text{ m}}{\pi^2 \text{ m s}^{-2}}} = 2{\cdot}0 \text{ s}.$$

**Simple Pendulum as a Linear Simple Harmonic Oscillator**

If the amplitude of oscillation is small, the path of the particle is approximately a straight line and the motion can be described as a linear simple harmonic motion. We rederive expression (12.23) for the time period using this approach.

Consider the situation shown in figure (12.11).

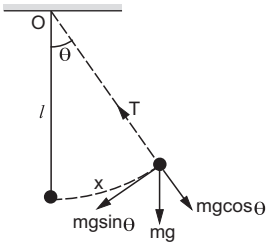



Figure 12.11

Suppose the string makes an angle θ with the vertical at time $t$. The distance of the particle from the equilibrium position along the arc is $x = l\theta$. The speed of the particle at time $t$ is

$$v = \frac{dx}{dt}$$

and the tangential acceleration is

$$a_t = \frac{dv}{dt} = \frac{d^2x}{dt^2}. \qquad \ldots \text{ (i)}$$

Forces acting on the particle are (a) the weight $mg$ and (b) the tension $T$. The component of $mg$ along the tangent to the path is $-mg\sin\theta$ and that of $T$ is zero. Thus, the total tangential force on the particle is $-mg\sin\theta$. Using (i) we get

$$-mg\sin\theta = m\frac{d^2x}{dt^2}$$

or,
$$\frac{d^2x}{dt^2} = -g\sin\theta. \qquad \ldots \text{ (ii)}$$

If the amplitude of oscillation is small, $\sin\theta \approx \theta = xl^{-1}$. Equation (ii) above thus becomes (for small oscillations)

$$\frac{d^2x}{dt^2} = -\frac{g}{l}x$$

or,
$$\frac{d^2x}{dt^2} = -\omega^2 x$$

where
$$\omega = \sqrt{gl^{-1}}.$$

This equation represents a simple harmonic motion of the particle along the arc of the circle in which it moves. The angular frequency is $\omega = \sqrt{gl^{-1}}$ and the time period is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{lg^{-1}}$$

which is same as in equation (12.23).

**Determination of *g* in Laboratory**

A simple pendulum provides an easy method to measure the value of '$g$' in a laboratory. A small spherical ball with a hook is suspended from a clamp through a light thread as shown in figure (12.12).

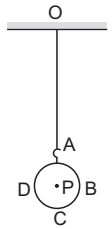



Figure 12.12

The lengths $AC$ and $BD$ are measured with slide callipers. The length $OA$ of the thread is measured with a meter scale. The effective length is

$$OP = OA + AP = OA + AC - \frac{BD}{2}.$$

The bob is slightly pulled aside and gently released from rest. The pendulum starts making oscillations. The time for a number of oscillations (say 20 or 50) is measured with a stop watch and the time period is obtained. The value of $g$ is calculated by equation (12.23). The length of the thread is varied and the experiment is repeated a number of times to minimise the effect of random errors.

***Example 12.8***

*In a laboratory experiment with simple pendulum it was found that it took* 36 s *to complete* 20 *oscillations when the effective length was kept at* 80 cm. *Calculate the acceleration due to gravity from these data.*

***Solution*** : The time period of a simple pendulum is given by

$$T = 2\pi\sqrt{lg^{-1}}$$

or,
$$g = \frac{4\pi^2 l}{T^2}. \qquad \ldots \text{ (i)}$$

In the experiment described in the question, the time period is

$$T = \frac{36 \text{ s}}{20} = 1{\cdot}8 \text{ s}.$$

Thus, by (i),

$$g = \frac{4\pi^2 \times 0{\cdot}80 \text{ m}}{(1{\cdot}8 \text{ s})^2} = 9{\cdot}75 \text{ m s}^{-2}.$$

## 12.9 PHYSICAL PENDULUM

Any rigid body suspended from a fixed support constitutes a physical pendulum. A circular ring suspended on a nail in a wall, a heavy metallic rod suspended through a hole in it, etc., are examples of physical pendulum. Figure (12.13) shows a physical pendulum. A rigid body is suspended through a hole at *O*. When the centre of mass *C* is vertically below *O*, the body may remain at rest. We call this position $\theta = 0$. When the body is pulled aside and released, it executes oscillations.

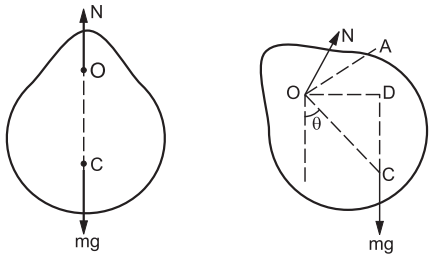



Figure 12.13

The body rotates about a horizontal axis through *O* and perpendicular to the plane of motion. Let this axis be *OA*. Suppose the angular displacement of the body is θ at time *t*. The line *OC* makes an angle θ with the vertical at this instant.

Forces on the body are (a) the weight *mg* and (b) the contact force $\mathcal{N}$ by the support at *O*.

The torque of $\mathcal{N}$ about *OA* is zero as the force $\mathcal{N}$ acts through the point *O*. The torque of *mg* has magnitude

$$|\Gamma| = mg\,(OD)$$
$$= mg\,(OC)\sin\theta = mgl\sin\theta$$

where $l = OC$ is the separation between the point of suspension and the centre of mass. This torque tries to bring the body back towards $\theta = 0$. Thus, we can write

$$\Gamma = -mgl\sin\theta.$$

If the moment of inertia of the body about *OA* is *I,* the angular acceleration becomes

$$\alpha = \frac{\Gamma}{I} = -\frac{mgl}{I}\sin\theta. \qquad \ldots \text{ (i)}$$

We see that the angular acceleration is not proportional to the angular displacement and the motion is not strictly simple harmonic. However, for small displacements $\sin\theta \approx \theta$ so that equation (i) becomes

$$\alpha = -\omega^2\theta$$

where $\omega^2 = mgl I^{-1}$.

Thus, for small oscillations, the motion is nearly simple harmonic. The time period is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgl}}\,. \qquad \ldots \text{ (12.24)}$$

***Example 12.9***

*A uniform rod of length* 1·00 m *is suspended through an end and is set into oscillation with small amplitude under gravity. Find the time period of oscillation.*

***Solution*** **:** For small amplitude the angular motion is nearly simple harmonic and the time period is given by

$$T = 2\pi\sqrt{\frac{I}{mgl}} = 2\pi\sqrt{\frac{(mL^2/3)}{mgL/2}}$$

$$= 2\pi\sqrt{\frac{2L}{3g}} = 2\pi\sqrt{\frac{2 \times 1{\cdot}00 \text{ m}}{3 \times 9{\cdot}80 \text{ m s}^{-2}}} = 1{\cdot}64 \text{ s}.$$

## 12.10 TORSIONAL PENDULUM

In torsional pendulum, an extended body is suspended by a light thread or a wire. The body is rotated through an angle about the wire as the axis of rotation (figure 12.14).

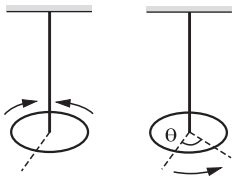



Figure 12.14

The wire remains vertical during this motion but a twist is produced in the wire. The lower end of the wire is rotated through an angle with the body but the upper end remains fixed with the support. Thus, a twist θ is produced. The twisted wire exerts a restoring torque on the body to bring it back to its original position in which the twist θ in the wire is zero. This torque has a magnitude proportional to the angle of twist which is equal to the angle rotated by the body. The proportionality constant is called the *torsional constant* of the wire. Thus, if the torsional constant of the wire is *k* and the body is rotated through an angle θ, the torque produced is $\Gamma = -k\theta$.

If *I* be the moment of inertia of the body about the vertical axis, the angular acceleration is

$$\alpha = \frac{\Gamma}{I} = -\frac{k}{I}\theta$$

$$= -\omega^2 \theta$$

where $\omega = \sqrt{\frac{k}{I}}$.

Thus, the motion of the body is simple harmonic and the time period is

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{I}{k}} \cdot \quad \ldots (12.25)$$

***Example 12.10***

*A uniform disc of radius* 5·0 cm *and mass* 200 g *is fixed at its centre to a metal wire, the other end of which is fixed with a clamp. The hanging disc is rotated about the wire through an angle and is released. If the disc makes torsional oscillations with time period* 0·20 s*, find the torsional constant of the wire.*

***Solution*** **:** The situation is shown in figure (12.15). The moment of inertia of the disc about the wire is

$$I = \frac{mr^2}{2} = \frac{(0{\cdot}200 \text{ kg})\,(5{\cdot}0 \times 10^{-2} \text{ m})^2}{2}$$

$$= 2{\cdot}5 \times 10^{-4} \text{ kg-m}^2.$$

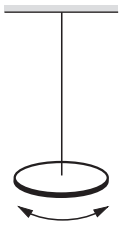


Figure 12.15

The time period is given by

$$T = 2\pi \sqrt{\frac{I}{k}}$$

or, $$k = \frac{4\pi^2 I}{T^2}$$

$$= \frac{4\pi^2 (2{\cdot}5 \times 10^{-4} \text{ kg-m}^2)}{(0{\cdot}20 \text{ s})^2}$$

$$= 0{\cdot}25 \frac{\text{kg-m}^2}{\text{s}^2} \cdot$$

## 12.11 COMPOSITION OF TWO SIMPLE HARMONIC MOTIONS

A simple harmonic motion is produced when a restoring force proportional to the displacement acts on a particle. If the particle is acted upon by two separate forces each of which can produce a simple harmonic motion, the resultant motion of the particle is a combination of two simple harmonic motions.

Let $\vec{r}_1$ denote the position of the particle at time $t$ if the force $\vec{F}_1$ alone acts on it. Similarly, let $\vec{r}_2$ denote the position at time $t$ if the force $\vec{F}_2$ alone acts on it. Newton's second law gives,

$$m \frac{d^2 \vec{r}_1}{dt^2} = \vec{F}_1$$

and $$m \frac{d^2 \vec{r}_2}{dt^2} = \vec{F}_2.$$

Adding them,

$$m \frac{d^2 \vec{r}_1}{dt^2} + m \frac{d^2 \vec{r}_2}{dt^2} = \vec{F}_1 + \vec{F}_2$$

or, $$m \frac{d^2}{dt^2} (\vec{r}_1 + \vec{r}_2) = \vec{F}_1 + \vec{F}_2. \quad \ldots (i)$$

But $\vec{F}_1 + \vec{F}_2$ is the resultant force acting on the particle and so the position $\vec{r}$ of the particle when both the forces act, is given by

$$m \frac{d^2 \vec{r}}{dt^2} = \vec{F}_1 + \vec{F}_2. \quad \ldots (ii)$$

Comparing (i) and (ii) we can show that

$$\vec{r} = \vec{r}_1 + \vec{r}_2$$

and $$\vec{u} = \vec{u}_1 + \vec{u}_2$$

if these conditions are met at $t = 0$.

Thus, if two forces $\vec{F}_1$ and $\vec{F}_2$ act together on a particle, its position at any instant can be obtained as follows. Assume that only the force $\vec{F}_1$ acts and find the position $\vec{r}_1$ at that instant. Then assume that only the force $\vec{F}_2$ acts and find the position $\vec{r}_2$ at that same instant. The actual position will be the vector sum of $\vec{r}_1$ and $\vec{r}_2$.

### (A) Composition of two Simple Harmonic Motions in Same Direction

Suppose two forces act on a particle, the first alone would produce a simple harmonic motion given by

$$x_1 = A_1 \sin \omega t$$

and the second alone would produce a simple harmonic motion given by

$$x_2 = A_2 \sin(\omega t + \delta).$$

Both the motions are along the $x$-direction. The amplitudes may be different and their phases differ by $\delta$. Their frequency is assumed to be same. The resultant position of the particle is then given by

$$x = x_1 + x_2$$

$$= A_1 \sin \omega t + A_2 \sin(\omega t + \delta)$$

$$= A_1 \sin \omega t + A_2 \sin \omega t \cos \delta + A_2 \cos \omega t \sin \delta$$

$$= (A_1 + A_2 \cos \delta) \sin \omega t + (A_2 \sin \delta) \cos \omega t$$

$$= C \sin\omega t + D \cos\omega t$$

$$= \sqrt{C^2 + D^2}\left[\frac{C}{\sqrt{C^2 + D^2}} \sin\omega t + \frac{D}{\sqrt{C^2 + D^2}} \cos\omega t\right] \quad \ldots \text{(i)}$$

where $C = A_1 + A_2 \cos\delta$ and $D = A_2 \sin\delta$.

Now $\frac{C}{\sqrt{C^2 + D^2}}$ and $\frac{D}{\sqrt{C^2 + D^2}}$ both have magnitudes less than 1 and the sum of their squares is 1. Thus, we can find an angle $\varepsilon$ between 0 and $2\pi$ such that

$$\sin\varepsilon = \frac{D}{\sqrt{C^2 + D^2}} \text{ and } \cos\varepsilon = \frac{C}{\sqrt{C^2 + D^2}}.$$

Equation (i) then becomes

$$x = \sqrt{C^2 + D^2}\,(\cos\varepsilon \sin\omega t + \sin\varepsilon \cos\omega t)$$

or, $$x = A \sin(\omega t + \varepsilon) \quad \ldots (12.26)$$

where

$$A = \sqrt{C^2 + D^2}$$

$$= \sqrt{(A_1 + A_2 \cos\delta)^2 + (A_2 \sin\delta)^2}$$

$$= \sqrt{A_1^2 + 2A_1 A_2 \cos\delta + A_2^2 \cos^2\delta + A_2^2 \sin^2\delta}$$

$$= \sqrt{A_1^2 + 2A_1A_2 \cos\delta + A_2^2} \quad \ldots (12.27)$$

and $$\tan\varepsilon = \frac{D}{C} = \frac{A_2 \sin\delta}{A_1 + A_2 \cos\delta}. \quad \ldots (12.28)$$

Equation (12.26) shows that the resultant of two simple harmonic motions along the same direction is itself a simple harmonic motion. The amplitude and phase of the resultant simple harmonic motion depend on the amplitudes of the two component simple harmonic motions as well as the phase difference between them.

**Amplitude of The Resultant Simple Harmonic Motion**

The amplitude of the resultant simple harmonic motion is given by equation (12.27),

$$A = \sqrt{A_1^2 + 2A_1A_2 \cos\delta + A_2^2}.$$

If $\delta = 0$, the two simple harmonic motions are in phase

$$A = \sqrt{A_1^2 + 2A_1A_2 + A_2^2} = A_1 + A_2.$$

The amplitude of the resultant motion is equal to the sum of amplitudes of the individual motions. This is the maximum possible amplitude.

If $\delta = \pi$, the two simple harmonic motions are out of phase and

$$A = \sqrt{A_1^2 - 2A_1A_2 + A_2^2} = A_1 - A_2 \text{ or } A_2 - A_1.$$

As the amplitude is always positive we can write $A = |A_1 - A_2|$. If $A_1 = A_2$ the resultant amplitude is zero and the particle does not oscillate at all.

For any value of $\delta$ other than 0 and $\pi$ the resultant amplitude is between $|A_1 - A_2|$ and $A_1 + A_2$.

***Example 12.11***

*Find the amplitude of the simple harmonic motion obtained by combining the motions*

$$x_1 = (2{\cdot}0 \text{ cm}) \sin\omega t$$

*and* $$x_2 = (2{\cdot}0 \text{ cm}) \sin(\omega t + \pi/3).$$

***Solution*** : The two equations given represent simple harmonic motions along $X$-axis with amplitudes $A_1 = 2{\cdot}0$ cm and $A_2 = 2{\cdot}0$ cm. The phase difference between the two simple harmonic motions is $\pi/3$. The resultant simple harmonic motion will have an amplitude $A$ given by

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2 \cos\delta}$$

$$= \sqrt{(2{\cdot}0 \text{ cm})^2 + (2{\cdot}0 \text{ cm})^2 + 2\,(2{\cdot}0 \text{ cm})^2 \cos\frac{\pi}{3}}$$

$$= 3{\cdot}5 \text{ cm}.$$

**Vector Method of Combining Two Simple Harmonic Motions**

There is a very useful method to remember the equations of resultant simple harmonic motion when two simple harmonic motions of same frequency and in same direction combine. Suppose the two individual motions are represented by

$$x_1 = A_1 \sin\omega t$$

and $$x_2 = A_2 \sin(\omega t + \delta).$$

Let us for a moment represent the first simple harmonic motion by a vector of magnitude $A_1$ and the second simple harmonic motion by another vector of magnitude $A_2$. We draw these vectors in figure (12.16). The vector $A_2$ is drawn at an angle $\delta$ with $A_1$ to represent that the second simple harmonic motion has a phase difference of $\delta$ with the first simple harmonic motion.

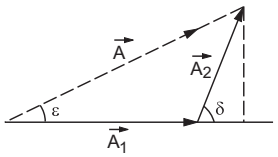



Figure 12.16

The resultant $\vec{A}$ of these two vectors will represent the resultant simple harmonic motion. As we know from vector algebra, the magnitude of the resultant vector is

$$A = \sqrt{A_1^2 + 2A_1A_2 \cos\delta + A_2^2}$$

which is same as equation (12.27). The resultant $\vec{A}$ makes an angle ε with $\vec{A_1}$, where

$$\tan\varepsilon = \frac{A_2 \sin\delta}{A_1 + A_2 \cos\delta}$$

which is same as equation (12.28).

This method can easily be extended to more than two vectors. Figure (12.17) shows the construction for adding three simple harmonic motions in the same direction.

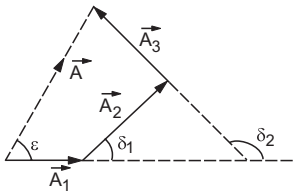



Figure 12.17

$$x_1 = A_1 \sin\omega t$$
$$x_2 = A_2 \sin(\omega t + \delta_1)$$
$$x_3 = A_3 \sin(\omega t + \delta_2).$$

The resultant motion is given by $x = A \sin(\omega t + \varepsilon)$.

### (B) Composition of Two Simple Harmonic Motions in Perpendicular Directions

Suppose two forces act on a particle, the first alone would produce a simple harmonic motion in $x$-direction given by

$$x = A_1 \sin\omega t \qquad \ldots \text{ (i)}$$

and the second would produce a simple harmonic motion in $y$-direction given by

$$y = A_2 \sin(\omega t + \delta). \qquad \ldots \text{ (ii)}$$

The amplitudes $A_1$ and $A_2$ may be different and their phases differ by δ. The frequencies of the two simple harmonic motions are assumed to be equal. The resultant motion of the particle is a combination of the two simple harmonic motions. The position of the particle at time $t$ is $(x, y)$ where $x$ is given by equation (i) and $y$ is given by (ii). The motion is thus two-dimensional and the path of the particle is in general an ellipse. The equation of the path may be obtained by eliminating $t$ from (i) and (ii).

By (i),

$$\sin\omega t = \frac{x}{A_1}.$$

Thus, $$\cos\omega t = \sqrt{1 - \frac{x^2}{A_1^2}}.$$

Putting in (ii)

$$y = A_2 [\sin\omega t \cos\delta + \cos\omega t \sin\delta]$$

$$= A_2 \left[ \frac{x}{A_1} \cos\delta + \sqrt{1 - \frac{x^2}{A_1^2}} \sin\delta \right]$$

or, $$\left( \frac{y}{A_2} - \frac{x}{A_1} \cos\delta \right)^2 = \left( 1 - \frac{x^2}{A_1^2} \right) \sin^2\delta$$

or, $$\frac{y^2}{A_2^2} - \frac{2xy}{A_1 A_2} \cos\delta + \frac{x^2}{A_1^2} \cos^2\delta = \sin^2\delta - \frac{x^2}{A_1^2} \sin^2\delta$$

or, $$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - \frac{2xy \cos\delta}{A_1 A_2} = \sin^2\delta. \qquad \ldots \text{ (12.29)}$$

This is an equation of an ellipse and hence the particle moves in ellipse. Equation (i) shows that $x$ remains between $-A_1$ and $+A_1$ and (ii) shows that $y$ remains between $A_2$ and $-A_2$. Thus, the particle always remains inside the rectangle defined by

$$x = \pm A_1, \; y = \pm A_2.$$

The ellipse given by (12.29) is traced inside this rectangle and touches it on all the four sides (figure 12.18).

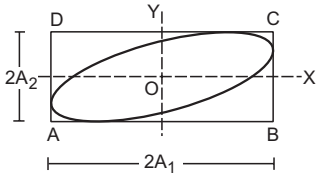



Figure 12.18

#### Special Cases

**(a) $\delta = 0$**

The two simple harmonic motions are in phase. When the $x$-coordinate of the particle crosses the value 0, the $y$-coordinate also crosses the value 0. When $x$-coordinate reaches its maximum value $A_1$, the $y$-coordinate also reaches its maximum value $A_2$. Similarly, when $x$-coordinate reaches its minimum value $-A_1$, the $y$-coordinate reaches its minimum value $-A_2$.

If we substitute $\delta = 0$ in equation (12.29) we get

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - \frac{2xy}{A_1 A_2} = 0$$

or, $$\left( \frac{x}{A_1} - \frac{y}{A_2} \right)^2 = 0$$

or, $$y = \frac{A_2}{A_1} x \qquad \ldots \text{ (iii)}$$

which is the equation of a straight line passing through the origin and having a slope $\tan^{-1}\frac{A_2}{A_1}$. Figure (12.19) shows the path. Equation (iii) represents the diagonal *AC* of the rectangle. The particle moves on this diagonal.

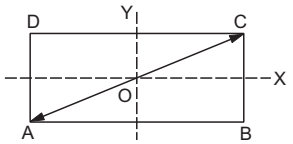



Figure 12.19

Equation (iii) can be directly obtained by dividing (i) by (ii) and putting $\delta = 0$. The displacement of the particle on this straight line at time $t$ is

$$r = \sqrt{x^2 + y^2} = \sqrt{(A_1 \sin\omega t)^2 + (A_2 \sin\omega t)^2}$$

$$= \sqrt{(A_1^2 + A_2^2)}\ \sin\omega t.$$

Thus, the resultant motion is a simple harmonic motion with same frequency and phase as the component motions. The amplitude of the resultant simple harmonic motion is $\sqrt{A_1^2 + A_2^2}$ as is also clear from figure (12.19).

**(b) $\delta = \pi$**

The two simple harmonic motions are out of phase in this case. When the $x$-coordinate of the particle reaches its maximum value $A_1$, the $y$-coordinate reaches its minimum value $-A_2$. Similarly, when the $x$-coordinate reaches its minimum value $-A_1$, the $y$-coordinate takes its maximum value $A_2$.

Putting $\delta = \pi$ in equation (12.29) we get

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} + \frac{2xy}{A_1 A_2} = 0$$

or, $$\left(\frac{x}{A_1} + \frac{y}{A_2}\right)^2 = 0$$

or, $$y = -\frac{A_2}{A_1} \cdot x$$

which is the equation of the line *BD* in figure (12.20).

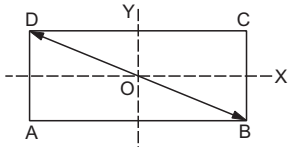



Figure 12.20

Thus the particle oscillates on the diagonal *BD* of the rectangle as shown in figure (12.20).

The displacement on this line at time $t$ may be obtained from equation (i) and (ii) (with $\delta = \pi$).

$$r = \sqrt{x^2 + y^2} = \sqrt{[A_1 \sin\omega t]^2 + [A_2 \sin(\omega t + \pi)]^2}$$

$$= \sqrt{A_1^2 \sin^2\omega t + A_2^2 \sin^2\omega t} = \sqrt{A_1^2 + A_2^2}\ \sin\omega t.$$

Thus the resultant motion is a simple harmonic motion with amplitude $\sqrt{A_1^2 + A_2^2}$.

**(c) $\delta = \pi/2$**

The two simple harmonic motions differ in phase by $\pi/2$. Equations (i) and (ii) may be written as

$$x = A_1 \sin\omega t$$

$$y = A_2 \sin(\omega t + \pi/2) = A_2 \cos\omega t.$$

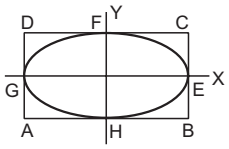



Figure 12.21

The $x$-coordinate takes its maximum value $x = A_1$ when $\sin\omega t = 1$. Then $\cos\omega t = 0$ and hence, the $y$-coordinate is zero. The particle is at the point *E* in figure (12.21). When $x$-coordinate reduces to 0, $\sin\omega t = 0$, and $\cos\omega t$ becomes 1. Then $y$-coordinate takes its maximum value $A_2$ so that the particle reaches the point *F*. Then $x$ reduces to $-A_1$ and $y$ becomes 0. This corresponds to the point *G* of figure (12.21). As $x$ increases to 0 again, $y$ takes its minimum value $-A_2$, the particle is at the point *H*. The motion of the particle is along an ellipse *EFGHE* inscribed in the rectangle shown. The major and the minor axes of the ellipse are along the *X* and *Y*-axes.

Putting $\delta = \pi/2$ in equation (12.29) we get

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} = 1$$

which is the standard equation of an ellipse with its axes along *X* and *Y*-axes and with its centre at the origin. The length of the major and minor axes are $2A_1$ and $2A_2$.

If $A_1 = A_2 = A$ together with $\delta = \pi/2$, the rectangle of figure (12.21) becomes a square and the ellipse becomes a circle. Equation (12.29) becomes

$$x^2 + y^2 = A^2$$

which represents a circle.

Thus, the combination of two simple harmonic motions of equal amplitude in perpendicular directions differing in phase by $\pi/2$ is a circular motion.

The circular motion may be clockwise or anticlockwise, depending on which component leads the other.

## 12.12 DAMPED HARMONIC MOTION

A particle will execute a simple harmonic motion with a constant amplitude if the resultant force on it is proportional to the displacement and is directed opposite to it. Nature provides a large number of situations in which such restoring force acts. The spring-mass system and the simple pendulum are examples. However, in many of the cases some kind of damping force is also present with the restoring force. The damping force may arise due to friction between the moving parts, air resistance or several other causes. The damping force is a function of speed of the moving system and is directed opposite to the velocity. Energy is lost due to the negative work done by the damping force and the system comes to a halt in due course.

The damping force may be a complicated function of speed. In several cases of practical interest the damping force is proportional to the speed. This force may then be written as

$$F = -bv.$$

The equation of motion is

$$m\frac{dv}{dt} = -kx - bv.$$

This equation can be solved using standard methods of calculus. For small damping the solution is of the form

$$x = A_0 e^{-\frac{bt}{2m}} \sin(\omega' t + \delta) \qquad \ldots (12.30)$$

where $\omega' = \sqrt{(k/m) - (b/2m)^2} = \sqrt{\omega_0^2 - (b/2m)^2}$.

For small $b$, the angular frequency $\omega' \approx \sqrt{k/m} = \omega_0$. Thus, the system oscillates with almost the natural angular frequency $\sqrt{k/m}$ (with which the system will oscillate if there is no damping) and with amplitude decreasing with time according to the equation

$$A = A_0 e^{-\frac{bt}{2m}}.$$

The amplitude decreases with time and finally becomes zero. Figure (12.22) shows qualitatively the displacement of the particle as a function of time.

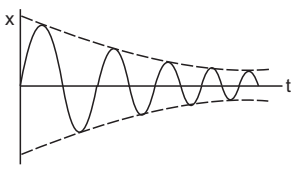



Figure 12.22

If the damping is large the system may not oscillate at all. If displaced, it will go towards the mean position and stay there without overshooting on the other side. The damping for which the oscillation just ceases is called *critical damping*.

## 12.13 FORCED OSCILLATION AND RESONANCE

In certain situations apart from the restoring force and the damping force, there is yet another force applied on the body which itself changes periodically with time. As a simplest case suppose a force $F = F_0 \sin\omega t$ is applied to a body of mass $m$ on which a restoring force $-kx$ and a damping force $-bv$ is acting. The equation of motion for such a body is

$$m\frac{dv}{dt} = -kx - bv + F_0 \sin\omega t.$$

The motion is somewhat complicated for some time and after this the body oscillates with the frequency $\omega$ of the applied periodic force. The displacement is given by

$$x = A\sin(\omega t + \phi).$$

Such an oscillation is called *forced oscillation*. The amplitude of the oscillation is given by

$$A = \frac{F_0/m}{\sqrt{(\omega^2 - \omega_0^2)^2 + (b\omega/m)^2}} \qquad \ldots (12.31)$$

where $\omega_0 = \sqrt{k/m}$ is the natural angular frequency.

In forced oscillation the energy lost due to the damping force is compensated by the work done by the applied force. The oscillations with constant amplitude are, therefore, sustained.

If we vary the angular frequency $\omega$ of the applied force, this amplitude changes and becomes maximum when $\omega = \omega' = \sqrt{\omega_0^2 - b^2/(2m)^2}$. This condition is called *resonance*. For small damping $\omega' \approx \omega_0$ and the resonance occurs when the applied frequency is (almost) equal to the natural frequency.

Figure (12.23) shows the amplitude as a function of the applied frequency. We see that the amplitude is large if the damping is small. Also the resonance is sharp in this case, that is the amplitude rapidly falls if $\omega$ is different from $\omega_0$.

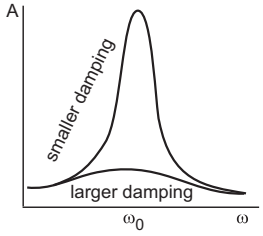



Figure 12.23

If the damping were ideally zero, the amplitude of the forced vibration at resonance would be infinity by equation (12.31). Some damping is always present in mechanical systems and the amplitude remains finite.

However, the amplitude may become very large if the damping is small and the applied frequency is close to the natural frequency. This effect is important in designing bridges and other civil constructions. On July 1, 1940, the newly constructed Tacoma Narrows Bridge (Washington) was opened for traffic. Only four months after this, a mild wind set up the bridge in resonant vibrations. In a few hours the amplitude became so large that the bridge could not stand the stress and a part broke off and went into the water below.

### *Worked Out Examples*

**1.** *The equation of a particle executing simple harmonic motion is* $x = (5\text{ m})\sin\left[(\pi\text{ s}^{-1})t + \frac{\pi}{3}\right]$. *Write down the amplitude, time period and maximum speed. Also find the velocity at* $t = 1$ s.

***Solution*** : Comparing with equation $x = A\sin(\omega t + \delta)$, we see that

the amplitude = 5 m,

and time period $= \frac{2\pi}{\omega} = \frac{2\pi}{\pi\text{ s}^{-1}} = 2$ s.

The maximum speed $= A\omega = 5\text{ m} \times \pi\text{ s}^{-1} = 5\pi\text{ m s}^{-1}$.

The velocity at time $t = \frac{dx}{dt} = A\omega\cos(\omega t + \delta)$.

At $t = 1$ s,

$$v = (5\text{ m})(\pi\text{ s}^{-1})\cos\left(\pi + \frac{\pi}{3}\right) = -\frac{5\pi}{2}\text{ m s}^{-1}.$$

**2.** *A block of mass* 5 kg *executes simple harmonic motion under the restoring force of a spring. The amplitude and the time period of the motion are* 0·1 m *and* 3·14 s *respectively. Find the maximum force exerted by the spring on the block.*

***Solution*** : The maximum force exerted on the block is $kA$ when the block is at the extreme position.

The angular frequency $\omega = \frac{2\pi}{T} = 2\text{ s}^{-1}$.

The spring constant $= k = m\omega^2$

$= (5\text{ kg})(4\text{ s}^{-2}) = 20\text{ N m}^{-1}$.

Maximum force $= kA = (20\text{ N m}^{-1})(0{\cdot}1\text{ m}) = 2$ N.

**3.** *A particle executing simple harmonic motion has angular frequency* 6·28 $\text{s}^{-1}$ *and amplitude* 10 cm. *Find* (*a*) *the time period,* (*b*) *the maximum speed,* (*c*) *the maximum acceleration,* (*d*) *the speed when the displacement is* 6 cm *from the mean position,* (*e*) *the speed at* $t = 1/6$ s *assuming that the motion starts from rest at* $t = 0$.

***Solution*** :

(a) Time period $= \frac{2\pi}{\omega} = \frac{2\pi}{6{\cdot}28}\text{ s} = 1$ s.

(b) Maximum speed $= A\omega = (0{\cdot}1\text{ m})(6{\cdot}28\text{ s}^{-1})$

$= 0{\cdot}628\text{ m s}^{-1}$.

(c) Maximum acceleration $= A\omega^2$

$= (0{\cdot}1\text{ m})(6{\cdot}28\text{ s}^{-1})^2$

$= 4\text{ m s}^{-2}$.

(d) $v = \omega\sqrt{A^2 - x^2} = (6{\cdot}28\text{ s}^{-1})\sqrt{(10\text{ cm})^2 - (6\text{ cm})^2}$

$= 50{\cdot}2\text{ cm s}^{-1}$.

(e) At $t = 0$, the velocity is zero, i.e., the particle is at an extreme. The equation for displacement may be written as

$$x = A\cos\omega t.$$

The velocity is $v = -A\,\omega\sin\omega t$.

At $t = \frac{1}{6}$ s, $v = -(0{\cdot}1\text{ m})(6{\cdot}28\text{ s}^{-1})\sin\left(\frac{6{\cdot}28}{6}\right)$

$= (-0{\cdot}628\text{ m s}^{-1})\sin\frac{\pi}{3}$

$= -54{\cdot}4\text{ cm s}^{-1}$.

**4.** *A particle executes a simple harmonic motion of time period T. Find the time taken by the particle to go directly from its mean position to half the amplitude.*

***Solution*** : Let the equation of motion be $x = A\sin\omega t$.

At $t = 0$, $x = 0$ and hence the particle is at its mean position. Its velocity is

$$v = A\,\omega\cos\omega t = A\,\omega$$

which is positive. So it is going towards $x = A/2$.

The particle will be at $x = A/2$, at a time $t$, where

$$\frac{A}{2} = A\sin\omega t$$

or, $\sin\omega t = 1/2$

or, $\omega t = \pi/6$.

Here minimum positive value of $\omega t$ is chosen because we are interested in finding the time taken by the particle to directly go from $x = 0$ to $x = A/2$.

Thus, $t = \frac{\pi}{6\,\omega} = \frac{\pi}{6(2\pi/T)} = \frac{T}{12}$.

**5.** *A block of mass m hangs from a vertical spring of spring constant k. If it is displaced from its equilibrium position, find the time period of oscillations.*

***Solution*** **:** Suppose the length of the spring is stretched by a length $\Delta l$. The tension in the spring is $k\,\Delta l$ and this is the force by the spring on the block. The other force on the block is $mg$ due to gravity. For equilibrium, $mg = k\,\Delta l$ or $\Delta l = mg/k$. Take this position of the block as $x = 0$. If the block is further displaced by $x$, the resultant force is $k\left(\frac{mg}{k} + x\right) - mg = kx$.

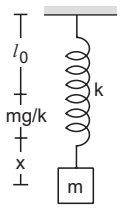



Figure 12-W1

Thus, the resultant force is proportional to the displacement. The motion is simple harmonic with a time period $T = 2\pi\sqrt{\frac{m}{k}}$.

We see that in vertical oscillations, gravity has no effect on time period. The only effect it has is to shift the equilibrium position by a distance $mg/k$ as if the natural length is increased (or decreased if the lower end of the spring is fixed) by $mg/k$.

**6.** *A particle suspended from a vertical spring oscillates* 10 *times per second. At the highest point of oscillation the spring becomes unstretched. (a) Find the maximum speed of the block. (b) Find the speed when the spring is stretched by* 0·20 cm. *Take* $g = \pi^2\ \text{m s}^{-2}$.

***Solution*** **:**

(a) The mean position of the particle during vertical oscillations is $mg/k$ distance away from its position when the spring is unstretched. At the highest point, i.e., at an extreme position, the spring is unstretched.

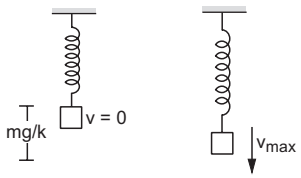



Figure 12-W2

Hence the amplitude is

$$A = \frac{mg}{k}. \qquad \ldots \text{(i)}$$

The angular frequency is

$$\omega = \sqrt{\frac{k}{m}} = 2\pi\nu = (20\pi)\ \text{s}^{-1} \qquad \ldots \text{(ii)}$$

or, $\frac{m}{k} = \frac{1}{400\,\pi^2}\,\text{s}^2$.

Putting in (i), the amplitude is

$$A = \left(\frac{1}{400\,\pi^2}\,\text{s}^2\right)\left(\pi^2\,\frac{\text{m}}{\text{s}^2}\right) = \frac{1}{400}\,\text{m} = 0{\cdot}25\ \text{cm}.$$

The maximum speed $= A\,\omega$

$$= (0{\cdot}25\ \text{cm})\ (20\,\pi\ \text{s}^{-1}) = 5\,\pi\ \text{cm s}^{-1}.$$

(b) When the spring is stretched by 0·20 cm, the block is 0·25 cm – 0·20 cm = 0·05 cm above the mean position. The speed at this position will be

$$v = \omega\sqrt{A^2 - x^2} = (20\,\pi\ \text{s}^{-1})\sqrt{(0{\cdot}25\ \text{cm})^2 - (0{\cdot}05\ \text{cm})^2} \cong 15{\cdot}4\ \text{cm s}^{-1}.$$

**7.** *The pulley shown in figure (12-W3) has a moment of inertia I about its axis and mass m. Find the time period of vertical oscillation of its centre of mass. The spring has spring constant k and the string does not slip over the pulley.*

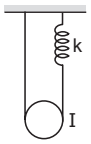



Figure 12-W3

***Solution*** **:** Let us first find the equilibrium position. For rotational equilibrium of the pulley, the tensions in the two strings should be equal. Only then the torque on the pulley will be zero. Let this tension be $T$. The extension of the spring will be $y = T/k$, as the tension in the spring will be the same as the tension in the string. For translational equilibrium of the pulley,

$$2\,T = mg \text{ or, } 2\,ky = mg \quad \text{or,} \quad y = \frac{mg}{2\,k}.$$

The spring is extended by a distance $\frac{mg}{2\,k}$ when the pulley is in equilibrium.

Now suppose, the centre of the pulley goes down further by a distance $x$. The total increase in the length of the string plus the spring is $2x$ ($x$ on the left of the pulley and $x$ on the right). As the string has a constant length, the extension of the spring is $2x$. The energy of the system is

$$U = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2 - mgx + \frac{1}{2}k\left(\frac{mg}{2\,k} + 2\,x\right)^2$$

$$= \frac{1}{2}\left(\frac{I}{r^2} + m\right) v^2 + \frac{m^2 g^2}{8k} + 2kx^2.$$

As the system is conservative, $\frac{dU}{dt} = 0$,

giving $0 = \left(\frac{I}{r^2} + m\right) v \frac{dv}{dt} + 4kxv$

or, $\frac{dv}{dt} = -\frac{4kx}{\left(\frac{I}{r^2} + m\right)}$

or, $a = -\omega^2 x$, where $\omega^2 = \frac{4k}{\left(\frac{I}{r^2} + m\right)}$.

Thus, the centre of mass of the pulley executes a simple harmonic motion with time period

$$T = 2\pi \sqrt{\left(\frac{I}{r^2} + m\right)/(4k)}\ .$$

**8.** *The friction coefficient between the two blocks shown in figure (12-W4) is* $\mu$ *and the horizontal plane is smooth. (a) If the system is slightly displaced and released, find the time period. (b) Find the magnitude of the frictional force between the blocks when the displacement from the mean position is x. (c) What can be the maximum amplitude if the upper block does not slip relative to the lower block ?*

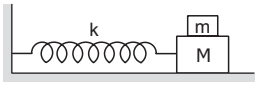



Figure 12-W4

***Solution*** **:**

(a) For small amplitude, the two blocks oscillate together. The angular frequency is

$$\omega = \sqrt{\frac{k}{M+m}}$$

and so the time period $T = 2\pi \sqrt{\frac{M+m}{k}}$ .

(b) The acceleration of the blocks at displacement $x$ from the mean position is

$$a = -\omega^2 x = \frac{-kx}{M+m}.$$

The resultant force on the upper block is, therefore,

$$ma = \frac{-mkx}{M+m}.$$

This force is provided by the friction of the lower block. Hence, the magnitude of the frictional force is $\frac{mk\,|\,x\,|}{M+m}$.

(c) Maximum force of friction required for simple harmonic motion of the upper block is $\frac{m\,k\,A}{M+m}$ at the extreme positions. But the maximum frictional force can only be $\mu\ mg$. Hence

$$\frac{m\,k\,A}{M+m} = \mu\ mg \ \text{ or, } \ A = \frac{\mu\,(M+m)\,g}{k}.$$

**9.** *The left block in figure (12-W5) collides inelastically with the right block and sticks to it. Find the amplitude of the resulting simple harmonic motion.*

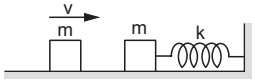



Figure 12-W5

***Solution*** **:** Assuming the collision to last for a small interval only, we can apply the principle of conservation of momentum. The common velocity after the collision is $\frac{v}{2}$. The kinetic energy $= \frac{1}{2}(2m)\left(\frac{v}{2}\right)^2 = \frac{1}{4}mv^2$. This is also the total energy of vibration as the spring is unstretched at this moment. If the amplitude is $A$, the total energy can also be written as $\frac{1}{2}kA^2$. Thus

$$\frac{1}{2}kA^2 = \frac{1}{4}mv^2, \text{ giving } A = \sqrt{\frac{m}{2k}}\ v.$$

**10.** *Describe the motion of the mass m shown in figure (*12-W6*). The walls and the block are elastic.*

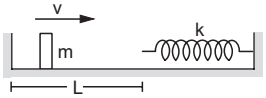



Figure 12-W6

***Solution*** **:** The block reaches the spring with a speed $v$. It now compresses the spring. The block is decelerated due to the spring force, comes to rest when $\frac{1}{2}mv^2 = \frac{1}{2}kx^2$ and returns back. It is accelerated due to the spring force till the spring acquires its natural length. The contact of the block with the spring is now broken. At this instant it has regained its speed $v$ (towards left) as the spring is unstretched and no potential energy is stored. This process takes half the period of oscillation, i.e., $\pi\sqrt{m/k}$. The block strikes the left wall after a time $L/v$ and as the collision is elastic, it rebounds with the same speed $v$. After a time $L/v$, it again reaches the spring and the process is repeated. The block thus undergoes periodic motion with time period $\pi\sqrt{m/k} + \frac{2L}{v}$.

**11.** *A block of mass m is suspended from the ceiling of a stationary standing elevator through a spring of spring constant k. Suddenly, the cable breaks and the elevator starts falling freely. Show that the block now executes a*

*simple harmonic motion of amplitude $mg/k$ in the elevator.*

***Solution*** **:** When the elevator is stationary, the spring is stretched to support the block. If the extension is $x$, the tension is $kx$ which should balance the weight of the block.

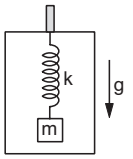



Figure 12-W7

Thus, $x = mg/k$. As the cable breaks, the elevator starts falling with acceleration '$g$'. We shall work in the frame of reference of the elevator. Then we have to use a pseudo force $mg$ upward on the block. This force will 'balance' the weight. Thus, the block is subjected to a net force $kx$ by the spring when it is at a distance $x$ from the position of unstretched spring. Hence, its motion in the elevator is simple harmonic with its mean position corresponding to the unstretched spring. Initially, the spring is stretched by $x = mg/k$, where the velocity of the block (with respect to the elevator) is zero. Thus, the amplitude of the resulting simple harmonic motion is $mg/k$.

**12.** *The spring shown in figure (12-W8) is kept in a stretched position with extension $x_0$ when the system is released. Assuming the horizontal surface to be frictionless, find the frequency of oscillation.*

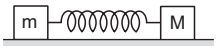



Figure 12-W8

***Solution*** **:** Considering "the two blocks plus the spring" as a system, there is no external resultant force on the system. Hence the centre of mass of the system will remain at rest. The mean positions of the two simple harmonic motions occur when the spring becomes unstretched. If the mass $m$ moves towards right through a distance $x$ and the mass $M$ moves towards left through a distance $X$ before the spring acquires natural length,

$$x + X = x_0. \qquad \ldots \text{(i)}$$

$x$ and $X$ will be the amplitudes of the two blocks $m$ and $M$ respectively. As the centre of mass should not change during the motion, we should also have

$$mx = MX. \qquad \ldots \text{(ii)}$$

From (i) and (ii), $x = \dfrac{Mx_0}{M+m}$ and $X = \dfrac{mx_0}{M+m}$.

Hence, the left block is $x = \dfrac{Mx_0}{M+m}$ distance away from its mean position in the beginning of the motion. The force by the spring on this block at this instant is equal to the tension of spring, i.e., $T = kx_0$.

Now $x = \dfrac{Mx_0}{M+m}$ or, $x_0 = \dfrac{M+m}{M}x$

Thus, $T = \dfrac{k(M+m)}{M}x$ or, $a = \dfrac{T}{m} = \dfrac{k(M+m)}{Mm}x$.

The angular frequency is, therefore, $\omega = \sqrt{\dfrac{k(M+m)}{Mm}}$

and the frequency is $\nu = \dfrac{\omega}{2\pi} = \dfrac{1}{2\pi}\sqrt{\dfrac{k(M+m)}{Mm}}$ .

**13.** *Assume that a narrow tunnel is dug between two diametrically opposite points of the earth. Treat the earth as a solid sphere of uniform density. Show that if a particle is released in this tunnel, it will execute a simple harmonic motion. Calculate the time period of this motion.*

***Solution*** **:**

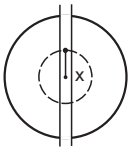



Figure 12-W9

Consider the situation shown in figure (12-W9). Suppose at an instant $t$ the particle in the tunnel is at a distance $x$ from the centre of the earth. Let us draw a sphere of radius $x$ with its centre at the centre of the earth. Only the part of the earth within this sphere will exert a net attraction on the particle. Mass of this part is

$$M' = \frac{\frac{4}{3}\pi x^3}{\frac{4}{3}\pi R^3} M = \frac{x^3}{R^3} M.$$

The force of attraction is, therefore,

$$F = \frac{G(x^3/R^3)\,Mm}{x^2} = \frac{GMm}{R^3}x.$$

This force acts towards the centre of the earth. Thus, the resultant force on the particle is opposite to the displacement from the centre of the earth and is proportional to it. The particle, therefore, executes a simple harmonic motion in the tunnel with the centre of the earth as the mean position.

The force constant is $k = \dfrac{GMm}{R^3}$, so that the time period is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{R^3}{GM}}.$$

**14.** *A simple pendulum of length* 40 cm *oscillates with an angular amplitude of* 0·04 rad. *Find (a) the time period, (b) the linear amplitude of the bob, (c) the speed of the bob when the string makes* 0·02 rad *with the vertical and (d) the angular acceleration when the bob is in momentary rest. Take* $g = 10\ \text{m s}^{-2}$.

***Solution :***

(a) The angular frequency is

$$\omega = \sqrt{g/l} = \sqrt{\frac{10\ \text{m s}^{-2}}{0{\cdot}4\ \text{m}}} = 5\ \text{s}^{-1}.$$

The time period is

$$\frac{2\pi}{\omega} = \frac{2\pi}{5\ \text{s}^{-1}} = 1{\cdot}26\ \text{s}.$$

(b) Linear amplitude $= 40\ \text{cm} \times 0{\cdot}04 = 1{\cdot}6\ \text{cm}$.

(c) Angular speed at displacement 0·02 rad is

$$\Omega = (5\ \text{s}^{-1})\sqrt{(0{\cdot}04)^2 - (0{\cdot}02)^2}\ \text{rad} = 0{\cdot}17\ \text{rad s}^{-1}.$$

Linear speed of the bob at this instant

$$= (40\ \text{cm}) \times 0{\cdot}17\ \text{s}^{-1} = 6{\cdot}8\ \text{cm s}^{-1}.$$

(d) At momentary rest, the bob is in extreme position. Thus, the angular acceleration

$$\alpha = (0{\cdot}04\ \text{rad})\ (25\ \text{s}^{-2}) = 1\ \text{rad s}^{-2}.$$

**15.** *A simple pendulum having a bob of mass m undergoes small oscillations with amplitude* $\theta_0$. *Find the tension in the string as a function of the angle made by the string with the vertical. When is this tension maximum, and when is it minimum* ?

***Solution*** : Suppose the speed of the bob at angle $\theta$ is $v$. Using conservation of energy between the extreme position and the position with angle $\theta$,

$$\frac{1}{2}mv^2 = mgl\ (\cos\theta - \cos\theta_0). \qquad \ldots\ (i)$$

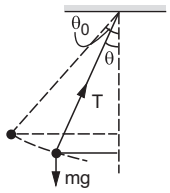



Figure 12-W10

As the bob moves in a circular path, the force towards the centre should be equal to $mv^2/l$. Thus,

$$T - mg\cos\theta = mv^2/l.$$

Using (i),

$$T - mg\cos\theta = 2\,mg\,(\cos\theta - \cos\theta_0)$$

or, $$T = 3\,mg\cos\theta - 2\,mg\cos\theta_0.$$

Now $\cos\theta$ is maximum at $\theta = 0$ and decreases as $|\theta|$ increases (for $|\theta| < 90°$).

Thus, the tension is maximum when $\theta = 0$, i.e., at the mean position and is minimum when $\theta = \pm\theta_0$, i.e., at extreme positions.

**16.** *A simple pendulum is taken at a place where its separation from the earth's surface is equal to the radius of the earth. Calculate the time period of small oscillations if the length of the string is* 1·0 m. *Take* $g = \pi^2\ \text{m s}^{-2}$ *at the surface of the earth.*

***Solution :*** At a height $R$ (radius of the earth) the acceleration due to gravity is

$$g' = \frac{GM}{(R+R)^2} = \frac{1}{4}\frac{GM}{R^2} = g/4.$$

The time period of small oscillations of the simple pendulum is

$$T = 2\pi\sqrt{l/g'} = 2\pi\sqrt{\frac{1{\cdot}0\ \text{m}}{\frac{1}{4} \times \pi^2\ \text{m s}^{-2}}} = 2\pi\left(\frac{2}{\pi}\ \text{s}\right) = 4\ \text{s}.$$

**17.** *A simple pendulum is suspended from the ceiling of a car accelerating uniformly on a horizontal road. If the acceleration is* $a_0$ *and the length of the pendulum is l, find the time period of small oscillations about the mean position.*

***Solution :*** We shall work in the car frame. As it is accelerated with respect to the road, we shall have to apply a pseudo force $ma_0$ on the bob of mass m.

For mean position, the acceleration of the bob with respect to the car should be zero. If $\theta$ be the angle made by the string with the vertical, the tension, weight and the pseudo force will add to zero in this position.

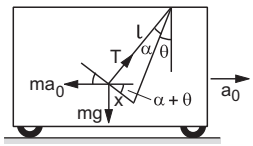



Figure 12-W11

Suppose, at some instant during oscillation, the string is further deflected by an angle $\alpha$ so that the displacement of the bob is $x$. Taking the components perpendicular to the string,

component of $T = 0$,

component of $mg = mg\sin(\alpha + \theta)$ and

component of $ma_0 = -ma_0\cos(\alpha + \theta)$.

Thus, the resultant component $F$

$$= m[g\sin(\alpha + \theta) - a_0\cos(\alpha + \theta)].$$

Expanding the sine and cosine and putting $\cos\alpha \approx 1$, $\sin\alpha \approx \alpha = x/l$, we get

$$F = m\left[g\sin\theta - a_0\cos\theta + (g\cos\theta + a_0\sin\theta)\frac{x}{l}\right]. \qquad \ldots\ (i)$$

At $x = 0$, the force $F$ on the bob should be zero, as this is the mean position. Thus by (i),

$$0 = m[g \sin\theta - a_0 \cos\theta] \quad \ldots \text{(ii)}$$

giving $\tan\theta = \dfrac{a_0}{g}$

Thus, $\sin\theta = \dfrac{a_0}{\sqrt{a_0^2 + g^2}}$ … (iii)

$$\cos\theta = \frac{g}{\sqrt{a_0^2 + g^2}}. \quad \ldots \text{(iv)}$$

Putting (ii), (iii) and (iv) in (i), $F = m\sqrt{g^2 + a_0^2}\ \dfrac{x}{l}$

or, $F = m\,\omega^2 x$, where $\omega^2 = \dfrac{\sqrt{g^2 + a_0^2}}{l}$.

This is an equation of simple harmonic motion with time period

$$t = \frac{2\pi}{\omega} = 2\pi \frac{\sqrt{l}}{\left(g^2 + a_0^2\right)^{1/4}}.$$

An easy working rule may be found out as follows. In the mean position, the tension, the weight and the pseudo force balance.

From figure (12-W12), the tension is

$$T = \sqrt{(ma_0)^2 + (mg)^2}$$

or, $\dfrac{T}{m} = \sqrt{a_0^2 + g^2}$.

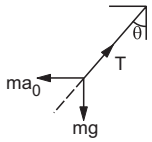



Figure 12-W12

This plays the role of effective '$g$'. Thus the time period is

$$t = 2\pi\sqrt{\frac{l}{T/m}} = 2\pi \frac{\sqrt{l}}{[g^2 + a_0^2]^{1/4}}.$$

**18.** *A uniform meter stick is suspended through a small pin hole at the* 10 cm *mark. Find the time period of small oscillation about the point of suspension.*

***Solution*** **:** Let the mass of the stick be $m$. The moment of inertia of the stick about the axis of rotation through the point of suspension is

$$I = \frac{ml^2}{12} + md^2,$$

where $l = 1$ m and $d = 40$ cm.

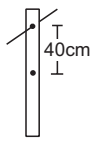



Figure 12-W13

The separation between the centre of mass of the stick and the point of suspension is $d = 40$ cm. The time period of this physical pendulum is

$$T = 2\pi\sqrt{\frac{I}{mgd}}$$

$$= 2\pi\sqrt{\left(\frac{ml^2}{12} + md^2\right)/(mgd)}$$

$$= 2\pi\left[\sqrt{\left(\frac{1}{12} + 0{\cdot}16\right)/4}\right] \text{s} = 1{\cdot}55 \text{ s}.$$

**19.** *The moment of inertia of the disc used in a torsional pendulum about the suspension wire is* 0·2 kg-m$^2$. *It oscillates with a period of* 2 s. *Another disc is placed over the first one and the time period of the system becomes* 2·5 s. *Find the moment of inertia of the second disc about the wire.*

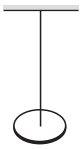


Figure 12-W14

***Solution*** **:**

Let the torsional constant of the wire be $k$. The moment of inertia of the first disc about the wire is 0·2 kg-m$^2$. Hence, the time period is

$$2 \text{ s} = 2\pi\sqrt{\frac{I}{K}}$$

$$= 2\pi\sqrt{\frac{0{\cdot}2 \text{ kg-m}^2}{k}}. \quad \ldots \text{(i)}$$

When the second disc having moment of inertia $I_1$ about the wire is added, the time period is

$$2{\cdot}5 \text{ s} = 2\pi\sqrt{\frac{0{\cdot}2 \text{ kg-m}^2 + I_1}{k}} \quad \ldots \text{(ii)}$$

From (i) and (ii), $\dfrac{6{\cdot}25}{4} = \dfrac{0{\cdot}2 \text{ kg-m}^2 + I_1}{0{\cdot}2 \text{ kg-m}^2}$.

This gives $I_1 \approx 0{\cdot}11$ kg-m$^2$.

**20.** *A uniform rod of mass m and length l is suspended through a light wire of length l and torsional constant k as shown in figure (12-W15). Find the time period if the system makes (a) small oscillations in the vertical plane about the suspension point and (b) angular oscillations in the horizontal plane about the centre of the rod.*

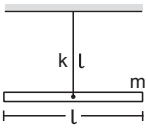



Figure 12-W15

***Solution*** :

(a) The oscillations take place about the horizontal line through the point of suspension and perpendicular to the plane of the figure. The moment of inertia of the rod about this line is

$$\frac{ml^2}{12} + ml^2 = \frac{13}{12} ml^2.$$

The time period $= 2\pi \sqrt{\frac{I}{mgl}} = 2\pi \sqrt{\frac{13\, ml^2}{12\, mgl}}$

$$= 2\pi \sqrt{\frac{13\, l}{12\, g}}\ .$$

(b) The angular oscillations take place about the suspension wire. The moment of inertia about this line is $ml^2/12$. The time period is

$$2\pi \sqrt{\frac{I}{k}} = 2\pi \sqrt{\frac{ml^2}{12\, k}}\ .$$

**21.** *A particle is subjected to two simple harmonic motions*

$$x_1 = A_1 \sin\omega t$$

*and* $x_2 = A_2 \sin(\omega t + \pi/3).$

*Find (a) the displacement at $t = 0$, (b) the maximum speed of the particle and (c) the maximum acceleration of the particle.*

***Solution*** :

(a) At $t = 0$, $x_1 = A_1 \sin\omega t = 0$

and $x_2 = A_2 \sin(\omega t + \pi/3)$

$$= A_2 \sin(\pi/3) = \frac{A_2 \sqrt{3}}{2}\ .$$

Thus, the resultant displacement at $t = 0$ is

$$x = x_1 + x_2 = \frac{A_2 \sqrt{3}}{2}\ .$$

(b) The resultant of the two motions is a simple harmonic motion of the same angular frequency ω. The amplitude of the resultant motion is

$$A = \sqrt{A_1^2 + A_2^2 + 2\, A_1 A_2 \cos(\pi/3)}$$

$$= \sqrt{A_1^2 + A_2^2 + A_1 A_2}\ .$$

The maximum speed is

$$v_{max} = A\, \omega = \omega \sqrt{A_1^2 + A_2^2 + A_1 A_2}\ .$$

(c) The maximum acceleration is

$$a_{max} = A\, \omega^2 = \omega^2 \sqrt{A_1^2 + A_2^2 + A_1 A_2}\ .$$

**22.** *A particle is subjected to two simple harmonic motions in the same direction having equal amplitudes and equal frequency. If the resultant amplitude is equal to the amplitude of the individual motions, find the phase difference between the individual motions.*

***Solution*** : Let the amplitudes of the individual motions be $A$ each. The resultant amplitude is also $A$. If the phase difference between the two motions is δ,

$$A = \sqrt{A^2 + A^2 + 2A\ .\ A\ .\ \cos\delta}$$

or, $$= A \sqrt{2(1 + \cos\delta)} = 2A \cos\frac{\delta}{2}$$

or, $$\cos\frac{\delta}{2} = \frac{1}{2}$$

or, $$\delta = 2\pi/3.$$

□

## QUESTIONS FOR SHORT ANSWER

**1.** A person goes to bed at sharp 10·00 pm every day. Is it an example of periodic motion ? If yes, what is the time period ? If no, why ?

**2.** A particle executing simple harmonic motion comes to rest at the extreme positions. Is the resultant force on the particle zero at these positions according to Newton's first law ?

**3.** Can simple harmonic motion take place in a noninertial frame? If yes, should the ratio of the force applied with the displacement be constant ?

**4.** A particle executes simple harmonic motion. If you are told that its velocity at this instant is zero, can you say what is its displacement ? If you are told that its velocity at this instant is maximum, can you say what is its displacement ?

**5.** A small creature moves with constant speed in a vertical circle on a bright day. Does its shadow formed by the sun on a horizontal plane move in a simple harmonic motion ?

**6.** A particle executes simple harmonic motion. Let $P$ be a point near the mean position and $Q$ be a point near an extreme. The speed of the particle at $P$ is larger than the speed at $Q$. Still the particle crosses $P$ and $Q$ equal number of times in a given time interval. Does it make you unhappy ?

**7.** In measuring time period of a pendulum, it is advised to measure the time between consecutive passage through the mean position in the same direction. This is said to result in better accuracy than measuring time between consecutive passage through an extreme position. Explain.

**8.** It is proposed to move a particle in simple harmonic motion on a rough horizontal surface by applying an external force along the line of motion. Sketch the graph of the applied force against the position of the particle. Note that the applied force has two values for a given position depending on whether the particle is moving in positive or negative direction.

**9.** Can the potential energy in a simple harmonic motion be negative ? Will it be so if we choose zero potential energy at some point other than the mean position ?

**10.** The energy of a system in simple harmonic motion is given by $E = \frac{1}{2} m \omega^2 A^2$. Which of the following two statements is more appropriate ?
(A) The energy is increased because the amplitude is increased.
(B) The amplitude is increased because the energy is increased.

**11.** A pendulum clock gives correct time at the equator. Will it gain time or loose time as it is taken to the poles ?

**12.** Can a pendulum clock be used in an earth-satellite ?

**13.** A hollow sphere filled with water is used as the bob of a pendulum. Assume that the equation for simple pendulum is valid with the distance between the point of suspension and centre of mass of the bob acting as the effective length of the pendulum. If water slowly leaks out of the bob, how will the time period vary ?

**14.** A block of known mass is suspended from a fixed support through a light spring. Can you find the time period of vertical oscillation only by measuring the extension of the spring when the block is in equilibrium ?

**15.** A platoon of soldiers marches on a road in steps according to the sound of a marching band. The band is stopped and the soldiers are ordered to break the steps while crossing a bridge. Why ?

**16.** The force acting on a particle moving along $X$-axis is $F = -k(x - v_0 t)$ where $k$ is a positive constant. An observer moving at a constant velocity $v_0$ along the $X$-axis looks at the particle. What kind of motion does he find for the particle ?

## OBJECTIVE I

**1.** A student says that he had applied a force $F = -k\sqrt{x}$ on a particle and the particle moved in simple harmonic motion. He refuses to tell whether $k$ is a constant or not. Assume that he has worked only with positive $x$ and no other force acted on the particle.
(a) As $x$ increases $k$ increases.
(b) As $x$ increases $k$ decreases.
(c) As $x$ increases $k$ remains constant.
(d) The motion cannot be simple harmonic.

**2.** The time period of a particle in simple harmonic motion is equal to the time between consecutive appearances of the particle at a particular point in its motion. This point is
(a) the mean position (b) an extreme position
(c) between the mean position and the positive extreme
(d) between the mean position and the negative extreme.

**3.** The time period of a particle in simple harmonic motion is equal to the smallest time between the particle acquiring a particular velocity $\vec{v}$. The value of $v$ is
(a) $v_{max}$ (b) 0
(c) between 0 and $v_{max}$ (d) between 0 and $-v_{max}$.

**4.** The displacement of a particle in simple harmonic motion in one time period is
(a) $A$ (b) $2A$ (c) $4A$ (d) zero.

**5.** The distance moved by a particle in simple harmonic motion in one time period is
(a) $A$ (b) $2A$ (c) $4A$ (d) zero.

**6.** The average acceleration in one time period in a simple harmonic motion is
(a) $A\omega^2$ (b) $A\omega^2/2$ (c) $A\omega^2/\sqrt{2}$ (d) zero.

**7.** The motion of a particle is given by $x = A\sin\omega t + B\cos\omega t$. The motion of the particle is
(a) not simple harmonic
(b) simple harmonic with amplitude $A + B$
(c) simple harmonic with amplitude $(A + B)/2$
(d) simple harmonic with amplitude $\sqrt{A^2 + B^2}$.

**8.** The displacement of a particle is given by $\vec{r} = A(\vec{i}\cos\omega t + \vec{j}\sin\omega t)$. The motion of the particle is
(a) simple harmonic (b) on a straight line
(c) on a circle (d) with constant acceleration.

**9.** A particle moves on the $X$-axis according to the equation $x = A + B\sin\omega t$. The motion is simple harmonic with amplitude
(a) $A$ (b) $B$ (c) $A + B$ (d) $\sqrt{A^2 + B^2}$.

**10.** Figure (12-Q1) represents two simple harmonic motions.

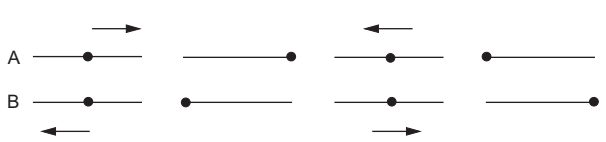



Figure 12-Q1

The parameter which has different values in the two motions is
(a) amplitude (b) frequency
(c) phase (d) maximum velocity.

**11.** The total mechanical energy of a spring-mass system in simple harmonic motion is $E = \frac{1}{2} m \omega^2 A^2$. Suppose the oscillating particle is replaced by another particle of double the mass while the amplitude $A$ remains the same. The new mechanical energy will
(a) become $2E$ (b) become $E/2$
(c) become $\sqrt{2}E$ (d) remain $E$.

**12.** The average energy in one time period in simple harmonic motion is
(a) $\frac{1}{2} m \omega^2 A^2$ (b) $\frac{1}{4} m \omega^2 A^2$
(c) $m \omega^2 A^2$ (d) zero.

**13.** A particle executes simple harmonic motion with a frequency $\nu$. The frequency with which the kinetic energy oscillates is
(a) $\nu/2$ (b) $\nu$ (c) $2\nu$ (d) zero.

**14.** A particle executes simple harmonic motion under the restoring force provided by a spring. The time period is $T$. If the spring is divided in two equal parts and one part is used to continue the simple harmonic motion, the time period will
(a) remain $T$ (b) become $2T$
(c) become $T/2$ (d) become $T/\sqrt{2}$.

**15.** Two bodies $A$ and $B$ of equal mass are suspended from two separate massless springs of spring constant $k_1$ and $k_2$ respectively. If the bodies oscillate vertically such that their maximum velocities are equal, the ratio of the amplitude of $A$ to that of $B$ is
(a) $k_1/k_2$ (b) $\sqrt{k_1/k_2}$
(c) $k_2/k_1$ (d) $\sqrt{k_2/k_1}$.

**16.** A spring-mass system oscillates with a frequency $\nu$. If it is taken in an elevator slowly accelerating upward, the frequency will
(a) increase (b) decrease
(c) remain same (d) become zero.

**17.** A spring-mass system oscillates in a car. If the car accelerates on a horizontal road, the frequency of oscillation will
(a) increase (b) decrease
(c) remain same (d) become zero.

**18.** A pendulum clock that keeps correct time on the earth is taken to the moon. It will run
(a) at correct rate (b) 6 times faster
(c) $\sqrt{6}$ times faster (d) $\sqrt{6}$ times slower.

**19.** A wall clock uses a vertical spring-mass system to measure the time. Each time the mass reaches an extreme position, the clock advances by a second. The clock gives correct time at the equator. If the clock is taken to the poles it will
(a) run slow (b) run fast
(c) stop working (d) give correct time.

**20.** A pendulum clock keeping correct time is taken to high altitudes,
(a) it will keep correct time
(b) its length should be increased to keep correct time
(c) its length should be decreased to keep correct time
(d) it cannot keep correct time even if the length is changed.

**21.** The free end of a simple pendulum is attached to the ceiling of a box. The box is taken to a height and the pendulum is oscillated. When the bob is at its lowest point, the box is released to fall freely. As seen from the box during this period, the bob will
(a) continue its oscillation as before
(b) stop
(c) will go in a circular path
(d) move on a straight line.

## OBJECTIVE II

**1** Select the correct statements.
(a) A simple harmonic motion is necessarily periodic.
(b) A simple harmonic motion is necessarily oscillatory.
(c) An oscillatory motion is necessarily periodic.
(d) A periodic motion is necessarily oscillatory.

**2.** A particle moves in a circular path with a uniform speed. Its motion is
(a) periodic (b) oscillatory
(c) simple harmonic (d) angular simple harmonic.

**3.** A particle is fastened at the end of a string and is whirled in a vertical circle with the other end of the string being fixed. The motion of the particle is
(a) periodic (b) oscillatory
(c) simple harmonic (d) angular simple harmonic.

**4.** A particle moves in a circular path with a continuously increasing speed. Its motion is
(a) periodic (b) oscillatory
(c) simple harmonic (d) none of them.

**5.** The motion of a torsional pendulum is
(a) periodic (b) oscillatory
(c) simple harmonic (d) angular simple harmonic.

**6.** Which of the following quantities are always negative in a simple harmonic motion ?
(a) $\vec{F} \cdot \vec{a}$. (b) $\vec{v} \cdot \vec{r}$. (c) $\vec{a} \cdot \vec{r}$. (d) $\vec{F} \cdot \vec{r}$.

**7.** Which of the following quantities are always positive in a simple harmonic motion ?
(a) $\vec{F} \cdot \vec{a}$. (b) $\vec{v} \cdot \vec{r}$. (c) $\vec{a} \cdot \vec{r}$. (d) $\vec{F} \cdot \vec{r}$.

**8.** Which of the following quantities are always zero in a simple harmonic motion ?
(a) $\vec{F} \times \vec{a}$. (b) $\vec{v} \times \vec{r}$. (c) $\vec{a} \times \vec{r}$. (d) $\vec{F} \times \vec{r}$.

**9.** Suppose a tunnel is dug along a diameter of the earth. A particle is dropped from a point, a distance $h$ directly

above the tunnel. The motion of the particle as seen from the earth is
(a) simple harmonic (b) parabolic
(c) on a straight line (d) periodic.

**10.** For a particle executing simple harmonic motion, the acceleration is proportional to
(a) displacement from the mean position
(b) distance from the mean position
(c) distance travelled since $t = 0$
(d) speed.

**11.** A particle moves in the $X$-$Y$ plane according to the equation

$$\vec{r} = (\vec{i} + 2\vec{j})\, A \cos\omega t.$$

The motion of the particle is
(a) on a straight line (b) on an ellipse
(c) periodic (d) simple harmonic.

**12.** A particle moves on the $X$-axis according to the equation $x = x_0 \sin^2 \omega t$. The motion is simple harmonic
(a) with amplitude $x_0$ (b) with amplitude $2x_0$
(c) with time period $\frac{2\pi}{\omega}$ (d) with time period $\frac{\pi}{\omega}$.

**13.** In a simple harmonic motion
(a) the potential energy is always equal to the kinetic energy
(b) the potential energy is never equal to the kinetic energy
(c) the average potential energy in any time interval is equal to the average kinetic energy in that time interval
(d) the average potential energy in one time period is equal to the average kinetic energy in this period.

**14.** In a simple harmonic motion
(a) the maximum potential energy equals the maximum kinetic energy
(b) the minimum potential energy equals the minimum kinetic energy
(c) the minimum potential energy equals the maximum kinetic energy
(d) the maximum potential energy equals the minimum kinetic energy.

**15.** An object is released from rest. The time it takes to fall through a distance $h$ and the speed of the object as it falls through this distance are measured with a pendulum clock. The entire apparatus is taken on the moon and the experiment is repeated
(a) the measured times are same
(b) the measured speeds are same
(c) the actual times in the fall are equal
(d) the actual speeds are equal.

**16.** Which of the following will change the time period as they are taken to moon ?
(a) A simple pendulum (b) A physical pendulum
(c) A torsional pendulum (d) A spring-mass system

## EXERCISES

**1.** A particle executes simple harmonic motion with an amplitude of 10 cm and time period 6 s. At $t = 0$ it is at position $x = 5$ cm going towards positive $x$-direction. Write the equation for the displacement $x$ at time $t$. Find the magnitude of the acceleration of the particle at $t = 4$ s.

**2.** The position, velocity and acceleration of a particle executing simple harmonic motion are found to have magnitudes 2 cm, 1 m s$^{-1}$ and 10 m s$^{-2}$ at a certain instant. Find the amplitude and the time period of the motion.

**3.** A particle executes simple harmonic motion with an amplitude of 10 cm. At what distance from the mean position are the kinetic and potential energies equal ?

**4.** The maximum speed and acceleration of a particle executing simple harmonic motion are 10 cm s$^{-1}$ and 50 cm s$^{-2}$. Find the position(s) of the particle when the speed is 8 cm s$^{-1}$.

**5.** A particle having mass 10 g oscillates according to the equation $x = (2{\cdot}0 \text{ cm}) \sin[(100 \text{ s}^{-1})t + \pi/6]$. Find (a) the amplitude, the time period and the spring constant (b) the position, the velocity and the acceleration at $t = 0$.

**6.** The equation of motion of a particle started at $t = 0$ is given by $x = 5 \sin (20\, t + \pi/3)$, where $x$ is in centimetre and $t$ in second. When does the particle
(a) first come to rest
(b) first have zero acceleration
(c) first have maximum speed ?

**7.** Consider a particle moving in simple harmonic motion according to the equation

$$x = 2{\cdot}0 \cos(50\, \pi\, t + \tan^{-1} 0{\cdot}75)$$

where $x$ is in centimetre and $t$ in second. The motion is started at $t = 0$. (a) When does the particle come to rest for the first time ? (b) When does the acceleration have its maximum magnitude for the first time ? (c) When does the particle come to rest for the second time ?

**8.** Consider a simple harmonic motion of time period $T$. Calculate the time taken for the displacement to change value from half the amplitude to the amplitude.

**9.** The pendulum of a clock is replaced by a spring-mass system with the spring having spring constant 0·1 N m$^{-1}$. What mass should be attached to the spring ?

**10.** A block suspended from a vertical spring is in equilibrium. Show that the extension of the spring equals the length of an equivalent simple pendulum, i.e., a pendulum having frequency same as that of the block.

**11.** A block of mass 0·5 kg hanging from a vertical spring executes simple harmonic motion of amplitude 0·1 m and time period 0·314 s. Find the maximum force exerted by the spring on the block.

**12.** A body of mass 2 kg suspended through a vertical spring executes simple harmonic motion of period 4 s. If the

oscillations are stopped and the body hangs in equilibrium, find the potential energy stored in the spring.

**13.** A spring stores 5 J of energy when stretched by 25 cm. It is kept vertical with the lower end fixed. A block fastened to its other end is made to undergo small oscillations. If the block makes 5 oscillations each second, what is the mass of the block ?

**14.** A small block of mass $m$ is kept on a bigger block of mass $M$ which is attached to a vertical spring of spring constant $k$ as shown in the figure. The system oscillates vertically. (a) Find the resultant force on the smaller block when it is displaced through a distance $x$ above its equilibrium position. (b) Find the normal force on the smaller block at this position. When is this force smallest in magnitude ? (c) What can be the maximum amplitude with which the two blocks may oscillate together ?

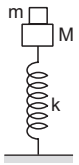



Figure 12-E1

**15.** The block of mass $m_1$ shown in figure (12-E2) is fastened to the spring and the block of mass $m_2$ is placed against it. (a) Find the compression of the spring in the equilibrium position. (b) The blocks are pushed a further distance $(2/k)\ (m_1 + m_2)g \sin\theta$ against the spring and released. Find the position where the two blocks separate. (c) What is the common speed of blocks at the time of separation ?

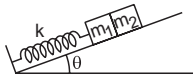



Figure 12-E2

**16.** In figure (12-E3) $k = 100\ \text{N m}^{-1}$, $M = 1$ kg and $F = 10$ N. (a) Find the compression of the spring in the equilibrium position. (b) A sharp blow by some external agent imparts a speed of 2 m s$^{-1}$ to the block towards left. Find the sum of the potential energy of the spring and the kinetic energy of the block at this instant. (c) Find the time period of the resulting simple harmonic motion. (d) Find the amplitude. (e) Write the potential energy of the spring when the block is at the left extreme. (f) Write the potential energy of the spring when the block is at the right extreme.

The answers of (b), (e) and (f) are different. Explain why this does not violate the principle of conservation of energy.

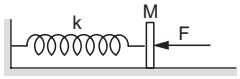



Figure 12-E3

**17.** Find the time period of the oscillation of mass $m$ in figures 12-E4 a, b, c. What is the equivalent spring constant of the pair of springs in each case ?

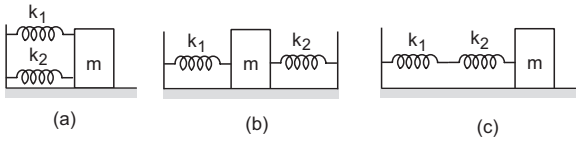



Figure 12-E4

**18.** The spring shown in figure (12-E5) is unstretched when a man starts pulling on the cord. The mass of the block is $M$. If the man exerts a constant force $F$, find (a) the amplitude and the time period of the motion of the block, (b) the energy stored in the spring when the block passes through the equilibrium position and (c) the kinetic energy of the block at this position.

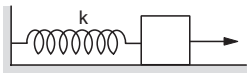



Figure 12-E5

**19.** A particle of mass $m$ is attatched to three springs $A$, $B$ and $C$ of equal force constants $k$ as shown in figure (12-E6). If the particle is pushed slightly against the spring $C$ and released, find the time period of oscillation.

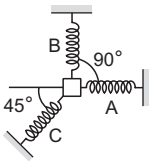



Figure 12-E6

**20.** Repeat the previous exercise if the angle between each pair of springs is 120° initially.

**21.** The springs shown in the figure (12-E7) are all unstretched in the beginning when a man starts pulling the block. The man exerts a constant force $F$ on the block. Find the amplitude and the frequency of the motion of the block.

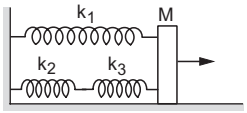



Figure 12-E7

**22.** Find the elastic potential energy stored in each spring shown in figure (12-E8), when the block is in equilibrium. Also find the time period of vertical oscillation of the block.

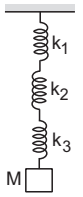



Figure 12-E8

**23.** The string, the spring and the pulley shown in figure (12-E9) are light. Find the time period of the mass $m$.

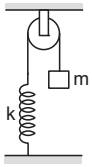



Figure 12-E9

**24.** Solve the previous problem if the pulley has a moment of inertia I about its axis and the string does not slip over it.

**25.** Consider the situation shown in figure (12-E10). Show that if the blocks are displaced slightly in opposite directions and released, they will execute simple harmonic motion. Calculate the time period.

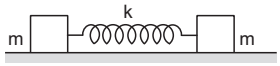



Figure 12-E10

**26.** A rectangular plate of sides $a$ and $b$ is suspended from a ceiling by two parallel strings of length $L$ each (figure 12-E11). The separation between the strings is $d$. The plate is displaced slightly in its plane keeping the strings tight. Show that it will execute simple harmonic motion. Find the time period.

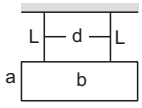



Figure 12-E11

**27.** A 1 kg block is executing simple harmonic motion of amplitude 0·1 m on a smooth horizontal surface under the restoring force of a spring of spring constant 100 N $m^{-1}$. A block of mass 3 kg is gently placed on it at the instant it passes through the mean position. Assuming that the two blocks move together, find the frequency and the amplitude of the motion.

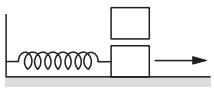


Figure 12-E12

**28.** The left block in figure (12-E13) moves at a speed $v$ towards the right block placed in equilibrium. All collisions to take place are elastic and the surfaces are frictionless. Show that the motions of the two blocks are periodic. Find the time period of these periodic motions. Neglect the widths of the blocks.

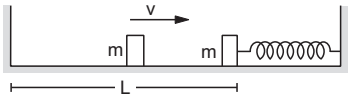



Figure 12-E13

**29.** Find the time period of the motion of the particle shown in figure (12-E14). Neglect the small effect of the bend near the bottom.

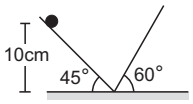



Figure 12-E14

**30.** All the surfaces shown in figure (12-E15) are frictionless. The mass of the car is $M$, that of the block is $m$ and the spring has spring constant $k$. Initially, the car and the block are at rest and the spring is stretched through a length $x_0$ when the system is released. (a) Find the amplitudes of the simple harmonic motion of the block and of the car as seen from the road. (b) Find the time period(s) of the two simple harmonic motions.

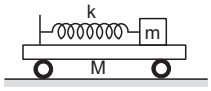



Figure 12-E15

**31.** A uniform plate of mass $M$ stays horizontally and symmetrically on two wheels rotating in opposite directions (figure 12-E16). The separation between the wheels is $L$. The friction coefficient between each wheel and the plate is μ. Find the time period of oscillation of the plate if it is slightly displaced along its length and released.

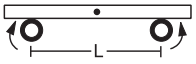



Figure 12-E16

**32.** A pendulum having time period equal to two seconds is called a seconds pendulum. Those used in pendulum clocks are of this type. Find the length of a seconds pendulum at a place where $g = \pi^2$ m $s^{-2}$.

**33.** The angle made by the string of a simple pendulum with the vertical depends on time as $\theta = \frac{\pi}{90} \sin[(\pi\ s^{-1})\ t]$. Find the length of the pendulum if $g = \pi^2$ m $s^{-2}$.

**34**. The pendulum of a certain clock has time period 2·04 s. How fast or slow does the clock run during 24 hours ?

**35**. A pendulum clock giving correct time at a place where $g = 9{\cdot}800\ \text{m s}^{-2}$ is taken to another place where it loses 24 seconds during 24 hours. Find the value of $g$ at this new place.

**36**. A simple pendulum is constructed by hanging a heavy ball by a 5·0 m long string. It undergoes small oscillations. (a) How many oscillations does it make per second ? (b) What will be the frequency if the system is taken on the moon where acceleration due to gravitation of the moon is $1{\cdot}67\ \text{m s}^{-2}$ ?

**37**. The maximum tension in the string of an oscillating pendulum is double of the minimum tension. Find the angular amplitude.

**38**. A small block oscillates back and forth on a smooth concave surface of radius $R$ (figure 12-E17). Find the time period of small oscillation.

Figure 12-E17

**39.** A spherical ball of mass $m$ and radius $r$ rolls without slipping on a rough concave surface of large radius $R$. It makes small oscillations about the lowest point. Find the time period.

**40**. A simple pendulum of length 40 cm is taken inside a deep mine. Assume for the time being that the mine is 1600 km deep. Calculate the time period of the pendulum there. Radius of the earth = 6400 km.

**41.** Assume that a tunnel is dug across the earth (radius = $R$) passing through its centre. Find the time a particle takes to cover the length of the tunnel if (a) it is projected into the tunnel with a speed of $\sqrt{gR}$ (b) it is released from a height $R$ above the tunnel (c) it is thrown vertically upward along the length of tunnel with a speed of $\sqrt{gR}$.

**42.** Assume that a tunnel is dug along a chord of the earth, at a perpendicular distance $R/2$ from the earth's centre where $R$ is the radius of the earth. The wall of the tunnel is frictionless. (a) Find the gravitational force exerted by the earth on a particle of mass $m$ placed in the tunnel at a distance $x$ from the centre of the tunnel. (b) Find the component of this force along the tunnel and perpendicular to the tunnel. (c) Find the normal force exerted by the wall on the particle. (d) Find the resultant force on the particle. (e) Show that the motion of the particle in the tunnel is simple harmonic and find the time period.

**43**. A simple pendulum of length $l$ is suspended through the ceiling of an elevator. Find the time period of small oscillations if the elevator (a) is going up with an acceleration $a_0$ (b) is going down with an acceleration $a_0$ and (c) is moving with a uniform velocity.

**44.** A simple pendulum of length 1 feet suspended from the ceiling of an elevator takes $\pi/3$ seconds to complete one oscillation. Find the acceleration of the elevator.

**45**. A simple pendulum fixed in a car has a time period of 4 seconds when the car is moving uniformly on a horizontal road. When the accelerator is pressed, the time period changes to 3·99 seconds. Making an approximate analysis, find the acceleration of the car.

**46**. A simple pendulum of length $l$ is suspended from the ceiling of a car moving with a speed $v$ on a circular horizontal road of radius $r$. (a) Find the tension in the string when it is at rest with respect to the car. (b) Find the time period of small oscillation.

**47.** The ear-ring of a lady shown in figure (12-E18) has a 3 cm long light suspension wire. (a) Find the time period of small oscillations if the lady is standing on the ground. (b) The lady now sits in a merry-go-round moving at $4\ \text{m s}^{-1}$ in a circle of radius 2 m. Find the time period of small oscillations of the ear-ring.

Figure 12-E18

**48.** Find the time period of small oscillations of the following systems. (a) A metre stick suspended through the 20 cm mark. (b) A ring of mass $m$ and radius $r$ suspended through a point on its periphery. (c) A uniform square plate of edge $a$ suspended through a corner. (d) A uniform disc of mass $m$ and radius $r$ suspended through a point $r/2$ away from the centre.

**49.** A uniform rod of length $l$ is suspended by an end and is made to undergo small oscillations. Find the length of the simple pendulum having the time period equal to that of the rod.

**50.** A uniform disc of radius $r$ is to be suspended through a small hole made in the disc. Find the minimum possible time period of the disc for small oscillations. What should be the distance of the hole from the centre for it to have minimum time period ?

**51**. A hollow sphere of radius 2 cm is attached to an 18 cm long thread to make a pendulum. Find the time period of oscillation of this pendulum. How does it differ from the time period calculated using the formula for a simple pendulum ?

**52**. A closed circular wire hung on a nail in a wall undergoes small oscillations of amplitude $2^0$ and time period 2 s. Find (a) the radius of the circular wire, (b) the speed of the particle farthest away from the point of suspension as it goes through its mean position, (c) the acceleration of this particle as it goes through its mean position and (d) the acceleration of this particle when it is at an extreme position. Take $g = \pi^2\ \text{m s}^{-2}$.

**53**. A uniform disc of mass $m$ and radius $r$ is suspended through a wire attached to its centre. If the time period of the torsional oscillations be $T$, what is the torsional constant of the wire?

54. Two small balls, each of mass $m$ are connected by a light rigid rod of length $L$. The system is suspended from its centre by a thin wire of torsional constant $k$. The rod is rotated about the wire through an angle $\theta_0$ and released. Find the force exerted by the rod on one of the balls as the system passes through the mean position.

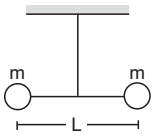



Figure 12-E19

55. A particle is subjected to two simple harmonic motions of same time period in the same direction. The amplitude of the first motion is 3·0 cm and that of the second is 4·0 cm. Find the resultant amplitude if the phase difference between the motions is (a) 0°, (b) 60°, (c) 90°.

56. Three simple harmonic motions of equal amplitudes $A$ and equal time periods in the same direction combine. The phase of the second motion is 60° ahead of the first and the phase of the third motion is 60° ahead of the second. Find the amplitude of the resultant motion.

57. A particle is subjected to two simple harmonic motions given by

$$x_1 = 2{\cdot}0 \sin(100\,\pi\, t) \text{ and } x_2 = 2{\cdot}0 \sin(120\,\pi\, t + \pi/3),$$

where $x$ is in centimeter and $t$ in second. Find the displacement of the particle at (a) $t = 0{\cdot}0125$, (b) $t = 0{\cdot}025$.

58. A particle is subjected to two simple harmonic motions, one along the $X$-axis and the other on a line making an angle of 45° with the $X$-axis. The two motions are given by

$$x = x_0 \sin\omega t \text{ and } s = s_0 \sin\omega t$$

Find the amplitude of the resultant motion.

□

## ANSWERS

### OBJECTIVE I

1. (a) 2. (b) 3. (a) 4. (d) 5. (c) 6. (d)
7. (d) 8. (c) 9. (b) 10. (c) 11. (d) 12. (a)
13. (c) 14. (d) 15. (d) 16. (c) 17. (c) 18. (d)
19. (d) 20. (c) 21. (c).

### OBJECTIVE II

1. (a), (b) 2. (a) 3. (a)
4. (d) 5. (a), (b), (d) 6. (c), (d)
7. (a) 8. all 9. (c), (d)
10. (a) 11. (a), (c), (d) 12. (d)
13. (d) 14. (a), (b) 15. (a), (b)
16. (a), (b).

### EXERCISES

1. $x = (10 \text{ cm}) \sin\left(\frac{2\,\pi}{6 \text{ s}}\, t + \frac{\pi}{6}\right)$, $\approx 11 \text{ cm s}^{-2}$
2. 4·9 cm, 0·28 s
3. $5\sqrt{2}$ cm
4. ± 1·2 cm from the mean position
5. (a) 2·0 cm, 0·063 s, 100 N m$^{-1}$
   (b) 1·0 cm, 1·73 m s$^{-1}$, 100 m s$^{-2}$
6. (a) $\frac{\pi}{120}$ s (b) $\frac{\pi}{30}$ s (c) $\frac{\pi}{30}$ s
7. (a) $1{\cdot}6 \times 10^{-2}$ s (b) $1{\cdot}6 \times 10^{-2}$ s (c) $3{\cdot}6 \times 10^{-2}$ s
8. $T/6$
9. ≈ 10 g
11. 25 N
12. 40 J
13. 0·16 kg
14. (a) $\frac{mkx}{M+m}$ (b) $mg - \frac{mkx}{M+m}$, at the highest point
    (c) $g\,\frac{(M+m)}{k}$
15. (a) $\frac{(m_1+m_2)g \sin\theta}{k}$
    (b) When the spring acquires its natural length
    (c) $\sqrt{\frac{3}{k}(m_1+m_2)}\; g \sin\theta$
16. (a) 10 cm (b) 2·5 J (c) $\pi/5$ s
    (d) 20 cm (e) 4·5 J (f) 0·5 J
17. (a) $2\pi\sqrt{\frac{m}{k_1+k_2}}$ (b) $2\pi\sqrt{\frac{m}{k_1+k_2}}$ (c) $2\pi\sqrt{\frac{m(k_1+k_2)}{k_1k_2}}$
18. (a) $\frac{F}{k}$, $2\,\pi\sqrt{\frac{M}{k}}$, (b) $\frac{F^2}{2\,k}$ (c) $\frac{F^2}{2\,k}$
19. $2\,\pi\sqrt{\frac{m}{2\,k}}$
20. $2\,\pi\sqrt{\frac{2\,m}{3\,k}}$

21. $\frac{F(k_2+k_3)}{k_1k_2+k_2k_3+k_3k_1}$, $\frac{1}{2\pi}\sqrt{\frac{k_1k_2+k_2k_3+k_3k_1}{M(k_2+k_3)}}$

22. $\frac{M^2g^2}{2k_1}$, $\frac{M^2g^2}{2k_2}$ and $\frac{M^2g^2}{2k_3}$ from above, time period
$= 2\pi\sqrt{M\left(\frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3}\right)}$

23. $2\pi\sqrt{\frac{m}{k}}$

24. $2\pi\sqrt{\frac{(m+I/r^2)}{k}}$

25. $2\pi\sqrt{\frac{m}{2k}}$

26. $2\pi\sqrt{\frac{L}{g}}$

27. $\frac{5}{2\pi}$ Hz, 5 cm

28. $\left(\pi\sqrt{\frac{m}{k}}+\frac{2L}{v}\right)$

29. $\approx 0{\cdot}73$ s

30. (a) $\frac{Mx_0}{M+m}$, $\frac{mx_0}{M+m}$ (b) $2\pi\sqrt{\frac{mM}{k(M+m)}}$

31. $2\pi\sqrt{\frac{L}{2\mu g}}$

32. 1 m

33. 1 m

34. 28·3 minutes slow

35. 9·795 m s$^{-2}$

36. (a) $0{\cdot}70/\pi$ (b) $1/(2\pi\sqrt{3})$ Hz

37. $\cos^{-1}(3/4)$

38. $2\pi\sqrt{R/g}$

39. $2\pi\sqrt{\frac{7(R-r)}{5g}}$

40. 1·47 s

41. $\frac{\pi}{2}\sqrt{\frac{R}{g}}$ in each case

42. (a) $\frac{GMm}{R^3}\sqrt{x^2+R^2/4}$ (b) $\frac{GMm}{R^3}x$, $\frac{GMm}{2R^2}$
(c) $\frac{GMm}{2R^2}$, (d) $\frac{GMm}{R^3}x$ (e) $2\pi\sqrt{R^3/(GM)}$

43. (a) $2\pi\sqrt{\frac{l}{g+a_0}}$ (b) $2\pi\sqrt{\frac{l}{g-a_0}}$ (c) $2\pi\sqrt{\frac{l}{g}}$

44. 4 f s$^{-2}$ upwards

45. $g/10$

46. (a) $ma$ (b) $2\pi\sqrt{l/a}$, where $a=\left[g^2+\frac{v^4}{r^2}\right]^{1/2}$

47. (a) 0·34 s (b) 0·30 s

48. (a) 1·51 s (b) $2\pi\sqrt{\frac{2r}{g}}$ (c) $2\pi\sqrt{\frac{\sqrt{8}a}{3g}}$ (d) $2\pi\sqrt{\frac{3r}{2g}}$

49. $2l/3$

50. $2\pi\sqrt{\frac{r\sqrt{2}}{g}}$, $r/\sqrt{2}$

51. 0·89 s, it is about 0·3% larger than the calculated value

52. (a) 50 cm (b) 11 cm s$^{-1}$
(c) 1·2 cm s$^{-2}$ towards the point of suspension
(d) 34 cm s$^{-2}$ towards the mean position

53. $\frac{2\pi^2mr^2}{T^2}$

54. $\left[\frac{k^2\theta_0^4}{L^2}+m^2g^2\right]^{1/2}$

55. (a) 7·0 cm (b) 6·1 cm (c) 5·0 cm

56. $2A$

57. (a) – 2·41 cm (b) 0·27 cm

58. $\left[x_0^2+s_0^2+\sqrt{2}\,x_0s_0\right]^{1/2}$

□

## CHAPTER 13

# FLUID MECHANICS

### 13.1 FLUIDS

Matter is broadly divided into three categories, solid, liquid and gas. The intermolecular forces are strong in solids, so that the shape and size of solids do not easily change. This force is comparatively less in liquids and so the shape is easily changed. Although the shape of a liquid can be easily changed, the volume of a given mass of a liquid is not so easy to change. It needs quite a good effort to change the density of liquids. In gases, the intermolecular forces are very small and it is simple to change both the shape and the density of a gas. Liquids and gases together are called fluids, i.e., that which can flow.

In this chapter we shall largely deal with liquids. The equations derived may be applicable to gases in many cases with some modifications. We shall assume that the liquids we deal with are *incompressible and nonviscous*. The first condition means that the density of the liquid is independent of the variations in pressure and always remains constant. The second condition means that parts of the liquid in contact do not exert any tangential force on each other. The force by one part of the liquid on the other part is perpendicular to the surface of contact. Thus, there is no friction between the adjacent layers of a liquid.

### 13.2 PRESSURE IN A FLUID

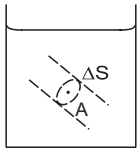



Figure 13.1

Consider a point $A$ in the fluid (figure 13.1). Imagine a small area $\Delta S$ containing the point $A$. The fluid on one side of the area presses the fluid on the other side and vice versa. Let the common magnitude of the forces be $F$. We define the pressure of the fluid at the point $A$ as

$$P = \lim_{\Delta S \to 0} \frac{F}{\Delta S} \qquad \ldots \ (13.1)$$

For a homogeneous and nonviscous fluid, this quantity does not depend on the orientation of $\Delta S$ and hence we talk of pressure *at a point*. For such a fluid, pressure is a scalar quantity having only magnitude.

#### Unit of Pressure

The SI unit of pressure is N m $^{-2}$ called *pascal* and abbreviated as Pa.

#### Variation of Pressure with Height

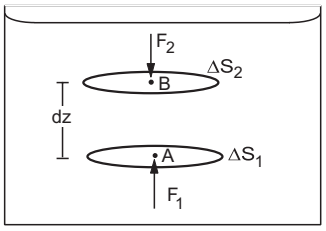



Figure 13.2

Let us consider two points $A$ and $B$ (figure 13.2) separated by a small vertical height $dz$. Imagine a horizontal area $\Delta S_1$ containing $A$ and an identical horizontal area $\Delta S_2$ containing $B$. The area $\Delta S_1 = \Delta S_2 = \Delta S$. Consider the fluid enclosed between the two surfaces $\Delta S_1, \Delta S_2$ and the vertical boundary joining them. The vertical forces acting on this fluid are

(a) $F_1$, vertically upward by the fluid below it

(b) $F_2$, vertically downward by the fluid above it and

(c) weight $W$, vertically downward.

Let the pressure at the surface $A$ be $P$ and the pressure at $B$ be $P + dP$. Then

$$F_1 = P\,\Delta S$$

and $$F_2 = (P + dP)\Delta S.$$

The volume of the fluid considered is $(\Delta S)\,(dz)$. If the density of the fluid at $A$ is $\rho$, the mass of the fluid considered is $\rho(\Delta S)\,(dz)$ and hence its weight $W$ is

$$W = \rho\,(\Delta S)\,(dz)\,g.$$

For vertical equilibrium,

$$F_1 = F_2 + W$$

or, $$P\,\Delta S = (P + dP)\,\Delta S + \rho\,g(dz)\,\Delta S$$

or, $$dP = -\rho g\,dz\,. \qquad \ldots\ (13.2)$$

As we move up through a height $dz$ the pressure decreases by $\rho g\,dz$, where $\rho$ is the density of the fluid at that point.

Now consider two points at $z = 0$ and $z = h$. If the pressure at $z = 0$ is $P_1$ and that at $z = h$ is $P_2$, then from equation (13.2)

$$\int_{P_1}^{P_2} dP = \int_0^z -\rho g\,dz$$

or, $$P_2 - P_1 = \int_0^z -\rho g\,dz\,.$$

If the density is same everywhere

$$P_2 - P_1 = -\rho g z$$

or, $$P_1 = P_2 + \rho g z\,. \qquad \ldots\ (13.3)$$

Next consider two points $A$ and $B$ in the same horizontal line inside a fluid. Imagine a small vertical area $\Delta S_1$ containing the point $A$ and a similar vertical area $\Delta S_2$ containing the point $B$.

Figure 13.3

The area $\Delta S_1 = \Delta S_2 = \Delta S$. Consider the liquid contained in the horizontal cylinder bounded by $\Delta S_1$ and $\Delta S_2$. If the pressures at $A$ and $B$ are $P_1$ and $P_2$ respectively, the forces in the direction $AB$ are

(a) $P_1\Delta S$ towards right and

(b) $P_2\Delta S$ towards left.

If the fluid remains in equilibrium,

$$P_1\Delta S = P_2\Delta S\,.$$

or, $$P_1 = P_2$$

Thus, the pressure is same at two points in the same horizontal level.

## 13.3 PASCAL'S LAW

We have seen in the previous section that the pressure difference between two points in a liquid at rest depends only on the difference in vertical height between the points. The difference is in fact $\rho gz$, where $\rho$ is the density of the liquid (assumed constant) and $z$ is the difference in vertical height. Suppose by some means the pressure at one point of the liquid is increased. The pressure at all other points of the liquid must also increase by the same amount because the pressure difference must be the same between two given points. This is the content of Pascal's law which may be stated as follows :

*If the pressure in a liquid is changed at a particular point, the change is transmitted to the entire liquid without being diminished in magnitude.*

Figure 13.4

As an example, suppose a flask fitted with a piston is filled with a liquid as shown in figure (13.4). Let an external force $F$ be applied on the piston. If the cross-sectional area of the piston is $A$, the pressure just below the piston is increased by $F/A$. By Pascal's law, the pressure at any point $B$ will also increase by the same amount $F/A$. This is because the pressure at $B$ has to be $\rho gz$ more than the pressure at the piston, where $z$ is the vertical distance *of B* below the piston. By applying the force we do not appreciably change $z$ (as the liquid is supposed to be incompressible) and hence the pressure difference remains unchanged. As the pressure at the piston is increased by $F/A$, the pressure at $B$ also increases by the same amount.

Pascal's law has several interesting applications. Figure (13.5) shows the principle of a *hydraulic lift* used to raise heavy loads such as a car.

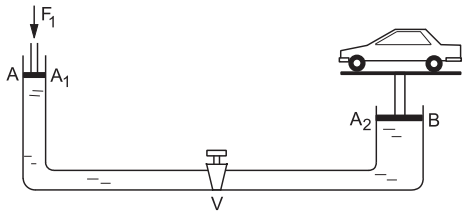



Figure 13.5

It consists of two vertical cylinders $A$ and $B$ of different cross-sectional areas $A_1$ and $A_2$ connected by a horizontal tube. Pistons are fitted in both the cylinder. The load is kept on a platform fixed with the piston of larger area. A liquid is filled in the equipment. A valve $V$ is fitted in the horizontal tube which allows the liquid to go from $A$ to $B$ when pressed from the $A$-side. The piston $A$ is pushed by a force $F_1$. The pressure in the liquid increases everywhere by an amount $F_1/A_1$. The valve $V$ is open and the liquid flows into the cylinder $B$. It exerts an extra force $F_2 = A_2\left(\frac{F_1}{A_1}\right)$ on the larger piston in the upward direction which raises the load upward.

The advantage of this method is that if $A_2$ is much larger than $A_1$, even a small force $F_1$ is able to generate a large force $F_2$ which can raise the load. It may be noted that there is no gain in terms of work. The work done by $F_1$ is same as that by $F_2$. The piston $A$ has to traverse a larger downward distance as compared to the height raised by $B$.

### 13.4 ATMOSPHERIC PRESSURE AND BAROMETER

The atmosphere of the earth is spread up to a height of about 200 km. This atmosphere presses the bodies on the surface of the earth. The force exerted by the air on any body is perpendicular to the surface of the body. We define *atmospheric pressure* as follows. Consider a small surface $\Delta S$ in contact with air. If the force exerted by the air on this part is $F$, the atmospheric pressure is

$$P_0 = \lim_{\Delta S \to 0} \frac{F}{\Delta S}.$$

Atmospheric pressure at the top of the atmosphere is zero as there is nothing above it to exert the force. The pressure at a distance $z$ below the top will be $\int_0^z \rho g\, dz$. Remember, neither $\rho$ nor $g$ can be treated as constant over large variations in heights. However, the density of air is quite small and so the atmospheric pressure does not vary appreciably over small distances. Thus, we say that the atmospheric pressure at Patna is 76 cm of mercury without specifying whether it is at Gandhi Maidan or at the top of Golghar.

Torricelli devised an ingenious way to measure the atmospheric pressure. The instrument is known as *barometer*.

In this, a glass tube open at one end and having a length of about a meter is filled with mercury. The open end is temporarily closed (by a thumb or otherwise) and the tube is inverted in a cup of mercury. With the open end dipped into the cup, the temporary closure is removed. The mercury column in the tube falls down a little and finally stays there.

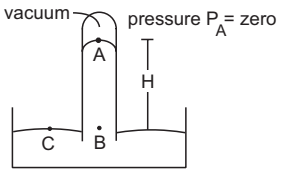



Figure 13.6

Figure (13.6) shows schematically the situation. The upper part of the tube contains vacuum as the mercury goes down and no air is allowed in. Thus, the pressure at the upper end $A$ of the mercury column inside the tube is $P_A$ = zero. Let us consider a point $C$ on the mercury surface in the cup and another point $B$ in the tube at the same horizontal level. The pressure at $C$ is equal to the atmospheric pressure. As $B$ and $C$ are in the same horizontal level, the pressures at $B$ and $C$ are equal. Thus, the pressure at $B$ is equal to the atmospheric pressure $P_0$ in the lab.

Suppose the point $B$ is at a depth $H$ below $A$. If $\rho$ be the density of mercury,

$$P_B = P_A + \rho g H$$

or, $$P_0 = \rho g H. \quad \ldots (13.4)$$

The height $H$ of the mercury column in the tube above the surface in the cup is measured. Knowing the density of mercury and the acceleration due to gravity, the atmospheric pressure can be calculated using equation (13.4).

The atmospheric pressure is often given as the length of mercury column in a barometer. Thus, a pressure of 76 cm of mercury means

$$P_0 = (13{\cdot}6 \times 10^3 \text{ kg m}^{-3})\,(9{\cdot}8 \text{ m s}^{-2})\,(0{\cdot}76 \text{ m})$$
$$= 1{\cdot}01 \times 10^5 \text{ Pa}.$$

This pressure is written as 1 atm. If the tube is insufficient in length, the mercury column will not fall down and no vacuum will be created. The inner surface of the tube will be in contact with the mercury at the top and will exert a pressure $P_A$ on it.

***Example 13.1***

*Water is filled in a flask up to a height of* 20 cm. *The bottom of the flask is circular with radius* 10 cm. *If the atmospheric pressure is* $1{\cdot}01 \times 10^5$ Pa, *find the force exerted by the water on the bottom. Take* $g = 10 \text{ m s}^{-2}$ *and density of water* $= 1000 \text{ kg m}^{-3}$.

***Solution*** : The pressure at the surface of the water is equal to the atmospheric pressure $P_0$. The pressure at the bottom is

$$P = P_0 + h\rho g$$
$$= 1{\cdot}01 \times 10^5 \text{ Pa} + (0{\cdot}20 \text{ m})\,(1000 \text{ kg m}^{-3})\,(10 \text{ m s}^{-2})$$
$$= 1{\cdot}01 \times 10^5 \text{ Pa} + 0{\cdot}02 \times 10^5 \text{ Pa}$$
$$= 1{\cdot}03 \times 10^5 \text{ Pa}.$$

The area of the bottom $= \pi r^2 = 3{\cdot}14 \times (0{\cdot}1 \text{ m})^2$
$$= 0{\cdot}0314 \text{ m}^2.$$

The force on the bottom is, therefore,
$$F = P \pi r^2$$
$$= (1{\cdot}03 \times 10^5 \text{ Pa}) \times (0{\cdot}0314 \text{ m}^2) = 3230 \text{ N}.$$

**Manometer**

Manometer is a simple device to measure the pressure in a closed vessel containing a gas. It consists of a U-tube having some liquid. One end of the tube is open to the atmosphere and the other end is connected to the vessel (figure 13.7).

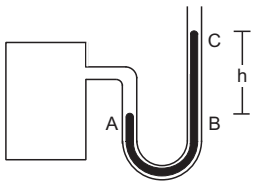



Figure 13.7

The pressure of the gas is equal to the pressure at $A$
$$= \text{ pressure at } B$$
$$= \text{ pressure at } C + h\rho g$$
$$= P_0 + h\rho g$$

when $P_0$ is the atmospheric pressure, $h = BC$ is the difference in levels of the liquid in the two arms and $\rho$ is the density of the liquid.

The excess pressure $P - P_0$ is called the *quage pressure.*

## 13.5 ARCHIMEDES' PRINCIPLE

When a body is partially or fully dipped into a fluid, the fluid exerts forces on the body. At any small portion of the surface of the body, the force by the fluid is perpendicular to the surface and is equal to the pressure at that point multiplied by the area (figure 13.8). The resultant of all these contact forces is called the force of *buoyancy* or *buoyant force.*

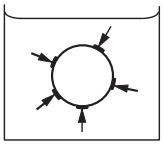


Figure 13.8

Archimedes' principle states that *when a body is partially or fully dipped into a fluid at rest, the fluid exerts an upward force of buoyancy equal to the weight of the displaced fluid.*

Archimedes' principle is not an independent principle and may be deduced from Newton's laws of motion.

Consider the situation shown in figure (13.8) where a body is shown dipped into a fluid. Suppose the body dipped in the fluid is replaced by the same fluid of equal volume. As the entire fluid now becomes homogeneous, all parts will remain in equilibrium. The part of the fluid substituting the body also remains in equilibrium. Forces acting on this substituting fluid are

(a) the weight $mg$ of this part of the fluid, and

(b) the resultant $B$ of the contact forces by the remaining fluid.

As the substituting fluid is in equilibrium, these two should be equal and opposite. Thus,
$$B = mg \qquad \ldots \quad (13.5)$$
and it acts in the vertically upward direction.

Now the substituting fluid just occupies the space which was previously occupied by the body. Hence, the shape of the boundary of the substituting fluid is same as the boundary of the body. Thus, the magnitude and direction of the force due to the pressure on any small area of the boundary is same for the body as for the substituting fluid. The force of buoyancy on the body is, therefore, same as the force of buoyancy $B$ on the substituting fluid.

From equation (13.5) the force of buoyancy on a dipped body is equal to the weight $mg$ of the displaced fluid and acts along the vertically upward direction. This is Archimedes' principle.

Note that in this derivation we have assumed that the fluid is in equilibrium in an inertial frame. If it is not so, the force of buoyancy may be different from the weight of the displaced fluid.

**Floatation**

When a solid body is dipped into a fluid, the fluid exerts an upward force of buoyancy on the solid. If the force of buoyancy equals the weight of the solid, the solid will remain in equilibrium. This is called *floatation.* When the overall density of the solid is smaller than the density of the fluid, the solid floats with a part of it in the fluid. The fraction dipped is such that the weight of the displaced fluid equals the weight of the solid.

***Example 13.2***

*A* 700 g *solid cube having an edge of length* 10 cm *floats in water. How much volume of the cube is outside the water* ? *Density of water* = 1000 kg $m^{-3}$.

***Solution*** **:** The weight of the cube is balanced by the buoyant force. The buoyant force is equal to the weight of the water displaced. If a volume $V$ of the cube is inside the water, the weight of the displaced water $= V\rho g$, where $\rho$ is the density of water. Thus,

$$V\rho g = (0{\cdot}7 \text{ kg})\, g$$

or, $$V = \frac{0{\cdot}7 \text{ kg}}{\rho} = \frac{0{\cdot}7 \text{ kg}}{1000 \text{ kg m}^{-3}} = 7 \times 10^{-4} \text{ m}^3 = 700 \text{ cm}^3.$$

The total volume of the cube $= (10 \text{ cm})^3 = 1000 \text{ cm}^3$.

The volume outside the water is

$$1000 \text{ cm}^3 - 700 \text{ cm}^3 = 300 \text{ cm}^3.$$

## 13.6 PRESSURE DIFFERENCE AND BUOYANT FORCE IN ACCELERATING FLUIDS

Equations (13.3) and (13.5) were derived by assuming that the fluid under consideration is in equilibrium in an inertial frame. If this is not the case, the equations must be modified. We shall discuss some special cases of accelerating fluids.

### A Liquid Placed in an Elevator

#### (a) Pressure Difference

Suppose a beaker contains some liquid and it is placed in an elevator which is going up with an acceleration $a_0$ (figure 13.9). Let $A$ and $B$ be two points in the liquid, $B$ being at a vertical height $z$ above $A$. Construct a small horizontal area $\Delta S$ around $A$ and an equal horizontal area around $B$. Construct a vertical cylinder with the two areas as the faces. Consider the motion of the liquid contained within this cylinder. Let $P_1$ be the pressure at $A$ and $P_2$ be the pressure at $B$.

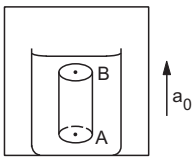



Figure 13.9

Forces acting on the liquid contained in the cylinder, in the vertical direction, are :

(a) $P_1\Delta S$, upward due to the liquid below it

(b) $P_2\Delta S$, downward due to the liquid above it and

(c) weight $mg = (\Delta S)z\rho g$ downward, where $\rho$ is the density of the liquid.

Under the action of these three forces the liquid is accelerating upward with an acceleration $a_0$. From Newton's second law

$$P_1\Delta S - P_2\Delta S - mg = ma_0$$

or, $$(P_1 - P_2)\Delta S = m(g + a_0) = (\Delta S)z\rho(g + a_0)$$

or, $$P_1 - P_2 = \rho(g + a_0)z \; . \qquad \ldots \; (13.6)$$

#### (b) Buoyant Force

Now suppose a body is dipped inside a liquid of density $\rho$ placed in an elevator going up with an acceleration $a_0$. Let us calculate the force of buoyancy $B$ on this body. As was done earlier, let us suppose that we substitute the body into the liquid by the same liquid of equal volume. The entire liquid becomes a homogenous mass and hence the substituted liquid is at rest with respect to the rest of the liquid. Thus, the substituted liquid is also going up with an acceleration $a_0$ together with the rest of the liquid.

The forces acting on the substituted liquid are

(a) the buoyant force $B$ and

(b) the weight $mg$ of the substituted liquid.

From Newton's second law

$$B - mg = ma_0$$

or, $$B = m(g + a_0) \qquad \ldots \; (13.7)$$

Equation (13.6) and (13.7) are similar to the corresponding equations for unaccelerated liquid with the only difference that $g + a$ takes the role of $g$.

### B Free Surface of a Liquid in Horizontal Acceleration

Consider a liquid placed in a beaker which is accelerating horizontally with an acceleration $a_0$ (figure 13.10). Let $A$ and $B$ be two points in the liquid at a separation $l$ in the same horizontal line along the acceleration $a_0$. We shall first obtain the pressure difference between the points $A$ and $B$.

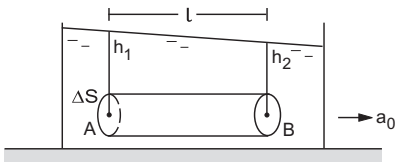



Figure 13.10

Construct a small vertical area $\Delta S$ around $A$ and an equal area around $B$. Consider the liquid contained in the horizontal cylinder with the two areas as the flat faces. Let the pressure at $A$ be $P_1$ and the pressure at $B$ be $P_2$. The forces along the line $AB$ are

(a) $P_1\Delta S$ towards right due to the liquid on the left and

(b) $P_2\Delta S$ towards left due to the liquid on the right.

Under the action of these forces, the liquid contained in the cylinder is accelerating towards right. From Newton's second law,

$$P_1\Delta S - P_2\Delta S = ma_0$$

or, $$(P_1 - P_2)\Delta S = (\Delta S)l\rho a_0$$

or, $$P_1 - P_2 = l\rho a_0 . \quad \ldots (13.8)$$

The two points in the same horizontal line do not have equal pressure if the liquid is accelerated horizontally.

As there is no vertical acceleration, the equation (13.3) is valid. If the atmospheric pressure is $P_0$, the pressure at $A$ is $P_1 = P_0 + h_1\rho g$ and the pressure at $B$ is $P_2 = P_0 + h_2\rho g$, where $h_1$ and $h_2$ are the depths of $A$ and $B$ from the free surface. Substituting in (13.8)

$$h_1\rho g - h_2\rho g = l\rho a_0$$

or, $$\frac{h_1 - h_2}{l} = \frac{a_0}{g}$$

or, $$\tan\theta = \frac{a_0}{g}$$

where $\theta$ is the inclination of the free surface with the horizontal.

## 13.7 FLOW OF FLUIDS

The flow of fluid is in general a complex branch of mechanics. If you look at the motion of water in a fall (like Rallah fall near Manali or Kemti fall near Moussurie) the view is very pleasant. The water falls from a height and then proceeds on a flat bed or a slope with thumping, jumping and singing if you can appreciate the music. But if you try to analyse the motion of each particle on the basis of laws of mechanics, the task is tremendously difficult. Other examples of fluid flow are the sailing of clouds and the motion of smoke when a traditional Chulha using coal, wood or goitha (prepared from cowdung) in an Indian village is lit. The motion of each smoke particle is governed by the same Newton's laws but to predict the motion of a particular particle is not easy.

## 13.8 STEADY AND TURBULENT FLOW

Consider a liquid passing through a glass tube (figure 13.11). Concentrate on a particular point $A$ in the tube and look at the particles arriving at $A$. If the velocity of the liquid is small, all the particles which come to $A$ will have same speed and will move in same direction. As a particle goes from $A$ to another point $B$, its speed and direction may change, but all the particles reaching $A$ will have the same speed at $A$ and all the particles reaching $B$ will have the same speed at $B$. Also, if one particle passing through $A$ has gone through $B$, then all the particles passing through $A$ go through $B$. Such a flow of fluid is called a *steady flow*.

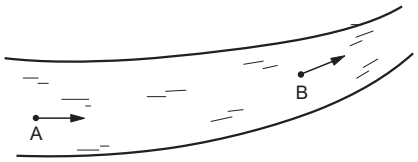



Figure 13.11

In steady flow the velocity of fluid particles reaching a particular point is the same at all time. Thus, each particle follows the same path as taken by a previous particle passing through that point.

If the liquid is pushed in the tube at a rapid rate, the flow may become turbulent. In this case, the velocities of different particles passing through the same point may be different and change erratically with time. The motion of water in a high fall or a fast flowing river is, in general, turbulent.

Steady flow is also called *streamline flow*.

### Line of Flow : Streamline

The path taken by a particle in flowing fluid is called its *line of flow*. The tangent at any point on the line of flow gives the direction of motion of that particle at that point. In the case of steady flow, all the particles passing through a given point follow the same path and hence we have a unique line of flow passing through a given point. In this case, the line of flow is also called a *streamline*. Thus, the tangent to the streamline at any point gives the direction of all the particles passing through that point. It is clear that two streamlines cannot intersect, otherwise, the particle reaching at the intersection will have two different directions of motion.

### Tube of Flow

Consider an area $S$ in a fluid in steady flow. Draw streamlines from all the points of the periphery of $S$. These streamlines enclose a tube, of which $S$ is a cross-section. Such a tube is called a *tube of flow*. As the streamlines do not cross each other, fluid flowing through differnt tubes of flow cannot intermix, although there is no physical partition between the tubes. When a liquid is passed slowly through a pipe, the pipe itself is one tube of flow.

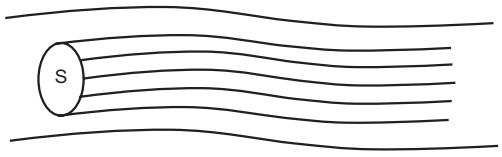



Figure 13.12

## 13.9 IRROTATIONAL FLOW OF AN INCOMPRESSIBLE AND NONVISCOUS FLUID

The analysis of the flow of a fluid becomes much simplified if we consider the fluid to be incompressible and nonviscous and that the flow is irrotational. Incompressibility means that the density of the fluid is same at all the points and remains constant as time passes. This assumption is quite good for liquids and is valid in certain cases of flow of gases. Viscosity of a fluid is related to the internal friction when a layer of fluid slips over another layer. Mechanical energy is lost against such viscous forces. The assumption of a nonviscous fluid will mean that we are neglecting the effect of such internal friction. Irrotational flow means there is no net angular velocity of fluid particles. When you put some washing powder in a bucket containing water and mix it by rotating your hand in circular path along the wall of the bucket, the water comes into rotational motion. Quite often water flowing in rivers show small vortex formation where it goes in rotational motion about a centre. Now onwards we shall consider only the irrotational motion of an incompressible and nonviscous fluid.

## 13.10 EQUATION OF CONTINUITY

We have seen that the fluid going through a tube of flow does not intermix with fluid in other tubes. The total mass of fluid going into the tube through any cross section should, therefore, be equal to the total mass coming out of the same tube from any other cross section in the same time. This leads to the equation of continuity.

Let us consider two cross sections of a tube of flow at the points $A$ and $B$ (figure 13.13). Let the area of cross section at $A$ be $A_1$ and that at $B$ be $A_2$. Let the speed of the fluid be $v_1$ at $A$ and $v_2$ at $B$.

Figure 13.13

How much fluid goes into the tube through the cross section at $A$ in a time interval $\Delta t$ ? Let us construct a cylinder of length $v_1\Delta t$ at $A$ as shown in the figure. As the fluid at $A$ has speed $v_1$, all the fluid included in this cylinder will cross through $A_1$ in the time interval $\Delta t$. Thus, the volume of the fluid going into the tube through the cross section at $A$ is $A_1v_1\Delta t$. Similarly, the volume of the fluid going out of the tube through the cross section at $B$ is $A_2v_2\Delta t$. If the fluid is incompressible, we must have

$$A_1v_1\Delta t = A_2v_2\,\Delta t$$

or, $$A_1v_1 = A_2v_2\,. \qquad \ldots\ (13.9)$$

*The product of the area of cross section and the speed remains the same at all points of a tube of flow.* This is called the *equation of continuity* and expresses the law of conservation of mass in fluid dynamics.

***Example 13.3***

*Figure (13.14) shows a liquid being pushed out of a tube by pressing a piston. The area of cross section of the piston is* $1{\cdot}0\ \text{cm}^2$ *and that of the tube at the outlet is* $20\ \text{mm}^2$. *If the piston is pushed at a speed of* $2\ \text{cm s}^{-1}$, *what is the speed of the outgoing liquid* ?

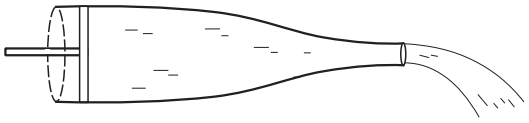


Figure 13.14

***Solution*** : From the equation of continuity

$$A_1v_1 = A_2v_2$$

or, $$(1{\cdot}0\ \text{cm}^2)\,(2\ \text{cm s}^{-1}) = (20\ \text{mm}^2)v_2$$

or, $$v_2 = \frac{1{\cdot}0\ \text{cm}^2}{20\ \text{mm}^2} \times 2\ \text{cm s}^{-1}$$

$$= \frac{100\ \text{mm}^2}{20\ \text{mm}^2} \times 2\ \text{cm s}^{-1} = 10\ \text{cm s}^{-1}\,.$$

## 13.11 BERNOULLI EQUATION

Bernoulli equation relates the speed of a fluid at a point, the pressure at that point and the height of that point above a reference level. It is just the application of work–energy theorem in the case of fluid flow.

We shall consider the case of irrotational and steady flow of an incompressible and nonviscous liquid. Figure (13.15) shows such a flow of a liquid in a tube of varying cross section and varying height. Consider the liquid contained between the cross sections $A$ and $B$ of the tube. The heights of $A$ and $B$ are $h_1$ and $h_2$ respectively from a reference level. This liquid advances into the tube and after a time $\Delta t$ is contained between the cross sections $A'$ and $B'$ as shown in figure.

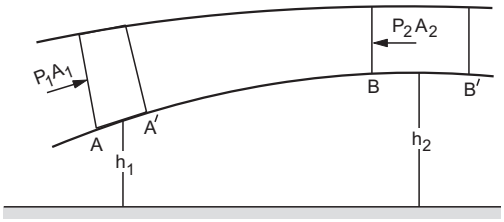



Figure 13.15

Suppose the area of cross section at $A = A_1$
the area of cross section at $B = A_2$
the speed of the liquid at $A = v_1$
the speed of the liquid at $B = v_2$
the pressure at $A = P_1$
the pressure at $B = P_2$
and the density of the liquid = ρ.

The distance $AA' = v_1\Delta t$ and the distance $BB' = v_2\Delta t$. The volume between $A$ and $A'$ is $A_1v_1\Delta t$ and the volume between $B$ and $B'$ is $A_2v_2\Delta t$. By the equation of continuity,

$$A_1v_1\Delta t = A_2v_2\Delta t \ .$$

The mass of this volume of liquid is

$$\Delta m = \rho A_1v_1\Delta t = \rho A_2v_2\Delta t. \quad \ldots \text{ (i)}$$

Let us calculate the total work done on the part of the liquid just considered.

The forces acting on this part of the liquid are

(a) $P_1A_1$, by the liquid on the left

(b) $P_2A_2$, by the liquid on the right

(c) $(\Delta m)g$, the weight of the liquid considered and

(d) $\mathscr{N}$, contact forces by the walls of the tube.

In time $\Delta t$, the point of application of $P_1A_1$ is displaced by $AA' = v_1\Delta t$. Thus, the work done by $P_1A_1$ in time $\Delta t$ is

$$W_1 = (P_1A_1)\,(v_1\Delta t) = P_1\left(\frac{\Delta m}{\rho}\right).$$

Similarly, the work done by $P_2A_2$ in time $\Delta t$ is

$$W_2 = -(P_2A_2)\,(v_2\Delta t) = -P_2\left(\frac{\Delta m}{\rho}\right).$$

The work done by the weight is equal to the negative of the change in gravitational potential energy.

The change in potential energy (P.E.) in time $\Delta t$ is

$$\begin{aligned} &P.E. \text{ of } A'BB' - P.E. \text{ of } AA'B \\ &= P.E. \text{ of } A'B + P.E. \text{ of } BB' \\ &\qquad - P.E. \text{ of } AA' - P.E. \text{ of } A'B \\ &= P.E. \text{ of } BB' - P.E. \text{ of } AA' \\ &= (\Delta m)gh_2 - (\Delta m)gh_1. \end{aligned}$$

Thus, the work done by the weight in time $\Delta t$ is

$$W_3 = (\Delta m)gh_1 - (\Delta m)gh_2.$$

The contact force $\mathscr{N}$ does no work on the liquid because it is perpendicular to the velocity.

The total work done on the liquid considered, in the time interval $\Delta t$ *is*

$$\begin{aligned} W &= W_1 + W_2 + W_3 \\ &= P_1\left(\frac{\Delta m}{\rho}\right) - P_2\left(\frac{\Delta m}{\rho}\right) + (\Delta m)gh_1 - (\Delta m)gh_2 \quad \ldots \text{ (ii)} \end{aligned}$$

The change in kinetic energy (K.E.) of the same liquid in time $\Delta t$ is

$$\begin{aligned} &\text{K.E. of } A'BB' - \text{K.E. of } AA'B \\ &= \text{K.E. of } A'B + \text{K.E. of } BB' - \text{K.E. of } AA' - \text{K.E. of } A'B \\ &= \text{K.E. of } BB' - \text{K.E. of } AA' \\ &= \frac{1}{2}(\Delta m)\,v_2^2 - \frac{1}{2}(\Delta m)v_1^2. \quad \ldots \text{ (iii)} \end{aligned}$$

Since the flow is assumed to be steady, the speed at any point remains constant in time and hence the K.E. of the part $A'B$ is same at initial and final time and cancels out when change in kinetic energy of the system is considered.

By the work–energy theorem, the total work done on the system is equal to the change in its kinetic energy. Thus,

$$P_1\left(\frac{\Delta m}{\rho}\right) - P_2\left(\frac{\Delta m}{\rho}\right) + (\Delta m)gh_1 - (\Delta m)gh_2 = \frac{1}{2}(\Delta m)v_2^2 - \frac{1}{2}(\Delta m)v_1^2$$

or, $$\frac{P_1}{\rho} + gh_1 + \frac{1}{2}v_1^2 = \frac{P_2}{\rho} + gh_2 + \frac{1}{2}v_2^2$$

or, $$P_1 + \rho gh_1 + \frac{1}{2}\rho v_1^2 = P_2 + \rho gh_2 + \frac{1}{2}\rho v_2^2 \quad \ldots \text{ (13.10)}$$

or, $$P + \rho gh + \frac{1}{2}\rho v^2 = \text{constant} \quad \ldots \text{ (13.11)}$$

This is known as *Bernoulli equation.*

***Example13.4***

*Figure (13.16) shows a liquid of density* $1200\ \text{kg m}^{-3}$ *flowing steadily in a tube of varying cross section. The cross section at a point A is* $1{\cdot}0\ \text{cm}^2$ *and that at B is* $20\ \text{mm}^2$*, the points A and B are in the same horizontal plane. The speed of the liquid at A is* $10\ \text{cm s}^{-1}$*. Calculate the difference in pressures at A and B.*

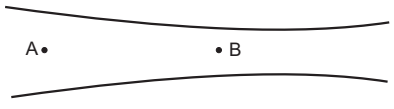



Figure 13.16

***Solution*** **:** From equation of continuity, the speed $v_2$ at $B$ is given by,

$$A_1v_1 = A_2v_2$$

or, $$(1{\cdot}0\ \text{cm}^2)\,(10\ \text{cm s}^{-1}) = (20\ \text{mm}^2)v_2$$

or, $$v_2 = \frac{1{\cdot}0\ \text{cm}^2}{20\ \text{mm}^2} \times 10\ \text{cm s}^{-1} = 50\ \text{cm s}^{-1}.$$

By Bernoulli equation,

$$P_1 + \rho gh_1 + \frac{1}{2}\rho v_1^2 = P_2 + \rho gh_2 + \frac{1}{2}\rho v_2^2.$$

Here $h_1 = h_2$. Thus,

$$P_1 - P_2 = \frac{1}{2}\rho v_2^2 - \frac{1}{2}\rho v_1^2$$

$$= \frac{1}{2} \times (1200 \text{ kg m}^{-3})\,(2500 \text{ cm}^2 \text{ s}^{-2} - 100 \text{ cm}^2 \text{ s}^{-2})$$

$$= 600 \text{ kg m}^{-3} \times 2400 \text{ cm}^2 \text{ s}^{-2} = 144 \text{ Pa}.$$

## 13.12 APPLICATIONS OF BERNOULLI EQUATION

### (a) Hydrostatics

If the speed of the fluid is zero everywhere, we get the situation of hydrostatics. Putting $v_1 = v_2 = 0$ in the Bernoulli equation (13.10)

$$P_1 + \rho g h_1 = P_2 + \rho g h_2$$

or, $$P_1 - P_2 = \rho\, g(h_2 - h_1)$$

as expected from hydrostatics.

### (b) Speed of Efflux

Consider a liquid of density ρ filled in a tank of large cross-sectional area $A_1$. There is a hole of cross-sectional area $A_2$ at the bottom and the liquid flows out of the tank through the hole. The situation is shown in figure (13.17). Suppose $A_2 << A_1$.

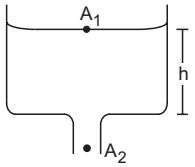



Figure 13.17

Let $v_1$ and $v_2$ be the speeds of the liquid at $A_1$ and $A_2$. As both the cross sections are open to the atmosphere, the pressures there equals the atmospheric pressure $P_0$. If the height of the free surface above the hole is $h$, Bernoulli equation gives

$$P_0 + \frac{1}{2}\rho v_1^2 + \rho g h = P_0 + \frac{1}{2}\rho v_2^2. \qquad \ldots \text{ (i)}$$

By the equation of continuity

$$A_1 v_1 = A_2 v_2 .$$

Putting $v_1$ in terms of $v_2$ in (i),

$$\frac{1}{2}\rho\left(\frac{A_2}{A_1}\right)^2 v_2^2 + \rho g h = \frac{1}{2}\rho v_2^2$$

or, $$\left[1 - \left(\frac{A_2}{A_1}\right)^2\right] v_2^2 = 2gh.$$

If $A_2 << A_1$, this equation reduces to $v_2^2 = 2gh$

or, $v_2 = \sqrt{2gh}$ .

*The speed of liquid coming out through a hole at a depth h below the free surface is the same as that of a particle fallen freely through the height h under gravity.* This is known as *Torricelli's theorem.* The speed of the liquid coming out is called the *speed of efflux.*

***Example 13.5***

*A water tank is constructed on the top of a building. With what speed will the water come out of a tap* 6·0 m *below the water level in the tank ? Assume steady flow and that the pressure above the water level is equal to the atmospheric pressure.*

***Solution*** : The velocity is given by Torricelli's theorem

$$v = \sqrt{2gh}$$

$$= \sqrt{2 \times (9{\cdot}8 \text{ m s}^{-2}) \times (6{\cdot}0 \text{ m})} \approx 11 \text{ m s}^{-1}.$$

### (c) Ventury Tube

A ventury tube is used to measure the flow speed of a fluid in a tube. It consists of a constriction or a throat in the tube. As the fluid passes through the constriction, its speed increases in accordance with the equation of continuity. The pressure thus decreases as required by Bernoulli equation.

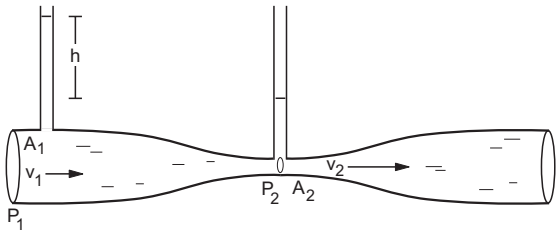



Figure 13.18

Figure (13.18) shows a ventury tube through which a liquid of density ρ is flowing. The area of cross section is $A_1$ at the wider part and $A_2$ at the constriction. Let the speeds of the liquid at $A_1$ and $A_2$ be $v_1$ and $v_2$ and the pressures at $A_1$ and $A_2$ be $P_1$ and $P_2$ respectively. By the equation of continuity

$$A_1 v_1 = A_2 v_2 \qquad \ldots \text{ (i)}$$

and by Bernoulli equation,

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

or, $$(P_1 - P_2) = \frac{1}{2}\rho\left(v_2^2 - v_1^2\right). \qquad \ldots \text{ (ii)}$$

Figure (13.18) also shows two vertical tubes connected to the ventury tube at $A_1$ and $A_2$. If the difference in heights of the liquid levels in these tubes is $h$, we have

$$P_1 - P_2 = \rho g h.$$

Putting in (ii),

$$2gh = v_2^2 - v_1^2 \quad \ldots \text{(iii)}$$

Knowing $A_1$ and $A_2$, one can solve equations (i) and (iii) so as to get $v_1$ and $v_2$. This allows one to know the rate of flow of liquid past a cross section.

**(d) Aspirator Pump**

When a fluid passes through a region at a large speed, the pressure there decreases. This fact finds a number of useful applications. In an aspirator pump a barrel $A$ terminates in a small constriction $B$ (figure 13.19). A narrow tube $C$ connects the constriction to a vessel containing the liquid to be sprayed. The air in the barrel $A$ is pushed by the operator through a piston. As the air passes through the constriction $B$, its speed is considerably increased and consequently the pressure drops. Due to reduced pressure in the constriction $B$, the liquid is raised from the vessel and is sprayed with the expelled air.

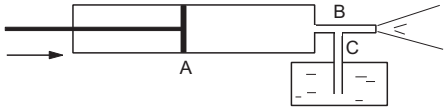



Figure 13.19

**(e) Change of Plane of Motion of a Spinning Ball**

Quite often when swing bowlers of cricket deliver the ball, the ball changes its plane of motion in air. Such a deflection from the plane of projection may be explained on the basis of Bernoulli equation.

Suppose a ball spinning about the vertical direction is going ahead with some velocity in the horizontal direction in otherwise still air. Let us work in a frame in which the centre of the ball is at rest. In this frame the air moves past the ball at a speed $v$ in the opposite direction. The situation is shown in (13.20).

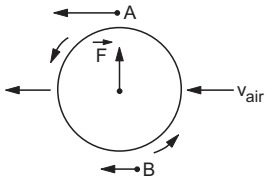



Figure 13.20

The plane of the figure represents horizontal plane. The air that goes from the $A$ side of the ball in the figure is dragged by the spin of the ball and its speed increases. The air that goes from the $B$ side of the ball in the figure suffers an opposite drag and its speed decreases. The pressure of air is reduced on the $A$ side and is increased on the $B$ side as required by the Bernoulli's theorem. As a result, a net force $F$ acts on the ball from the $B$ side to the $A$ side due to this pressure difference. This force causes the deviation of the plane of motion.

## *Worked Out Examples*

**1.** *A beaker of circular cross section of radius* 4 cm *is filled with mercury up to a height of* 10 cm. *Find the force exerted by the mercury on the bottom of the beaker. The atmospheric pressure* $= 10^5$ N m$^{-2}$. *Density of mercury* $= 13600$ kg m$^{-3}$. *Take* $g = 10$ m s$^{-2}$.

***Solution*** **:** The pressure at the surface

$$= \text{atmospheric pressure}$$

$$= 10^5 \text{ N m}^{-2}.$$

The pressure at the bottom

$$= 10^5 \text{ N m}^{-2} + h\rho g$$

$$= 10^5 \text{ N m}^{-2} + (0{\cdot}1 \text{ m})\,(13600 \text{ kg m}^{-3})\,(10 \text{ m s}^{-2})$$

$$= 10^5 \text{ N m}^{-2} + 13600 \text{ N m}^{-2}$$

$$= 1{\cdot}136 \times 10^5 \text{ N m}^{-2}.$$

The force exerted by the mercury on the bottom

$$= (1{\cdot}136 \times 10^5 \text{ N m}^{-2}) \times (3{\cdot}14 \times 0{\cdot}04 \text{ m} \times 0{\cdot}04 \text{ m})$$

$$= 571 \text{ N}.$$

**2.** *The density of air near earth's surface is* 1·3 kg m$^{-3}$ *and the atmospheric pressure is* $1{\cdot}0 \times 10^5$ N m$^{-2}$. *If the atmosphere had uniform density, same as that observed at the surface of the earth, what would be the height of the atmosphere to exert the same pressure* ?

***Solution*** **:** Let the uniform density be ρ and atmospheric height be $h$. The pressure at the surface of the earth would be

$$p = \rho g h$$

or, $\quad 1{\cdot}0 \times 10^5 \text{ N m}^{-2} = (1{\cdot}3 \text{ kg m}^{-3})\,(9{\cdot}8 \text{ m s}^{-2})\,h$

or,
$$h = \frac{1{\cdot}0 \times 10^5 \text{ N m}^{-2}}{(1{\cdot}3 \text{ kg m}^{-3})\,(9{\cdot}8 \text{ m s}^{-2})} = 7850 \text{ m}.$$

Even Mount Everest (8848 m) would have been outside the atmosphere.

**3.** *The liquids shown in figure (13-W1) in the two arms are mercury (specific gravity* = 13·6*) and water. If the difference of heights of the mercury columns is* 2 cm, *find the height h of the water column.*

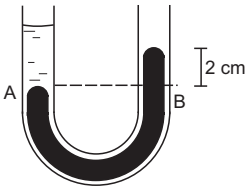



Figure 13-W1

***Solution*** **:** Suppose the atmospheric pressure $= P_0$ .

Pressure at $A = P_0 + h\ (1000\ \text{kg m}^{-3})\ g$ .

Pressure at $B = P_0 + (0{\cdot}02\ \text{m})(13600\ \text{kg m}^{-3})\ g$ .

These pressures are equal as $A$ and $B$ are at the same horizontal level. Thus,

$$h = (0{\cdot}02\ \text{m})\ \ 13{\cdot}6$$

$$\approx 0{\cdot}27\ \text{m} = 27\ \text{cm}.$$

**4.** *A cylindrical vessel containing a liquid is closed by a smooth piston of mass m as shown in the figure. The area of cross section of the piston is A. If the atmospheric pressure is* $P_0$ *, find the pressure of the liquid just below the piston.*

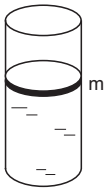



Figure 13-W2

***Solution*** **:** Let the pressure of the liquid just below the piston be $P$. The forces acting on the piston are

(a) its weight, $mg$ (downward)

(b) force due to the air above it, $P_0 A$ (downward)

(c) force due to the liquid below it, $PA$ (upward).

If the piston is in equilibrium,

$$PA = P_0 A + mg$$

or,
$$P = P_0 + \frac{mg}{A}\ .$$

**5.** *The area of cross section of the two arms of a hydraulic press are* 1 cm$^2$ and 10 cm$^2$ *respectively (figure 13-W3). A force of* 5 N *is applied on the water in the thinner arm. What force should be applied on the water in the thicker arm so that the water may remain in equilibrium* ?

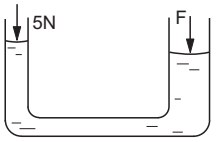



Figure 13-W3

***Solution*** **:** In equilibrium, the pressures at the two surfaces should be equal as they lie in the same horizontal level. If the atmospheric pressure is $P_0$ and a force $F$ is applied to maintain the equilibrium, the pressures are

$$P_0 + \frac{5\ \text{N}}{1\ \text{cm}^2} \text{ and } P_0 + \frac{F}{10\ \text{cm}^2} \text{ respectively.}$$

This gives $F = 50$ N.

**6.** *A copper piece of mass* 10 g *is suspended by a vertical spring. The spring elongates* 1 cm *over its natural length to keep the piece in equilibrium. A beaker containing water is now placed below the piece so as to immerse the piece completely in water. Find the elongation of the spring. Density of copper* = 9000 kg m$^{-3}$. *Take* $g$ = 10 m s$^{-2}$.

***Solution*** **:** Let the spring constant be $k$. When the piece is hanging in air, the equilibrium condition gives

$$k(1\ \text{cm}) = (0{\cdot}01\ \text{kg})\ (10\ \text{m s}^{-2})$$

or $\qquad k(1\ \text{cm}) = 0{\cdot}1\ \text{N}\ .$ $\qquad \ldots$ (i)

The volume of the copper piece

$$= \frac{0{\cdot}01\ \text{kg}}{9000\ \text{kg m}^{-3}} = \frac{1}{9} \times 10^{-5}\ \text{m}^3.$$

This is also the volume of water displaced when the piece is immersed in water. The force of buoyancy

= weight of the liquid displaced

$$= \frac{1}{9} \times 10^{-5}\ \text{m}^3 \times (1000\ \text{kg m}^{-3}) \times (10\ \text{m s}^{-2})$$

$$= 0{\cdot}011\ \text{N}.$$

If the elongation of the spring is $x$ when the piece is immersed in water, the equilibrium condition of the piece gives,

$$kx = 0{\cdot}1\ \text{N} - 0{\cdot}011\ \text{N} = 0{\cdot}089\ \text{N}. \qquad \ldots \text{(ii)}$$

By (i) and (ii),

$$x = \frac{0{\cdot}089}{0{\cdot}1}\ \text{cm} = 0{\cdot}89\ \text{cm}.$$

**7.** *A cubical block of wood of edge* 3 cm *floats in water. The lower surface of the cube just touches the free end of a vertical spring fixed at the bottom of the pot. Find the maximum weight that can be put on the block without wetting it. Density of wood* = 800 kg m$^{-3}$ *and spring constant of the spring* = 50 N m$^{-1}$. *Take* $g$ = 10 m s$^{-2}$.

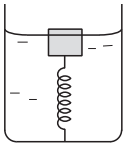


Figure 13-W4

***Solution*** **:** The specific gravity of the block = 0·8. Hence the height inside water = 3 cm × 0·8 = 2·4 cm. The height outside water = 3 cm – 2·4 = 0·6 cm. Suppose the maximum weight that can be put without wetting it is *W*. The block in this case is completely immersed in the water. The volume of the displaced water

= volume of the block = $27 \times 10^{-6}$ m$^3$.

Hence, the force of buoyancy

$= (27 \times 10^{-6}\ \text{m}^3) \times (1000\ \text{kg m}^{-3}) \times (10\ \text{m s}^{-2})$

$= 0{\cdot}27$ N .

The spring is compressed by 0·6 cm and hence the upward force exerted by the spring

$= 50\ \text{N m}^{-1} \times 0{\cdot}6\ \text{cm} = 0{\cdot}3$ N.

The force of buoyancy and the spring force taken together balance the weight of the block plus the weight *w* put on the block. The weight of the block is

$W = (27 \times 10^{-6}\ \text{m}) \times (800\ \text{kg m}^{-3}) \times (10\ \text{m s}^{-2})$

$= 0{\cdot}22$ N .

Thus, $w = 0{\cdot}27\ \text{N} + 0{\cdot}3\ \text{N} - 0{\cdot}22\ \text{N}$

$= 0{\cdot}35$ N .

**8.** *A wooden plank of length* 1 m *and uniform cross section is hinged at one end to the bottom of a tank as shown in figure (13-W5). The tank is filled with water up to a height of* 0·5 m. *The specific gravity of the plank is* 0·5. *Find the angle* θ *that the plank makes with the vertical in the equilibrium position. (Exclude the case* $\theta = 0$.*)*

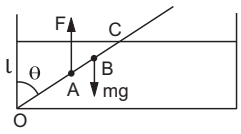



Figure 13-W5

***Solution*** **:** The forces acting on the plank are shown in the figure. The height of water level is $l = 0{\cdot}5$ m. The length of the plank is 1·0 m = 2*l*. The weight of the plank acts through the centre *B* of the plank. We have $OB = l$. The buoyant force *F* acts through the point *A* which is the middle point of the dipped part *OC* of the plank.

We have $OA = \dfrac{OC}{2} = \dfrac{l}{2\cos\theta}$.

Let the mass per unit length of the plank be ρ.

Its weight $mg = 2l\rho g$.

The mass of the part *OC* of the plank $= \left(\dfrac{l}{\cos\theta}\right)\rho$.

The mass of water displaced $= \dfrac{1}{0{\cdot}5}\dfrac{l}{\cos\theta}\rho = \dfrac{2l\rho}{\cos\theta}$.

The buoyant force *F* is, therefore, $F = \dfrac{2l\rho g}{\cos\theta}$.

Now, for equilibrium, the torque of *mg* about *O* should balance the torque of *F* about *O*.

So, $mg(OB)\sin\theta = F(OA)\sin\theta$

or, $(2l\rho)l = \left(\dfrac{2l\rho}{\cos\theta}\right)\left(\dfrac{l}{2\cos\theta}\right)$

or, $\cos^2\theta = \dfrac{1}{2}$

or, $\cos\theta = \dfrac{1}{\sqrt{2}}$, or, $\theta = 45°$.

**9.** *A cylindrical block of wood of mass M is floating in water with its axis vertical. It is depressed a little and then released. Show that the motion of the block is simple harmonic and find its frequency.*

***Solution*** **:** Suppose a height *h* of the block is dipped in the water in equilibrium position. If *r* be the radius of the cylindrical block, the volume of the water displaced $= \pi r^2 h$. For floating in equilibrium

$$\pi r^2 h\rho g = W \qquad \ldots (i)$$

where ρ is the density of water and *W* the weight of the block.

Now suppose during the vertical motion, the block is further dipped through a distance *x* at some instant. The volume of the displaced water is $\pi r^2(h + x)$. The forces acting on the block are, the weight *W* vertically downward and the buoyancy $\pi r^2(h + x)\rho g$ vertically upward.

Net force on the block at displacement *x* from the equilibrium position is

$$F = W - \pi r^2(h + x)\rho g$$
$$= W - \pi r^2 h\rho g - \pi r^2 \rho x g$$

Using (i),

$F = -\pi r^2 \rho g x = -kx$, where $k = \pi r^2 \rho g$.

Thus, the block executes SHM with frequency

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{M}} = \frac{1}{2\pi}\sqrt{\frac{\pi r^2 \rho g}{M}}.$$

**10.** *Water flows in a horizontal tube as shown in figure (13-W6). The pressure of water changes by* 600 N m$^{-2}$

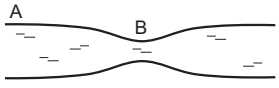



Figure 13-W6

*between A and B where the areas of cross section are* $30\ cm^2$ *and* $15\ cm^2$ *respectively. Find the rate of flow of water through the tube.*

***Solution*** : Let the velocity at $A = v_A$ and that at $B = v_B$

By the equation of continuity, $\frac{v_B}{v_A} = \frac{30\ cm^2}{15\ cm^2} = 2$ .

By Bernoulli equation,

$$P_A + \frac{1}{2}\rho v_A^2 = P_B + \frac{1}{2}\rho v_B^2$$

or, $$P_A - P_B = \frac{1}{2}\rho(2v_A)^2 - \frac{1}{2}\rho v_A^2 = \frac{3}{2}\rho v_A^2 ,$$

or, $$600\ N\ m^{-2} = \frac{3}{2}(1000\ kg\ m^{-3})\ v_A^2$$

or, $$v_A = \sqrt{0{\cdot}4\ m^2 s^{-2}} = 0{\cdot}63\ m\ s^{-1}$$

The rate of flow = $(30\ cm^2)$ $(0{\cdot}63\ m\ s^{-1})$ = $1890\ cm^3 s^{-1}$.

**11.** *The area of cross section of a large tank is* $0{\cdot}5\ m^2$. *It has an opening near the bottom having area of cross section* $1\ cm^2$. *A load of* 20 kg *is applied on the water at the top. Find the velocity of the water coming out of the opening at the time when the height of water level is* 50 cm *above the bottom. Take* $g = 10\ m\ s^{-2}$.

***Solution*** :

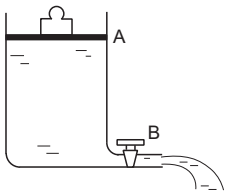



Figure 13-W7

As the area of cross section of the tank is large compared to that of the opening, the speed of water in the tank will be very small as compared to the speed at the opening. The pressure at the surface of water in the tank is that due to the atmosphere plus due to the load.

$$P_A = P_0 + \frac{(20\ kg)\ (10\ m\ s^{-2})}{0{\cdot}5\ m^2} = P_0 + 400\ N\ m^{-2}.$$

At the opening, the pressure is that due to the atmosphere.

Using Bernoulli equation

$$P_A + \rho g h + \frac{1}{2}\rho v_A^2 = P_B + \frac{1}{2}\rho v_B^2$$

or, $$P_0 + 400\ N\ m^{-2} + (1000\ kg\ m^{-3})\ (10\ m\ s^{-2})\ (0{\cdot}5\ m) + 0$$

$$= P_0 + \frac{1}{2}(1000\ kg\ m^{-3}) v_B^2$$

or, $$5400\ N\ m^{-2} = (500\ kg\ m^{-3})^3\ v_B^2$$

or, $$v_B \approx 3{\cdot}3\ m\ s^{-1}.$$

□

## QUESTIONS FOR SHORT ANSWER

**1.** Is it always true that the molecules of a dense liquid are heavier than the molecules of a lighter liquid ?

**2.** If someone presses a pointed needle against your skin, you are hurt. But if someone presses a rod against your skin with the same force, you easily tolerate. Explain.

**3.** In the derivation of $P_1 - P_2 = \rho g z$, it was assumed that the liquid is incompressible. Why will this equation not be strictly valid for a compressible liquid ?

**4.** Suppose the density of air at Madras is $\rho_0$ and atmospheric pressure is $P_0$. If we go up, the density and the pressure both decrease. Suppose we wish to calculate the pressure at a height 10 km above Madras. If we use the equation $P_0 - P = \rho_0 g z$, will we get a pressure more than the actual or less than the actual ? Neglect the variation in $g$. Does your answer change if you also consider the variation in $g$ ?

**5.** The free surface of a liquid resting in an inertial frame is horizontal. Does the normal to the free surface pass through the centre of the earth ? Think separately if the liquid is (a) at the equator (b) at a pole (c) somewhere else.

**6.** A barometer tube reads 76 cm of mercury. If the tube is gradually inclined keeping the open end immersed in the mercury reservoir, will the length of mercury column be 76 cm, more than 76 cm or less than 76 cm ?

**7.** A one meter long glass tube is open at both ends. One end of the tube is dipped into a mercury cup, the tube is kept vertical and the air is pumped out of the tube by connecting the upper end to a suction pump. Can mercury be pulled up into the pump by this process ?

**8.** A satellite revolves round the earth. Air pressure inside the satellite is maintained at 76 cm of mercury. What will be the height of mercury column in a barometer tube 1 m long placed in the satellite ?

**9.** Consider the barometer shown in figure (13-Q1). If a small hole is made at a point $P$ in the barometer tube, will the mercury come out from this hole ?

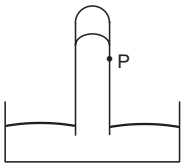



Figure 13-Q1

**10.** Is Archimedes' principle valid in an elevator accelerating up ? In a car accelerating on a level road ?

**11.** Why is it easier to swim in sea water than in fresh water ?

**12.** A glass of water has an ice cube floating in water. The water level just touches the rim of the glass. Will the water overflow when the ice melts ?

**13.** A ferry boat loaded with rocks has to pass under a bridge. The maximum height of the rocks is slightly more than the height of the bridge so that the boat just fails to pass under the bridge. Should some of the rocks be removed or some more rocks be added ?

**14.** Water is slowly coming out from a vertical pipe. As the water descends after coming out, its area of cross section reduces. Explain this on the basis of the equation of continuity.

**15.** While watering a distant plant, a gardener partially closes the exit hole of the pipe by putting his finger on it. Explain why this results in the water stream going to a larger distance.

**16.** A Gipsy car has a canvass top. When the car runs at high speed, the top bulges out. Explain.

## OBJECTIVE I

**1.** A liquid can easily change its shape but a solid can not because
(a) the density of a liquid is smaller than that of a solid
(b) the forces between the molecules is stronger in solid than in liquids
(c) the atoms combine to form bigger molecules in a solid
(d) the average separation between the molecules is larger in solids.

**2.** Consider the equations

$$P = \underset{\Delta s \to 0}{\text{Lim}} \frac{F}{\Delta S} \text{ and } P_1 - P_2 = \rho g z.$$

In an elevator accelerating upward
(a) both the equations are valid
(b) the first is valid but not the second
(c) the second is valid but not the first
(d) both are invalid.

**3.** The three vessels shown in figure (13-Q2) have same base area. Equal volumes of a liquid are poured in the three vessels. The force on the base will be
(a) maximum in vessel $A$ (b) maximum in vessel $B$
(c) maximum in vessel $C$ (d) equal in all the vessels.

Figure 13-Q2

**4.** Equal mass of three liquids are kept in three identical cylindrical vessels $A$, $B$ and $C$. The densities are $\rho_A$, $\rho_B$, $\rho_C$ with $\rho_A < \rho_B < \rho_C$. The force on the base will be
(a) maximum in vessel $A$ (b) maximum in vessel $B$
(c) maximum in vessel $C$ (d) equal in all the vessels.

**5.** Figure (13-Q3) shows a siphon. The liquid shown is water. The pressure difference $P_B - P_A$ between the points $A$ and $B$ is
(a) 400 N m$^{-2}$ (b) 3000 N m$^{-2}$
(c) 1000 N m$^{-2}$ (d) zero.

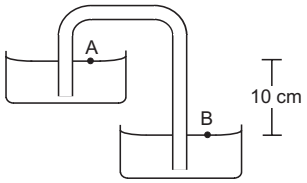



Figure 13-Q3

**6.** A beaker containing a liquid is kept inside a big closed jar. If the air inside the jar is continuously pumped out, the pressure in the liquid near the bottom of the liquid will
(a) increase (b) decrease (c) remain constant
(d) first decrease and then increase.

**7.** The pressure in a liquid at two points in the same horizontal plane are equal. Consider an elevator accelerating upward and a car accelerating on a horizontal road. The above statement is correct in
(a) the car only
(b) the elevator only
(c) both of them
(d) neither of them.

**8.** Suppose the pressure at the surface of mercury in a barometer tube is $P_1$ and the pressure at the surface of mercury in the cup is $P_2$.
(a) $P_1 = 0$, $P_2$ = atmospheric pressure
(b) $P_1$ = atmospheric pressure, $P_2 = 0$
(c) $P_1 = P_2$ = atmospheric pressure
(d) $P_1 = P_2 = 0$.

**9.** A barometer kept in an elevator reads 76 cm when it is at rest. If the elevator goes up with increasing speed,

the reading will be
(a) zero (b) 76 cm (c) < 76 cm (d) > 76 cm.

**10.** A barometer kept in an elevator accelerating upward reads 76 cm. The air pressure in the elevator is
(a) 76 cm (b) < 76 cm (c) > 76 cm (d) zero.

**11.** To construct a barometer, a tube of length 1 m is filled completely with mercury and is inverted in a mercury cup. The barometer reading on a particular day is 76 cm. Suppose a 1 m tube is filled with mercury up to 76 cm and then closed by a cork. It is inverted in a mercury cup and the cork is removed. The height of mercury column in the tube over the surface in the cup will be
(a) zero (b) 76 cm (c) > 76 cm (d) < 76 cm.

**12.** A 20 N metal block is suspended by a spring balance. A beaker containing some water is placed on a weighing machine which reads 40 N. The spring balance is now lowered so that the block gets immersed in the water. The spring balance now reads 16 N. The reading of the weighing machine will be
(a) 36 N (b) 60 N (c) 44 N (d) 56 N.

**13.** A piece of wood is floating in water kept in a bottle. The bottle is connected to an air pump. Neglect the compressibility of water. When more air is pushed into the bottle from the pump, the piece of wood will float with
(a) larger part in the water (b) lesser part in the water
(c) same part in the water (d) it will sink.

**14.** A metal cube is placed in an empty vessel. When water is filled in the vessel so that the cube is completely immersed in the water, the force on the bottom of the vessel in contact with the cube
(a) will increase (b) will decrease
(c) will remain the same (d) will become zero.

**15.** A wooden object floats in water kept in a beaker. The object is near a side of the beaker (figure 13-Q4). Let $P_1, P_2, P_3$ be the pressures at the three points *A, B* and *C* of the bottom as shown in the figure.

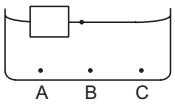



Figure 13-Q4

(a) $P_1 = P_2 = P_3$. (b) $P_1 < P_2 < P_3$.
(c) $P_1 > P_2 > P_3$. (d) $P_2 = P_3 \neq P_1$.

**16.** A closed cubical box is completely filled with water and is accelerated horizontally towards right with an acceleration *a*. The resultant normal force by the water on the top of the box
(a) passes through the centre of the top
(b) passes through a point to the right of the centre
(c) passes through a point to the left of the centre
(d) becomes zero.

**17.** Consider the situation of the previous problem. Let the water push the left wall by a force $F_1$ and the right wall by a force $F_2$.
(a) $F_1 = F_2$ (b) $F_1 > F_2$ (c) $F_1 < F_2$
(d) the information is insufficient to know the relation between $F_1$ and $F_2$.

**18.** Water enters through end *A* with a speed $v_1$ and leaves through end *B* with a speed $v_2$ of a cylindrical tube *AB*. The tube is always completely filled with water. In case *I* the tube is horizontal, in case *II* it is vertical with the end *A* upward and in case *III* it is vertical with the end *B* upward. We have $v_1 = v_2$ for
(a) case *I* (b) case *II* (c) case *III* (d) each case.

**19.** Bernoulli theorem is based on conservation of
(a) momentum (b) mass
(c) energy (d) angular momentum.

**20.** Water is flowing through a long horizontal tube. Let $P_A$ and $P_B$ be the pressures at two points *A* and *B* of the tube.
(a) $P_A$ must be equal to $P_B$.
(b) $P_A$ must be greater than $P_B$.
(c) $P_A$ must be smaller than $P_B$.
(d) $P_A = P_B$ only if the cross-sectional area at *A* and *B* are equal.

**21.** Water and mercury are filled in two cylindrical vessels up to same height. Both vessels have a hole in the wall near the bottom. The velocity of water and mercury coming out of the holes are $v_1$ and $v_2$ respectively.
(a) $v_1 = v_2$. (b) $v_1 = 13{\cdot}6\, v_2$.
(c) $v_1 = v_2/13{\cdot}6$. (d) $v_1 = \sqrt{13{\cdot}6}\, v_2$.

**22.** A large cylindrical tank has a hole of area *A* at its bottom. Water is poured in the tank by a tube of equal cross-sectional area *A* ejecting water at the speed *v*.
(a) The water level in the tank will keep on rising.
(b) No water can be stored in the tank
(c) The water level will rise to a height $v^2/2g$ and then stop.
(d) The water level will oscillate.

## OBJECTIVE II

**1.** A solid floats in a liquid in a partially dipped position.
(a) The solid exerts a force equal to its weight on the liquid.
(b) The liquid exerts a force of buoyancy on the solid which is equal to the weight of the solid.
(c) The weight of the displaced liquid equals the weight of the solid.
(d) The weight of the dipped part of the solid is equal to the weight of the displaced liquid.

**2.** The weight of an empty balloon on a spring balance is $W_1$. The weight becomes $W_2$ when the balloon is filled with air. Let the weight of the air itself be $w$. Neglect the thickness of the balloon when it is filled with air. Also neglect the difference in the densities of air inside and outside the balloon.
(a) $W_2 = W_1$. (b) $W_2 = W_1 + w$.
(c) $W_2 < W_1 + w$. (d) $W_2 > W_1$.

**3.** A solid is completely immersed in a liquid. The force exerted by the liquid on the solid will
(a) increase if it is pushed deeper inside the liquid
(b) change if its orientation is changed
(c) decrease if it is taken partially out of the liquid
(d) be in the vertically upward direction.

**4.** A closed vessel is half filled with water. There is a hole near the top of the vessel and air is pumped out from this hole.
(a) The water level will rise up in the vessel.
(b) The pressure at the surface of the water will decrease.
(c) The force by the water on the bottom of the vessel will decrease.
(d) The density of the liquid will decrease.

**5.** In a streamline flow,
(a) the speed of a particle always remains same
(b) the velocity of a particle always remains same
(c) the kinetic energies of all the particles arriving at a given point are the same
(d) the momenta of all the particles arriving at a given point are the same.

**6.** Water flows through two identical tubes $A$ and $B$. A volume $V_0$ of water passes through the tube $A$ and $2V_0$ through $B$ in a given time. Which of the following may be correct ?
(a) Flow in both the tubes are steady.
(b) Flow in both the tubes are turbulent.
(c) Flow is steady in $A$ but turbulent in $B$.
(d) Flow is steady in $B$ but turbulent in $A$.

**7.** Water is flowing in streamline motion through a tube with its axis horizontal. Consider two points $A$ and $B$ in the tube at the same horizontal level.
(a) The pressures at $A$ and $B$ are equal for any shape of the tube.
(b) The pressures are never equal.
(c) The pressures are equal if the tube has a uniform cross section.
(d) The pressures may be equal even if the tube has a nonuniform cross section.

**8.** There is a small hole near the bottom of an open tank filled with a liquid. The speed of the water ejected does not depend on
(a) area of the hole (b) density of the liquid
(c) height of the liquid from the hole
(d) acceleration due to gravity.

## EXERCISES

**1.** The surface of water in a water tank on the top of a house is 4 m above the tap level. Find the pressure of water at the tap when the tap is closed. Is it necessary to specify that the tap is closed ? Take $g = 10\ \text{m s}^{-2}$.

**2.** The heights of mercury surfaces in the two arms of the manometer shown in figure (13-E1) are 2 cm and 8 cm. Atmospheric pressure $= 1{\cdot}01 \times 10^5\ \text{N m}^{-2}$. Find (a) the pressure of the gas in the cylinder and (b) the pressure of mercury at the bottom of the U tube.

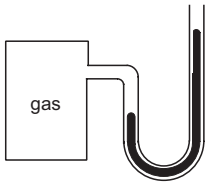



Figure 13-E1

**3.** The area of cross section of the wider tube shown in figure (13-E2) is 900 $\text{cm}^2$. If the boy standing on the piston weighs 45 kg, find the difference in the levels of water in the two tubes.

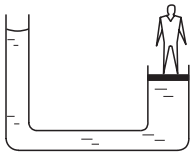


Figure 13-E2

**4.** A glass full of water has a bottom of area 20 $\text{cm}^2$, top of area 20 $\text{cm}^2$, height 20 cm and volume half a litre.
(a) Find the force exerted by the water on the bottom.
(b) Considering the equilibrium of the water, find the resultant force exerted by the sides of the glass on the water. Atmospheric pressure $= 1{\cdot}0 \times 10^5\ \text{N m}^{-2}$. Density of water $= 1000\ \text{kg m}^{-3}$ and $g = 10\ \text{m s}^{-2}$. Take all numbers to be exact.

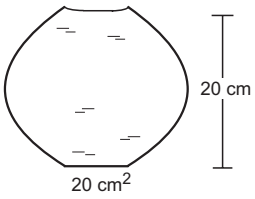



Figure 13-E3

**5.** Suppose the glass of the previous problem is covered by a jar and the air inside the jar is completely pumped out. (a) What will be the answers to the problem ? (b) Show that the answers do not change if a glass of different shape is used provided the height, the bottom area and the volume are unchanged.

**6.** If water be used to construct a barometer, what would be the height of water column at standard atmospheric pressure (76 cm of mercury) ?

**7.** Find the force exerted by the water on a $2\ m^2$ plane surface of a large stone placed at the bottom of a sea 500 m deep. Does the force depend on the orientation of the surface ?

**8.** Water is filled in a rectangular tank of size $3\ m \times 2\ m \times 1\ m$. (a) Find the total force exerted by the water on the bottom surface of the tank. (b) Consider a vertical side of area $2\ m \times 1\ m$. Take a horizontal strip of width $\delta x$ metre in this side, situated at a depth of $x$ metre from the surface of water. Find the force by the water on this strip. (c) Find the torque of the force calculated in part (b) about the bottom edge of this side. (d) Find the total force by the water on this side. (e) Find the total torque by the water on the side about the bottom edge. Neglect the atmospheric pressure and take $g = 10\ m\ s^{-2}$.

**9.** An ornament weighing 36 g in air, weighs only 34 g in water. Assuming that some copper is mixed with gold to prepare the ornament, find the amount of copper in it. Specific gravity of gold is 19·3 and that of copper is 8·9.

**10.** Refer to the previous problem. Suppose, the goldsmith argues that he has not mixed copper or any other material with gold, rather some cavities might have been left inside the ornament. Calculate the volume of the cavities left that will allow the weights given in that problem.

**11.** A metal piece of mass 160 g lies in equilibrium inside a glass of water (figure 13-E4). The piece touches the bottom of the glass at a small number of points. If the density of the metal is $8000\ kg\ m^{-3}$, find the normal force exerted by the bottom of the glass on the metal piece.

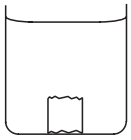


Figure 13-E4

**12.** A ferry boat has internal volume $1\ m^3$ and weight 50 kg. (a) Neglecting the thickness of the wood, find the fraction of the volume of the boat immersed in water. (b) If a leak develops in the bottom and water starts coming in, what fraction of the boat's volume will be filled with water before water starts coming in from the sides ?

**13.** A cubical block of ice floating in water has to support a metal piece weighing 0·5 kg. What can be the minimum edge of the block so that it does not sink in water ? Specific gravity of ice = 0·9.

**14.** A cube of ice floats partly in water and partly in K.oil (figure 13-E5). Find the ratio of the volume of ice immersed in water to that in K.oil. Specific gravity of K.oil is 0·8 and that of ice is 0·9.

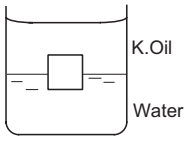



Figure 13-E5

**15.** A cubical box is to be constructed with iron sheets 1 mm in thickness. What can be the minimum value of the external edge so that the cube does not sink in water ? Density of iron $= 8000\ kg\ m^{-3}$ and density of water $= 1000\ kg\ m^{-3}$.

**16.** A cubical block of wood weighing 200 g has a lead piece fastened underneath. Find the mass of the lead piece which will just allow the block to float in water. Specific gravity of wood is 0·8 and that of lead is 11·3.

**17.** Solve the previous problem if the lead piece is fastened on the top surface of the block and the block is to float with its upper surface just dipping into water.

**18.** A cubical metal block of edge 12 cm floats in mercury with one fifth of the height inside the mercury. Water is poured till the surface of the block is just immersed in it. Find the height of the water column to be poured. Specific gravity of mercury = 13·6.

**19.** A hollow spherical body of inner and outer radii 6 cm and 8 cm respectively floats half-submerged in water. Find the density of the material of the sphere.

**20.** A solid sphere of radius 5 cm floats in water. If a maximum load of 0·1 kg can be put on it without wetting the load, find the specific gravity of the material of the sphere.

**21.** Find the ratio of the weights, as measured by a spring balance, of a 1 kg block of iron and a 1 kg block of wood. Density of iron $= 7800\ kg\ m^{-3}$, density of wood $= 800\ kg\ m^{-3}$ and density of air $= 1{\cdot}293\ kg\ m^{-3}$.

**22.** A cylindrical object of outer diameter 20 cm and mass 2 kg floats in water with its axis vertical. If it is slightly depressed and then released, find the time period of the resulting simple harmonic motion of the object.

**23.** A cylindrical object of outer diameter 10 cm, height 20 cm and density $8000\ kg\ m^{-3}$ is supported by a vertical spring and is half dipped in water as shown in figure(13-E6). (a) Find the elongation of the spring in equilibrium condition. (b) If the object is slightly depressed and released, find the time period of resulting oscillations of the object. The spring constant $= 500\ N\ m^{-1}$.

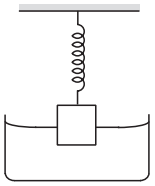


Figure 13-E6

**24.** A wooden block of mass 0·5 kg and density 800 kg m$^{-3}$ is fastened to the free end of a vertical spring of spring constant 50 N m$^{-1}$ fixed at the bottom. If the entire system is completely immersed in water, find (a) the elongation (or compression) of the spring in equilibrium and (b) the time-period of vertical oscillations of the block when it is slightly depressed and released.

**25.** A cube of ice of edge 4 cm is placed in an empty cylindrical glass of inner diameter 6 cm. Assume that the ice melts uniformly from each side so that it always retains its cubical shape. Remembering that ice is lighter than water, find the length of the edge of the ice cube at the instant it just leaves contact with the bottom of the glass.

**26.** A U-tube containing a liquid is accelerated horizontally with a constant acceleration $a_0$. If the separation between the vertical limbs is $l$, find the difference in the heights of the liquid in the two arms.

**27.** At Deoprayag (Garhwal) river Alaknanda mixes with the river Bhagirathi and becomes river Ganga. Suppose Alaknanda has a width of 12 m, Bhagirathi has a width of 8 m and Ganga has a width of 16 m. Assume that the depth of water is same in the three rivers. Let the average speed of water in Alaknanda be 20 km h$^{-1}$ and in Bhagirathi be 16 km h$^{-1}$. Find the average speed of water in the river Ganga.

**28.** Water flows through a horizontal tube of variable cross section (figure 13-E7). The area of cross section at $A$ and $B$ are 4 mm$^2$ and 2 mm$^2$ respectively. If 1 cc of water enters per second through $A$, find (a) the speed of water at $A$, (b) the speed of water at $B$ and (c) the pressure difference $P_A - P_B$.

Figure 13-E7

**29.** Suppose the tube in the previous problem is kept vertical with $A$ upward but the other conditions remain the same. The separation between the cross sections at $A$ and $B$ is 15/16 cm. Repeat parts (a), (b) and (c) of the previous problem. Take $g = 10$ m s$^{-2}$.

**30.** Suppose the tube in the previous problem is kept vertical with $B$ upward. Water enters through $B$ at the rate of 1 cm$^3$ s$^{-1}$. Repeat parts (a), (b) and (c). Note that the speed decreases as the water falls down.

**31.** Water flows through a tube shown in figure (13-E8). The areas of cross section at $A$ and $B$ are 1 cm$^2$ and 0·5 cm$^2$ respectively. The height difference between $A$ and $B$ is 5 cm. If the speed of water at $A$ is 10 cm s$^{-1}$, find (a) the speed at $B$ and (b) the difference in pressures at $A$ and $B$.

Figure 13-E8

**32.** Water flows through a horizontal tube as shown in figure (13-E9). If the difference of heights of water column in the vertical tubes is 2 cm, and the areas of cross section at $A$ and $B$ are 4 cm$^2$ and 2 cm$^2$ respectively, find the rate of flow of water across any section.

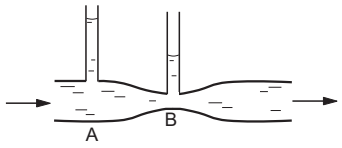



Figure 13-E9

**33.** Water flows through the tube shown in figure (13-E10). The areas of cross section of the wide and the narrow portions of the tube are 5 cm$^2$ and 2 cm$^2$ respectively. The rate of flow of water through the tube is 500 cm$^3$ s$^{-1}$. Find the difference of mercury levels in the U-tube.

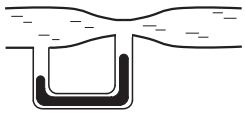


Figure 13-E10

**34.** Water leaks out from an open tank through a hole of area 2 mm$^2$ in the bottom. Suppose water is filled up to a height of 80 cm and the area of cross section of the tank is 0·4 m$^2$. The pressure at the open surface and at the hole are equal to the atmospheric pressure. Neglect the small velocity of the water near the open surface in the tank. (a) Find the initial speed of water coming out of the hole. (b) Find the speed of water coming out when half of water has leaked out. (c) Find the volume of water leaked out during a time interval $dt$ after the height remained is $h$. Thus find the decrease in height $dh$ in terms of $h$ and $dt$.
(d) From the result of part (c) find the time required for half of the water to leak out.

**35.** Water level is maintained in a cylindrical vessel up to a fixed height $H$. The vessel is kept on a horizontal plane. At what height above the bottom should a hole be made in the vessel so that the water stream coming out of the hole strikes the horizontal plane at the greatest distance from the vessel (figure 13-E11)?

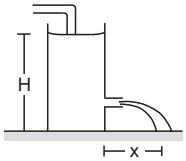



Figure 13.11

## ANSWERS

### OBJECTIVE I

1. (b) 2. (b) 3. (c) 4. (d) 5. (d) 6. (b)
7. (b) 8. (a) 9. (c) 10. (c) 11. (d) 12. (c)
13. (c) 14. (c) 15. (a) 16. (c) 17. (b) 18. (d)
19. (c) 20. (d) 21. (a) 22. (c)

### OBJECTIVE II

1. (a), (b), (c) 2. (a), (c) 3. (c), (d)
4. (b), (c) 5. (c), (d) 6. (a), (b), (c)
7. (c), (d) 8. (a), (b)

### EXERCISES

1. 40000 $N/m^2$, Yes
2. (a) $1{\cdot}09 \times 10^5$ $N/m^2$ (b) $1{\cdot}12 \times 10^5$ $N/m^2$
3. 50 cm
4. (a) 204 N (b) 1 N upward
5. 4 N, 1 N upward
6. 1033·6 cm
7. $10^7$ N, No
8. (a) 60000 N, (b) $20000\, x\, \delta x$ N
   (c) $20000 \times (1 - x)\delta x$ N–m (d) 10000 N,
   (e) 10000/3 N–m
9. 2·2 g
10. 0·112 $cm^3$
11. 1·4 N
12. (a) $\frac{1}{20}$ (b) $\frac{19}{20}$
13. 17 cm
14. 1 : 1
15. 4·8 cm
16. 54·8 g
17. 50 g
18. 10·4 cm
19. 865 kg $m^{-3}$
20. 0·8
21. 1·0015
22. 0·5 s
23. (a) 23·5 cm (b) 0·93 s
24. (a) 2·5 cm (b) $\pi / 5$ s
25. 2·26 cm
26. $a_0 l / g$
27. 23 km/h
28. (a) 25 cm/s, (b) 50 cm/s (c) 94 $N/m^2$
29. (a) 25 cm/s, (b) 50 cm/s, (d) zero
30. (a) 25 cm/s, (b) 50 cm/s, (c) 188 $N/m^2$
31. (a) 20 cm/s, (b) 485 $N/m^2$
32. 146 cc/s
33. 2·13 cm
34. (a) 4 m/s, (b) $\sqrt{8}$ m/s
   (c) $(2\ mm^2)\sqrt{2gh}\, dt$, $\sqrt{2gh} \times 5 \times 10^{-6}\, dt$
   (d) 6·5 hours
35. $H/2$.

□

CHAPTER 14

# SOME MECHANICAL PROPERTIES OF MATTER

## 14.1 MOLECULAR STRUCTURE OF A MATERIAL

Matter is made of molecules and atoms. An atom is made of a nucleus and electrons. The nucleus contains positively charged protons and neutrons, collectively called nucleons. Nuclear forces operating between different nucleons are responsible for the structure of the nucleus. Electromagnetic forces operate between a pair of electrons and between an electron and the nucleus. These forces are responsible for the structure of an atom. The forces between different atoms are responsible for the structure of a molecule and the forces between the molecules are responsible for the structure of the material as seen by us.

### Interatomic and Intermolecular Forces

The force between two atoms can be typically represented by the potential energy curve shown in figure (14.1). The horizontal axis represents the separation between the atoms. The zero of potential energy is taken when the atoms are widely separated ($r = \infty$) .

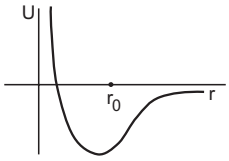



Figure 14.1

As the separation between the atoms is decreased from a large value, the potential energy also decreases, becoming negative. This shows that the force between the atoms is attractive in this range. As the separation is decreased to a particular value $r_0$, the potential energy is minimum. At this separation, the force is zero and the atoms can stay in equilibrium. If the separation is further decreased, the potential energy increases. This means a repulsive force acts between the atoms at small separations.

A polyatomic molecule is formed when the atoms are arranged in such a fashion that the total potential energy of the system is minimum.

The force between two molecules has the same general nature as shown in figure (14.1). At large separation, the force between two molecules is weak and attractive. The force increases as the separation is decreased to a particular value and then decreases to zero at $r = r_0$ . If the separation is further decreased, the force becomes repulsive.

### Bonds

The atoms form molecules primarily due to the electrostatic interaction between the electrons and the nuclei. These interactions are described in terms of different kinds of *bonds*. We shall briefly discuss two important bonds that frequently occur in materials.

#### Ionic Bond

In an *ionic bond* two atoms come close to each other and an electron is completely transferred from one atom to the other. This leaves the first atom positively charged and the other one negatively charged. There is an electrostatic attraction between the ions which keeps them bound. For example, when a sodium atom comes close to a chlorine atom, an electron of the sodium atom is completely transferred to the chlorine atom. The positively charged sodium ion and the negatively charged chlorine ion attract each other to form an ionic bond resulting in sodium chloride molecule.

#### Covalent Bond

In many of the cases a complete transfer of electron from one atom to another does not take place to form

a bond. Rather, electrons from neighbouring atoms are made available for sharing between the atoms. Such bonds are called *covalent bond.* When two hydrogen atoms come close to each other, both the electrons are available to both the nuclei. In other words, each electron moves through the total space occupied by the two atoms. Each electron is pulled by both the nuclei. Chlorine molecule is also formed by this mechanism. Two chlorine atoms share a pair of electrons to form the bond. Another example of covalent bond is hydrogen chloride (HCl) molecule.

**Three States of Matter**

If two molecules are kept at a separation $r = r_0$, they will stay in equilibrium. If they are slightly pulled apart so that $r > r_0$, an attractive force will operate between them. If they are slightly pushed so that $r < r_0$, a repulsive force will operate. Thus, if a molecule is slightly displaced from its equilibrium position, it will oscillate about its mean position. This is the situation in a solid. The molecules are close to each other, very nearly at the equilibrium separations. The amplitude of vibrations is very small and the molecules remain almost fixed at their positions. This explains why a solid has a fixed shape if no external forces act to deform it.

In liquids, the average separation between the molecules is somewhat larger. The attractive force is weak and the molecules are more free to move inside the whole mass of the liquid. In gases, the separation is much larger and the molecular force is very weak.

**Solid State**

In solids, the intermolecular forces are so strong that the molecules or ions remain almost fixed at their equilibrium positions. Quite often these equilibrium positions have a very regular three-dimensional arrangement which we call *crystal.* The positions occupied by the molecules or the ions are called *lattice points.* Because of this long range ordering, the molecules or ions combine to form large rigid solids.

The crystalline solids are divided into four categories depending on the nature of the bonding between the basic units.

**Molecular Solid**

In a molecular solid, the molecules are formed due to covalent bonds between the atoms. The bonding between the molecules depends on whether the molecules are *polar* or *nonpolar* as discussed below. If the centre of negative charge in a molecule coincides with the centre of the positive charge, the molecule is called *nonpolar.* Molecules of hydrogen, oxygen, chlorine, etc., are of this type. Otherwise, the molecule is called a *polar* molecule. Water molecule is polar. The bond between polar molecules is called a *dipole–dipole bond.* The bond between nonpolar molecules is called a *van der Waals bond.* Molecular solids are usually soft and have low melting point. They are poor conductors of electricity.

**Ionic Solid**

In an ionic solid, the lattice points are occupied by positive and negative ions. The electrostatic attraction between these ions binds the solid. These attraction forces are quite strong so that the material is usually hard and has fairly high melting point. They are poor conductors of electricity.

**Covalent Solid**

In a covalent solid, atoms are arranged in the crystalline form. The neighbouring atoms are bound by shared electrons. Such covalent bonds extend in space so as to form a large solid structure. Diamond, silicon, etc., are examples of covalent solids. Each carbon atom is bonded to four neighbouring carbon atoms in a diamond structure. They are quite hard, have high melting point and are poor conductors of electricity.

**Metallic Solid**

In a metallic solid, positive ions are situated at the lattice points. These ions are formed by detaching one or more electrons from the constituent atoms. These electrons are highly mobile and move throughout the solid just like a gas. They are very good conductors of electricity.

**Amorphous or Glassy State**

There are several solids which do not exhibit a long range ordering. However, they still show a local ordering so that some molecules (say 4–5) are bonded together to form a structure. Such independent units are randomly arranged to form the extended solid. In this respect the amorphous solid is similar to a liquid which also lacks any long range ordering. However, the intermolecular forces in amorphous solids are much stronger than those in liquids. This prevents the amorphous solid to flow like a fluid. A typical example is glass made of silicon and oxygen together with some other elements like calcium and sodium. The structure contains strong Si–O–Si bonds, but the structure does not extend too far in space.

The amorphous solids do not have a well-defined melting point. Different bonds have different strengths and as the material is heated the weaker bonds break earlier starting the melting process. The stronger bonds break at higher temperatures to complete the melting process.

## 14.2 ELASTICITY

We have used the concept of a rigid solid body in which the distance between any two particles is always fixed. Real solid bodies do not exactly fulfil this condition. When external forces are applied, the body may get deformed. When deformed, internal forces develop which try to restore the body in its original shape. The extent to which the shape of a body is restored when the deforming forces are removed varies from material to material. The property to restore the natural shape or to oppose the deformation is called *elasticity*. If a body completely gains its natural shape after the removal of the deforming forces, it is called a perfectly *elastic* body. If a body remains in the deformed state and does not even partially regain its original shape after the removal of the deforming forces, it is called a perfectly *inelastic* or *plastic* body. Quite often, when the deforming forces are removed, the body partially regains the original shape. Such bodies are partially elastic.

### Microscopic Reason of Elasticity

A solid body is composed of a great many molecules or atoms arranged in a particular fashion. Each molecule is acted upon by the forces due to the neighbouring molecules. The solid takes such a shape that each molecule finds itself in a position of stable equilibrium. When the body is deformed, the molecules are displaced from their original positions of stable equilibrium. The intermolecular distances change and restoring forces start acting on the molecules which drive them back to their original positions and the body takes its natural shape.

One can compare this situation to a spring-mass system. Consider a particle connected to several particles through springs. If this particle is displaced a little, the springs exert a resultant force which tries to bring the particle towards its natural position. In fact, the particle will oscillate about this position. In due course, the oscillations will be damped out and the particle will regain its original position.

## 14.3 STRESS

### Longitudinal and Shearing Stress

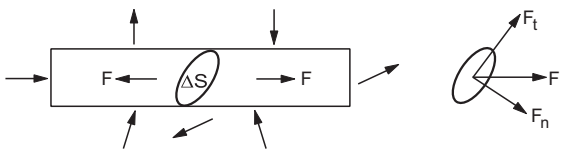



Figure 14.2

Consider a body (figure 14.2) on which several forces are acting. The resultant of these forces is zero so that the centre of mass remains at rest. Due to the forces, the body gets deformed and internal forces appear. Consider any cross-sectional area $\Delta S$ of the body. The parts of the body on the two sides of $\Delta S$ exert forces $\vec{F}$, $-\vec{F}$ on each other. These internal forces $\vec{F}$, $-\vec{F}$ appear because of the deformation.

The force $\vec{F}$ may be resolved in two components, $F_n$ normal to $\Delta S$ and $F_t$ tangential to $\Delta S$. We define the *normal stress* or *longitudinal stress* over the area as

$$\Gamma_n = \frac{F_n}{\Delta S} \qquad \ldots \quad (14.1)$$

and the *tangential stress* or *shearing stress* over the area as

$$\Gamma_t = \frac{F_t}{\Delta S} \cdot \qquad \ldots \quad (14.2)$$

The longitudinal stress can be of two types. The two parts of the body on the two sides of $\Delta S$ may pull each other. The longitudinal stress is then called the *tensile stress*. This is the case when a rod or a wire is stretched by equal and opposite forces (figure 14.3). In case of tensile stress in a wire or a rod, the force $F_n$ is just the tension.

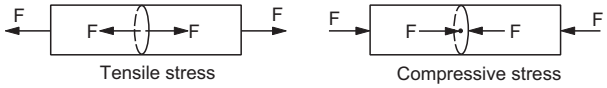



Figure 14.3

If the rod is pushed at the two ends with equal and opposite forces, it will be under compression. Taking any cross section $\Delta S$ of the rod the two parts on the two sides push each other. The longitudinal stress is then called *compressive stress*.

If the area is not specifically mentioned, a cross section perpendicular to the length is assumed.

***Example 14.1***

*A load of* 4·0 kg *is suspended from a ceiling through a steel wire of radius* 2·0 mm. *Find the tensile stress developed in the wire when equilibrium is achieved. Take* $g = 3{\cdot}1\pi$ m s$^{-2}$.

***Solution*** : Tension in the wire is

$$F = 4{\cdot}0 \times 3{\cdot}1\pi \text{ N} .$$

The area of cross section is

$$A = \pi r^2 = \pi \times (2{\cdot}0 \times 10^{-3} \text{ m})^2$$

$$= 4{\cdot}0\, \pi \times 10^{-6} \text{ m}^2 .$$

Thus, the tensile stress developed

$$= \frac{F}{A} = \frac{4{\cdot}0 \times 3{\cdot}1\,\pi}{4{\cdot}0\,\pi \times 10^{-6}}\ \text{N m}^{-2}$$

$$= 3{\cdot}1 \times 10^{6}\ \text{N m}^{-2}.$$

**Volume Stress**

Another type of stress occurs when a body is acted upon by forces acting everywhere on the surface in such a way that (a) the force at any point is normal to the surface and (b) the magnitude of the force on any small surface area is proportional to the area. This is the case when a small solid body is immersed in a fluid. If the pressure at the location of the solid is $P$, the force on any area $\Delta S$ is $P\Delta S$ directed perpendicularly to the area. The force per unit area is then called *volume stress* (figure 14.4). It is

$$\Gamma_v = \frac{F}{A} \qquad \ldots \ (14.3)$$

which is same as the pressure.

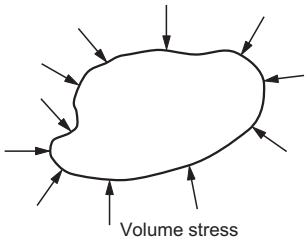



Figure 14.4

## 14.4 STRAIN

Associated with each type of stress defined above, there is a corresponding type of strain.

**Longitudinal Strain**

Consider a rod of length $l$ being pulled by equal and opposite forces. The length of the rod increases from its natural value $L$ to $L + \Delta L$. The fractional change $\Delta L/L$ is called the *longitudinal strain*.

$$\text{Longitudinal strain} = \Delta L/L\,. \qquad \ldots \ (14.4)$$

If the length increases from its natural length, the longitudinal strain is called *tensile strain*. If the length decreases from its natural length, the longitudinal strain is called *compressive strain.*

**Shearing Strain**

This type of strain is produced when a shearing stress is present over a section. Consider a body with square cross section and suppose forces parallel to the surfaces are applied as shown in figure (14.5). Note that the resultant of the four forces shown is zero as well as the total torque of the four forces is zero.

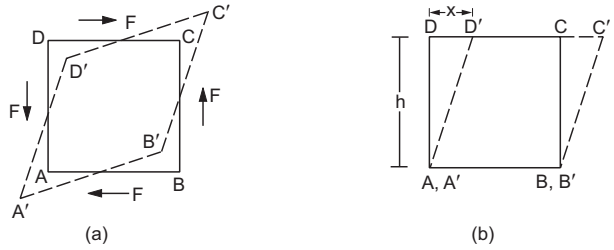



Figure 14.5

This ensures that the body remains in translational and rotational equilibrium after the deformation. Because of the tangential forces parallel to the faces, these faces are displaced. The shape of the cross section changes from a square to a parallelogram. In figure (14.5a) the dotted area represents the deformed cross section. To measure the deformation, we redraw the dotted area by rotating it a little so that one edge $A'B'$ coincides with its undeformed position $AB$. The drawing is presented in part (b) of figure (14.5).

We define the shearing strain as the displacement of a layer divided by its distance from the fixed layer. In the situation of figure (14.5),

$$\text{Shearing strain} = DD'/DA = x/h\,.$$

Shearing strain is also called *shear*.

**Volume Strain**

When a body is subjected to a volume stress, its volume changes. The *volume strain* is defined as the fractional change in volume. If $V$ is the volume of unstressed body and $V + \Delta V$ is the volume when the volume stress exists, the volume strain is defined as

$$\text{Volume strain} = \Delta V/V.$$

## 14.5 HOOKE'S LAW AND THE MODULII OF ELASTICITY

*If the deformation is small, the stress in a body is proportional to the corresponding strain.*

This fact is known as Hooke's law. Thus, if a rod is stretched by equal and opposite forces $F$ each, a tensile stress $F/A$ is produced in the rod where $A$ is the area of cross section. The length of the rod increases from its natural value $L$ to $L + \Delta L$. Tensile strain is $\Delta L/L$.

By Hooke's law, for small deformations,

$$\frac{\text{Tensile stress}}{\text{Tensile strain}} = Y \qquad \ldots \ (14.5)$$

is a constant for the given material. This ratio of tensile stress over tensile strain is called *Young modulus* for the material. In the situation described above, the Young modulus is

$$Y = \frac{F/A}{\Delta L/L} = \frac{FL}{A\Delta L} . \quad \ldots (14.6)$$

If the rod is compressed, compressive stress and compressive strain appear. Their ratio $Y$ is same as that for the tensile case.

***Example 14.2***

*A load of* 4·0 kg *is suspended from a ceiling through a steel wire of length* 20 m *and radius* 2·0 mm. *It is found that the length of the wire increases by* 0·031 mm *as equilibrium is achieved. Find Young modulus of steel. Take* $g = 3{\cdot}1\ \pi\ \text{m s}^{-2}$.

***Solution*** : The longitudinal stress $= \dfrac{(4{\cdot}0\ \text{kg})\ (3{\cdot}1\ \pi\ \text{m s}^{-2})}{\pi\ (2{\cdot}0 \times 10^{-3}\ \text{m})^{2}}$

$= 3{\cdot}1 \times 10^{6}\ \text{N m}^{-2}$ .

The longitudinal strain $= \dfrac{0{\cdot}031 \times 10^{-3}\ \text{m}}{2{\cdot}0\ \text{m}}$

$= 0{\cdot}0155 \times 10^{-3}$

Thus $Y = \dfrac{3{\cdot}1 \times 10^{6}\ \text{N m}^{-2}}{0{\cdot}0155 \times 10^{-3}} = 2{\cdot}0 \times 10^{11}\ \text{N m}^{-2}$.

The ratio of shearing stress over shearing strain is called the *Shear modulus, Modulus of rigidity* or *Torsional modulus*. In the situation of figure (14.5) the shear modulus is

$$\eta = \frac{F/A}{x/h} = \frac{Fh}{Ax} . \quad \ldots (14.7)$$

The ratio of volume stress over volume strain is called *Bulk modulus*. If $P$ be the volume stress (same as pressure) and $\Delta V$ be the increase in volume, the Bulk modulus is defined as

$$B = -\frac{P}{\Delta V/V} . \quad \ldots (14.8)$$

The minus sign makes $B$ positive as volume actually decreases on applying pressure. Quite often, the change in volume is measured corresponding to a change in pressure. The bulk modulus is then defined as

$$B = -\frac{\Delta P}{\Delta V/V} = -V\frac{dP}{dV} .$$

*Compressibility* $K$ is defined as the reciprocal of the bulk modulus.

$$K = \frac{1}{B} = -\frac{1}{V}\frac{dV}{dP} . \quad \ldots (14.9)$$

Yet another kind of modulus of elasticity is associated with the longitudinal stress and strain. When a rod or a wire is subjected to a tensile stress, its length increases in the direction of the tensile force. At the same time the length perpendicular to the tensile force decreases. For a cylindrical rod, the length increases and the diameter decreases when the rod is stretched (Figure 14.6).

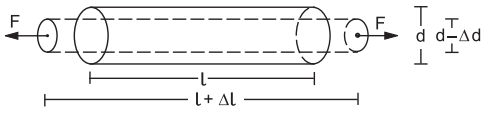



Figure 14.6

The fractional change in the transverse length is proportional to the fractional change in the longitudinal length. The constant of proportionality is called *Poisson ratio*. Thus, Poisson ratio is

$$\sigma = -\frac{\Delta d/d}{\Delta L/L} . \quad \ldots (14.10)$$

The minus sign ensures that σ is positive. Table (14.1) lists the elastic constants of some of the common materials. Table (14.2) lists compressibilities of some liquids.

**Table 14.1 : *Elastic constants***

| Material | Young Modulus $Y$ $10^{11}\ \text{N m}^{-2}$ | Shear Modulus $\eta$ $10^{11}\ \text{N m}^{-2}$ | Bulk Modulus $B$ $10^{11}\ \text{N m}^{-2}$ | Poisson ratio σ |
|---|---|---|---|---|
| Aluminium | 0·70 | 0·30 | 0·70 | 0·16 |
| Brass | 0·91 | 0·36 | 0·61 | 0·26 |
| Copper | 1·1 | 0·42 | 1·4 | 0·32 |
| Iron | 1·9 | 0·70 | 1·0 | 0·27 |
| Steel | 2·0 | 0·84 | 1·6 | 0·19 |
| Tungsten | 3·6 | 1·5 | 2·0 | 0·20 |

**Table 14.2 : *Compressibilities of liquids***

| Liquid | Compressibility $K$ $10^{-11}\ \text{m}^{2}\ \text{N}^{-1}$ |
|---|---|
| Carbon disulphide | 64 |
| Ethyl alcohol | 110 |
| Glycerine | 21 |
| Mercury | 3·7 |
| Water | 49 |

## 14.6 RELATION BETWEEN LONGITUDINAL STRESS AND STRAIN

For a small deformation, the longitudinal stress is proportional to the longitudinal strain. What happens if the deformation is not small ? The relation of stress and strain is much more complicated in such a case and the nature depends on the material under study. We describe here the behaviour for two representative materials, a metal wire and a rubber piece.

**Metal Wire**

Suppose a metal wire is stretched by equal forces at the ends so that its length increases from its natural value. Figure (14.7) shows qualitatively the relation between the stress and the strain as the deformation gradually increases.

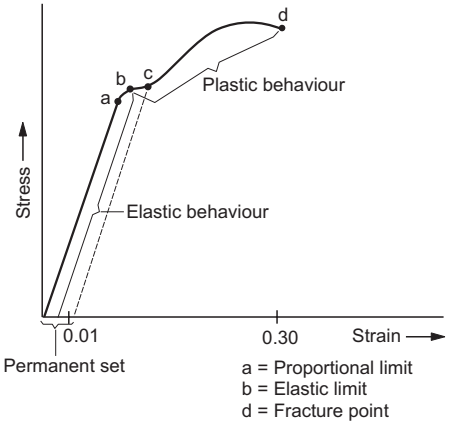



Figure 14.7

When the strain is small (say < 0·01), the stress is proportional to the strain. This is the region where Hooke's law is valid and where Young's modulus is defined. The point *a* on the curve represents the *proportional limit* up to which stress and strain are proportional.

If the strain is increased a little bit, the stress is not proportional to the strain. However, the wire still remains elastic. This means, if the stretching force is removed, the wire acquires its natural length. This behaviour is shown up to a point *b* on the curve known as the *elastic limit* or the *yield point.* If the wire is stretched beyond the elastic limit, the strain increases much more rapidly. If the stretching force is removed, the wire does not come back to its natural length. Some permanent increase in length takes place. In figure (14.7), we have shown this behaviour by the dashed line from *c*. The behaviour of the wire is now plastic. If the deformation is increased further, the wire breaks at a point *d* known as *fracture point*. The stress corresponding to this point is called *breaking stress.*

If large deformation takes place between the elastic limit and the fracture point, the material is called *ductile*. If it breaks soon after the elastic limit is crossed, it is called *brittle.*

**Rubber**

A distinctly different stress–strain relation exists for vulcanized rubber, the behaviour is qualitatively shown in figure (14.8). The material remains elastic even when it is stretched to over several times its original length. In the case shown in figure (14.8), the length is increased to 8 times its natural length, even then if the stretching forces are removed, it will come back to its original length.

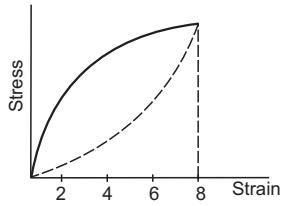



Figure 14.8

In this respect rubber is more elastic than a ductile metal like steel. However, the magnitude of stress for a given strain is much larger in steel than in rubber. This means large internal forces appear if the steel wire is deformed. In this sense, steel is more elastic than rubber. There are two important phenomena to note from figure (14.8). Firstly, in no part of this large deformation stress is proportional to strain. There is almost no region of proportionality. Secondly, when the deforming force is removed the original curve is not retraced although the sample finally acquires its natural length. The work done by the material in returning to its original shape is less than the work done by the deforming force when it was deformed. A particular amount of energy is, thus, absorbed by the material in the cycle which appears as heat. This phenomenon is called *elastic hysteresis.*

Elastic hysteresis has an important application in shock absorbers. If a padding of vulcanized rubber is given between a vibrating system and, say, a flat board, the rubber is compressed and released in every cycle of vibration. As energy is absorbed in the rubber in each cycle, only a part of the energy of vibrations is transmitted to the board.

## 14.7 ELASTIC POTENTIAL ENERGY OF A STRAINED BODY

When a body is in its natural shape, its potential energy corresponding to the molecular forces is minimum. We may take the potential energy in this state to be zero. When deformed, internal forces appear and work has to be done against these forces. Thus, the potential energy of the body is increased. This is called the *elastic potential energy*. We shall derive an expression for the increase in elastic potential energy when a wire is stretched from its natural length.

Suppose a wire having natural length $L$ and cross-sectional area $A$ is fixed at one end and is stretched

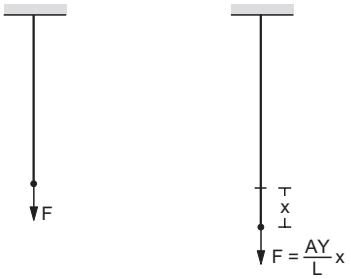



Figure 14.9

by an external force applied at the other end (figure 14.9). The force is so adjusted that the wire is only slowly stretched. This ensures that at any time during the extension the external force equals the tension in the wire. When the extension is $x$, the wire is under a longitudinal stress $F/A$, where $F$ is the tension at this time. The strain is $x/L$.

If Young modulus is $Y$,

$$\frac{F/A}{x/L} = Y$$

or, $$F = \frac{AY}{L} x. \qquad \ldots \text{(i)}$$

The work done by the external force in a further extension $dx$ is

$$dW = F\,dx.$$

Using (i),

$$dW = \frac{AY}{L} x\,dx\,.$$

The total work by the external force in an extension 0 to $l$ is

$$W = \int_0^l \frac{AY}{L} x\,dx$$

$$= \frac{AY}{2L} l^2.$$

This work is stored into the wire as its elastic potential energy.

Thus, the elastic potential energy of the stretched wire is,

$$U = \frac{AY}{2L} l^2\,. \qquad \ldots \text{(14.11)}$$

This may be written as

$$U = \frac{1}{2}\left(AY\frac{l}{L}\right) l$$

$$= \frac{1}{2} \text{(maximum stretching force) (extension).}$$

Equation (14.11) may also be written as

$$U = \frac{1}{2}\left(Y\frac{l}{L}\right)\frac{l}{L}(AL)$$

or, $$\textit{Potential energy} = \frac{1}{2} \times \textit{stress} \times \textit{strain} \times \textit{volume}.$$

$$\ldots \text{(14.12)}$$

***Example 14.3***

*A steel wire of length* 2·0 m *is stretched through* 2·0 mm. *The cross-sectional area of the wire is* 4·0 $mm^2$. *Calculate the elastic potential energy stored in the wire in the stretched condition. Young modulus of steel* = $2{\cdot}0 \times 10^{11}$ N $m^{-2}$.

***Solution*** : The strain in the wire $\frac{\Delta l}{l} = \frac{2{\cdot}0 \text{ mm}}{2{\cdot}0 \text{ m}} = 10^{-3}$.

The stress in the wire = $Y \times$ strain

$$= 2{\cdot}0 \times 10^{11} \text{ N m}^{-2} \times 10^{-3} = 2{\cdot}0 \times 10^8 \text{ N m}^{-2}.$$

The volume of the wire = $(4 \times 10^{-6} \text{ m}^2) \times (2{\cdot}0 \text{ m})$

$$= 8{\cdot}0 \times 10^{-6} \text{ m}^3.$$

The elastic potential energy stored

$$= \frac{1}{2} \times \text{stress} \times \text{strain} \times \text{volume}$$

$$= \frac{1}{2} \times 2{\cdot}0 \times 10^8 \text{ N m}^{-2} \times 10^{-3} \times 8{\cdot}0 \times 10^{-6} \text{ m}^3$$

$$= 0{\cdot}8 \text{ J}.$$

## 14.8 DETERMINATION OF YOUNG MODULUS IN LABORATORY

Figure (14.10) shows the experimental set up of a simple method to determine Young modulus in a laboratory. A long wire A (say 2–3 m) is suspended from a fixed support. It carries a fixed graduated scale and below it a heavy fixed load. This load keeps the wire straight and free from kinks. The wire itself serves as a reference. The experimental wire B of almost equal length is also suspended from the same support close to the reference wire. A vernier scale is attached at the free end of the experimental wire. This vernier scale can slide against the main scale attached to the reference wire.

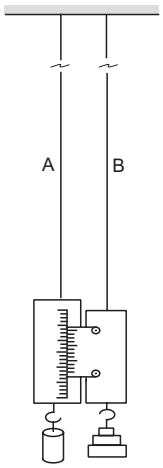



Figure 14.10

A hanger is attached at the lower end of the vernier scale. A number of slotted half kilogram or one kilogram weights may be slipped into the hanger.

First of all, the radius of the experimental wire is measured at several places with a screw gauge. From the average radius *r,* the breaking weight is determined using the standard value of the breaking stress for the material. Half of this breaking weight is the permissible weight.

Some initial load, say 1 kg or 2 kg, is kept on the hanger (this should be much smaller than the permissible weight). This keeps the experimental wire straight and kink-free. The reading of the main scale and vernier coincidence are noted. A known weight say 1/2 kg or 1 kg is slipped into the hanger. The set up is left for about a minute so that the elongation takes place fully. The reading on the scale are noted. The difference of the scale readings gives the extension due to the extra weight put. The weight is gradually increased up to the permissible weight and every time the extension is noted.

The experiment is repeated in reverse order decreasing the weight gradually in the same steps and everytime noting the extension.

From the data, extension versus load curve is plotted. This curve should be a straight line passing through the origin (figure 14.11). The slope of this line gives

$$\tan\theta = \frac{l}{Mg} \cdot$$

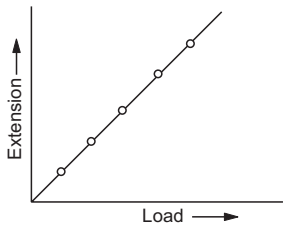



Figure 14.11

Now the stress due to the weight $Mg$ at the end is

$$\text{stress} = \frac{Mg}{\pi r^2}$$

and $$\text{strain} = \frac{l}{L} \cdot$$

Thus, $$Y = \frac{MgL}{\pi r^2 l} = \frac{L}{\pi r^2 \tan\theta} \cdot$$

All the quantities on the right-hand side are known and hence Young modulus $Y$ may be calculated.

### 14.9 SURFACE TENSION

The properties of a surface are quite often markedly different from the properties of the bulk material. A molecule well inside a body is surrounded by similar particles from all sides. But a molecule on the surface has particles of one type on one side and of a different type on the other side. Figure (14.12) shows an example. A molecule of water well inside the bulk experiences forces from water molecules from all sides but a molecule at the surface interacts with air molecules from above and water molecules from below. This asymmetric force distribution is responsible for surface tension.

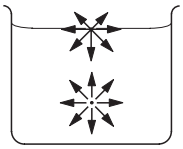


Figure 14.12

By a surface we shall mean a layer approximately 10–15 molecular diameters. The force between two molecules decreases as the separation between them increases. The force becomes negligible if the separation exceeds 10–15 molecular diameters. Thus, if we go 10–15 molecular diameters deep, a molecule finds equal forces from all directions.

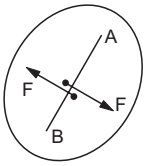



Figure 14.13

Imagine a line $AB$ drawn on the surface of a liquid (figure 14.13). The line divides the surface in two parts, surface on one side and the surface on the other side of the line. Let us call them surface to the left of the line and surface to the right of the line. It is found that the two parts of the surface pull each other with a force proportional to the length of the line $AB$. These forces of pull are perpendicular to the line separating the two parts and are tangential to the surface. In this respect the surface of the liquid behaves like a stretched rubber sheet. The rubber sheet which is stretched from all sides is in the state of tension. Any part of the sheet pulls the adjacent part towards itself.

Let $F$ be the common magnitude of the forces exerted on each other by the two parts of the surface across a line of length $l$. We define the surface tension $S$ of the liquid as

$$S = F/l. \qquad \ldots (14.13)$$

The SI unit of surface tension is N $m^{-1}$.

***Example 14.4***

*Water is kept in a beaker of radius* 5·0 cm. *Consider a diameter of the beaker on the surface of the water. Find the force by which the surface on one side of the diameter pulls the surface on the other side. Surface tension of water* = 0·075 N $m^{-1}$.

***Solution*** : The length of the diameter is

$$l = 2r = 10 \text{ cm}$$

$$= 0{\cdot}1 \text{ m}.$$

The surface tension is $S = F/l$. Thus,

$$F = Sl$$

$$= (0{\cdot}075 \text{ N m}^{-1}) \times (0{\cdot}1 \text{ m}) = 7{\cdot}5 \times 10^{-3} \text{ N}.$$

The fact that a liquid surface has the property of surface tension can be demonstrated by a number of simple experiments.

(a) Take a ring of wire and dip it in soap solution. When the ring is taken out, a soap film bounded by the ring is formed. Now take a loop of thread, wet it and place it gently on the soap film. The loop stays on the film in an irregular fashion as it is placed. Now prick a hole in the film inside the loop with a needle. The thread is radially pulled by the film surface outside and it takes a circular shape (figure 14.14).

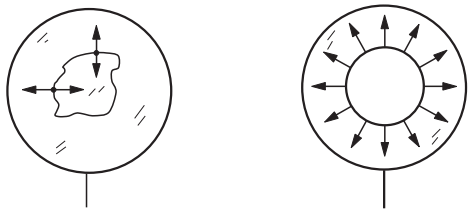


Figure 14.14

Before the pricking, there were surfaces both inside and outside the thread loop. Taking any small part of the thread, surfaces on both sides pulled it and the net force was zero. The thread could remain in any shape. Once the surface inside was punctured, the outside surface pulled the thread to take the circular shape.

Figure 14.15

(b) Take a U-shaped frame of wire on which a light wire can slide (figure 14.15). Dip the frame in a soap solution and take it out. A soap film is formed between the frame and the sliding wire. If the frame is kept in a horizontal position and the friction is negligible, the sliding wire quickly slides towards the closing arm of the frame. This shows that the soap surface in contact with the wire pulls it parallel to the surface. If the frame is kept vertical with the sliding wire at the lower position, one can hang some weight from it to keep it in equilibrium. The force due to surface tension by the surface in contact with the sliding wire balances the weight.

**Tendency to Decrease the Surface Area**

The property of surface tension may also be described in terms of the tendency of a liquid to decrease its surface area. Because of the existence of forces across any line in the surface, the surface tends to shrink whenever it gets a chance to do so. The two demonstrations described above may help us in understanding the relation between the force of surface tension and the tendency to shrink the surface.

In the first example, the soap film is pricked in the middle. The remaining surface readjusts its shape so that a circular part bounded by the thread loop is excluded. The loop has a fixed length and the largest area that can be formed with a fixed periphery is a circle. This ensures that the surface of the soap solution takes the minimum possible area.

In the second example, the wire can slide on the frame. When kept in horizontal position, the wire slides to the closing arm of the U-shaped frame so that the surface shrinks.

There are numerous examples which illustrate that the surface of a liquid tries to make its area minimum. When a painting brush is inside a liquid, the bristles of the brush wave freely. When the brush is taken out of the liquid, surfaces are formed between the bristles. To minimise the area of these surfaces, they stick together.

A small drop of liquid takes a nearly spherical shape. This is because, for a given volume, the sphere assume the smallest surface area. Because of gravity there is some deviation from the spherical shape, but for small drops this may be neglected.

Table 14.3 gives the values of surface tension of some liquids.

**Table 14.3 :** ***Surface tension***

| Liquid | Surface tension N $m^{-1}$ | Liquid | Surface Tension N $m^{-1}$ |
|---|---|---|---|
| Mercury | 0·465 | Glycerine | 0·063 |
| Water | 0·075 | Carbon tetra chloride | 0·027 |
| Soap solution | 0·030 | Ethyl alcohol | 0·022 |

## 14.10 SURFACE ENERGY

We have seen that a molecule well within the volume of a liquid is surrounded by the similar liquid molecules from all sides and hence there is no resultant force on it (figure 14.12). On the other hand, a molecule in the surface is surrounded by similar liquid molecules only on one side of the surface while on the other side it may be surrounded by air molecules or the molecules of the vapour of the liquid etc. These vapours having much less density exert only a small force. Thus, there is a resultant inward force on a molecule in the surface. This force tries to pull the molecule into the liquid. Thus, the surface layer remains in microscopic turbulence. Molecules are pulled back from the surface layer to the bulk and new molecules from the bulk go to the surface to fill the empty space.

When a molecule is taken from the inside to the surface layer, work is done against the inward resultant force while moving up in the layer. The potential energy is increased due to this work. A molecule in the surface has greater potential energy than a molecule well inside the liquid. The extra energy that a surface layer has is called the *surface energy*. The surface energy is related to the surface tension as discussed below.

Consider a U-shaped frame with a sliding wire on its arm. Suppose it is dipped in a soap solution, taken out and placed in a horizontal position (figure 14.16).

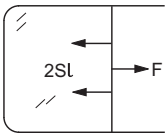



Figure 14.16

The soap film that is formed may look quite thin, but on the molecular scale its thickness is not small. It may have several hundred thousands molecular layers. So it has two surfaces enclosing a bulk of soap solution. Both the surfaces are in contact with the sliding wire and hence exert forces of surface tension on it. If $S$ be the surface tension of the solution and $l$ be the length of the sliding wire, each surface will pull the wire parallel to itself with a force $Sl$. The net force of pull $F$ on the wire due to both the surfaces is

$$F = 2\,Sl\,.$$

One has to apply an external force equal and opposite to $F$ so as to keep the wire in equilibrium.

Now suppose the wire is slowly pulled out by the external force through a distance $x$ so that the area of the frame is increased by $lx$. As there are two surfaces of the solution, a new surface area $2lx$ is created. The liquid from the inside is brought to create the new surface.

The work done by the external force in the displacement is

$$W = F\,x = 2Sl\,x = S\,(2lx)\,.$$

As there is no change in kinetic energy, the work done by the external force is stored as the potential energy of the new surface.

The increase in surface energy is

$$U = W = S\,(2lx)\,.$$

Thus, $$\frac{U}{(2lx)} = S$$

or, $$\frac{U}{A} = S. \qquad \ldots\ (14.14)$$

We see that the surface tension of a liquid is equal to the surface energy per unit surface area.

In this interpretation, the SI unit of surface tension may be written as J $m^{-2}$. It may be verified that N $m^{-1}$ is equivalent to J $m^{-2}$.

***Example 14.5***

*A water drop of radius* $10^{-2}$ m *is broken into* 1000 *equal droplets. Calculate the gain in surface energy. Surface tension of water is* 0·075 N $m^{-1}$.

***Solution*** : The volume of the original drop is

$$V = \frac{4}{3}\pi R^3, \text{ where } R = 10^{-2}\text{ m}.$$

If $r$ is the radius of each broken droplet, the volume is also

$$V = 1000 \times \frac{4}{3}\pi r^3.$$

Thus, $$1000\,r^3 = R^3$$

or, $$r = R/10.$$

The surface area of the original drop is $A_1 = 4\pi R^2$ and the surface area of the 1000 droplets is

$$A_2 = 1000 \times 4\,\pi\,r^2 = 40\,\pi\,R^2.$$

The increase in area is

$$\Delta A = A_2 - A_1 = 40\,\pi R^2 - 4\,\pi R^2 = 36\,\pi R^2.$$

The gain in surface energy is

$$\Delta U = (\Delta A)\,S = 36\,\pi R^2 S$$
$$= 36 \times 3{\cdot}14 \times (10^{-4}\text{ m}^2) \times (0{\cdot}075\text{ N m}^{-1})$$
$$= 8{\cdot}5 \times 10^{-4}\text{ J}.$$

## 14.11 EXCESS PRESSURE INSIDE A DROP

Let us consider a spherical drop of liquid of radius $R$ (figure 14.17). If the drop is small, the effect of gravity may be neglected and the shape may be assumed to be spherical.

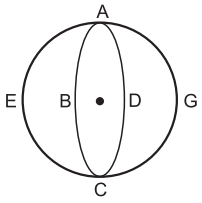



Figure 14.17

Imagine a diametric cross section *ABCD* of the drop which divides the drop in two hemispheres. The surfaces of the two hemispheres touch each other along the periphery *ABCDA*. Each hemispherical surface pulls the other hemispherical surface due to the surface tension.

Consider the equilibrium of the hemispherical surface *ABCDE*. Forces acting on this surface are

(i) $F_1$, due to the surface tension of the surface *ABCDG* in contact,

(ii) $F_2$, due to the air outside the surface *ABCDE* and

(iii) $F_3$, due to the liquid inside the surface *ABCDE*.

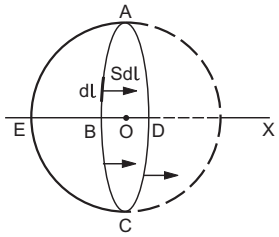



Figure 14.18

The force due to surface tension acts on the points of the periphery *ABCDA*. The force on any small part *dl* of this periphery is $S\,dl$ (figure 14.18) and acts parallel to the symmetry axis *OX*. The resultant of all these forces due to surface tension is

$$F_1 = 2\pi RS$$

along *OX*.

Now consider the forces due to the air outside the surface *ABCDE*. Consider a small part $\Delta S$ of the surface as shown in figure (14.19).

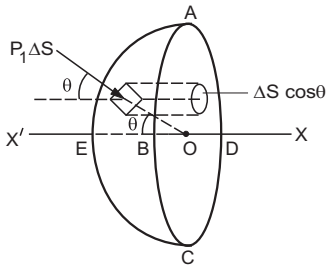



Figure 14.19

If the pressure just outside the surface is $P_1$, the force on this surface $\Delta S$ is $P_1\Delta S$ along the radial direction. By symmetry, the resultant of all such forces acting on different parts of the hemispherical surface must be along *OX*.

If the radius through $\Delta S$ makes an angle $\theta$ with *OX*, the component of $P_1\Delta S$ along *OX* will be $P_1\Delta S\cos\theta$. If we project the area $\Delta S$ on the diametric plane *ABCD,* the area of projection will be $\Delta S\cos\theta$. Thus, we can write,

component of $P_1\Delta S$ along *OX*

$= P_1$ (projection of $\Delta S$ on the plane *ABCD*).

When components of all the forces $P_1\Delta S$ on different $\Delta S$ are added, we get the resultant force due to the air outside the hemispherical surface. This resultant is then

$F_2 = P_1$ (Projection of the hemispherical surface *ABCDE* on the plane *ABCD*).

The projection of the hemispherical surface on the plane *ABCD* is the circular disc *ABCD* itself, having an area $\pi R^2$. Thus,

$$F_2 = P_1\pi R^2.$$

Similarly, the resultant force on this surface due to the liquid inside is $F_3 = P_2 \cdot \pi R^2$, where $P_2$ is the pressure just inside the surface. This force will be in the direction *OX′*.

For equilibrium of the hemispherical surface *ABCDE* we should have,

$$F_1 + F_2 = F_3$$

or, $$2\pi RS + P_1\pi R^2 = P_2\pi R^2$$

or, $$P_2 - P_1 = 2S/R\,. \qquad \ldots\ (14.15)$$

*The pressure inside the surface is greater than the pressure outside the surface by an amount $2S/R$.*

In the case of a drop, there is liquid on the concave side of the surface and air on the convex side. The pressure on the concave side is greater than the pressure on the convex side. This result is true in all cases. If we have an air bubble inside a liquid (figure 14.20), a single surface is formed. There is air on the concave side and liquid on the convex side. The pressure in the concave side (that is in the air) is greater than the pressure in the convex side (that is in the liquid) by an amount $2S/R$.

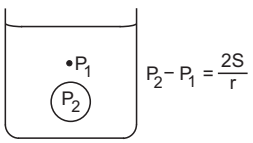



Figure 14.20

***Example 14.6***

*Find the excess pressure inside a mercury drop of radius* 2·0 mm. *The surface tension of mercury* = 0·464 N $m^{-1}$.

***Solution*** : The excess pressure inside the drop is

$$P_2 - P_1 = 2S/R$$

$$= \frac{2 \times 0{\cdot}464 \text{ N m}^{-1}}{2{\cdot}0 \times 10^{-3} \text{ m}} = 464 \text{ N m}^{-2}.$$

## 14.12 EXCESS PRESSURE IN A SOAP BUBBLE

Soap bubbles can be blown by dipping one end of a glass tube in a soap solution for a short time and then blowing air in it from the other end. Such a bubble has a small thickness and there is air both inside the bubble and outside the bubble. The thickness of the bubble may look small to eye but it still has hundreds of thousands of molecular layers. So it has two surface layers, one towards the outside air and the other towards the enclosed air. Between these two surface layers there is bulk soap solution.

Figure 14.21

Let the pressure of the air outside the bubble be $P_1$, that within the soap solution be $P'$ and that of the air inside the bubble be $P_2$. Looking at the outer surface, the solution is on the concave side of the surface, hence

$$P' - P_1 = 2S/R$$

where $R$ is the radius of the bubble. As the thickness of the bubble is small on a macroscopic scale, the difference in the radii of the two surfaces will be negligible.

Similarly, looking at the inner surface, the air is on the concave side of the surface, hence

$$P_2 - P' = 2S/R.$$

Adding the two equations

$$P_2 - P_1 = 4S/R. \qquad \ldots \quad (14.16)$$

*The pressure inside a bubble is greater than the pressure outside by an amount* $4S/R$.

***Example 14.7***

*A* 0·02 cm *liquid column balances the excess pressure inside a soap bubble of radius* 7·5 mm. *Determine the density of the liquid. Surface tension of soap solution* = 0·03 N $m^{-1}$.

***Solution*** : The excess pressure inside a soap bubble is

$$\Delta P = 4S/R = \frac{4 \times 0{\cdot}03 \text{ N m}^{-1}}{7{\cdot}5 \times 10^{-3} \text{ m}} = 16 \text{ N m}^{-2}.$$

The pressure due to 0·02 cm of the liquid column is

$$\Delta P = h\rho g$$

$$= (0{\cdot}02 \times 10^{-2} \text{ m})\ \rho\ (9{\cdot}8 \text{ m s}^{-2}).$$

Thus, $16 \text{ N m}^{-2} = (0{\cdot}02 \times 10^{-2} \text{ m})\ \rho\ (9{\cdot}8 \text{ m s}^{-2})$

or, $\rho = 8{\cdot}2 \times 10^{3} \text{ kg m}^{-3}$.

## 14.13 CONTACT ANGLE

When a liquid surface touches a solid surface, the shape of the liquid surface near the contact is generally curved. When a glass plate is immersed in water, the surface near the plate becomes concave as if the water is pulled up by the plate (figure 14.22). On the other hand, if a glass plate is immersed in mercury, the surface is depressed near the plate.

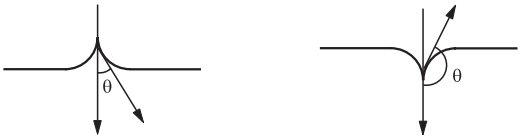



Figure 14.22

The angle between the tangent planes at the solid surface and the liquid surface at the contact is called the *contact angle.* In this the tangent plane to the solid surface is to be drawn towards the liquid and the tangent plane to the liquid is to be drawn away from the solid. Figure (14.22) shows the construction of contact angle. For the liquid that rises along the solid surface, the contact angle is smaller than 90°. For the liquid that is depressed along the solid surface, the contact angle is greater than 90°. Table (14.4) gives the contact angles for some of the pairs of solids and liquids.

**Table 14.4 : *Contact angles***

| Substance | Contact angle | Substance | Contact angle |
|---|---|---|---|
| Water with glass | 0° | Water with paraffin | 107° |
| Mercury with glass | 140° | Methylene iodide with glass | 29° |

Let us now see why the liquid surface bends near the contact with a solid. A liquid in equilibrium cannot sustain tangential stress. The resultant force on any small part of the surface layer must be perpendicular

to the surface there. Consider a small part of the liquid surface near its contact with the solid (figure 14.23).

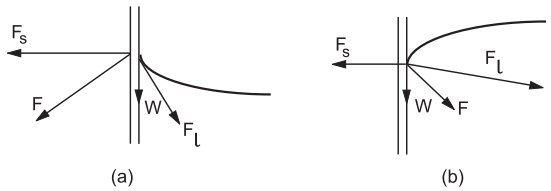



Figure 14.23

The forces acting on this part are

(a) $F_s$, attraction due to the molecules of the solid surface near it,

(b) $F_l$, the force due to the liquid molecules near this part, and

(c) $W$, the weight of the part considered.

The force between the molecules of the same material is known as *cohesive force* and the force between the molecules of different kinds of material is called *adhesive force*. Here $F_s$ is adhesive force and $F_l$ is cohesive force.

As is clear from the figure, the adhesive force $F_s$ is perpendicular to the solid surface and is into the solid. The cohesive force $F_l$ is in the liquid, its direction and magnitude depends on the shape of the liquid surface as this determines the distribution of the molecules attracting the part considered. Of course, $F_s$ and $F_l$ depend on the nature of the substances especially on their densities.

The direction of the resultant of $F_s$, $F_l$ and $W$ decides the shape of the surface near the contact. The liquid rests in such a way that the surface is perpendicular to this resultant. If the resultant passes through the solid (figure 14.23a), the surface is concave upward and the liquid rises along the solid. If the resultant passes through the liquid (figure 14.23b), the surface is convex upward and the liquid is depressed near the solid.

If a solid surface is just dipped in liquid (figure 14.24) so that it is not projected out, the force $F_s$ will not be perpendicular to the solid. The actual angle between the solid surface and the liquid surface may be different from the standard contact angle for the pair.

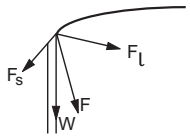



Figure 14.24

## 14.14 RISE OF LIQUID IN A CAPILLARY TUBE

When one end of a tube of small radius (known as a capillary tube) is dipped into a liquid, the liquid rises or is depressed in the tube. If the contact angle is less than 90°, the liquid rises. If it is greater than 90°, it is depressed.

Suppose a tube of radius $r$ is dipped into a liquid of surface tension $S$ and density $\rho$. Let the angle of contact between the solid and the liquid be $\theta$. If the radius of the tube is small, the surface in the tube is nearly spherical. Figure (14.25) shows the situation.

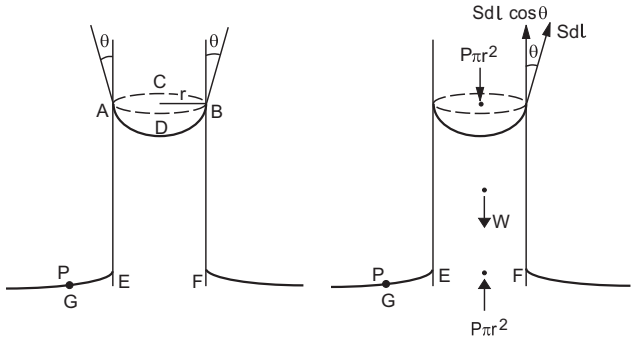



Figure 14.25

Consider the equilibrium of the part of liquid raised in the tube. In figure (14.25) this liquid is contained in the volume *ABEF*. Forces on this part of the liquid are

(a) $F_1$, by the surface of the tube on the surface *ABCD* of the liquid,

(b) $F_2$, due to the pressure of the air above the surface *ABCD*,

(c) $F_3$, due to the pressure of the liquid below EF and

(d) the weight $W$ of the liquid *ABEF*.

*ABCD* is the surface of the liquid inside the capillary tube. It meets the wall of the tube along a circle of radius $r$. The angle made by the liquid surface with the surface of the tube is equal to the contact angle $\theta$.

Consider a small part $dl$ of the periphery $2\pi r$ along which the surface of the liquid and the tube meet. The liquid surface across this pulls the tube surface by a force $S\,dl$ tangentially along the liquid surface. From Newton's third law, the tube surface across this small part pulls the liquid surface by an equal force $S\,dl$ in opposite direction. The vertical component of this force is $S\,dl\cos\theta$. The total force exerted on the liquid surface by the tube surface across the contact circle is

$$F_1 = \int S\,dl\cos\theta$$

$$= S\cos\theta\int dl$$

$$= 2\pi r\,S\cos\theta. \qquad \ldots \text{(i)}$$

The horizontal component $Sdl \sin\theta$ adds to zero when summed over the entire periphery.

The force $F_2$ due to the pressure of the air outside the surface $ABCD$ is $P.\pi r^2$ where $P$ is the atmospheric pressure. (This result was derived for hemispherical surface while deducing the excess pressure inside a drop. Same derivation works here.)

This force acts vertically downward. The pressure at $EF$ is equal to the atmospheric pressure $P$. This is because $EF$ is in the same horizontal plane as the free surface outside the tube and the pressure there is $P$. The force due to the liquid below $EF$ is, therefore, $P\pi r^2$ in vertically upward direction.

Thus, $F_2$ and $F_3$ cancel each other and the force $F_1 = 2\pi r S \cos\theta$ balances the weight $W$ in equilibrium. If the height raised in the tube is $h$ and if we neglect the weight of the liquid contained in the meniscus, the volume of the liquid raised is $\pi r^2 h$. The weight of this part is then

$$W = \pi r^2 h \rho g. \quad \ldots \text{(ii)}$$

Thus, $\pi r^2 h \rho g = 2\pi r S \cos\theta$

so that $$h = \frac{2S\cos\theta}{r\rho g}. \quad \ldots (14.17)$$

We see that the height raised is inversely proportional to the radius of the capillary. If the contact angle $\theta$ is greater than 90°, the term $\cos\theta$ is negative and hence $h$ is negative. The expression then gives the depression of the liquid in the tube.

The correction due to the weight of the liquid contained in the meniscus can be easily made if the contact angle is zero. This is the case with water rising in a glass capillary. The meniscus is then hemispherical (Figure 14.26).

Figure 14.26

The volume of the shaded part is

$$(\pi r^2)r - \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) = \frac{1}{3}\pi r^3.$$

The weight of the liquid contained in the meniscus is $\frac{1}{3}\pi r^3 \rho g$. Equation (ii) is then replaced by

$$\pi r^2 h \rho g + \frac{1}{3}\pi r^3 \rho g = 2\pi r S$$

or, $$h = \frac{2S}{r\rho g} - \frac{r}{3}. \quad \ldots (14.18)$$

***Example 14.8***

*A capillary tube of radius* 0·20 mm *is dipped vertically in water. Find the height of the water column raised in the tube. Surface tension of water* $= 0{\cdot}075\ \text{N m}^{-1}$ *and density of water* $= 1000\ \text{kg m}^{-3}$. *Take* $g = 10\ \text{m s}^{-2}$.

***Solution*** : We have,

$$h = \frac{2S\cos\theta}{r\rho g}$$

$$= \frac{2 \times 0{\cdot}075\ \text{N m}^{-1} \times 1}{(0{\cdot}20 \times 10^{-3}\ \text{m}) \times (1000\ \text{kg m}^{-3})\ (10\ \text{m s}^{-2})}$$

$$= 0{\cdot}075\ \text{m} = 7{\cdot}5\ \text{cm}.$$

**Tube of Insufficient Length**

Equation (14.17) or (14.18) gives the height raised in a capillary tube. If the tube is of a length less than $h$, the liquid does not overflow. The angle made by the liquid surface with the tube changes in such a way that the force $2\pi r S \cos\theta$ equals the weight of the liquid raised.

## 14.15 VISCOSITY

When a layer of a fluid slips or tends to slip on another layer in contact, the two layers exert tangential forces on each other. The directions are such that the relative motion between the layers is opposed. This property of a fluid to oppose relative motion between its layers is called *viscosity*. The forces between the layers opposing relative motion between them are known as the *forces of viscosity*. Thus, viscosity may be thought of as the internal friction of a fluid in motion.

If a solid surface is kept in contact with a fluid and is moved, forces of viscosity appear between the solid surface and the fluid layer in contact. The fluid in contact is dragged with the solid. If the viscosity is sufficient, the layer moves with the solid and there is no relative slipping. When a boat moves slowly on the water of a calm river, the water in contact with the boat is dragged with it, whereas the water in contact with the bed of the river remains at rest. Velocities of different layers are different. Let $v$ be the velocity of the layer at a distance $z$ from the bed and $v + dv$ be the velocity at a distance $z + dz$. (figure 14.27).

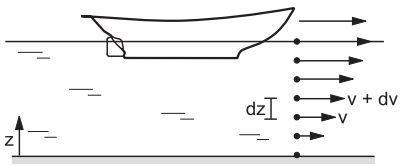



Figure 14.27

Thus, the velocity differs by $dv$ in going through a distance $dz$ perpendicular to it. The quantity $dv/dz$ is called the *velocity gradient.*

The force of viscosity between two layers of a fluid is proportional to the velocity gradient in the direction perpendicular to the layers. Also, the force is proportional to the area of the layer.

Thus, if $F$ is the force exerted by a layer of area $A$ on a layer in contact,

$$F \propto A \text{ and } F \propto dv/dz$$

or, $$F = -\eta A\, dv/dz. \quad \ldots (14.19)$$

The negative sign is included as the force is frictional in nature and opposes relative motion. The constant of proportionality $\eta$ is called the *coefficient of viscosity*.

The SI unit of viscosity can be easily worked out from equation (14.19). It is N–s $m^{-2}$. However, the corresponding CGS unit dyne–s $cm^{-2}$ is in common use and is called a poise in honour of the French scientist Poiseuille. We have

$$1 \text{ poise} = 0{\cdot}1 \text{ N–s m}^{-2}.$$

**Dimensions of the Coefficient of Viscosity**

Writing dimensions of different variables in equation (14.19),

$$MLT^{-2} = [\eta]\, L^2 \cdot \frac{L/T}{L}$$

or, $$[\eta] = \frac{MLT^{-2}}{L^2 T^{-1}}$$

or, $$[\eta] = ML^{-1}T^{-1}. \quad \ldots (14.20)$$

The coefficient of viscosity strongly depends on temperature. Table (14.5) gives the values for some of the commom fluids.

**Table 14.5 : *Coefficient of viscosity***

| Temperature °C | Viscosity of castor oil, poise | Viscosity of water, centi-poise | Viscosity of air, micro-poise |
|---|---|---|---|
| 0 | 53 | 1·792 | 171 |
| 20 | 9·86 | 1·005 | 181 |
| 40 | 2·31 | 0·656 | 190 |
| 60 | 0·80 | 0·469 | 200 |
| 80 | 0·30 | 0·357 | 209 |
| 100 | 0·17 | 0·284 | 218 |

## 14.16 FLOW THROUGH A NARROW TUBE : POISEUILLE'S EQUATION

Suppose a fluid flows through a narrow tube in steady flow. Because of viscosity, the layer in contact with the wall of the tube remains at rest and the layers away from the wall move fast. Poiseuille derived a formula for the rate of flow of viscous fluid through a cylindrical tube. We shall try to obtain the formula using dimensional analysis.

Suppose a fluid having coefficient of viscosity $\eta$ and density $\rho$ is flowing through a cylindrical tube of radius $r$ and length $l$. Let $P$ be the pressure difference in the liquid at the two ends. It is found that the volume of the liquid flowing per unit time through the tube depends on the pressure gradient $P/l$, the coefficient of viscosity $\eta$ and the radius $r$. If $V$ be the volume flowing in time $t$, we guess that

$$\frac{V}{t} = k \left(\frac{P}{l}\right)^a \eta^{\,b} r^{\,c} \quad \ldots \text{(i)}$$

where $k$ is a dimensionless constant.

Taking dimensions,

$$L^3 T^{-1} = \left(\frac{ML^{-1}T^{-2}}{L}\right)^a (ML^{-1}T^{-1})^{\,b} L^c$$

or, $$L^3 T^{-1} = M^{a+b} L^{-2a-b+c} T^{-2a-b}$$

Equating the exponents of M, L and T we get,

$$0 = a + b$$
$$3 = -2a - b + c$$
$$-1 = -2a - b.$$

Solving these equations,

$$a = 1,\ b = -1 \text{ and } c = 4.$$

Thus,

$$\frac{V}{t} = k \frac{Pr^4}{\eta l}.$$

The dimensionless constant $k$ is equal to $\pi/8$ and hence the rate of flow is

$$\frac{V}{t} = \frac{\pi P r^4}{8\eta l}. \quad \ldots (14.21)$$

This is *Poiseuille's formula.*

## 14.17 STOKES' LAW

When a solid body moves through a fluid, the fluid in contact with the solid is dragged with it. Relative velocities are established between the layers of the fluid near the solid so that the viscous forces start operating. The fluid exerts viscous force on the solid to oppose the motion of the solid. The magnitude of the viscous force depends on the shape and size of the solid body, its speed and the coefficient of viscosity of the fluid.

Suppose a spherical body of radius $r$ moves at a speed $v$ through a fluid of viscosity $\eta$. The viscous force $F$ acting on the body depends on $r$, $v$ and $\eta$. Assuming that the force is proportional to various powers of these quantities, we can obtain the dependence through dimensional analysis.

Let $F = k\, r^{a}\, v^{b}\, \eta^{c}$ … (i)

where $k$ is a dimensionless constant. Taking dimensions on both sides,

$$M\,L\,T^{-2} = L^{a}\,(LT^{-1})^{b}\,(ML^{-1}T^{-1})^{c}.$$

Comparing the exponents of M, L and T,

$$1 = c$$
$$1 = a + b - c$$
$$-2 = -b - c$$

Solving these equations, $a = 1$, $b = 1$ and $c = 1$.

Thus, by (i), $F = k\, r\, v\, \eta$.

The dimensionless constant $k$ equals $6\pi$, so that the equation becomes

$$F = 6\,\pi\,\eta\,r\,v. \quad \text{… (14.22)}$$

This equation is known as *Stokes' law*.

***Example 14.9***

*An air bubble of diameter* 2 mm *rises steadily through a solution of density* 1750 kg m$^{-3}$ *at the rate of* 0·35 cm s$^{-1}$. *Calculate the coefficient of viscosity of the solution. The density of air is negligible.*

***Solution*** : The force of buoyancy $B$ is equal to the weight of the displaced liquid. Thus,

$$B = \frac{4}{3}\,\pi\,r^{3}\sigma g.$$

This force is upward. The viscous force acting downward is

$$F = 6\,\pi\,\eta r v.$$

The weight of the air bubble may be neglected as the density of air is small. For uniform velocity

$$F = B$$

or, $6\,\pi\,\eta r v = \frac{4}{3}\,\pi\,r^{3}\,\sigma g$

or, $$\eta = \frac{2r^{2}\sigma g}{9v}$$

$$= \frac{2 \times (1 \times 10^{-3}\ \text{m})^{2} \times (1750\ \text{kg m}^{-3})\,(9{\cdot}8\ \text{m s}^{-2})}{9 \times (0{\cdot}35 \times 10^{-2}\ \text{m s}^{-1})}$$

$$\approx 11 \text{ poise.}$$

This appears to be a highly viscous liquid.

## 14.18 TERMINAL VELOCITY

The viscous force on a solid moving through a fluid is proportional to its velocity. When a solid is dropped in a fluid, the forces acting on it are

(a) weight $W$ acting vertically downward,

(b) the viscous force $F$ acting vertically upward and

(c) the buoyancy force $B$ acting vertically upward.

The weight $W$ and the buoyancy $B$ are constant but the force $F$ is proportional to the velocity $v$. Initially, the velocity and hence the viscous force $F$ is zero and the solid is accelerated due to the force $W$–$B$. Because of the acceleration the velocity increases. Accordingly, the viscous force also increases. At a certain instant the viscous force becomes equal to $W$–$B$. The net force then becomes zero and the solid falls with constant velocity. This constant velocity is known as the terminal velocity.

Consider a spherical body falling through a liquid. Suppose the density of the body = $\rho$, density of the liquid = $\sigma$, radius of the sphere = $r$ and the terminal velocity = $v_0$. The viscous force is

$$F = 6\pi\,\eta r v_0.$$

The weight $W = \frac{4}{3}\,\pi\,r^{3}\rho g$

and the buoyancy force $B = \frac{4}{3}\,\pi\,r^{3}\sigma g$.

We have

$$6\pi\,\eta r v_0 = W - B = \frac{4}{3}\,\pi r^{3}\rho g - \frac{4}{3}\,\pi r^{3}\sigma g$$

or, $$v_0 = \frac{2\,r^{2}(\rho - \sigma)g}{9\,\eta}. \quad \text{… (14.23)}$$

## 14.19 MEASURING COEFFICIENT OF VISCOSITY BY STOKES' METHOD

Viscosity of a liquid may be determined by measuring the terminal velocity of a solid sphere in it. Figure (14.28) shows the apparatus. A test tube $A$ contains the experimental liquid and is fitted into a water bath $B$. A thermometer $T$ measures the temperature of the bath. A tube $C$ is fitted in the cork of the test tube $A$. There are three equidistant marks $P$, $Q$ and $R$ on the test tube well below the tube $C$.

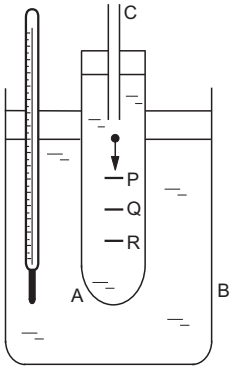



Figure 14.28

A spherical metal ball is dropped in the tube $C$. The time interval taken by the ball to pass through the length $PQ$ and through the length $QR$ are noted with the help of a stop watch. If these two are not

equal, a smaller metal ball is tried. The process is repeated till the two time intervals are the same. In this case the ball has achieved its terminal velocity before passing through the mark $P$. The radius of the ball is determined by a screw guage. Its mass $m$ is determined by weighing it. The length $PQ = QR$ is measured with a scale.

Let $r$ = radius of the spherical ball
$m$ = mass of the ball
$t$ = time interval in passing through the length $PQ$ or $QR$
$d$ = length $PQ = QR$
$\eta$ = coefficient of viscosity of the liquid
$\sigma$ = density of the liquid.

The density of the solid is $\rho = \dfrac{m}{\frac{4}{3}\pi r^3}$ and the terminal velocity is $v_0 = d/t$. Using equation (14.23)

$$\eta = \frac{2}{9}\frac{(\rho - \sigma)gr^2}{d/t}.$$

This method is useful for a highly viscous liquid such as Castor oil.

## 14.20 CRITICAL VELOCITY AND REYNOLDS NUMBER

When a fluid flows in a tube with small velocity, the flow is steady. As the velocity is gradually increased, at one stage the flow becomes turbulent. The largest velocity which allows a steady flow is called the *critical velocity.*

Whether the flow will be steady or turbulent mainly depends on the density, velocity and the coefficient of viscosity of the fluid as well as the diameter of the tube through which the fluid is flowing. The quantity

$$N = \frac{\rho v D}{\eta} \qquad \ldots \quad (14.24)$$

is called the *Reynolds number* and plays a key role in determining the nature of flow. It is found that if the Reynolds number is less than 2000, the flow is steady. If it is greater than 3000, the flow is turbulent. If it is between 2000 and 3000, the flow is unstable. In this case it may be steady and may suddenly change to turbulent or it may be turbulent and may suddenly change to steady.

### *Worked Out Examples*

**1** *One end of a wire* 2 m *long and* 0·2 $cm^2$ *in cross section is fixed in a ceiling and a load of* 4·8 kg *is attached to the free end. Find the extension of the wire. Young modulus of steel* = $2{\cdot}0 \times 10^{11}$ N $m^{-2}$. *Take* $g = 10$ m $s^{-2}$.

***Solution*** : We have

$$Y = \frac{\text{stress}}{\text{strain}} = \frac{T/A}{l/L}$$

with symbols having their usual meanings. The extension is

$$l = \frac{TL}{AY}.$$

As the load is in equilibrium after the extension, the tension in the wire is equal to the weight of the load

$$= 4{\cdot}8 \text{ kg} \times 10 \text{ m s}^{-2} = 48 \text{ N}$$

$$\text{Thus, } l = \frac{(48 \text{ N})\ (2 \text{ m})}{(0{\cdot}2 \times 10^{-4} \text{ m}^2) \times (2{\cdot}0 \times 10^{11} \text{ N m}^{-2})}$$

$$= 2{\cdot}4 \times 10^{-5} \text{ m}.$$

**2**. *One end of a nylon rope of length* 4·5 m *and diameter* 6 mm *is fixed to a tree-limb. A monkey weighing* 100 N *jumps to catch the free end and stays there. Find the elongation of the rope and the corresponding change in the diameter. Young modulus of nylon* = $4{\cdot}8 \times 10^{11}$ N $m^{-2}$ *and Poisson ratio of nylon* = 0·2.

***Solution*** : As the monkey stays in equilibrium, the tension in the rope equals the weight of the monkey. Hence,

$$Y = \frac{\text{stress}}{\text{strain}} = \frac{T/A}{l/L}$$

or, $$l = \frac{TL}{AY}$$

$$\text{or, elongation} = l = \frac{(100 \text{ N}) \times (4{\cdot}5 \text{ m})}{(\pi \times 9 \times 10^{-6} \text{ m}^2) \times (4{\cdot}8 \times 10^{11} \text{ N m}^{-2})}$$

$$= 3{\cdot}32 \times 10^{-5} \text{ m}.$$

$$\text{Again, Poisson ratio} = \frac{\Delta d/d}{l/L} = \frac{(\Delta d)L}{ld}$$

$$\text{or,} \qquad 0{\cdot}2 = \frac{\Delta d \times 4{\cdot}5 \text{ m}}{(3{\cdot}32 \times 10^{-5} \text{ m}) \times (6 \times 10^{-3} \text{ m})}$$

$$\text{or,} \qquad \Delta d = \frac{0{\cdot}2 \times 6 \times 3{\cdot}32 \times 10^{-8} \text{ m}}{4{\cdot}5}$$

$$= 8{\cdot}8 \times 10^{-9} \text{ m}.$$

**3.** *Two blocks of masses* 1 kg *and* 2 kg *are connected by a metal wire going over a smooth pulley as shown in figure (14-W1). The breaking stress of the metal is* $2 \times 10^9$ N $m^{-2}$. *What should be the minimum radius of the wire used if it is not to break* ? *Take* $g = 10$ m $s^{-2}$.

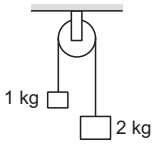


Figure 14-W1

***Solution*** : The stress in the wire $= \dfrac{\text{Tension}}{\text{Area of cross section}}$.

To avoid breaking, this stress should not exceed the breaking stress.

Let the tension in the wire be $T$. The equations of motion of the two blocks are,

$$T - 10\ \text{N} = (1\ \text{kg})\ a$$

and $$20\ \text{N} - T = (2\ \text{kg})\ a.$$

Eliminating $a$ from these equations,

$$T = (40/3)\ \text{N}.$$

$$\text{The stress} = \frac{(40/3)\ \text{N}}{\pi r^2}.$$

If the minimum radius needed to avoid breaking is $r$,

$$2 \times 10^9\ \frac{\text{N}}{\text{m}^2} = \frac{(40/3)\ \text{N}}{\pi r^2}$$

Solving this,

$$r = 4{\cdot}6 \times 10^{-5}\ \text{m}.$$

**4.** *Two wires of equal cross section but one made of steel and the other of copper, are joined end to end. When the combination is kept under tension, the elongations in the two wires are found to be equal. Find the ratio of the lengths of the two wires. Young modulus of steel* $= 2{\cdot}0 \times 10^{11}\ \text{N m}^{-2}$ *and that of copper* $= 1{\cdot}1 \times 10^{11}\ \text{N m}^{-2}$.

***Solution*** : As the cross sections of the wires are equal and same tension exists in both, the stresses developed are equal. Let the original lengths of the steel wire and the copper wire be $L_s$ and $L_c$ respectively and the elongation in each wire be $l$.

$$\frac{l}{L_s} = \frac{\text{stress}}{2{\cdot}0 \times 10^{11}\ \text{N m}^{-2}} \quad \ldots\ (\text{i})$$

and $$\frac{l}{L_c} = \frac{\text{stress}}{1{\cdot}1 \times 10^{11}\ \text{N m}^{-2}}. \quad \ldots\ (\text{ii})$$

Dividing (ii) by (i),

$$L_s/L_c = 2{\cdot}0/1{\cdot}1 = 20 : 11.$$

**5.** *Find the decrease in the volume of a sample of water from the following data. Initial volume* $= 1000\ \text{cm}^3$, *initial pressure* $= 10^5\ \text{N m}^{-2}$, *final pressure* $= 10^6\ \text{N m}^{-2}$, *compressibility of water* $= 50 \times 10^{-11}\ \text{m}^2\ \text{N}^{-1}$.

***Solution*** : The change in pressure

$$= \Delta P = 10^6\ \text{N m}^{-2} - 10^5\ \text{N m}^{-2}$$

$$= 9 \times 10^5\ \text{N m}^{-2}.$$

$$\text{Compressibility} = \frac{1}{\text{Bulk modulus}} = -\frac{\Delta V/V}{\Delta P}$$

or, $$50 \times 10^{-11}\ \text{m}^2\ \text{N}^{-1} = -\frac{\Delta V}{(10^{-3}\ \text{m}^3) \times (9 \times 10^5\ \text{N m}^{-2})}$$

or, $$\Delta V = -50 \times 10^{-11} \times 10^{-3} \times 9 \times 10^5\ \text{m}^3$$

$$= -4{\cdot}5 \times 10^{-7}\ \text{m}^3 = -0{\cdot}45\ \text{cm}^3.$$

Thus the decrease in volume is $0{\cdot}45\ \text{cm}^3$.

**6.** *One end of a metal wire is fixed to a ceiling and a load of* 2 kg *hangs from the other end. A similar wire is attached to the bottom of the load and another load of* 1 kg *hangs from this lower wire. Find the longitudinal strain in both the wires. Area of cross section of each wire is* $0{\cdot}005\ \text{cm}^2$ *and Young modulus of the metal is* $2{\cdot}0 \times 10^{11}\ \text{N m}^{-2}$. *Take* $g = 10\ \text{m s}^{-2}$.

***Solution*** : The situation is described in figure (14-W2). As the 1 kg mass is in equilibrium, the tension in the lower wire equals the weight of the load.

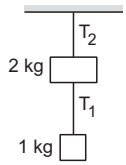



Figure 14-W2

Thus $$T_1 = 10\ \text{N}$$

$$\text{Stress} = 10\ \text{N}/0{\cdot}005\ \text{cm}^2$$

$$= 2 \times 10^7\ \text{N m}^{-2}.$$

$$\text{Longitudinal strain} = \frac{\text{stress}}{Y} = \frac{2 \times 10^7\ \text{N m}^{-2}}{2 \times 10^{11}\ \text{N m}^{-2}} = 10^{-4}.$$

Considering the equilibrium of the upper block, we can write,

$$T_2 = 20\ \text{N} + T_1, \quad \text{or,} \quad T_2 = 30\ \text{N}.$$

$$\text{Stress} = 30\ \text{N}/0{\cdot}005\ \text{cm}^2$$

$$= 6 \times 10^7\ \text{N m}^{-2}.$$

$$\text{Longitudinal strain} = \frac{6 \times 10^7\ \text{N m}^{-2}}{2 \times 10^{11}\ \text{N m}^{-2}} = 3 \times 10^{-4}.$$

**7.** *Each of the three blocks P, Q and R shown in figure (14-W3) has a mass of* 3 kg. *Each of the wires A and B has cross-sectional area* $0{\cdot}005\ \text{cm}^2$ *and Young modulus* $2 \times 10^{11}\ \text{N m}^{-2}$. *Neglect friction. Find the longitudinal strain developed in each of the wires. Take* $g = 10\ \text{m s}^{-2}$.

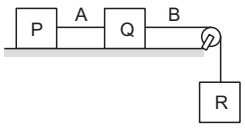



Figure 14-W3

***Solution*** : The block $R$ will descend vertically and the blocks $P$ and $Q$ will move on the frictionless horizontal table. Let the common magnitude of the acceleration be $a$. Let the tensions in the wires $A$ and $B$ be $T_A$ and $T_B$ respectively.

Writing the equations of motion of the blocks $P$, $Q$ and $R$, we get,

$$T_A = (3\ \text{kg})\ a \quad \ldots\ (\text{i})$$

$$T_B - T_A = (3\text{ kg})\, a \qquad \ldots \text{(ii)}$$

and $(3\text{ kg})\, g - T_B = (3\text{ kg})\, a.$ $\ldots$ (iii)

By (i) and (ii),

$$T_B = 2\, T_A.$$

By (i) and (iii),

$$T_A + T_B = (3\text{ kg})\, g = 30\text{ N}$$

or, $3\, T_A = 30\text{ N}$

or, $T_A = 10\text{ N}$ and $T_B = 20\text{ N}.$

$$\text{Longitudinal strain} = \frac{\text{Longitudinal stress}}{\text{Young modulus}}$$

$$\text{Strain in wire } A = \frac{10\text{ N}/0{\cdot}005\text{ cm}^2}{2 \times 10^{11}\text{ N m}^{-2}} = 10^{-4}$$

$$\text{and strain in wire } B = \frac{20\text{ N}/0{\cdot}005\text{ cm}^2}{2 \times 10^{11}\text{ N m}^{-2}} = 2 \times 10^{-4}.$$

**8.** *A wire of area of cross section* $3{\cdot}0\text{ mm}^2$ *and natural length* 50 cm *is fixed at one end and a mass of* $2{\cdot}1$ kg *is hung from the other end. Find the elastic potential energy stored in the wire in steady state. Young modulus of the material of the wire* $= 1{\cdot}9 \times 10^{11}\text{ N m}^{-2}$. *Take* $g = 10\text{ m s}^{-2}$.

***Solution*** **:** The volume of the wire is

$$V = (3{\cdot}0\text{ mm}^2)\,(50\text{ cm})$$

$$= (3{\cdot}0 \times 10^{-6}\text{ m}^2)\,(0{\cdot}50\text{ m}) = 1{\cdot}5 \times 10^{-6}\text{ m}^3.$$

Tension in the wire is

$$T = mg$$

$$= (2{\cdot}1\text{ kg})\,(10\text{ m s}^{-2}) = 21\text{ N}.$$

$$\text{The stress} = T/A$$

$$= \frac{21\text{ N}}{3{\cdot}0\text{ mm}^2} = 7{\cdot}0 \times 10^6\text{ N m}^{-2}.$$

$$\text{The strain} = \text{stress}/Y$$

$$= \frac{7{\cdot}0 \times 10^6\text{ N m}^{-2}}{1{\cdot}9 \times 10^{11}\text{ N m}^{-2}} = 3{\cdot}7 \times 10^{-5}.$$

The elastic potential energy of the wire is

$$U = \frac{1}{2}\,(\text{stress})\,(\text{strain})\,(\text{volume})$$

$$= \frac{1}{2}\,(7{\cdot}0 \times 10^6\text{ N m}^{-2})\,(3{\cdot}7 \times 10^{-5})\,(1{\cdot}5 \times 10^{-6}\text{ m}^3)$$

$$= 1{\cdot}9 \times 10^{-4}\text{ J}.$$

**9.** *A block of weight* 10 N *is fastened to one end of a wire of cross-sectional area* 3 mm$^2$ *and is rotated in a vertical circle of radius* 20 cm. *The speed of the block at the bottom of the circle is* 2 m s$^{-1}$. *Find the elongation of the wire when the block is at the bottom of the circle. Young modulus of the material of the wire* $= 2 \times 10^{11}\text{ N m}^{-2}$.

***Solution*** **:** Forces acting on the block are (a) the tension $T$ and (b) the weight $W$. At the lowest point, the resultant force is $T - W$ towards the centre. As the block is going in a circle, the net force towards the centre should be $mv^2/r$ with usual symbols. Thus,

$$T - W = mv^2/r$$

or, $T = W + mv^2/r$

$$= 10\text{ N} + \frac{(1\text{ kg})\,(2\text{ m s}^{-1})^2}{0{\cdot}2\text{ m}} = 30\text{ N}$$

We have $Y = \dfrac{T/A}{l/L}$

or, $l = \dfrac{TL}{AY}$

$$= \frac{30\text{ N} \times (20\text{ cm})}{(3 \times 10^{-6}\text{ m}^2) \times (2 \times 10^{11}\text{ N m}^{-2})}$$

$$= 5 \times 10^{-5} \times 20\text{ cm} = 10^{-3}\text{ cm}.$$

**10.** *A uniform heavy rod of weight W, cross-sectional area A and length L is hanging from a fixed support. Young modulus of the material of the rod is Y. Neglect the lateral contraction. Find the elongation of the rod.*

***Solution*** **:** Consider a small length $dx$ of the rod at a distance $x$ from the fixed end. The part below this small element has length $L - x$. The tension $T$ of the rod at the element equals the weight of the rod below it.

$$T = (L - x)\,\frac{W}{L}.$$

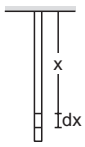



Figure 14-W4

Elongation in the element is given by

$$\text{elongation} = \text{original length} \times \text{stress}/Y$$

$$= \frac{T\,dx}{AY} = \frac{(L - x)\,W\,dx}{LAY}.$$

$$\text{The total elongation} = \int_0^L \frac{(L - x)\,W\,dx}{LAY}$$

$$= \frac{W}{LAY}\left(Lx - \frac{x^2}{2}\right)_0^L = \frac{WL}{2AY}.$$

**11.** *There is an air bubble of radius* $1{\cdot}0$ mm *in a liquid of surface tension* $0{\cdot}075\text{ N m}^{-1}$ *and density* $1000\text{ kg m}^{-3}$. *The bubble is at a depth of* 10 cm *below the free surface. By what amount is the pressure inside the bubble greater than the atmospheric pressure* ? *Take* $g = 9{\cdot}8\text{ m s}^{-2}$.

***Solution*** :

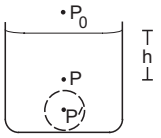



Figure 14-W5

Let the atmospheric pressure be $P_0$. The pressure of the liquid just outside the bubble is (figure 14-W5)

$$P = P_0 + h\rho g.$$

The pressure inside the bubble is

$$P' = P + \frac{2S}{r} = P_0 + h\rho g + \frac{2S}{r}$$

or, $P' - P_0$

$$= (10 \text{ cm})\,(1000 \text{ kg m}^{-3})\,(9{\cdot}8 \text{ m s}^{-2}) + \frac{2 \times 0{\cdot}075 \text{ N m}^{-1}}{1{\cdot}0 \times 10^{-3} \text{ m}}$$

$$= 980 \text{ N m}^{-2} + 150 \text{ N m}^{-2}$$

$$= 1130 \text{ Pa}.$$

**12.** *A light wire AB of length* 10 cm *can slide on a vertical frame as shown in figure (14-W6). There is a film of soap solution trapped between the frame and the wire. Find the load W that should be suspended from the wire to keep it in equilibrium. Neglect friction. Surface tension of soap solution* $= 25$ dyne cm$^{-1}$. *Take* $g = 10$ m s$^{-2}$.

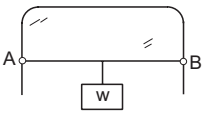



Figure 14-W6

***Solution*** : Soap solution film will be formed on both sides of the frame. Each film is in contact with the wire along a distance of 10 cm. The force exerted by the film on the wire

$$= 2 \times (10 \text{ cm}) \times (25 \text{ dyne cm}^{-1})$$

$$= 500 \text{ dyne} = 5 \times 10^{-3} \text{ N}.$$

This force acts vertically upward and should be balanced by the load. Hence the load that should be suspended is $5 \times 10^{-3}$ N. The mass of the load should be $\frac{5 \times 10^{-3} \text{ N}}{10 \text{ m s}^{-2}} = 5 \times 10^{-4}$ kg $= 0{\cdot}5$ g.

**13.** *The lower end of a capillary tube is dipped into water and it is seen that the water rises through* 7·5 cm *in the capillary. Find the radius of the capillary. Surface tension of water* $= 7{\cdot}5 \times 10^{-2}$ N m$^{-1}$. *Contact angle between water and glass* $= 0°$. *Take* $g = 10$ m s$^{-2}$.

***Solution*** : We have,

$$h = \frac{2S\cos\theta}{r\rho g}$$

or, $$r = \frac{2S\cos\theta}{h\rho g}$$

$$= \frac{2 \times (7{\cdot}5 \times 10^{-2} \text{ N m}^{-1}) \times 1}{(0{\cdot}075 \text{ m}) \times (1000 \text{ kg m}^{-3}) \times (10 \text{ m s}^{-2})}$$

$$= 2 \times 10^{-4} \text{ m} = 0{\cdot}2 \text{ mm}.$$

**14.** *Two mercury drops each of radius r merge to form a bigger drop. Calculate the surface energy released.*

***Solution*** :

Surface area of one drop before merging $= 4\pi r^2$.

Total surface area of both the drops $= 8\pi r^2$.

Hence, the surface energy before merging $= 8\pi r^2 S$.

When the drops merge, the volume of the bigger drop

$$= 2 \times \frac{4}{3}\pi r^3 = \frac{8}{3}\pi r^3.$$

If the radius of this new drop is $R$,

$$\frac{4}{3}\pi R^3 = \frac{8}{3}\pi r^3$$

or, $$R = 2^{1/3} r$$

or, $$4\pi R^2 = 4 \times 2^{2/3} \times \pi r^2.$$

Hence, the surface energy $= 4 \times 2^{2/3} \times \pi r^2 S$.

The released surface energy $= 8\pi r^2 S - 4 \times 2^{2/3} \pi r^2 S$

$$\approx 1{\cdot}65\, \pi r^2 S.$$

**15.** *A large wooden plate of area* 10 m$^2$ *floating on the surface of a river is made to move horizontally with a speed of* 2 m s$^{-1}$ *by applying a tangential force. If the river is* 1 m *deep and the water in contact with the bed is stationary, find the tangential force needed to keep the plate moving. Coefficient of viscosity of water at the temperature of the river* $= 10^{-2}$ poise.

***Solution*** : The velocity decreases from 2 m s$^{-1}$ to zero in 1 m of perpendicular length. Hence, velocity gradient

$$= dv/dx = 2 \text{ s}^{-1}.$$

Now, $$\eta = \left| \frac{F/A}{dv/dx} \right|$$

or, $$10^{-3} \frac{\text{N–s}}{\text{m}^2} = \frac{F}{(10 \text{ m}^2)\,(2 \text{ s}^{-1})}$$

or, $$F = 0{\cdot}02 \text{ N}.$$

**16.** *The velocity of water in a river is* 18 km h$^{-1}$ *near the surface. If the river is* 5 m *deep, find the shearing stress between the horizontal layers of water. The coefficient of viscosity of water* $= 10^{-2}$ poise.

***Solution*** : The velocity gradient in vertical direction is

$$\frac{dv}{dx} = \frac{18 \text{ km h}^{-1}}{5 \text{ m}} = 1{\cdot}0 \text{ s}^{-1}.$$

The magnitude of the force of viscosity is

$$F = \eta A \frac{dv}{dx}$$

The shearing stress is

$$F/A = \eta \frac{dv}{dx} = (10^{-2} \text{ poise}) (1{\cdot}0 \text{ s}^{-1}) = 10^{-3} \text{ N m}^{-2}.$$

**17.** *Find the terminal velocity of a rain drop of radius* 0·01 mm. *The coefficient of viscosity of air is* $1{\cdot}8 \times 10^{-5}$ N–s $m^{-2}$ *and its density is* 1·2 kg $m^{-3}$. *Density of water* = 1000 kg $m^{-3}$. *Take* $g = 10$ m $s^{-2}$.

***Solution*** **:** The forces on the rain drop are

(a) the weight $\frac{4}{3} \pi r^3 \rho g$ downward,

(b) the force of buoyancy $\frac{4}{3} \pi r^3 \sigma g$ upward,

(c) the force of viscosity $6\pi\eta rv$ upward.

Here ρ is the density of water and σ is the density of air. At terminal velocity the net force is zero. As the density of air is much smaller than the density of water, the force of buoyance may be neglected.

Thus, at terminal velocity

$$6\pi\eta rv = \frac{4}{3} \pi r^3 \rho g$$

or,
$$v = \frac{2 r^2 \rho g}{9\eta}.$$

$$= \frac{2 \times (0{\cdot}01 \text{ mm})^2 \times (1000 \text{ kg m}^{-3}) (10 \text{ m s}^{-2})}{9 \times (1{\cdot}8 \times 10^{-5} \text{ N–s m}^{-2})}$$

$$\approx 1{\cdot}2 \text{ cm s}^{-2}.$$

□

## QUESTIONS FOR SHORT ANSWER

**1.** The ratio stress/strain remains constant for small deformation of a metal wire. When the deformation is made larger, will this ratio increase or decrease ?

**2.** When a block of mass $M$ is suspended by a long wire of length $L$, the elastic potential energy stored in the wire is $\frac{1}{2}\times$ stress × strain × volume. Show that it is equal to $\frac{1}{2} Mgl$, where $l$ is the extension. The loss in gravitational potential energy of the mass earth system is $Mgl$. Where does the remaining $\frac{1}{2} Mgl$ energy go ?

**3.** When the skeleton of an elephant and the skeleton of a mouse are prepared in the same size, the bones of the elephant are shown thicker than those of the mouse. Explain why the bones of an elephant are thicker than proportionate. The bones are expected to withstand the stress due to the weight of the animal.

**4.** The yield point of a typical solid is about 1%. Suppose you are lying horizontally and two persons are pulling your hands and two persons are pulling your legs along your own length. How much will be the increase in your length if the strain is 1% ? Do you think your yield point is 1% or much less than that ?

**5.** When rubber sheets are used in a shock absorber, what happens to the energy of vibration ?

**6.** If a compressed spring is dissolved in acid, what happens to the elastic potential energy of the spring ?

**7.** A steel blade placed gently on the surface of water floats on it. If the same blade is kept well inside the water, it sinks. Explain.

**8.** When some wax is rubbed on a cloth, it becomes waterproof. Explain.

**9.** The contact angle between pure water and pure silver is 90°. If a capillary tube made of silver is dipped at one end in pure water, will the water rise in the capillary ?

**10.** It is said that a liquid rises or is depressed in a capillary due to the surface tension. If a liquid neither rises nor depresses in a capillary, can we conclude that the surface tension of the liquid is zero ?

**11.** The contact angle between water and glass is 0°. When water is poured in a glass to the maximum of its capacity, the water surface is convex upward. The angle of contact in such a situation is more than 90°. Explain.

**12.** A uniform vertical tube of circular cross section contains a liquid. The contact angle is 90°. Consider a diameter of the tube lying in the surface of the liquid. The surface to the right of this diameter pulls the surface on the left of it. What keeps the surface on the left in equilibrium ?

**13.** When a glass capillary tube is dipped at one end in water, water rises in the tube. The gravitational potential energy is thus increased. Is it a violation of conservation of energy ?

**14.** If a mosquito is dipped into water and released, it is not able to fly till it is dry again. Explain.

**15.** The force of surface tension acts tangentially to the surface whereas the force due to air pressure acts perpendicularly on the surface. How is then the force due to excess pressure inside a bubble balanced by the force due to the surface tension ?

**16.** When the size of a soap bubble is increased by pushing more air in it, the surface area increases. Does it mean that the average separation between the surface molecules is increased ?

**17.** Frictional force between solids operates even when they do not move with respect to each other. Do we have viscous force acting between two layers even if there is no relative motion ?

**18.** Water near the bed of a deep river is quiet while that near the surface flows. Give reasons.

**19.** If water in one flask and castor oil in other are violently shaken and kept on a table, which will come to rest earlier ?

## OBJECTIVE I

**1.** A rope 1 cm in diameter breaks if the tension in it exceeds 500 N. The maximum tension that may be given to a similar rope of diameter 2 cm is
(a) 500 N (b) 250 N (c) 1000 N (d) 2000 N.

**2.** The breaking stress of a wire depends on
(a) material of the wire (b) length of the wire
(c) radius of the wire (d) shape of the cross section.

**3.** A wire can sustain the weight of 20 kg before breaking. If the wire is cut into two equal parts, each part can sustain a weight of
(a) 10 kg (b) 20 kg (c) 40 kg (d) 80 kg.

**4.** Two wires $A$ and $B$ are made of same material. The wire $A$ has a length $l$ and diameter $r$ while the wire $B$ has a length $2l$ and diameter $r/2$. If the two wires are stretched by the same force, the elongation in $A$ divided by the elongation in $B$ is
(a) 1/8 (b) 1/4 (c) 4 (d) 8.

**5.** A wire elongates by 1·0 mm when a load $W$ is hung from it. If this wire goes over a pulley and two weights $W$ each are hung at the two ends, the elongation of the wire will be
(a) 0·5 m (b) 1·0 mm (c) 2·0 mm (d) 4·0 mm.

**6.** A heavy uniform rod is hanging vertically from a fixed support. It is stretched by its own weight. The diameter of the rod is
(a) smallest at the top and gradually increases down the rod
(b) largest at the top and gradually decreases down the rod
(c) uniform everywhere
(d) maximum in the middle.

**7.** When a metal wire is stretched by a load, the fractional change in its volume $\Delta V/V$ is proportional to
(a) $\frac{\Delta l}{l}$ (b) $\left(\frac{\Delta l}{l}\right)^2$ (c) $\sqrt{\Delta l/l}$ (d) none of these.

**8.** The length of a metal wire is $l_1$ when the tension in it is $T_1$ and is $l_2$ when the tension is $T_2$. The natural length of the wire is
(a) $\frac{l_1+l_2}{2}$ (b) $\sqrt{l_1 l_2}$ (c) $\frac{l_1T_2-l_2T_1}{T_2-T_1}$ (d) $\frac{l_1T_2+l_2T_1}{T_2+T_1}$.

**9.** A heavy mass is attached to a thin wire and is whirled in a vertical circle. The wire is most likely to break
(a) when the mass is at the highest point
(b) when the mass is at the lowest point
(c) when the wire is horizontal
(d) at an angle of $\cos^{-1}(1/3)$ from the upward vertical.

**10.** When a metal wire elongates by hanging a load on it, the gravitational potential energy is decreased.
(a) This energy completely appears as the increased kinetic energy of the block.
(b) This energy completely appears as the increased elastic potential energy of the wire.
(c) This energy completely appears as heat.
(d) None of these.

**11.** By a surface of a liquid we mean
(a) a geometrical plane like $x = 0$
(b) all molecules exposed to the atmosphere
(c) a layer of thickness of the order of $10^{-8}$ m
(d) a layer of thickness of the order of $10^{-4}$ m.

**12** An ice cube is suspended in vacuum in a gravity free hall. As the ice melts it
(a) will retain its cubical shape
(b) will change its shape to spherical
(c) will fall down on the floor of the hall
(d) will fly up.

**13.** When water droplets merge to form a bigger drop
(a) energy is liberated (b) energy is absorbed
(c) energy is neither liberated nor absobred
(d) energy may either be liberated or absorbed depending on the nature of the liquid.

**14.** The dimension $ML^{-1}T^{-2}$ can correspond to
(a) moment of a force (b) surface tension
(c) modulus of elasticity (d) coefficient of viscosity.

**15.** Air is pushed into a soap bubble of radius $r$ to double its radius. If the surface tension of the soap solution is $S$, the work done in the process is
(a) $8\pi r^2 S$ (b) $12\pi r^2 S$ (c) $16\pi r^2 S$ (d) $24\pi r^2 S$.

**16.** If more air is pushed in a soap bubble, the pressure in it
(a) decreases (b) increases
(c) remains same (d) becomes zero.

**17.** If two soap bubbles of different radii are connected by a tube,
(a) air flows from bigger bubble to the smaller bubble till the sizes become equal
(b) air flows from bigger bubble to the smaller bubble till the sizes are interchanged
(c) air flows from the smaller bubble to the bigger
(d) there is no flow of air.

**18.** Figure (14-Q1) shows a capillary tube of radius $r$ dipped into water. If the atmospheric pressure is $P_0$, the

pressure at point $A$ is

(a) $P_0$ (b) $P_0 + \frac{2S}{r}$ (c) $P_0 - \frac{2S}{r}$ (d) $P_0 - \frac{4S}{r}$.

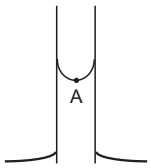



Figure 14-Q1

**19.** The excess pressure inside a soap bubble is twice the excess pressure inside a second soap bubble. The volume of the first bubble is $n$ times the volume of the second where $n$ is

(a) 4 (b) 2 (c) 1 (d) 0·125.

**20.** Which of the following graphs may represent the relation between the capillary rise $h$ and the radius $r$ of the capillary ?

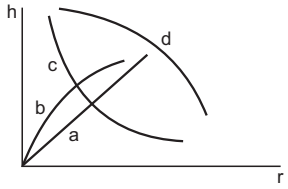



Figure 14-Q2

**21.** Water rises in a vertical capillary tube up to a length of 10 cm. If the tube is inclined at 45°, the length of water risen in the tube will be

(a) 10 cm (b) $10\sqrt{2}$ cm

(c) $10/\sqrt{2}$ cm (d) none of these.

**22.** A 20 cm long capillary tube is dipped in water. The water rises up to 8 cm. If the entire arrangement is put in a freely falling elevator, the length of water column in the capillary tube will be

(a) 8 cm (b) 6 cm (c) 10 cm (d) 20 cm.

**23.** Viscosity is a property of

(a) liquids only (b) solids only

(c) solids and liquids only (d) liquids and gases only.

**24.** The force of viscosity is

(a) electromagnetic (b) gravitational (c) nuclear (d) weak.

**25.** The viscous force acting between two layers of a liquid is given by $\frac{F}{A} = -\eta \frac{dv}{dz}$. This $F/A$ may be called

(a) pressure (b) longitudinal stress

(c) tangential stress (d) volume stress.

**26.** A raindrop falls near the surface of the earth with almost uniform velocity because

(a) its weight is negligible

(b) the force of surface tension balances its weight

(c) the force of viscosity of air balances its weight

(d) the drops are charged and atmospheric electric field balances its weight.

**27.** A piece of wood is taken deep inside a long column of water and released. It will move up

(a) with a constant upward acceleration

(b) with a decreasing upward acceleration

(c) with a deceleration

(d) with a uniform velocity.

**28.** A solid sphere falls with a terminal velocity of 20 m $s^{-1}$ in air. If it is allowed to fall in vacuum,

(a) terminal velocity will be 20 m $s^{-1}$

(b) terminal velocity will be less than 20 m $s^{-1}$

(c) terminal velocity will be more than 20 m $s^{-1}$

(d) there will be no terminal velocity.

**29.** A spherical ball is dropped in a long column of a viscous liquid. The speed of the ball as a function of time may be best represented by the graph

(a) $A$ (b) $B$ (c) $C$ (d) $D$.

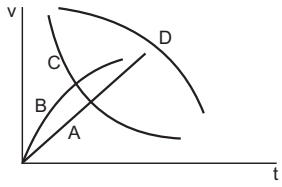



Figure 14-Q3

## OBJECTIVE II

**1.** A student plots a graph from his readings on the determination of Young modulus of a metal wire but forgets to put the labels (figure 14-Q4). The quantities on $X$ and $Y$-axes may be respectively

(a) weight hung and length increased

(b) stress applied and length increased

(c) stress applied and strain developed

(d) length increased and the weight hung.

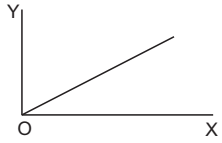



Figure 14-Q4

**2**. The properties of a surface are different from those of the bulk liquid because the surface molecules

(a) are smaller than other molelcules

(b) acquire charge due to collision from air molecules

(c) find different type of molecules in their range of influence

(d) feel a net force in one direction.

**3.** The rise of a liquid in a capillary tube depends on

(a) the material (b) the length

(c) the outer radius (d) the inner radius of the tube.

**4.** The contact angle between a solid and a liquid is a property of

(a) the material of the solid

(b) the material of the liquid

(c) the shape of the solid
(d) the mass of the solid.

**5.** A liquid is contained in a vertical tube of semicircular cross section (figure 14-Q5). The contact angle is zero. The force of surface tension on the curved part and on the flat part are in ratio
(a) 1 : 1 (b) 1 : 2 (c) $\pi : 2$ (d) $2 : \pi$.

Figure 14-Q5

**6.** When a capillary tube is dipped into a liquid, the liquid neither rises nor falls in the capillary.
(a) The surface tension of the liquid must be zero.
(b) The contact angle must be 90°.
(c) The surface tension may be zero.
(d) The contact angle may be 90°.

**7.** A solid sphere moves at a terminal velocity of 20 m $s^{-1}$ in air at a place where $g = 9{\cdot}8$ m $s^{-2}$. The sphere is taken in a gravity-free hall having air at the same pressure and pushed down at a speed of 20 m $s^{-1}$.
(a) Its initial acceleration will be $9{\cdot}8$ m $s^{-2}$ downward.
(b) Its initial acceleration will be $9{\cdot}8$ m $s^{-2}$ upward.
(c) The magnitude of acceleration will decrease as the time passes.
(d) It will eventually stop.

## EXERCISES

**1.** A load of 10 kg is suspended by a metal wire 3 m long and having a cross-sectional area 4 $mm^2$. Find (a) the stress (b) the strain and (c) the elongation. Young modulus of the metal is $2{\cdot}0 \times 10^{11}$ N $m^{-2}$.

**2.** A vertical metal cylinder of radius 2 cm and length 2 m is fixed at the lower end and a load of 100 kg is put on it. Find (a) the stress (b) the strain and (c) the compression of the cylinder. Young modulus of the metal $= 2 \times 10^{11}$ N $m^{-2}$.

**3.** The elastic limit of steel is $8 \times 10^8$ N $m^{-2}$ and its Young modulus $2 \times 10^{11}$ N $m^{-2}$. Find the maximum elongation of a half-metre steel wire that can be given without exceeding the elastic limit.

**4.** A steel wire and a copper wire of equal length and equal cross-sectional area are joined end to end and the combination is subjected to a tension. Find the ratio of (a) the stresses developed in the two wires and (b) the strains developed. $Y$ of steel $= 2 \times 10^{11}$ N $m^{-2}$. $Y$ of copper $= 1{\cdot}3 \times 10^{11}$ N $m^{-2}$.

**5.** In figure (14-E1) the upper wire is made of steel and the lower of copper. The wires have equal cross section. Find the ratio of the longitudinal strains developed in the two wires.

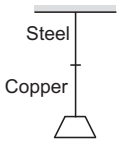



Figure 14-E1

**6.** The two wires shown in figure (14-E2) are made of the same material which has a breaking stress of $8 \times 10^8$ N $m^{-2}$. The area of cross section of the upper wire is $0{\cdot}006$ $cm^2$ and that of the lower wire is $0{\cdot}003$ $cm^2$. The mass $m_1 = 10$ kg, $m_2 = 20$ kg and the hanger is light. (a) Find the maximum load that can be put on the hanger without breaking a wire. Which wire will break first if the load is increased ? (b) Repeat the above part if $m_1 = 10$ kg and $m_2 = 36$ kg.

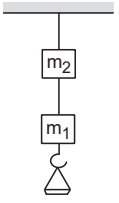



Figure 14-E2

**7.** Two persons pull a rope towards themselves. Each person exerts a force of 100 N on the rope. Find the Young modulus of the material of the rope if it extends in length by 1 cm. Original length of the rope = 2 m and the area of cross section = 2 $cm^2$.

**8.** A steel rod of cross-sectional area 4 $cm^2$ and length 2 m shrinks by $0{\cdot}1$ cm as the temperature decreases in night. If the rod is clamped at both ends during the day hours, find the tension developed in it during night hours. Young modulus of steel $= 1{\cdot}9 \times 10^{11}$ N $m^{-2}$.

**9.** Consider the situation shown in figure (14-E3). The force $F$ is equal to the $m_2 g/2$. If the area of cross section of the string is $A$ and its Young modulus $Y$, find the strain developed in it. The string is light and there is no friction anywhere.

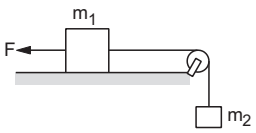



Figure 14-E3

**10.** A sphere of mass 20 kg is suspended by a metal wire of unstretched length 4 m and diameter 1 mm. When in equilibrium, there is a clear gap of 2 mm between the sphere and the floor. The sphere is gently pushed aside so that the wire makes an angle $\theta$ with the vertical and is released. Find the maximum value of $\theta$ so that the sphere does not rub the floor. Young modulus of the metal of the wire is $2{\cdot}0 \times 10^{11}$ N $m^{-2}$. Make appropriate approximations.

**11.** A steel wire of original length 1 m and cross-sectional area 4·00 $mm^2$ is clamped at the two ends so that it lies horizontally and without tension. If a load of 2·16 kg is suspended from the middle point of the wire, what would be its vertical depression ?

$Y$ of the steel = $2{\cdot}0 \times 10^{11}$ N $m^{-2}$. Take $g$ = 10 m $s^{-2}$.

**12.** A copper wire of cross-sectional area 0·01 $cm^2$ is under a tension of 20 N. Find the decrease in the cross-sectional area. Young modulus of copper = $1{\cdot}1 \times 10^{11}$ N $m^{-2}$ and Poisson ratio = 0·32.

[**Hint :** $\frac{\Delta A}{A} = 2\frac{\Delta r}{r}$]

**13.** Find the increase in pressure required to decrease the volume of a water sample by 0·01%. Bulk modulus of water = $2{\cdot}1 \times 10^9$ N $m^{-2}$.

**14.** Estimate the change in the density of water in ocean at a depth of 400 m below the surface. The density of water at the surface = 1030 kg $m^{-3}$ and the bulk modulus of water = $2 \times 10^9$ N $m^{-2}$.

**15.** A steel plate of face area 4 $cm^2$ and thickness 0·5 cm is fixed rigidly at the lower surface. A tangential force of 10 N is applied on the upper surface. Find the lateral displacement of the upper surface with respect to the lower surface. Rigidity modulus of steel = $8{\cdot}4 \times 10^{10}$ N $m^{-2}$.

**16.** A 5·0 cm long straight piece of thread is kept on the surface of water. Find the force with which the surface on one side of the thread pulls it. Surface tension of water = 0·076 N $m^{-1}$.

**17.** Find the excess pressure inside (a) a drop of mercury of radius 2 mm (b) a soap bubble of radius 4 mm and (c) an air bubble of radius 4 mm formed inside a tank of water. Surface tension of mercury, soap solution and water are 0·465 N $m^{-1}$, 0·03 N $m^{-1}$ and 0·076 N $m^{-1}$ respectively.

**18.** Consider a small surface area of 1 $mm^2$ at the top of a mercury drop of radius 4·0 mm. Find the force exerted on this area (a) by the air above it (b) by the mercury below it and (c) by the mercury surface in contact with it. Atmospheric pressure = $1{\cdot}0 \times 10^5$ Pa and surface tension of mercury = 0·465 N $m^{-1}$. Neglect the effect of gravity. Assume all numbers to be exact.

**19.** The capillaries shown in figure (14-E4) have inner radii 0·5 mm, 1·0 mm and 1·5 mm respectively. The liquid in the beaker is water. Find the heights of water level in the capillaries. The surface tension of water is $7{\cdot}5 \times 10^{-2}$ N $m^{-1}$.

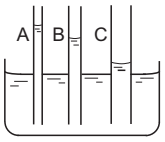



Figure 14-E4

**20.** The lower end of a capillary tube is immersed in mercury. The level of mercury in the tube is found to be 2 cm below the outer level. If the same tube is immersed in water, up to what height will the water rise in the capillary ?

**21.** A barometer is constructed with its tube having radius 1·0 mm. Assume that the surface of mercury in the tube is spherical in shape. If the atmospheric pressure is equal to 76 cm of mercury, what will be the height raised in the barometer tube? The contact angle of mercury with glass = 135° and surface tension of mercury = 0·465 N $m^{-1}$. Density of mercury = 13600 kg $m^{-3}$.

**22.** A capillary tube of radius 0·50 mm is dipped vertically in a pot of water. Find the difference between the pressure of the water in the tube 5·0 cm below the surface and the atmospheric pressure. Surface tension of water = 0·075 N $m^{-1}$.

**23.** Find the surface energy of water kept in a cylindrical vessel of radius 6·0 cm. Surface tension of water = 0·075 J $m^{-2}$.

**24.** A drop of mercury of radius 2 mm is split into 8 identical droplets. Find the increase in surface energy. Surface tension of mercury = 0·465 J $m^{-2}$.

**25.** A capillary tube of radius 1 mm is kept vertical with the lower end in water. (a) Find the height of water raised in the capillary. (b) If the length of the capillary tube is half the answer of part (a), find the angle $\theta$ made by the water surface in the capillary with the wall.

**26.** The lower end of a capillary tube of radius 1 mm is dipped vertically into mercury. (a) Find the depression of mercury column in the capillary. (b) If the length dipped inside is half the answer of part (a), find the angle made by the mercury surface at the end of the capillary with the vertical. Surface tension of mercury = 0·465 N $m^{-1}$ and the contact angle of mercury with glass = 135°.

**27.** Two large glass plates are placed vertically and parallel to each other inside a tank of water with separation between the plates equal to 1 mm. Find the rise of water in the space between the plates. Surface tension of water = 0·075 N $m^{-1}$.

**28.** Consider an ice cube of edge 1·0 cm kept in a gravity-free hall. Find the surface area of the water when the ice melts. Neglect the difference in densities of ice and water.

**29.** A wire forming a loop is dipped into soap solution and taken out so that a film of soap solution is formed. A loop of 6·28 cm long thread is gently put on the film and the film is pricked with a needle inside the loop. The thread loop takes the shape of a circle. Find the tension in the thread. Surface tension of soap solution = 0·030 N $m^{-1}$.

**30.** A metal sphere of radius 1 mm and mass 50 mg falls vertically in glycerine. Find (a) the viscous force exerted by the glycerine on the sphere when the speed of the sphere is 1 cm $s^{-1}$, (b) the hydrostatic force exerted by the glycerine on the sphere and (c) the terminal velocity with which the sphere will move down without acceleration. Density of glycerine = 1260 kg $m^{-3}$ and its coefficient of viscosity at room temperature = 8·0 poise.

**31.** Estimate the speed of vertically falling raindrops from the following data. Radius of the drops = 0·02 cm, viscosity of air = $1{\cdot}8 \times 10^{-4}$ poise, $g$ = 10 m $s^{-2}$ and density of water = 1000 kg $m^{-3}$.

**32.** Water flows at a speed of 6 cm $s^{-1}$ through a tube of radius 1 cm. Coefficient of viscosity of water at room temperature is 0·01 poise. Calculate the Reynolds number. Is it a steady flow ?

□

## ANSWERS

### OBJECTIVE I

| | | | | | |
|---|---|---|---|---|---|
| 1. (d) | 2. (a) | 3. (b) | 4. (a) | 5. (b) | 6. (a) |
| 7. (a) | 8. (c) | 9. (b) | 10. (d) | 11. (c) | 12. (b) |
| 13. (a) | 14. (c) | 15. (d) | 16. (a) | 17. (c) | 18. (c) |
| 19. (d) | 20. (c) | 21. (b) | 22. (d) | 23. (d) | 24. (a) |
| 25. (c) | 26. (c) | 27. (b) | 28. (d) | 29. (b) | |

### OBJECTIVE II

1. all
2. (c), (d)
3. (a), (b), (d)
4. (a), (b)
5. (c)
6. (c), (d)
7. (b), (c), (d)

### EXERCISES

1. (a) $2{\cdot}5 \times 10^{7}$ N $m^{-2}$ (b) $1{\cdot}25 \times 10^{-4}$ (c) $3{\cdot}75 \times 10^{-4}$ m
2. (a) $7{\cdot}96 \times 10^{5}$ N $m^{-2}$ (b) $4 \times 10^{-6}$ (c) $8 \times 10^{-6}$ m
3. 2 mm
4. (a) 1 (b) $\frac{\text{strain in copper wire}}{\text{strain in steel wire}} = \frac{20}{13}$
5. $\frac{\text{strain in copper wire}}{\text{strain in steel wire}} = 1{\cdot}54$
6. (a) 14 kg, lower (b) 2 kg, upper
7. $1 \times 10^{8}$ N $m^{-2}$
8. $3{\cdot}8 \times 10^{4}$ N
9. $\frac{m_2 g(2m_1 + m_2)}{2AY(m_1 + m_2)}$
10. 36·4°
11. 1·5 cm
12. $1{\cdot}164 \times 10^{-6}$ $cm^2$
13. $2{\cdot}1 \times 10^{6}$ N $m^{-2}$
14. 2 kg $m^{-3}$
15. $1{\cdot}5 \times 10^{-9}$ m
16. $3{\cdot}8 \times 10^{-3}$ N
17. (a) 465 N $m^{-2}$ (b) 30 N $m^{-2}$ (c) 38 N $m^{-2}$
18. (a) 0·1 N (b) 0·10023 N (c) 0·00023 N
19. 3 cm in $A$, 1·5 cm in $B$, 1 cm in $C$
20. 5·73 cm
21. 75·5 cm
22. 190 N $m^{-2}$
23. $8{\cdot}5 \times 10^{-4}$ J
24. 23·4 μ J
25. (a) 1·5 cm (b) 60°
26. (a) 4·93 mm (b) 111°
27. 1·5 cm
28. $(36\pi)^{1/3}$ $cm^2$
29. $6 \times 10^{-4}$ N
30. (a) $1{\cdot}5 \times 10^{-4}$ N (b) $5{\cdot}2 \times 10^{-5}$ N (c) 2·9 cm $s^{-1}$
31. 5 m $s^{-1}$
32. 1200, yes.

□

CHAPTER 15

# WAVE MOTION AND WAVES ON A STRING

## 15.1 WAVE MOTION

When a particle moves through space, it carries kinetic energy with itself. Wherever the particle goes, the energy goes with it. The energy is associated with the particle and is transported from one region of the space to the other together with the particle just like we ride a car and are taken from Lucknow to Varanasi with the car.

There is another way to transport energy from one part of space to the other without any bulk motion of material together with it. Sound is transmitted in air in this manner. When you say "Hello" to your friend, no material particle is ejected from your lips and falls on your friend's ear. You create some disturbance in the part of the air close to your lips. Energy is transferred to these air particles either by pushing them ahead or pulling them back. The density of the air in this part temporarily increases or decreases. These disturbed particles exert force on the next layer of air, transferring the disturbance to that layer. In this way, the disturbance proceeds in air and finally the air near the ear of the listener gets disturbed.

The disturbance produced in the air near the speaker travels in air, the air itself does not move. The air that is near the speaker at the time of uttering a word remains all the time near the speaker even when the message reaches the listener. This type of motion of energy is called a *wave motion.*

To give another example of propagation of energy without bulk motion of matter, suppose many persons are standing in a queue to buy cinema tickets from the ticket counter. It is not yet time, the counter is closed and the persons are getting annoyed. The last person in the queue is somewhat unruly, he leans forward pushing the man in front of him and then stands straight. The second last person, getting the jerk from behind, is forced to lean forward and push the man in front. This second last person manages to stand straight again but the third last person temporarily loses balance and leans forward. The jerk thus travels down the queue and finally the person at the front of the queue feels it. With the jerk, travels the energy down the queue from one end to another though the last person and the first person are still in their previous positions.

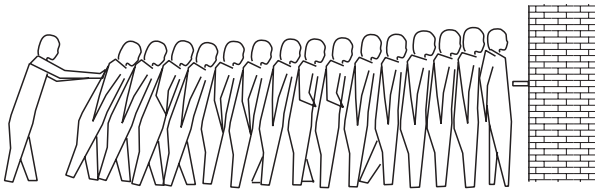


Figure 15.1

The world is full of examples of wave motion. When raindrops hit the surface of calm water, circular waves can be seen travelling on the surface. Any particle of water is only locally displaced for a short time but the disturbance spreads and the particles farther and farther get disturbed when the wave reaches them. Another common example of wave motion is the wave associated with light. One speciality about this wave is that it does not require any material medium for its propagation. The waves requiring a medium are called *mechanical waves* and those which do not require a medium are called *nonmechanical waves.*

In the present chapter, we shall study the waves on a stretched string, a mechanical wave in one dimension.

## 15.2 WAVE PULSE ON A STRING

Let us consider a long string with one end fixed to a wall and the other held by a person. The person pulls on the string keeping it tight. Suppose the person snaps his hand a little up and down producing a bump

in the string near his hand (Figure 15.2). The operation takes a very small time, say, one-tenth of a second after which the person stands still holding the string tight in his hand. What happens as time passes ?

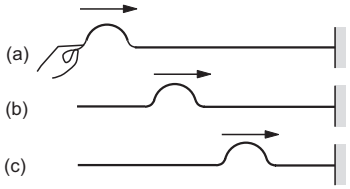



Figure 15.2

Experiments show that if the vertical displacement given is small, the disturbance travels down the string with constant speed. Figure (15.2) also shows the status of the string at successive instants. As time passes, the "bump" travels on the string towards right. For an elastic and homogeneous string, the bump moves with constant speed to cover equal distances in equal time. Also, the shape of the bump is not altered as it moves, provided the bump is small. Notice that no part of the string moves from left to right. The person is holding the left end tight and the string cannot slip from his hand. The part of the string, where the bump is present at an instant, is in up–down motion. As time passes, this part again regains its normal position. The person does some work on the part close to his hand giving some energy to that part. This disturbed part exerts elastic force on the part to the right and transfers the energy, the bump thus moves on to the right. In this way, different parts of the string are successively disturbed, transmitting the energy from left to right.

When a disturbance is localised only to a small part of the space at a time, we say that a *wave pulse* is passing through that part of the space. This happens when the source producing the disturbance (hand in this case) is active only for a short time. If the source is active for some extended time repeating its motion several times, we get a *wave train* or a *wave packet.* For example, if the person in figure (15.2) decides to vibrate his hand up and down 10 times and then stop, a wave train consisting of 10 loops will proceed on the string.

**Equation of a Travelling Wave**

Suppose, in the example of figure (15.2), the man starts snapping his hand at $t = 0$ and finishes his job at $t = \Delta t$. The vertical displacement $y$ of the left end of the string is a function of time. It is zero for $t < 0$, has nonzero value for $0 < t < \Delta t$ and is again zero for $t > \Delta t$. Let us represent this function by $f(t)$. Take the left end of the string as the origin and take the $X$-axis along the string towards right. The function $f(t)$ represents the displacement $y$ of the particle at $x = 0$ as a function of time

$$y(x = 0, t) = f(t).$$

The disturbance travels on the string towards right with a constant speed $v$. Thus, the displacement, produced at the left end at time $t$, reaches the point $x$ at time $t + x/v$. Similarly, the displacement of the particle at point $x$ at time $t$ was originated at the left end at the time $t - x/v$. But the displacement of the left end at time $t - x/v$ is $f(t - x/v)$. Hence,

$$y(x, t) = y(x = 0, t - x/v) = f(t - x/v).$$

The displacement of the particle at $x$ at time $t$, i.e., $y(x, t)$ is generally abbreviated as $y$ and the wave equation is written as

$$y = f(t - x/v). \quad \ldots (15.1)$$

Equation (15.1) represents a wave travelling in the positive $x$-direction with a constant speed $v$. Such a wave is called a *travelling wave* or a *progressive wave*. The function $f$ is arbitrary and depends on how the source moves. The time $t$ and the position $x$ must appear in the wave equation in the combination $t - x/v$ only. For example,

$$y = A \sin\frac{(t - x/v)}{T}, \quad y = A\, e^{-\frac{(t - x/v)}{T}}$$

are valid wave equations. They represent waves travelling in positive $x$-direction with constant speed. The equation $y = A \sin \frac{(x^2 - v^2 t^2)}{L^2}$ does not represent a wave travelling in $x$-direction with a constant speed.

If a wave travels in negative $x$-direction with speed $v$, its general equation may be written as

$$y = f(t + x/v). \quad \ldots (15.2)$$

The wave travelling in positive $x$-direction (equation 15.1) can also be written as

$$y = f\left(\frac{vt - x}{v}\right)$$

or, $$y = g(x - vt), \quad \ldots (15.3)$$

where $g$ is some other function having the following meaning. If we put $t = 0$ in equation (15.3), we get the displacement of various particles at $t = 0$, i.e.,

$$y(x,\ t = 0) = g(x).$$

Thus, $g(x)$ represents the shape of the string at $t = 0$. If the displacement of the different particles at $t = 0$ is represented by the function $g(x)$, the displacement of the particle at $x$ at time $t$ will be $y = g(x - vt)$. Similarly, if the wave is travelling along the negative $x$-direction and the displacement of

different particles at $t = 0$ is $g(x)$, the displacement of the particle at $x$ at time $t$ will be

$$y = g(x + vt). \quad \ldots (15.4)$$

Thus, the function $f$ in equations (15.1) and (15.2) represents the displacement of the point $x = 0$ as time passes and $g$ in (15.3) and (15.4) represents the displacement at $t = 0$ of different particles.

***Example 15.1***

*A wave is propagating on a long stretched string along its length taken as the positive x-axis. The wave equation is given as*

$$y = y_0 e^{-\left(\frac{t}{T} - \frac{x}{\lambda}\right)^2},$$

*where* $y_0 = 4$ *mm,* $T = 1{\cdot}0$ s *and* $\lambda = 4$ cm. *(a) Find the velocity of the wave. (b) Find the function f(t) giving the displacement of the particle at* $x = 0$. *(c) Find the function g(x) giving the shape of the string at* $t = 0$. *(d) Plot the shape g(x) of the string at* $t = 0$. *(e) Plot the shape of the string at* $t = 5$ s.

***Solution*** : (a) The wave equation may be written as

$$y = y_0 e^{-\frac{1}{T^2}\left(t - \frac{x}{\lambda/T}\right)^2}.$$

Comparing with the general equation $y = f(t - x/v)$, we see that

$$v = \frac{\lambda}{T} = \frac{4 \text{ cm}}{1{\cdot}0 \text{ s}} = 4 \text{ cm s}^{-1}.$$

(b) Putting $x = 0$ in the given equation,

$$f(t) = y_0 e^{-(t/T)^2}. \quad \ldots (i)$$

(c) Putting $t = 0$ in the given equation

$$g(x) = y_0 e^{-(x/\lambda)^2}. \quad \ldots (ii)$$

(d)

Figure 15.3(a)

(e)

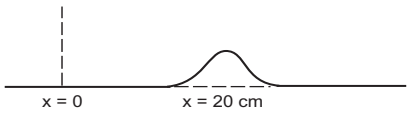



Figure 15.3(b)

## 15.3 SINE WAVE TRAVELLING ON A STRING

What happens if the person holding the string in figure (15.2) keeps waving his hand up and down continuously. He keeps doing work on the string and the energy is continuously supplied to the string. Any part of the string continues to vibrate up and down once the first disturbance has reached it. It receives energy from the left, transmits it to the right and the process continues till the person is not tired. The nature of vibration of any particle is similar to that of the left end, the only difference being that the motion is repeated after a time delay of $x/v$.

A very important special case arises when the person vibrates the left end $x = 0$ in a simple harmonic motion. The equation of motion of this end may then be written as

$$f(t) = A \sin \omega t, \quad \ldots (15.5)$$

where $A$ represents the amplitude and $\omega$ the angular frequency. The time period of oscillation is $T = 2\pi/\omega$ and the frequency of oscillation is $\nu = 1/T = \omega/2\pi$. The wave produced by such a vibrating source is called a *sine wave* or *sinusoidal wave*.

Since the displacement of the particle at $x = 0$ is given by (15.5), the displacement of the particle at $x$ at time $t$ will be

$$y = f(t - x/v)$$

or, $$y = A \sin \omega(t - x/v). \quad \ldots (15.6)$$

This follows from the fact that the wave moves along the string with a constant speed $v$ and the displacement of the particle at $x$ at time $t$ was originated at $x = 0$ at time $t - x/v$.

The velocity of the particle at $x$ at time $t$ is given by

$$\frac{\partial y}{\partial t} = A\,\omega \cos\omega(t - x/v). \quad \ldots (15.7)$$

The symbol $\frac{\partial}{\partial t}$ is used in place of $\frac{d}{dt}$ to indicate that while differentiating with respect to $t$, we should treat $x$ as constant. It is the same particle whose displacement should be considered as a function of time.

This velocity is totally different from the wave velocity $v$. The wave moves on the string at a constant velocity $v$ along the $x$-axis, but the particle moves up and down with velocity $\frac{\partial y}{\partial t}$ which changes with $x$ and $t$ according to (15.7).

Figure (15.4) shows the shape of the string as time passes. Each particle of the string vibrates in simple harmonic motion with the same amplitude $A$ and frequency $\nu$. The phases of the vibrations are, however, different. When a particle $P$ (figure 15.4) reaches its extreme position in upward direction, the particle $Q$ little to its right, is still coming up and the particle $R$ little to its left, has already crossed that phase and is going down. The phase difference is larger if the particles are separated farther.

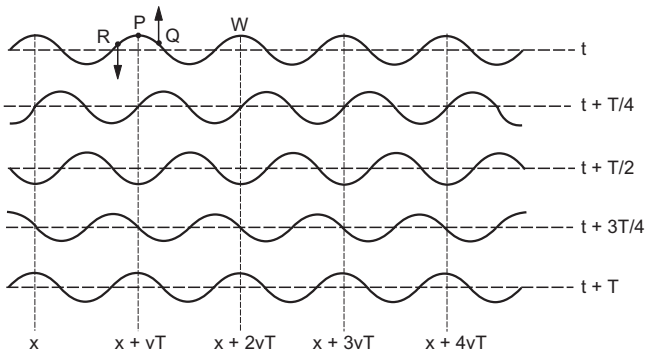



Figure 15.4

Each particle copies the motion of another particle at its left with a time delay of $x/v$, where $x$ is the separation between the two particles. For the particles $P$ and $W$, shown in figure (15.4), the separation is $\Delta x = vT$ and the particle $W$ copies the motion of $P$ after a time delay of $\Delta x/v = T$. But the motion of any particle at any instant is identical in all respects to its motion a time period $T$ later. So, a delay of one time period is equivalent to no delay and hence, the particles $P$ and $W$ vibrate in the same phase. They reach their extreme positions together, they cross their mean positions together, their displacements are identical and their velocities are identical at any instant. Same is true for any pair of particles separated by a distance $vT$. This separation is called the *wavelength* of the wave and is denoted by the Greek letter $\lambda$. Thus, $\lambda = vT$.

The above relation can easily be derived mathematically. Suppose, the particles at $x$ and $x + L$ vibrate in the same phase. By equation (15.6) and (15.7),

$$A \sin\left[\omega\left(t - \frac{x}{v}\right)\right] = A \sin\left[\omega\left(t - \frac{x+L}{v}\right)\right]$$

and $$A\,\omega \cos\left[\omega\left(t - \frac{x}{v}\right)\right] = A\,\omega \cos\left[\omega\left(t - \frac{x+L}{v}\right)\right].$$

This gives

$$\omega\left(t - \frac{x}{v}\right) = \omega\left(t - \frac{x+L}{v}\right) + 2\,n\,\pi,$$

where $n$ is an integer,

or, $$0 = -\frac{\omega L}{v} + 2\,n\,\pi$$

or, $$L = \frac{v}{\omega}\,2\,n\,\pi.$$

The minimum separation between the particles vibrating in same phase is obtained by putting $n = 1$ in the above equation. Thus, the wavelength is

$$\lambda = \frac{v}{\omega}\,2\pi = vT. \qquad \ldots\ (15.8)$$

Also, $$v = \lambda/T = \nu\lambda, \qquad \ldots\ (15.9)$$

where $\nu = 1/T$ is the frequency of the wave.

This represents an important relation between the three characteristic parameters of a sine wave namely, the wave velocity, the frequency and the wavelength.

The quantity $2\pi/\lambda$ is called the *wave number* and is generally denoted by the letter $k$.

Thus, $$k = \frac{2\,\pi}{\lambda} = \frac{2\pi\nu}{v} = \frac{\omega}{v}.$$

The segment where the disturbance is positive is called a *crest* of the wave and the segment where the disturbance is negative is called a *trough*. The separation between consecutive crests or between consecutive troughs is equal to the wavelength.

**Alternative Forms of Wave Equation**

We have written the wave equation of a wave travelling in $x$-direction as

$$y = A \sin \omega(t - x/v).$$

This can also be written in several other forms such as,

$$y = A \sin(\omega t - kx) \qquad \ldots\ (15.10)$$

$$= A \sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) \qquad \ldots\ (15.11)$$

$$= A \sin k(vt - x). \qquad \ldots\ (15.12)$$

Also, it should be noted that we have made our particular choice of $t = 0$ in writing equation (15.5) from which the wave equation is deduced. The origin of time is chosen at an instant when the left end $x = 0$ is crossing its mean position $y = 0$ and is going up. For a general choice of the origin of time, we will have to add a phase constant so that the equation will be

$$y = A \sin[\omega(t - x/v) + \phi]. \qquad \ldots\ (15.13)$$

The constant $\phi$ will be $\pi/2$ if we choose $t = 0$ at an instant when the left end reaches its extreme position $y = A$. The equation will then be

$$y = A \cos \omega(t - x/v). \qquad \ldots\ (15.14)$$

If $t = 0$ is taken at the instant when the left end is crossing the mean position from upward to downward direction, $\phi$ will be $\pi$ and the equation will be

$$y = A \sin \omega\left(\frac{x}{v} - t\right)$$

or, $$y = A \sin(kx - \omega t). \qquad \ldots\ (15.15)$$

***Example 15.2***

*Consider the wave* $y = (5\text{ mm}) \sin[(1\text{ cm}^{-1})x - (60\text{ s}^{-1})t]$. *Find (a) the amplitude, (b) the wave number, (c) the*

*wavelength, (d) the frequency, (e) the time period and (f) the wave velocity.*

***Solution*** : Comparing the given equation with equation (15.15), we find

(a) amplitude $A = 5$ mm

(b) wave number $k = 1\ \text{cm}^{-1}$

(c) wavelength $\lambda = \dfrac{2\pi}{k} = 2\pi$ cm

(d) frequency $\nu = \dfrac{\omega}{2\pi} = \dfrac{60}{2\pi}$ Hz

$= \dfrac{30}{\pi}$ Hz

(e) time period $T = \dfrac{1}{\nu} = \dfrac{\pi}{30}$ s

(f) wave velocity $v = \nu\lambda = 60\ \text{cm s}^{-1}$.

## 15.4 VELOCITY OF A WAVE ON A STRING

The velocity of a wave travelling on a string depends on the elastic and the inertia properties of the string. When a part of the string gets disturbed, it exerts an extra force on the neighbouring part because of the elastic property. The neighbouring part responds to this force and the response depends on the inertia property. The elastic force in the string is measured by its tension $F$ and the inertia by its mass per unit length. We have used the symbol $F$ for tension and not $T$ in order to avoid confusion with the time period.

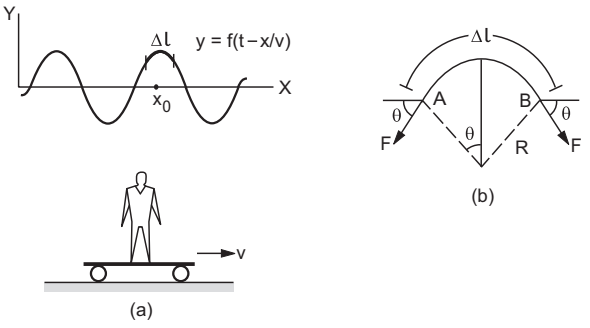



Figure 15.5

Suppose a wave $y = f\left(t - \dfrac{x}{v}\right)$ is travelling on the string in the positive $x$-direction with a speed $v$. Let us choose an observer who is riding on a car that moves along the $x$-direction with the same velocity $v$ (figure 15.5). Looking from this frame, the pattern of the string is at rest but the entire string is moving towards the negative $x$-direction with a speed $v$. If a crest is opposite to the observer at any instant, it will always remain opposite to him with the same shape while the string will pass through this crest in opposite direction like a snake.

Consider a small element $AB$ of the string of length $\Delta l$ at the highest point of a crest. Any small curve may be approximated by a circular arc. Suppose the small element $\Delta l$ forms an arc of radius $R$. The particles of the string in this element go in this circle with a speed $v$ as the string slides through this part. The general situation is shown in figure (15.5a) and the expanded view of the part near $\Delta l$ is shown in figure (15.5b).

We assume that the displacements are small so that the tension in the string does not appreciably change because of the disturbance. The element $AB$ is pulled by the parts of the string to its right and to its left. Resultant force on this element is in the downward direction as shown in figure (15.5b) and its magnitude is

$$F_r = F\sin\theta + F\sin\theta = 2F\sin\theta.$$

As $\Delta l$ is taken small, $\theta$ will be small and

$$\sin\theta \approx \frac{\Delta l/2}{R}$$

so that the resultant force on $\Delta l$ is

$$F_r = 2F\left(\frac{\Delta l/2}{R}\right) = F\Delta l/R.$$

If $\mu$ be the mass per unit length of the string, the element $AB$ has a mass $\Delta m = \Delta l\,\mu$. Its downward acceleration is

$$a = \frac{F_r}{\Delta m} = \frac{F\Delta l/R}{\mu\,\Delta l} = \frac{F}{\mu R}.$$

But the element is moving in a circle of radius $R$ with a constant speed $v$. Its acceleration is, therefore, $a = \dfrac{v^2}{R}$. The above equation becomes

$$\frac{v^2}{R} = \frac{F}{\mu R}$$

or, $$v = \sqrt{F/\mu}. \qquad \ldots\ (15.16)$$

The velocity of the wave on a string thus depends only on the tension $F$ and the linear mass density $\mu$. We have used the approximation that the tension $F$ remains almost unchanged as the part of the string vibrates up and down. This approximation is valid only for small amplitudes because as the string vibrates, the lengths of its parts change during the course of vibration and hence, the tension changes.

***Example 15.3***

*Figure (15.6) shows a string of linear mass density* $1{\cdot}0\ \text{g cm}^{-1}$ *on which a wave pulse is travelling. Find the*

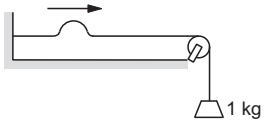



Figure 15.6

*time taken by the pulse in travelling through a distance of* 50 cm *on the string. Take* $g = 10\ \text{m s}^{-2}$.

***Solution*** : The tension in the string is $F = mg = 10$ N. The mass per unit length is $\mu = 1{\cdot}0\ \text{g cm}^{-1} = 0{\cdot}1\ \text{kg m}^{-1}$. The wave velocity is, therefore, $v = \sqrt{F/\mu} = \sqrt{\dfrac{10\ \text{N}}{0{\cdot}1\ \text{kg m}^{-1}}}$ $= 10\ \text{m s}^{-1}$. The time taken by the pulse in travelling through 50 cm is, therefore, 0·05 s.

## 15.5 POWER TRANSMITTED ALONG THE STRING BY A SINE WAVE

When a travelling wave is established on a string, energy is transmitted along the direction of propagation of the wave. Consider again a sine wave travelling along a stretched string in $x$-direction. The equation for the displacement in $y$-direction is

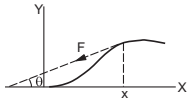



Figure 15.7

$$y = A \sin \omega(t - x/v). \qquad \ldots \text{ (i)}$$

Figure (15.7) shows a portion of the string at a time $t$ to the right of position $x$. The string on the left of the point $x$ exerts a force $F$ on this part. The direction of this force is along the tangent to the string at position $x$. The component of the force along the $Y$-axis is

$$F_y = -F \sin\theta \approx -F \tan\theta = -F \frac{\partial y}{\partial x}.$$

The power delivered by the force $F$ to the string on the right of position $x$ is, therefore,

$$P = \left(-F \frac{\partial y}{\partial x}\right) \frac{\partial y}{\partial t}.$$

By (i), it is

$$-F\left[\left(-\frac{\omega}{v}\right) A \cos \omega(t - x/v)\right] [\omega A \cos \omega(t - x/v)]$$
$$= \frac{\omega^2 A^2 F}{v} \cos^2 \omega (t - x/v).$$

This is the rate at which energy is being transmitted from left to right across the point at $x$. The $\cos^2$ term oscillates between 0 and 1 during a cycle and its average value is 1/2. The average power transmitted across any point is, therefore,

$$P_{av} = \frac{1}{2} \frac{\omega^2 A^2 F}{v} = 2\pi^2 \mu v A^2 \nu^2. \qquad \ldots \text{ (15.17)}$$

The power transmitted along the string is proportional to the square of the amplitude and square of the frequency of the wave.

***Example 15.4***

*The average power transmitted through a given point on a string supporting a sine wave is* 0·20 W *when the amplitude of the wave is* 2·0 mm. *What power will be transmitted through this point if the amplitude is increased to* 3·0 mm.

***Solution*** : Other things remaining the same, the power transmitted is proportional to the square of the amplitude.

Thus,

$$\frac{P_2}{P_1} = \frac{A_2^2}{A_1^2}$$

or, $$\frac{P_2}{0{\cdot}20\ \text{W}} = \frac{9}{4} = 2{\cdot}25$$

or, $$P_2 = 2{\cdot}25 \times 0{\cdot}20\ \text{W} = 0{\cdot}45\ \text{W}.$$

## 15.6 INTERFERENCE AND THE PRINCIPLE OF SUPERPOSITION

So far we have considered a single wave passing on a string. Suppose two persons are holding the string at the two ends and snap their hands to start a wave pulse each. One pulse starts from the left end and travels on the string towards right, the other starts at the right end and travels towards left. The pulses travel at same speed although their shapes depend on how the persons snap their hands. Figure (15.8) shows the shape of the string as time passes.

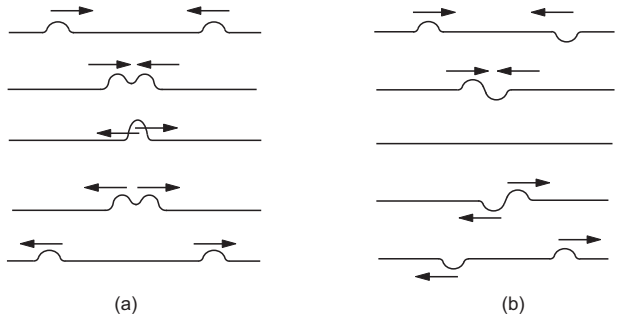



Figure 15.8

The pulses travel towards each other, overlap and recede from each other. The remarkable thing is that the shapes of the pulses, as they emerge after the overlap, are identical to their original shapes. Each pulse has passed the overlap region so smoothly as if the other pulse was not at all there. After the encounter, each pulse looks just as it looked before and each pulse travels just as it did before. The waves can pass through each other freely without being modified.

This is a unique property of the waves. The particles cannot pass through each other, they collide and their course of motion changes. How do we determine the shape of the string at the time when the pulses actually overlap ? The mechanism to know the resultant displacement of a particle which is acted upon by two or more waves simultaneously is very simple. The displacement of the particle is equal to the sum of the displacements the waves would have individually produced. If the first wave alone is travelling, let us say it displaces the particle by 0·2 cm upward and if the second wave alone is travelling, suppose the displacement of this same particle is 0·4 cm upward at that instant. The displacement of the particle at that instant will be 0·6 cm upward if both the waves pass through that particle simultaneously. The displacement of the particles, if the first wave alone were travelling, may be written as

$$y_1 = f_1(t - x/v)$$

and the displacement if the second wave alone were travelling may be written as

$$y_2 = f_2(t + x/v).$$

If both the waves are travelling on the string, the displacement of its different particles will be given by

$$y = y_1 + y_2 = f_1(t - x/v) + f_2(t + x/v).$$

The two individual displacements may be in opposite directions. The magnitude of the resulting displacement may be smaller than the magnitudes of the individual displacements.

If two wave pulses, approaching each other, are identical in shape except that one is inverted with respect to the other, at some instant the displacement of all the particles will be zero. However, the velocities of the particles will not be zero as the waves will emerge in the two directions shortly. Such a situation is shown in figure (15.8b). We see that there is an instant when the string is straight every where. But soon the wave pulses emerge which move away from each other.

Suppose one person snaps the end up and down whereas the other person snaps his end sideways. The displacements produced are at right angles to each other as indicated in figure (15.9). When the two waves overlap, the resultant displacement of any particle is the vector sum of the two individual displacements.

Figure 15.9

The above observations about the overlap of the waves may be summarised in the following statement which is known as the *principle of superposition.*

*When two or more waves simultaneously pass through a point, the disturbance at the point is given by the sum of the disturbances each wave would produce in absence of the other wave(s).*

In general, the principle of superposition is valid for small disturbances only. If the string is stretched too far, the individual displacements do not add to give the resultant displacement. Such waves are called *nonlinear waves.* In this course, we shall only be talking about linear waves which obey the superposition principle.

When two or more waves pass through the same region simultaneously we say that the waves interfere or the *interference of waves* takes place. The principle of superposition says that the phenomenon of wave interference is remarkably simple. Each wave makes its own contribution to the disturbance no matter what the other waves are doing.

## 15.7 INTERFERENCE OF WAVES GOING IN SAME DIRECTION

Suppose two identical sources send sinusoidal waves of same angular frequency ω in positive $x$-direction. Also, the wave velocity and hence, the wave number $k$ is same for the two waves. One source may be started a little later than the other or the two sources may be situated at different points. The two waves arriving at a point then differ in phase. Let the amplitudes of the two waves be $A_1$ and $A_2$ and the two waves differ in phase by an angle δ. Their equations may be written as

$$y_1 = A_1 \sin(kx - \omega t)$$

and $$y_2 = A_2 \sin(kx - \omega t + \delta).$$

According to the principle of superposition, the resultant wave is represented by

$$\begin{aligned} y = y_1 + y_2 &= A_1 \sin(kx - \omega t) + A_2 \sin(kx - \omega t + \delta) \\ &= A_1 \sin(kx - \omega t) + A_2 \sin(kx - \omega t)\cos\delta \\ &\qquad + A_2 \cos(kx - \omega t)\sin\delta \\ &= \sin(kx - \omega t)\,(A_1 + A_2\cos\delta) + \cos(kx - \omega t)\,(A_2\sin\delta). \end{aligned}$$

We can evaluate it using the method described in Chapter-12 to combine two simple harmonic motions.

If we write

$$A_1 + A_2\cos\delta = A\cos\varepsilon \qquad \ldots \text{(i)}$$

and $$A_2\sin\delta = A\sin\varepsilon, \qquad \ldots \text{(ii)}$$

we get

$$\begin{aligned} y &= A\,[\sin(kx - \omega t)\cos\varepsilon + \cos(kx - \omega t)\sin\varepsilon] \\ &= A\sin(kx - \omega t + \varepsilon). \end{aligned}$$

Thus, the resultant is indeed a sine wave of amplitude $A$ with a phase difference $\varepsilon$ with the first wave. By (i) and (ii),

$$A^2 = A^2 \cos^2\varepsilon + A^2 \sin^2\varepsilon$$
$$= (A_1 + A_2 \cos\delta)^2 + (A_2 \sin\delta)^2$$
$$= A_1^2 + A_2^2 + 2A_1 A_2 \cos\delta$$

or, $$A = \sqrt{A_1^2 + A_2^2 + 2A_1 A_2 \cos\delta}. \quad \ldots (15.18)$$

Also, $$\tan\varepsilon = \frac{A\sin\varepsilon}{A\cos\varepsilon} = \frac{A_2\sin\delta}{A_1 + A_2\cos\delta}. \quad \ldots (15.19)$$

As discussed in Chapter-12, these relations may be remembered by using a geometrical model. We draw a vector of length $A_1$ to represent $y_1 = A_1 \sin(kx - \omega t)$ and another vector of length $A_2$ at an angle $\delta$ with the first one to represent $y_2 = A_2 \sin(kx - \omega t + \delta)$. The resultant of the two vectors then represents the resultant wave $y = A \sin(kx - \omega t + \varepsilon)$. Figure (15.10) shows the construction.

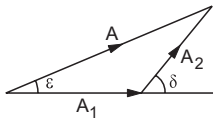



Figure 15.10

**Constructive and Destructive Interference**

We see from equation (15.18) that the resultant amplitude $A$ is maximum when $\cos\delta = +1$, or $\delta = 2n\pi$ and is minimum when $\cos\delta = -1$, or $\delta = (2n+1)\pi$, where $n$ is an integer. In the first case, the amplitude is $A_1 + A_2$ and in the second case, it is $|A_1 - A_2|$. The two cases are called *constructive* and *destructive* interferences respectively. The conditions may be written as,

constructive interference : $\delta = 2n\pi$
destructive interference : $\delta = (2n+1)\pi$ $\quad \ldots (15.20)$

***Example 15.5***

*Two waves are simultaneously passing through a string. The equations of the waves are given by*

$$y_1 = A_1 \sin k(x - vt)$$

*and*

$$y_2 = A_2 \sin k(x - vt + x_0),$$

*where the wave number* $k = 6{\cdot}28\ \text{cm}^{-1}$ *and* $x_0 = 1{\cdot}50$ cm. *The amplitudes are* $A_1 = 5{\cdot}0$ mm *and* $A_2 = 4{\cdot}0$ mm. *Find the phase difference between the waves and the amplitude of the resulting wave.*

***Solution*** : The phase of the first wave is $k(x - vt)$ and of the second is $k(x - vt + x_0)$.

The phase difference is, therefore,

$$\delta = k\,x_0 = (6{\cdot}28\ \text{cm}^{-1})\,(1{\cdot}50\ \text{cm}) = 2\pi \times 1{\cdot}5 = 3\pi.$$

The waves satisfy the condition of destructive interference. The amplitude of the resulting wave is given by

$$|A_1 - A_2| = 5{\cdot}0\ \text{mm} - 4{\cdot}0\ \text{mm} = 1{\cdot}0\ \text{mm}.$$

## 15.8 REFLECTION AND TRANSMISSION OF WAVES

In figure (15.2), a wave pulse was generated at the left end which travelled on the string towards right. When the pulse reaches a particular element, the forces on the element from the left part of the string and from the right part act in such a way that the element is disturbed according to the shape of the pulse.

The situation is different when the pulse reaches the right end which is clamped at the wall. The element at the right end exerts a force on the clamp and the clamp exerts equal and opposite force on the element. The element at the right end is thus acted upon by the force from the string left to it and by the force from the clamp. As this end remains fixed, the two forces are opposite to each other. The force from the left part of the string transmits the forward wave pulse and hence, the force exerted by the clamp sends a return pulse on the string whose shape is similar to the original pulse but is inverted. The original pulse tries to pull the element at the fixed end up and the return pulse sent by the clamp tries to pull it down. The resultant displacement is zero. Thus, the wave is reflected from the fixed end and the reflected wave is inverted with respect to the original wave. The shape of the string at any time, while the pulse is being reflected, can be found by adding an inverted image pulse to the incident pulse (figure 15.11).

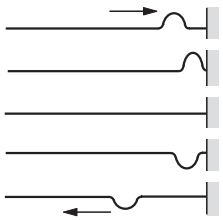


Figure 15.11

Let us now suppose that the right end of the string is attached to a light frictionless ring which can freely move on a vertical rod. A wave pulse is sent on the string from left (Figure 15.12). When the wave reaches the right end, the element at this end is acted on by the force from the left to go up. However, there is no corresponding restoring force from the right as the rod does not exert a vertical force on the ring. As a result, the right end is displaced in upward direction more

than the height of the pulse, i.e., it overshoots the normal maximum displacement. The lack of restoring force from right can be equivalently described in the following way. An extra force acts from right which sends a wave from right to left with its shape identical to the original one. The element at the end is acted upon by both the incident and the reflected wave and the displacements add. Thus, a wave is reflected by the free end without inversion.

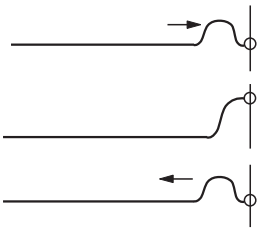


Figure 15.12

Quite often, the end point is neither completely fixed nor completely free to move. As an example, consider a light string attached to a heavier string as shown in figure (15.13). If a wave pulse is produced on the light string moving towards the junction, a part of the wave is reflected and a part is transmitted on the heavier string. The reflected wave is inverted with respect to the original one (figure 15.13a).

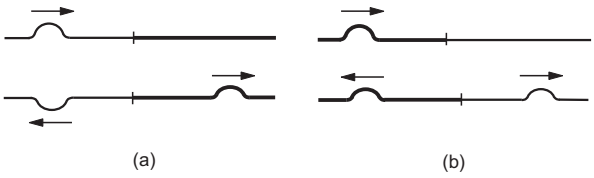



Figure 15.13

On the other hand, if the wave is produced on the heavier string, which moves towards the junction, a part will be reflected and a part transmitted, no inversion of wave shape will take place (figure 15.13b).

The rule about the inversion at reflection may be stated in terms of the wave velocity. The wave velocity is smaller for the heavier string ($v = \sqrt{F/\mu}$) and larger for the lighter string. The above observation may be stated as follows.

*If a wave enters a region where the wave velocity is smaller, the reflected wave is inverted. If it enters a region where the wave velocity is larger, the reflected wave is not inverted. The transmitted wave is never inverted.*

### 15.9 STANDING WAVES

Suppose two sine waves of equal amplitude and frequency propagate on a long string in opposite directions. The equations of the two waves are given by

$$y_1 = A \sin(\omega t - kx)$$

and $$y_2 = A \sin(\omega t + kx + \delta).$$

These waves interfere to produce what we call *standing waves*. To understand these waves, let us discuss the special case when $\delta = 0$.

The resultant displacements of the particles of the string are given by the principle of superposition as

$$\begin{aligned} y &= y_1 + y_2 \\ &= A\,[\sin(\omega t - kx) + \sin(\omega t + kx)] \\ &= 2A \sin \omega t \cos kx \end{aligned}$$

or, $$y = (2A \cos kx) \sin \omega t. \quad \ldots (15.21)$$

This equation can be interpreted as follows. Each particle of the string vibrates in a simple harmonic motion with an amplitude $|2A \cos kx|$. The amplitudes are not equal for all the particles. In particular, there are points where the amplitude $|2A \cos kx| = 0$. This will be the case when

$$\cos kx = 0$$

or, $$kx = \left(n + \frac{1}{2}\right)\pi$$

or, $$x = \left(n + \frac{1}{2}\right)\frac{\lambda}{2},$$

where $n$ is an integer.

For these particles, $\cos kx = 0$ and by equation (15.21) the displacement $y$ is zero all the time. Although these points are not physically clamped, they remain fixed as the two waves pass them simultaneously. Such points are known as *nodes*.

For the points where $|\cos kx| = 1$, the amplitude is maximum. Such points are known as *antinodes*.

We also see from equation (15.21) that at a time when $\sin \omega t = 1$, all the particles for which $\cos kx$ is positive reach their positive maximum displacement. At this particular instant, all the particles for which $\cos kx$ is negative, reach their negative maximum displacement. At a time when $\sin \omega t = 0$, all the particles cross their mean positions. Figure (15.14a) shows the change in the shape of the string as time passes. The time $t = 0$ in this figure corresponds to the instant when particles are at their maximum displacements. Figure (15.14b) shows the external appearance of the vibrating string. This type of wave is called a *standing wave* or a *stationary wave*. The particles at nodes do not move at all and the particles at the antinodes move with maximum amplitude.

It is clear that the separation between consecutive nodes or consecutive antinodes is $\lambda/2$. As the particles at the nodes do not move at all, energy cannot be transmitted across them. The main differences

between a standing wave and a travelling wave are summarised below.

1. In a travelling wave, the disturbance produced in a region propagates with a definite velocity but in a standing wave, it is confined to the region where it is produced.

2. In a travelling wave, the motion of all the particles are similar in nature. In a standing wave, different particles move with different amplitudes.

3. In a standing wave, the particles at nodes always remain in rest. In travelling waves, there is no particle which always remains in rest.

4. In a standing wave, all the particles cross their mean positions together. In a travelling wave, there is no instant when all the particles are at the mean positions together.

5. In a standing wave, all the particles between two successive nodes reach their extreme positions together, thus moving in phase. In a travelling wave, the phases of nearby particles are always different.

6. In a travelling wave, energy is transmitted from one region of space to other but in a standing wave, the energy of one region is always confined in that region.

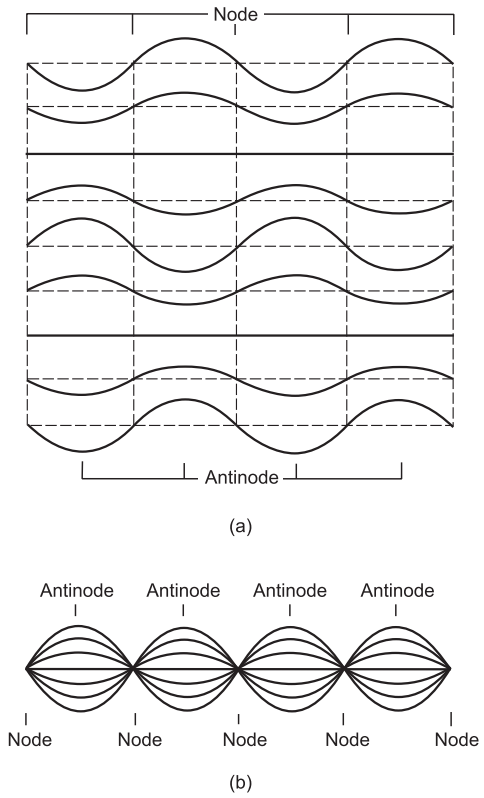



Figure 15.14

***Example 15.6***

*Two travelling waves of equal amplitudes and equal frequencies move in opposite directions along a string. They interfere to produce a standing wave having the equation*

$$y = A \cos kx \sin \omega t$$

*in which* $A = 1{\cdot}0$ mm, $k = 1{\cdot}57$ cm$^{-1}$ *and* $\omega = 78{\cdot}5$ s$^{-1}$. *(a) Find the velocity of the component travelling waves. (b) Find the node closest to the origin in the region* $x > 0$. *(c) Find the antinode closest to the origin in the region* $x > 0$. *(d) Find the amplitude of the particle at* $x = 2{\cdot}33$ cm.

***Solution*** : (a) The standing wave is formed by the superposition of the waves

$$y_1 = \frac{A}{2} \sin(\omega t - kx) \quad \text{and}$$

$$y_2 = \frac{A}{2} \sin(\omega t + kx).$$

The wave velocity (magnitude) of either of the waves is

$$v = \frac{\omega}{k} = \frac{78{\cdot}5 \text{ s}^{-1}}{1{\cdot}57 \text{ cm}^{-1}} = 50 \text{ cm s}^{-1}.$$

(b) For a node, $\cos kx = 0$.

The smallest positive $x$ satisfying this relation is given by

$$kx = \frac{\pi}{2}$$

or, $$x = \frac{\pi}{2k} = \frac{3{\cdot}14}{2 \times 1{\cdot}57 \text{ cm}^{-1}} = 1 \text{ cm}.$$

(c) For an antinode, $|\cos kx| = 1$.

The smallest positive $x$ satisfying this relation is given by

$$kx = \pi$$

or, $$x = \frac{\pi}{k} = 2 \text{ cm}.$$

(d) The amplitude of vibration of the particle at $x$ is given by $|A \cos kx|$. For the given point,

$$kx = (1{\cdot}57 \text{ cm}^{-1})\,(2{\cdot}33 \text{ cm}) = \frac{7}{6}\pi = \pi + \frac{\pi}{6}.$$

Thus, the amplitude will be

$$(1{\cdot}0 \text{ mm})\ |\cos(\pi + \pi/6)\ | = \frac{\sqrt{3}}{2} \text{ mm} = 0{\cdot}86 \text{ mm}.$$

## 15.10 STANDING WAVES ON A STRING FIXED AT BOTH ENDS (QUALITATIVE DISCUSSION)

Consider a string of length $L$ fixed at one end to a wall and the other end tied to a tuning fork which vibrates longitudinally with a small amplitude (figure 15.15). The fork produces sine waves of amplitude $A$ which travel on the string towards the fixed end and

get reflected from this end. The reflected waves which travel towards the fork are inverted in shape because they are reflected from a fixed end. These waves are again reflected from the fork. As the fork is heavy and vibrates longitudinally with a small amplitude, it acts like a fixed end and the waves reflected here are again inverted in shape. Therefore, the wave produced directly by the fork at this instant and the twice reflected wave have same shape, except that the twice reflected wave has already travelled a length $2L$.

Figure 15.15

Suppose the length of the string is such that $2L = \lambda$. The two waves interfere constructively and the resultant wave that proceeds towards right has an amplitude $2A$. This wave of amplitude $2A$ is again reflected by the wall and then by the fork. This twice reflected wave again interferes constructively with the oncoming new wave and a wave of amplitude $3A$ is produced. Thus, as time passes, the amplitude keeps on increasing. The string gets energy from the vibrations of the fork and the amplitude builds up. Same arguments hold if $2L$ is any integral multiple of $\lambda$ that is $L = n\lambda/2$, where $n$ is an integer.

In the above discussion, we have neglected any loss of energy due to air viscosity or due to lack of flexibility of string. In actual practice, energy is lost by several processes and the loss increases as the amplitude of vibration increases. Ultimately, a balance is reached when the rate of energy received from the fork equals the rate of energy lost due to various damping processes. In the steady state, waves of constant amplitude are present on the string from left to right as well as from right to left. These waves, propagating in opposite directions, produce standing waves on the string. Nodes and antinodes are formed and the amplitudes of vibration are large at antinodes. We say that the string is in resonance with the fork. The condition, $L = n\lambda/2$, for such a resonance may be stated in a different way. We have from equation (15.9)

$$v = \nu\lambda$$

or, $$\lambda = v/\nu.$$

The condition for resonance is, therefore,

$$L = n\frac{\lambda}{2}$$

or, $$L = \frac{n\,v}{2\,\nu}$$

or, $$\nu = \frac{n\,v}{2\,L} = \frac{n}{2\,L}\sqrt{F/\mu}. \qquad \ldots (15.22)$$

The lowest frequency with which a standing wave can be set up in a string fixed at both the ends is thus

$$\nu_0 = \frac{1}{2\,L}\sqrt{F/\mu}. \qquad \ldots (15.23)$$

This is called the *fundamental frequency* of the string. The other possible frequencies of standing waves are integral multiples of the fundamental frequency. The frequencies given by equation (15.22) are called the *natural frequencies, normal frequencies or resonant frequencies.*

***Example 15.7***

*A* 50 cm *long wire of mass* 20 g *supports a mass of* 1·6 kg *as shown in figure (15.16). Find the fundamental frequency of the portion of the string between the wall and the pulley. Take* $g = 10\ \text{m s}^{-2}$.

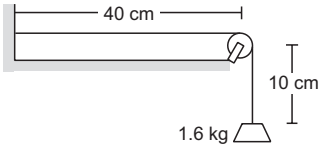



Figure 15.16

***Solution :*** The tension in the string is $F = (1{\cdot}6\ \text{kg})(10\ \text{m s}^{-2})$
$= 16\ \text{N}.$

The linear mass density is $\mu = \dfrac{20\ \text{g}}{50\ \text{cm}} = 0{\cdot}04\ \text{kg m}^{-1}$.

The fundamental frequency is

$$\nu_0 = \frac{1}{2\,L}\sqrt{F/\mu}$$

$$= \frac{1}{2 \times (0{\cdot}4\ \text{m})}\sqrt{\frac{16\ \text{N}}{0{\cdot}04\ \text{kg m}^{-1}}} = 25\ \text{Hz}.$$

What happens if the resonance condition (15.23) is not met? The phase difference between the twice reflected wave and the new wave is not an integral multiple of $2\,\pi$.

In fact, the phase difference with the new wave then depends on the number of reflections suffered by the original wave and hence, depends on time. At certain time instants, the amplitude is enhanced and at some other time instants, the amplitude is decreased. Thus, the average amplitude does not increase by interference and the vibrations are small. The string absorbs only a little amount of energy from the source.

## 15.11 ANALYTIC TREATMENT OF VIBRATION OF A STRING FIXED AT BOTH ENDS

Suppose a string of length $L$ is kept fixed at the ends $x = 0$ and $x = L$ and sine waves are produced on it. For certain wave frequencies, standing waves are set up in the string. Due to the multiple reflection at the ends and damping effects, waves going in the positive $x$-direction interfere to give a resultant wave

$$y_1 = A \sin(kx - \omega t).$$

Similarly, the waves going in the negative $x$-direction interfere to give the resultant wave

$$y_2 = A \sin(kx + \omega t + \delta).$$

The resultant displacement of the particle of the string at position $x$ and at time $t$ is given by the principle of superposition as

$$y = y_1 + y_2 = A \sin(kx - \omega t) + A \sin(kx + \omega t + \delta)$$

$$= 2A \sin\left(kx + \frac{\delta}{2}\right)\cos\left(\omega t + \frac{\delta}{2}\right). \quad \ldots \text{ (i)}$$

If standing waves are formed, the ends $x = 0$ and $x = L$ must be nodes because they are kept fixed. Thus, we have the boundary conditions

$y = 0$ at $x = 0$ for all $t$

and $y = 0$ at $x = L$ for all $t$.

The first boundary condition is satisfied by equation (i) if $\sin\frac{\delta}{2} = 0$,

or, $\delta = 0$.

Equation (i) then becomes

$$y = 2A \sin kx \cos \omega t. \quad \ldots \text{ (15.24)}$$

The second boundary condition will be satisfied if

$$\sin kL = 0$$

or, $kL = n\pi$, where $n = 1,2,3,4,5, \ldots$

or, $$\frac{2\pi L}{\lambda} = n\pi$$

or, $$L = \frac{n\lambda}{2}. \quad \ldots \text{ (15.25)}$$

If the length of the string is an integral multiple of $\lambda/2$, standing waves are produced. Again writing $\lambda = vT = \frac{v}{\nu}$, equation (15.25) becomes

$$\nu = \frac{n v}{2L} = \frac{n}{2L}\sqrt{F/\mu}$$

which is same as equation (15.22).

The lowest possible frequency is

$$\nu_0 = \frac{v}{2L} = \frac{1}{2L}\sqrt{F/\mu}. \quad \ldots \text{ (15.26)}$$

This is the fundamental frequency of the string.

The other natural frequencies with which standing waves can be formed on the string are

$$\nu_1 = 2\nu_0 = \frac{2}{2L}\sqrt{F/\mu}$$ 1st overtone, or 2nd harmonic,

$$\nu_2 = 3\nu_0 = \frac{3}{2L}\sqrt{F/\mu}$$ 2nd overtone, or 3rd harmonic,

$$\nu_3 = 4\nu_0 = \frac{4}{2L}\sqrt{F/\mu}$$ 3rd overtone, or 4th harmonic,

etc. In general, any integral multiple of the fundamental frequency is an allowed frequency. These higher frequencies are called *overtones*. Thus, $\nu_1 = 2\nu_0$ is the first overtone, $\nu_2 = 3\nu_0$ is the second overtone, etc. An integral multiple of a frequency is called its *harmonic*. Thus, *for a string fixed at both the ends, all the overtones are harmonics of the fundamental frequency and all the harmonics of the fundamental frequency are overtones.*

This property is unique to the string and makes it so valuable in musical instruments such as violin, guitar, sitar, santoor, sarod.

### Normal Modes of Vibration

When a string vibrates according to equation (15.24) with some natural frequency, it is said to vibrate in a *normal mode*. For the $n$th normal mode $k = \frac{n\pi}{L}$ and the equation for the displacement is, from equation (15.24),

$$y = 2A \sin\frac{n\pi x}{L}\cos \omega t. \quad \ldots \text{ (15.27)}$$

For fundamental mode, $n = 1$ and the equation of the standing wave is, from (15.27),

$$y = 2A \sin\frac{\pi x}{L}\cos \omega t.$$

The amplitude of vibration of the particle at $x$ is $2A \sin(\pi x/L)$ which is zero at $x = 0$ and at $x = L$. It is maximum at $x = L/2$ where $\sin(\pi x/L) = 1$. Thus, we have nodes at the ends and just one antinode at the middle point of the string.

In the first overtone, also known as the second harmonic, the constant $n$ is equal to 2 and equation(15.27) becomes

$$y = 2A \sin\frac{2\pi x}{L}\cos \omega t.$$

The amplitude $2A \sin\frac{2\pi x}{L}$ is zero at $x = 0$, $L/2$ and $L$ and is maximum at $L/4$ and $3L/4$. The middle point of the string is also a node and is not displaced during the vibration. The points $x = L/4$ and $x = 3L/4$ are the antinodes.

In the second overtone, $n = 3$ and equation(15.27) becomes

$$y = 2A \sin\frac{3\pi x}{L}\cos \omega t.$$

The nodes are at $x = 0$, $L/3$, $2L/3$ and $L$, where $\sin \frac{3\pi x}{L} = 0$. There are two nodes in between the ends. Antinodes occur midway between the nodes, i.e., at $x = L/6$, $L/2$ and $5L/6$.

Similarly, in the $n$th overtone, there are $n$ nodes between the ends and $n+1$ antinodes midway between the nodes. The shape of the string as it vibrates in a normal mode is shown in figure (15.17) for some of the normal modes.

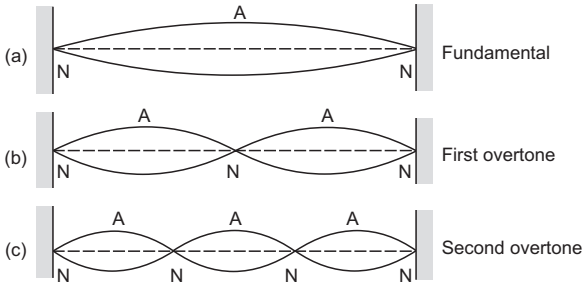



Figure 15.17

When the string of a musical instrument such as a sitar is plucked aside at some point, its shape does not correspond to any of the normal modes discussed above. In fact, the shape of the string is a combination of several normal modes and thus, a combination of frequencies are emitted.

## 15.12 VIBRATION OF A STRING FIXED AT ONE END

Standing waves can be produced on a string which is fixed at one end and whose other end is free to move in a transverse direction. Such a free end can be nearly achieved by connecting the string to a very light thread.

If the vibrations are produced by a source of "correct" frequency, standing waves are produced. If the end $x = 0$ is fixed and $x = L$ is free, the equation is again given by (15.24)

$$y = 2A \sin kx \cos \omega t$$

with the boundary condition that $x = L$ is an antinode. The boundary condition that $x = 0$ is a node is automatically satisfied by the above equation. For $x = L$ to be an antinode,

$$\sin kL = \pm 1$$

or, $$kL = \left(n + \frac{1}{2}\right)\pi$$

or, $$\frac{2\pi L}{\lambda} = \left(n + \frac{1}{2}\right)\pi$$

or, $$\frac{2L\nu}{v} = n + \frac{1}{2}$$

or, $$\nu = \left(n + \frac{1}{2}\right)\frac{v}{2L} = \frac{n + \frac{1}{2}}{2L}\sqrt{F/\mu}. \quad \ldots (15.28)$$

These are the normal frequencies of vibration. The fundamental frequency is obtained when $n = 0$, i.e.,

$$\nu_0 = v/4L.$$

The overtone frequencies are

$$\nu_1 = \frac{3v}{4L} = 3\nu_0,$$

$$\nu_2 = \frac{5v}{4L} = 5\nu_0,$$

$$\nu_3 = \frac{7v}{4L} = 7\nu_0, \quad \text{etc.}$$

We see that all the harmonics of the fundamental are not the allowed frequencies for the standing waves. Only the odd harmonics are the overtones. Figure (15.18) shows shapes of the string for some of the normal modes.

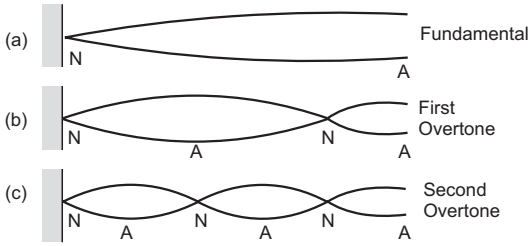



Figure 15.18

## 15.13 LAWS OF TRANSVERSE VIBRATIONS OF A STRING : SONOMETER

The fundamental frequency of vibration of a string fixed at both ends is given by equation (15.26). From this equation, one can immediately write the following statements known as "Laws of transverse vibrations of a string".

**(a) Law of length** – *The fundamental frequency of vibration of a string (fixed at both ends) is inversely proportional to the length of the string provided its tension and its mass per unit length remain the same.*

$\nu \propto 1/L$ if $F$ and $\mu$ are constants.

**(b) Law of tension** – *The fundamental frequency of a string is proportional to the square root of its tension provided its length and the mass per unit length remain the same.*

$\nu \propto \sqrt{F}$ if $L$ and $\mu$ are constants.

**(c) Law of mass** – *The fundamental frequency of a string is inversely proportional to the square root of the linear mass density, i.e., mass per unit length, provided the length and the tension remain the same.*

$$\nu \propto \frac{1}{\sqrt{\mu}} \text{ if } L \text{ and } F \text{ are constants.}$$

These laws may be experimentally studied with an apparatus called *sonometer*.

A typical design of a sonometer is shown in figure (15.19). One has a wooden box, also called the sound box, on which two bridges $A$ and $B$ are fixed at the ends. A metal wire $C$ is welded with the bridges and is kept tight. This wire $C$ is called the auxiliary wire. Another wire $D$, called the experimental wire is fixed at one end to the bridge $A$ and passes over the second bridge $B$ to hold a hanger $H$ on which suitable weights can be put. Two small movable bridges $C_1$ and $C_2$ may slide under the auxiliary wire and another two movable bridges $D_1$ and $D_2$ may slide under the experimental wire.

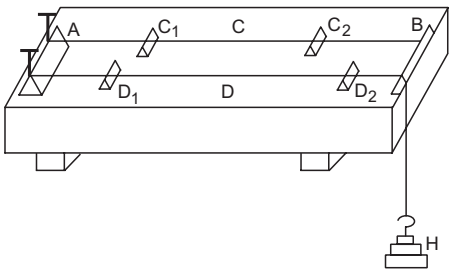



Figure 15.19

The portion of the wire between the movable bridges forms the "string" fixed at both ends. By sliding these bridges, the length of the wire may be changed. The tension of the experimental wire $D$ may be changed by changing the weights on the hanger. One can remove the experimental wire itself and put another wire in its place thereby changing the mass per unit length.

The waves can be produced on the wire by vibrating a tuning fork (by holding its stem and gently hitting a prong on a rubber pad) and pressing its stem on the platform of the sound box of the sonometer. The simple harmonic disturbance is transmitted to the wire through the bridges. The frequency of vibration is same as that of the tuning fork. If this frequency happens to be equal to one of the natural frequencies of the wire, standing waves with large amplitudes are set up on it. The tuning fork is then said to be in "resonance" or in "unison" with the wire.

How can one identify whether the tuning fork is in resonance with the wire or not ? A simple method is to place a small piece of paper (called a paper rider) at the middle point of the wire between the movable bridges. When vibrations in the wire are induced by putting the tuning fork in contact with the board, the paper-piece also vibrates. If the tuning fork is in resonance with the fundamental mode of vibration of the wire, the paper-piece is at the antinode. Because of the large amplitude of the wire there, it violently shakes and quite often jumps off the wire. Thus, the resonance can be detected just by visible inspection.

The paper-piece is also at an antinode if the wire is vibrating in its 3rd harmonic, although the amplitude will not be as large as it would be in the fundamental mode. The paper-piece may shake but not that violently.

Another good method to detect the resonance is based on the interference of sound waves of different frequencies. The tuning fork is sounded by gently hitting a prong on a rubber pad and the wire is plucked by hand. The resultant sound shows a periodic increase and decrease in intensity if the frequency of the fork is close (but not exactly equal) to one of the natural frequencies of the wire. This periodic variation in intensity is called *beats* that we shall study in the next chapter. The length is then only slightly varied till the beats disappear and that ensures resonance.

**Law of Length**

To study the law of length, only the experimental wire is needed. The wire is put under a tension by placing suitable weights (say 3 to 4 kg) on the hanger.

A tuning fork is vibrated and the length of the wire is adjusted by moving the movable bridges such that the fork is in resonance with the fundamental mode of vibration of the wire. The frequency $\nu$ of the tuning fork and the length $l$ of the wire resonating with it are noted. The experiment is repeated with different tuning forks and the product $\nu l$ is evaluated for each fork which should be a constant by the law of length.

**Law of Tension**

To study the law of tension, one may proceed as follows. A particular length of the experimental wire is selected by keeping the movable bridges $D_1$, $D_2$ fixed. The auxiliary wire is plucked. The vibration is transmitted to the experimental wire through the sound box. By adjusting the movable bridges $C_1$ and $C_2$, the fundamental frequency of the auxiliary wire is made equal to the fundamental frequency of the experimental wire by testing that the two wires resonate with each other. The tension in the experimental wire is changed and the length of the auxiliary wire is again adjusted to resonate with it. The experiment is repeated several times with different tensions and the corresponding lengths of the auxiliary wire are noted. Suppose $l'$ represents the length of the auxiliary wire resonating with the fixed length of the experimental wire when the tension in it is $T$. Also suppose $\nu$ is the frequency of vibration of

the wires in their fundamental modes in this situation. Then,

$$\nu \propto \frac{1}{l'} \text{ according to the law of length}$$

and $\nu \propto \sqrt{T}$ according to the law of tension.

Hence, $l' \propto 1/\sqrt{T}$.

The product $l'\sqrt{T}$ may be evaluated from the experiments which should be a constant.

Why do we have to use the auxiliary wire in the above scheme and not a tuning fork ? That is because, to adjust for the resonance, the variable quantity should be continuously changeable. As the length of the experimental wire is kept fixed and its frequency is to be compared as a function of tension, we need a source whose frequency can be continuously changed. Choosing different tuning forks to change the frequency will not work as the forks are available for discrete frequencies only.

**Law of Mass**

To study the law of mass, the length and the tension are to be kept constant and the mass per unit length is to be changed. Again, the auxiliary wire is used to resonate with the fixed length of the experimental wire as was suggested during the study of the law of tension. A fixed length of the experimental wire is chosen between the bridges $D_1$ and $D_2$ and a fixed tension is applied to it. The auxiliary wire is given a tension by hanging a certain load and its length is adjusted so that it resonates with the experimental wire. The experiment is repeated with different experimental wires keeping equal lengths between the movable bridges and applying equal tension. Each time the length $l'$ of the auxiliary wire is adjusted to bring it in resonance with the experimental wire. The mass per unit length of each experimental wire is obtained by weighing a known length of the wire. We have

$$\nu \propto 1/l' \quad \text{according to the law of length}$$

and $\nu \propto 1/\sqrt{\mu}$ according to the law of mass.

Thus, $l' \propto \sqrt{\mu}$.

The law of mass is thus studied by obtaining $\frac{l'}{\sqrt{\mu}}$ each time which should be a constant.

***Example 15.8***

*In a sonometer experiment, resonance is obtained when the experimental wire has a length of* 21 cm *between the bridges and the vibrations are excited by a tuning fork of frequency* 256 Hz. *If a tuning fork of frequency* 384 Hz *is used, what should be the length of the experimental wire to get the resonance.*

***Solution*** : By the law of length, $l_1 \nu_1 = l_2 \nu_2$

or, $$l_2 = \frac{\nu_1}{\nu_2} l_1 = \frac{256}{384} \times 21 \text{ cm} = 14 \text{ cm}.$$

## 15.14 TRANSVERSE AND LONGITUDINAL WAVES

The wave on a string is caused by the displacements of the particles of the string. These displacements are in a direction perpendicular to the direction of propagation of the wave. If the disturbance produced in a wave has a direction perpendicular to the direction of propagation of the wave, the wave is called a *transverse wave*. The wave on a string is a transverse wave. Another example of transverse wave is the light wave. It is the electric field which changes its value with space and time and the changes are propagated in space. The direction of the electric field is perpendicular to the direction of propagation of light when light travels in free space.

Sound waves are not transverse. The particles of the medium are pushed and pulled along the direction of propagation of sound. We shall study in some detail the mechanism of sound waves in the next chapter. If the disturbance produced as the wave passes is along the direction of the wave propagation, the wave is called a *longitudinal wave*. Sound waves are longitudinal.

All the waves cannot be characterised as either longitudinal or transverse. A very common example of a wave that is neither longitudinal nor transverse is a wave on the surface of water. When water in a steady lake is disturbed by shaking a finger in it, waves are produced on the water surface. The water particles move in elliptic or circular path as the wave passes through them. The elliptic motion has components both along and perpendicular to the direction of propagation of the wave.

## 15.15 POLARIZATION OF WAVES

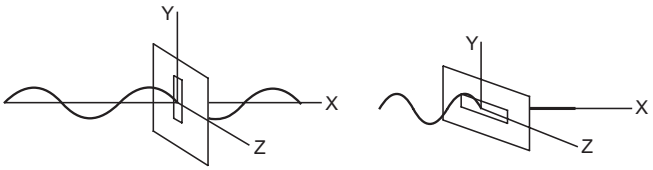



Figure 15.20

Suppose a stretched string goes through a slit made in a cardboard which is placed perpendicular to the string (figure 15.20). If we take the $X$-axis along the string, the cardboard will be in $Y$-$Z$ plane. Suppose the particles of the string are displaced in $y$-direction as the wave passes. If the slit in the cardboard is also along the $Y$-axis, the part of the string in the slit can

vibrate freely in the slit and the wave will pass through the slit. However, if the cardboard is rotated by 90° in its plane, the slit will point along the $Z$-axis. As the wave arrives at the slit, the part of the string in it tries to move along the $Y$-axis but the contact force by the cardboard does not allow it. The wave is not able to pass through the slit. If the slit is inclined to the $Y$-axis at some other angle, only a part of the wave is transmitted and in the transmitted wave the disturbance is produced parallel to the slit. Figure (15.21) suggests the same arrangement with two chairs.

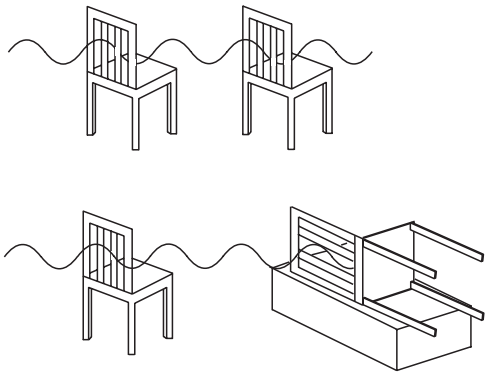


Figure 15.21

If the disturbance produced is always along a fixed direction, we say that the wave is linearly polarized in that direction. The waves considered in this chapter are linearly polarized in $y$-direction. Similarly, if a wave produces displacement along the $z$-direction, its equation is given by $z = A \sin \omega(t - x/v)$ and it is a linearly polarized wave, polarized in $z$-direction. Linearly polarized waves are also called *plane polarized.*

If each particle of the string moves in a small circle as the wave passes through it, the wave is called *circularly polarized.* If each particle goes in ellipse, the wave is called *elliptically polarized.*

Finally, if the particles are randomly displaced in the plane perpendicular to the direction of propagation, the wave is called *unpolarized.*

A circularly polarized or unpolarized wave passing through a slit does not show change in intensity as the slit is rotated in its plane. But the transmitted wave becomes linearly polarized in the direction parallel to the slit.

### *Worked Out Examples*

**1**. *The displacement of a particle of a string carrying a travelling wave is given by*

$$y = (3{\cdot}0 \text{ cm}) \sin 6{\cdot}28(0{\cdot}50x - 50\,t),$$

*where $x$ is in centimetre and $t$ in second. Find (a) the amplitude, (b) the wavelength, (c) the frequency and (d) the speed of the wave.*

***Solution*** : Comparing with the standard wave equation

$$y = A\sin(kx - \omega t)$$

$$= A\sin 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)$$

we see that,

amplitude $= A = 3{\cdot}0$ cm,

wavelength $= \lambda = \dfrac{1}{0{\cdot}50}$ cm $= 2{\cdot}0$ cm,

and the frequency $= \nu = \dfrac{1}{T} = 50$ Hz.

The speed of the wave is $v = \nu\lambda$

$$= (50 \text{ s}^{-1})\,(2{\cdot}0 \text{ cm})$$

$$= 100 \text{ cm s}^{-1}.$$

**2.** *The equation for a wave travelling in x-direction on a string is*

$$y = (3{\cdot}0 \text{ cm}) \sin[(3{\cdot}14 \text{ cm}^{-1})\,x - (314 \text{ s}^{-1})t].$$

*(a) Find the maximum velocity of a particle of the string.*

*(b) Find the acceleration of a particle at $x = 6{\cdot}0$ cm at time $t = 0{\cdot}11$ s.*

***Solution*** :

(a) The velocity of the particle at $x$ at time $t$ is

$$v = \frac{\partial y}{\partial t} = (3{\cdot}0 \text{ cm})\,(-314 \text{ s}^{-1}) \cos[(3{\cdot}14 \text{ cm}^{-1})\,x - (314 \text{ s}^{-1})t]$$

$$= (-9{\cdot}4 \text{ m s}^{-1}) \cos[(3{\cdot}14 \text{ cm}^{-1})\,x - (314 \text{ s}^{-1})t].$$

The maximum velocity of a particle will be

$$v = 9{\cdot}4 \text{ m s}^{-1}.$$

(b) The acceleration of the particle at $x$ at time $t$ is

$$a = \frac{\partial v}{\partial t} = -(9{\cdot}4 \text{ m s}^{-1})\,(314 \text{ s}^{-1}) \sin[(3{\cdot}14 \text{ cm}^{-1})\,x - (314 \text{ s}^{-1})t]$$

$$= -(2952 \text{ m s}^{-2}) \sin[(3{\cdot}14 \text{ cm}^{-1})\,x - (314 \text{ s}^{-1})t].$$

The acceleration of the particle at $x = 6{\cdot}0$ cm at time $t = 0{\cdot}11$ s is $a = -(2952 \text{ m s}^{-2}) \sin[6\pi - 11\pi] = 0$.

**3.** *A long string having a cross-sectional area* $0{\cdot}80\ \text{mm}^2$ *and density* $12{\cdot}5\ \text{g cm}^{-3}$*, is subjected to a tension of* 64 N *along the X-axis. One end of this string is attached to a vibrator moving in transverse direction at a frequency of* 20 Hz*. At* $t = 0$*, the source is at a maximum displacement* $y = 1{\cdot}0$ cm*. (a) Find the speed of the wave travelling on the string. (b) Write the equation for the wave. (c) What is the displacement of the particle of the string at* $x = 50$ cm *at time* $t = 0{\cdot}05$ *s ? (d) What is the velocity of this particle at this instant ?*

***Solution*** **:**

(a) The mass of 1 m long part of the string is

$$m = (0{\cdot}80\ \text{mm}^2) \times (1\ \text{m}) \times (12{\cdot}5\ \text{g cm}^{-3})$$
$$= (0{\cdot}80 \times 10^{-6}\ \text{m}^3) \times (12{\cdot}5 \times 10^3\ \text{kg m}^{-3})$$
$$= 0{\cdot}01\ \text{kg}.$$

The linear mass density is $\mu = 0{\cdot}01\ \text{kg m}^{-1}$. The wave speed is $v = \sqrt{F/\mu}$

$$= \sqrt{\frac{64\ \text{N}}{0{\cdot}01\ \text{kg m}^{-1}}} = 80\ \text{m s}^{-1}.$$

(b) The amplitude of the source is $A = 1{\cdot}0$ cm and the frequency is $\nu = 20$ Hz. The angular frequency is $\omega = 2\pi\nu = 40\pi\ \text{s}^{-1}$. Also at $t = 0$, the displacement is equal to its amplitude, i.e., at $t = 0, y = A$. The equation of motion of the source is, therefore,

$$y = (1{\cdot}0\ \text{cm}) \cos[(40\pi\ \text{s}^{-1})\, t] \qquad \ldots\ (\text{i})$$

The equation of the wave travelling on the string along the positive $X$-axis is obtained by replacing $t$ with $t - x/v$ in equation (i). It is, therefore,

$$y = (1{\cdot}0\ \text{cm}) \cos\left[(40\pi\ \text{s}^{-1})\left(t - \frac{x}{v}\right)\right]$$
$$= (1{\cdot}0\ \text{cm}) \cos\left[(40\pi\ \text{s}^{-1})t - \left(\frac{\pi}{2}\ \text{m}^{-1}\right)x\right], \qquad \ldots\ (\text{ii})$$

where the value of $v$ has been put from part (a).

(c) The displacement of the particle at $x = 50$ cm at time $t = 0{\cdot}05$ s is by equation (ii),

$$y = (1{\cdot}0\ \text{cm}) \cos[(40\pi\ \text{s}^{-1})\ (0{\cdot}05\ \text{s}) - \left(\frac{\pi}{2}\ \text{m}^{-1}\right)(0{\cdot}5\ \text{m})]$$
$$= (1{\cdot}0\ \text{cm}) \cos\left[2\pi - \frac{\pi}{4}\right]$$
$$= \frac{1{\cdot}0\ \text{cm}}{\sqrt{2}} = 0{\cdot}71\ \text{cm}.$$

(d) The velocity of the particle at position $x$ at time $t$ is, by equation (ii),

$$v = \frac{\partial y}{\partial t} = -\ (1{\cdot}0\ \text{cm})\ (40\pi\ \text{s}^{-1}) \sin\left[(40\pi\ \text{s}^{-1})\, t - \left(\frac{\pi}{2}\ \text{m}^{-1}\right)x\right].$$

Putting the values of $x$ and $t$,

$$v = -\ (40\pi\ \text{cm s}^{-1}) \sin\left(2\pi - \frac{\pi}{4}\right)$$
$$= \frac{40\pi}{\sqrt{2}}\ \text{cm s}^{-1} \approx 89\ \text{cm s}^{-1}.$$

**4.** *The speed of a transverse wave, going on a wire having a length* 50 cm *and mass* 5·0 g, is $80\ \text{m s}^{-1}$*. The area of cross section of the wire is* $1{\cdot}0\ \text{mm}^2$ *and its Young modulus is* $16 \times 10^{11}\ \text{N m}^{-2}$*. Find the extension of the wire over its natural length.*

***Solution*** **:** The linear mass density is

$$\mu = \frac{5 \times 10^{-3}\ \text{kg}}{50 \times 10^{-2}\ \text{m}} = 1{\cdot}0 \times 10^{-2}\ \text{kg m}^{-1}.$$

The wave speed is $v = \sqrt{F/\mu}$.

Thus, the tension is $F = \mu v^2$

$$= \left(1{\cdot}0 \times 10^{-2}\ \text{kg m}^{-1}\right) \times 6400\ \text{m}^2\ \text{s}^{-2} = 64\ \text{N}.$$

The Young modulus is given by

$$Y = \frac{F/A}{\Delta L/L}.$$

The extension is, therefore,

$$\Delta L = \frac{FL}{AY}$$
$$= \frac{(64\ \text{N})\ (0{\cdot}50\ \text{m})}{(1{\cdot}0 \times 10^{-6}\ \text{m}^2) \times (16 \times 10^{11}\ \text{N m}^{-2})} = 0{\cdot}02\ \text{mm}.$$

**5.** *A uniform rope of length* 12 m *and mass* 6 kg *hangs vertically from a rigid support. A block of mass* 2 kg *is attached to the free end of the rope. A transverse pulse of wavelength* 0·06 m *is produced at the lower end of the rope. What is the wavelength of the pulse when it reaches the top of the rope ?*

***Solution*** **:** As the rope is heavy, its tension will be different at different points. The tension at the free end will be $(2\ \text{kg})g$ and that at the upper end it will be $(8\ \text{kg})g$.

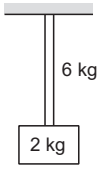



Figure 15-W1

We have, $v = \nu\lambda$

or, $\sqrt{F/\mu} = \nu\lambda$

or, $\sqrt{F}/\lambda = \nu\sqrt{\mu}.$ … (i)

The frequency of the wave pulse will be the same everywhere on the rope as it depends only on the frequency of the source. The mass per unit length is also the same throughout the rope as it is uniform. Thus, by (i), $\frac{\sqrt{F}}{\lambda}$ is constant.

Hence, $$\frac{\sqrt{(2\ \text{kg})g}}{0{\cdot}06\ \text{m}} = \frac{\sqrt{(8\ \text{kg})g}}{\lambda_1},$$

where $\lambda_1$ is the wavelength at the top of the rope. This gives $\lambda_1 = 0{\cdot}12$ m.

**6.** *Two waves passing through a region are represented by*

$$y = (1{\cdot}0 \text{ cm}) \sin[(3{\cdot}14 \text{ cm}^{-1})x - (157 \text{ s}^{-1})t]$$

*and* $y = (1{\cdot}5 \text{ cm}) \sin[(1{\cdot}57 \text{ cm}^{-1})x - (314 \text{ s}^{-1})t]$.

*Find the displacement of the particle at* $x = 4{\cdot}5$ cm *at time* $t = 5{\cdot}0$ ms.

***Solution*** **:** According to the principle of superposition, each wave produces its disturbance independent of the other and the resultant disturbance is equal to the vector sum of the individual disturbances. The displacements of the particle at $x = 4{\cdot}5$ cm at time $t = 5{\cdot}0$ ms due to the two waves are,

$$y_1 = (1{\cdot}0 \text{ cm}) \sin[(3{\cdot}14 \text{ cm}^{-1})(4{\cdot}5 \text{ cm}) - (157 \text{ s}^{-1})(5{\cdot}0 \times 10^{-3} \text{ s})]$$

$$= (1{\cdot}0 \text{ cm}) \sin\left[4{\cdot}5\pi - \frac{\pi}{4}\right]$$

$$= (1{\cdot}0 \text{ cm}) \sin[4\pi + \pi/4] = \frac{1{\cdot}0 \text{ cm}}{\sqrt{2}}$$

and

$$y_2 = (1{\cdot}5 \text{ cm}) \sin[(1{\cdot}57 \text{ cm}^{-1})(4{\cdot}5 \text{ cm}) - (314 \text{ s}^{-1})(5{\cdot}0 \times 10^{-3} \text{ s})]$$

$$= (1{\cdot}5 \text{ cm}) \sin\left[2{\cdot}25\pi - \frac{\pi}{2}\right]$$

$$= (1{\cdot}5 \text{ cm}) \sin[2\pi - \pi/4]$$

$$= -(1{\cdot}5 \text{ cm}) \sin\frac{\pi}{4} = -\frac{1{\cdot}5 \text{ cm}}{\sqrt{2}}.$$

The net displacement is

$$y = y_1 + y_2 = \frac{-0{\cdot}5 \text{ cm}}{\sqrt{2}} = -0{\cdot}35 \text{ cm}.$$

**7.** *The vibrations of a string fixed at both ends are described by the equation*

$$y = (5{\cdot}00 \text{ mm}) \sin[(1{\cdot}57 \text{ cm}^{-1})x] \sin[(314 \text{ s}^{-1})t]$$

*(a) What is the maximum displacement of the particle at* $x = 5{\cdot}66$ cm ? *(b) What are the wavelengths and the wave speeds of the two transverse waves that combine to give the above vibration* ? *(c) What is the velocity of the particle at* $x = 5{\cdot}66$ cm *at time* $t = 2{\cdot}00$ s ? *(d) If the length of the string is* 10·0 cm, *locate the nodes and the antinodes. How many loops are formed in the vibration* ?

***Solution*** **:**

(a) The amplitude of the vibration of the particle at position $x$ is

$$A = |(5{\cdot}00 \text{ mm}) \sin[(1{\cdot}57 \text{ cm}^{-1})x]|$$

For $x = 5{\cdot}66$ cm,

$$A = \left|(5{\cdot}00 \text{ mm}) \sin\left[\frac{\pi}{2} \times 5{\cdot}66\right]\right|$$

$$= \left|(5{\cdot}00 \text{ mm}) \sin\left(2{\cdot}5\,\pi + \frac{\pi}{3}\right)\right|$$

$$= \left|(5{\cdot}00 \text{ mm}) \cos\frac{\pi}{3}\right| = 2{\cdot}50 \text{ mm}.$$

(b) From the given equation, the wave number $k = 1{\cdot}57 \text{ cm}^{-1}$ and the angular frequency $\omega = 314 \text{ s}^{-1}$. Thus, the wavelength is

$$\lambda = \frac{2\pi}{k} = \frac{2 \times 3{\cdot}14}{1{\cdot}57 \text{ cm}^{-1}} = 4{\cdot}00 \text{ cm}$$

and the frequency is $\nu = \frac{\omega}{2\pi} = \frac{314 \text{ s}^{-1}}{2 \times 3{\cdot}14} = 50 \text{ s}^{-1}$.

The wave speed is $v = \nu\lambda = (50 \text{ s}^{-1})(4{\cdot}00 \text{ cm}) = 2{\cdot}00 \text{ m s}^{-1}$.

(c) The velocity of the particle at position $x$ at time $t$ is given by

$$v = \frac{\partial y}{\partial t} = (5{\cdot}00 \text{ mm}) \sin[(1{\cdot}57 \text{ cm}^{-1})x] [314 \text{ s}^{-1} \cos(314 \text{ s}^{-1})t]$$

$$= (157 \text{ cm s}^{-1}) \sin(1{\cdot}57 \text{ cm}^{-1})x \cos(314 \text{ s}^{-1})t.$$

Putting $x = 5{\cdot}66$ cm and $t = 2{\cdot}00$ s, the velocity of this particle at the given instant is

$$(157 \text{ cm s}^{-1}) \sin\left(\frac{5\pi}{2} + \frac{\pi}{3}\right)\cos(200\,\pi)$$

$$= (157 \text{ cm s}^{-1}) \times \cos\frac{\pi}{3} \times 1 = 78{\cdot}5 \text{ cm s}^{-1}.$$

(d) The nodes occur where the amplitude is zero, i.e.,

$$\sin(1{\cdot}57 \text{ cm}^{-1})x = 0.$$

or, $$\left(\frac{\pi}{2} \text{ cm}^{-1}\right)x = n\pi,$$

where $n$ is an integer.

Thus, $$x = 2n \text{ cm}.$$

The nodes, therefore, occur at $x = 0$, 2 cm, 4 cm, 6 cm, 8 cm and 10 cm. Antinodes occur in between them, i.e., at $x = 1$ cm, 3 cm, 5 cm, 7 cm and 9 cm. The string vibrates in 5 loops.

**8.** *A guitar string is* 90 cm *long and has a fundamental frequency of* 124 Hz. *Where should it be pressed to produce a fundamental frequency of* 186 Hz ?

***Solution*** **:** The fundamental frequency of a string fixed at both ends is given by

$$\nu = \frac{1}{2L}\sqrt{\frac{F}{\mu}}.$$

As $F$ and $\mu$ are fixed, $\frac{\nu_1}{\nu_2} = \frac{L_2}{L_1}$

or, $$L_2 = \frac{\nu_1}{\nu_2} L_1 = \frac{124 \text{ Hz}}{186 \text{ Hz}} (90 \text{ cm}) = 60 \text{ cm}.$$

Thus, the string should be pressed at 60 cm from an end.

**9.** *A sonometer wire has a total length of* 1 m *between the fixed ends. Where should the two bridges be placed below the wire so that the three segments of the wire have their fundamental frequencies in the ratio* 1 : 2 : 3 ?

***Solution*** **:** Suppose the lengths of the three segments are $L_1, L_2$ and $L_3$ respectively. The fundamental frequencies are

$$\nu_1 = \frac{1}{2L_1}\sqrt{F/\mu}$$

$$\nu_2 = \frac{1}{2L_2}\sqrt{F/\mu}$$

$$\nu_3 = \frac{1}{2L_3}\sqrt{F/\mu}$$

so that $\nu_1 L_1 = \nu_2 L_2 = \nu_3 L_3$. … (i)

As $\nu_1 : \nu_2 : \nu_3 = 1 : 2 : 3$, we have

$\nu_2 = 2\,\nu_1$ and $\nu_3 = 3\,\nu_1$ so that by (i)

$$L_2 = \frac{\nu_1}{\nu_2} L_1 = \frac{L_1}{2}$$

and $$L_3 = \frac{\nu_1}{\nu_3} L_1 = \frac{L_1}{3}.$$

As $L_1 + L_2 + L_3 = 1$ m,

we get $$L_1\left(1 + \frac{1}{2} + \frac{1}{3}\right) = 1 \text{ m}$$

or, $$L_1 = \frac{6}{11} \text{ m}.$$

Thus, $$L_2 = \frac{L_1}{2} = \frac{3}{11} \text{ m}$$

and $$L_3 = \frac{L_1}{3} = \frac{2}{11} \text{ m}.$$

One bridge should be placed at $\frac{6}{11}$ m from one end and the other should be placed at $\frac{2}{11}$ m from the other end.

**10.** *A wire having a linear mass density* $5{\cdot}0 \times 10^{-3}$ kg m$^{-1}$ *is stretched between two rigid supports with a tension of* 450 N. *The wire resonates at a frequency of* 420 Hz. *The next higher frequency at which the same wire resonates is* 490 Hz. *Find the length of the wire.*

***Solution*** **:** Suppose the wire vibrates at 420 Hz in its $n$th harmonic and at 490 Hz in its $(n + 1)$th harmonic.

$$420 \text{ s}^{-1} = \frac{n}{2L}\sqrt{F/\mu} \quad \text{… (i)}$$

and $$490 \text{ s}^{-1} = \frac{(n+1)}{2L}\sqrt{F/\mu}. \quad \text{… (ii)}$$

This gives $$\frac{490}{420} = \frac{n+1}{n}$$

or, $$n = 6.$$

Putting the value in (i),

$$420 \text{ s}^{-1} = \frac{6}{2L}\sqrt{\frac{450 \text{ N}}{5{\cdot}0 \times 10^{-3} \text{ kg m}^{-1}}} = \frac{900}{L} \text{ m s}^{-1}$$

or, $$L = \frac{900}{420} \text{ m} = 2{\cdot}1 \text{ m}.$$

□

## QUESTIONS FOR SHORT ANSWER

**1**. You are walking along a seashore and a mild wind is blowing. Is the motion of air a wave motion ?

**2.** The radio and TV programmes, telecast at the studio, reach our antenna by wave motion. Is it a mechanical wave or nonmechanical ?

**3.** A wave is represented by an equation $y = c_1 \sin(c_2 x + c_3 t)$. In which direction is the wave going ? Assume that $c_1, c_2$ and $c_3$ are all positive.

**4.** Show that the particle speed can never be equal to the wave speed in a sine wave if the amplitude is less than wavelength divided by $2\pi$.

**5.** Two wave pulses identical in shape but inverted with respect to each other are produced at the two ends of a stretched string. At an instant when the pulses reach the middle, the string becomes completely straight. What happens to the energy of the two pulses ?

**6.** Show that for a wave travelling on a string

$$\frac{y_{max}}{v_{max}} = \frac{v_{max}}{a_{max}},$$

where the symbols have usual meanings. Can we use componendo and dividendo taught in algebra to write

$$\frac{y_{max} + v_{max}}{y_{max} - v_{max}} = \frac{v_{max} + a_{max}}{v_{max} - a_{max}} \ ?$$

**7.** What is the smallest positive phase constant which is equivalent to $7{\cdot}5\,\pi$ ?

**8.** A string clamped at both ends vibrates in its fundamental mode. Is there any position (except the ends) on the string which can be touched without disturbing the motion ? What if the string vibrates in its first overtone ?

## OBJECTIVE I

**1.** A sine wave is travelling in a medium. The minimum distance between the two particles, always having same speed, is
(a) $\lambda/4$ (b) $\lambda/3$ (c) $\lambda/2$ (d) $\lambda$.

**2.** A sine wave is travelling in a medium. A particular particle has zero displacement at a certain instant. The particle closest to it having zero displacement is at a distance
(a) $\lambda/4$ (b) $\lambda/3$ (c) $\lambda/2$ (d) $\lambda$.

**3.** Which of the following equations represents a wave travelling along $Y$-axis ?
(a) $x = A \sin(ky - \omega t)$ (b) $y = A \sin(kx - \omega t)$
(c) $y = A \sin ky \cos \omega t$ (d) $y = A \cos ky \sin \omega t$.

**4.** The equation $y = A \sin^2(kx - \omega t)$ represents a wave motion with
(a) amplitude $A$, frequency $\omega/2\pi$
(b) amplitude $A/2$, frequency $\omega/\pi$
(c) amplitude $2A$, frequency $\omega/4\pi$
(d) does not represent a wave motion.

**5.** Which of the following is a mechanical wave ?
(a) Radio waves (b) $X$-rays
(c) Light waves (d) Sound waves.

**6.** A cork floating in a calm pond executes simple harmonic motion of frequency $\nu$ when a wave generated by a boat passes by it. The frequency of the wave is
(a) $\nu$ (b) $\nu/2$ (c) $2\nu$ (d) $\sqrt{2}\nu$.

**7.** Two strings $A$ and $B$, made of same material, are stretched by same tension. The radius of string $A$ is double of the radius of $B$. A transverse wave travels on $A$ with speed $v_A$ and on $B$ with speed $v_B$. The ratio $v_A/v_B$ is
(a) 1/2 (b) 2 (c) 1/4 (d) 4.

**8.** Both the strings, shown in figure (15-Q1), are made of same material and have same cross section. The pulleys are light. The wave speed of a transverse wave in the string $AB$ is $v_1$ and in $CD$ it is $v_2$. Then $v_1/v_2$ is
(a) 1 (b) 2 (c) $\sqrt{2}$ (d) $1/\sqrt{2}$.

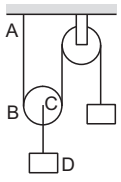



Figure 15-Q1

**9.** Velocity of sound in air is 332 m s$^{-1}$. Its velocity in vacuum will be
(a) $> 332$ m s$^{-1}$ (b) $= 332$ m s$^{-1}$
(c) $< 332$ m s$^{-1}$ (d) meaningless.

**10.** A wave pulse, travelling on a two-piece string, gets partially reflected and partially transmitted at the junction. The reflected wave is inverted in shape as compared to the incident one. If the incident wave has wavelength $\lambda$ and the transmitted wave $\lambda'$,
(a) $\lambda' > \lambda$ (b) $\lambda' = \lambda$ (c) $\lambda' < \lambda$
(d) nothing can be said about the relation of $\lambda$ and $\lambda'$.

**11.** Two waves represented by $y = a \sin(\omega t - kx)$ and $y = a \cos(\omega t - kx)$ are superposed. The resultant wave will have an amplitude
(a) $a$ (b) $\sqrt{2}a$ (c) $2a$ (d) 0.

**12.** Two wires $A$ and $B$, having identical geometrical construction, are stretched from their natural length by small but equal amount. The Young modulus of the wires are $Y_A$ and $Y_B$ whereas the densities are $\rho_A$ and $\rho_B$. It is given that $Y_A > Y_B$ and $\rho_A > \rho_B$. A transverse signal started at one end takes a time $t_1$ to reach the other end for $A$ and $t_2$ for $B$.
(a) $t_1 < t_2$ (b) $t_1 = t_2$ (c) $t_1 > t_2$
(d) the information is insufficient to find the relation between $t_1$ and $t_2$.

**13.** Consider two waves passing through the same string. Principle of superposition for displacement says that the net displacement of a particle on the string is sum of the displacements produced by the two waves individually. Suppose we state similar principles for the net velocity of the particle and the net kinetic energy of the particle. Such a principle will be valid for
(a) both the velocity and the kinetic energy
(b) the velocity but not for the kinetic energy
(c) the kinetic energy but not for the velocity
(d) neither the velocity nor the kinetic energy.

**14.** Two wave pulses travel in opposite directions on a string and approach each other. The shape of one pulse is inverted with respect to the other.
(a) The pulses will collide with each other and vanish after collision.
(b) The pulses will reflect from each other, i.e., the pulse going towards right will finally move towards left and vice versa.
(c) The pulses will pass through each other but their shapes will be modified.
(d) The pulses will pass through each other without any change in their shapes.

**15.** Two periodic waves of amplitudes $A_1$ and $A_2$ pass through a region. If $A_1 > A_2$, the difference in the maximum and minimum resultant amplitude possible is
(a) $2A_1$ (b) $2A_2$ (c) $A_1 + A_2$ (d) $A_1 - A_2$.

**16.** Two waves of equal amplitude $A$, and equal frequency travel in the same direction in a medium. The amplitude of the resultant wave is
(a) 0 (b) $A$ (c) $2A$ (d) between 0 and $2A$.

**17.** Two sine waves travel in the same direction in a medium. The amplitude of each wave is $A$ and the phase difference between the two waves is 120°. The resultant amplitude will be
(a) $A$ (b) $2A$ (c) $4A$ (d) $\sqrt{2}A$.

**18.** The fundamental frequency of a string is proportional to

(a) inverse of its length (b) the diameter
(c) the tension (d) the density.

**19.** A tuning fork of frequency 480 Hz is used to vibrate a sonometer wire having natural frequency 240 Hz. The wire will vibrate with a frequency of
(a) 240 Hz (b) 480 Hz
(c) 720 Hz (d) will not vibrate.

**20.** A tuning fork of frequency 480 Hz is used to vibrate a sonometer wire having natural frequency 410 Hz. The wire will vibrate with a frequency
(a) 410 Hz (b) 480 Hz (c) 820 Hz (d) 960 Hz.

**21.** A sonometer wire of length $l$ vibrates in fundamental mode when excited by a tuning fork of frequency 416 Hz. If the length is doubled keeping other things same, the string will
(a) vibrate with a frequency of 416 Hz
(b) vibrate with a frequency of 208 Hz
(c) vibrate with a frequency of 832 Hz
(d) stop vibrating.

**22.** A sonometer wire supports a 4 kg load and vibrates in fundamental mode with a tuning fork of frequency 416 Hz. The length of the wire between the bridges is now doubled. In order to maintain fundamental mode, the load should be changed to
(a) 1 kg (b) 2 kg (c) 8 kg (d) 16 kg.

## OBJECTIVE II

**1.** A mechanical wave propagates in a medium along the $X$-axis. The particles of the medium
(a) must move on the $X$-axis
(b) must move on the $Y$-axis
(c) may move on the $X$-axis
(d) may move on the $Y$-axis.

**2.** A transverse wave travels along the $Z$-axis. The particles of the medium must move
(a) along the $Z$-axis (b) along the $X$-axis
(c) along the $Y$-axis (d) in the $X$-$Y$ plane.

**3.** Longitudinal waves cannot
(a) have a unique wavelength (b) transmit energy
(c) have a unique wave velocity (d) be polarized.

**4.** A wave going in a solid
(a) must be longitudinal (b) may be longitudinal
(c) must be transverse (d) may be transverse.

**5.** A wave moving in a gas
(a) must be longitudinal (b) may be longitudinal
(c) must be transverse (d) may be transverse.

**6.** Two particles $A$ and $B$ have a phase difference of $\pi$ when a sine wave passes through the region.
(a) $A$ oscillates at half the frequency of $B$.
(b) $A$ and $B$ move in opposite directions.
(c) $A$ and $B$ must be separated by half of the wavelength.
(d) The displacements at $A$ and $B$ have equal magnitudes.

**7.** A wave is represented by the equation

$$y = (0{\cdot}001 \text{ mm}) \sin[(50 \text{ s}^{-1})t + (2{\cdot}0 \text{ m}^{-1})x].$$

(a) The wave velocity = 100 m s$^{-1}$.
(b) The wavelength = 2·0 m.
(c) The frequency = $25/\pi$ Hz.
(d) The amplitude = 0·001 mm.

**8.** A standing wave is produced on a string clamped at one end and free at the other. The length of the string
(a) must be an integral multiple of $\lambda/4$
(b) must be an integral multiple of $\lambda/2$
(c) must be an integral multiple of $\lambda$
(d) may be an integral multiple of $\lambda/2$.

**9.** Mark out the correct options.
(a) The energy of any small part of a string remains constant in a travelling wave.
(b) The energy of any small part of a string remains constant in a standing wave.
(c) The energies of all the small parts of equal length are equal in a travelling wave.
(d) The energies of all the small parts of equal length are equal in a standing wave.

**10.** In a stationary wave,
(a) all the particles of the medium vibrate in phase
(b) all the antinodes vibrate in phase
(c) the alternate antinodes vibrate in phase
(d) all the particles between consecutive nodes vibrate in phase.

## EXERCISES

**1.** A wave pulse passing on a string with a speed of 40 cm s$^{-1}$ in the negative $x$-direction has its maximum at $x = 0$ at $t = 0$. Where will this maximum be located at $t = 5$ s ?

**2.** The equation of a wave travelling on a string stretched along the $X$-axis is given by

$$y = A\, e^{-\left(\frac{x}{a}+\frac{t}{T}\right)^2}.$$

(a) Write the dimensions of $A$, $a$ and $T$. (b) Find the wave speed. (c) In which direction is the wave travelling ? (d) Where is the maximum of the pulse located at $t = T$ ? At $t = 2T$ ?

**3.** Figure (15-E1) shows a wave pulse at $t = 0$. The pulse moves to the right with a speed of 10 cm s$^{-1}$. Sketch the shape of the string at $t = 1$ s, 2 s and 3 s.

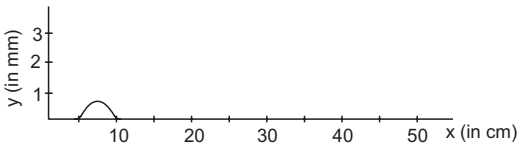


Figure 15-E1

**4**. A pulse travelling on a string is represented by the function

$$y = \frac{a^3}{(x - vt)^2 + a^2},$$

where $a = 5$ mm and $v = 20$ cm s$^{-1}$. Sketch the shape of the string at $t = 0$, 1 s and 2 s. Take $x = 0$ in the middle of the string.

**5.** The displacement of the particle at $x = 0$ of a stretched string carrying a wave in the positive $x$-direction is given by $f(t) = A \sin(t/T)$. The wave speed is $v$. Write the wave equation.

**6**. A wave pulse is travelling on a string with a speed $v$ towards the positive $X$-axis. The shape of the string at $t = 0$ is given by $g(x) = A \sin(x/a)$, where $A$ and $a$ are constants.
(a) What are the dimensions of $A$ and $a$ ? (b) Write the equation of the wave for a general time $t$, if the wave speed is $v$.

**7.** A wave propagates on a string in the positive $x$-direction at a velocity $v$. The shape of the string at $t = t_0$ is given by $g(x, t_0) = A \sin(x/a)$ . Write the wave equation for a general time $t$.

**8**. The equation of a wave travelling on a string is

$$y = (0{\cdot}10 \text{ mm}) \sin[(31{\cdot}4 \text{ m}^{-1})x + (314 \text{ s}^{-1})t].$$

(a) In which direction does the wave travel ? (b) Find the wave speed, the wavelength and the frequency of the wave. (c) What is the maximum displacement and the maximum speed of a portion of the string ?

**9.** A wave travels along the positive $x$-direction with a speed of 20 m s$^{-1}$. The amplitude of the wave is 0·20 cm and the wavelength 2·0 cm. (a) Write a suitable wave equation which describes this wave. (b) What is the displacement and velocity of the particle at $x = 2{\cdot}0$ cm at time $t = 0$ according to the wave equation written ? Can you get different values of this quantity if the wave equation is written in a different fashion ?

**10.** A wave is described by the equation

$$y = (1{\cdot}0 \text{ mm}) \sin \pi \left( \frac{x}{2{\cdot}0 \text{ cm}} - \frac{t}{0{\cdot}01 \text{ s}} \right).$$

(a) Find the time period and the wavelength ? (b) Write the equation for the velocity of the particles. Find the speed of the particle at $x = 1{\cdot}0$ cm at time $t = 0{\cdot}01$ s. (c) What are the speeds of the particles at $x = 3{\cdot}0$ cm, 5·0 cm and 7·0 cm at $t = 0{\cdot}01$ s ?
(d) What are the speeds of the particles at $x = 1{\cdot}0$ cm at $t = 0{\cdot}011$, 0·012, and 0·013 s ?

**11.** A particle on a stretched string supporting a travelling wave, takes 5·0 ms to move from its mean position to the extreme position. The distance between two consecutive particles, which are at their mean positions, is 2·0 cm. Find the frequency, the wavelength and the wave speed.

**12.** Figure (15-E2) shows a plot of the transverse displacements of the particles of a string at $t = 0$ through which a travelling wave is passing in the positive $x$-direction. The wave speed is 20 cm s$^{-1}$. Find (a) the amplitude, (b) the wavelength, (c) the wave number and (d) the frequency of the wave.

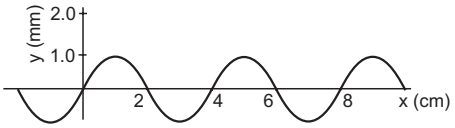



Figure 15-E2

**13.** A wave travelling on a string at a speed of 10 m s$^{-1}$ causes each particle of the string to oscillate with a time period of 20 ms. (a) What is the wavelength of the wave ? (b) If the displacement of a particle is 1·5 mm at a certain instant, what will be the displacement of a particle 10 cm away from it at the same instant ?

**14.** A steel wire of length 64 cm weighs 5 g. If it is stretched by a force of 8 N, what would be the speed of a transverse wave passing on it ?

**15.** A string of length 20 cm and linear mass density 0·40 g cm$^{-1}$ is fixed at both ends and is kept under a tension of 16 N. A wave pulse is produced at $t = 0$ near an end as shown in figure (15-E3), which travels towards the other end. (a) When will the string have the shape shown in the figure again ? (b) Sketch the shape of the string at a time half of that found in part (a).

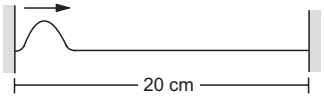



Figure 15-E3

**16.** A string of linear mass density 0·5 g cm$^{-1}$ and a total length 30 cm is tied to a fixed wall at one end and to a frictionless ring at the other end (figure 15-E4). The ring can move on a vertical rod. A wave pulse is produced on the string which moves towards the ring at a speed of 20 cm s$^{-1}$. The pulse is symmetric about its maximum which is located at a distance of 20 cm from the end joined to the ring. (a) Assuming that the wave is reflected from the ends without loss of energy, find the time taken by the string to regain its shape. (b) The shape of the string changes periodically with time. Find this time period. (c) What is the tension in the string ?

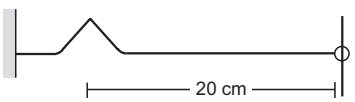



Figure 15-E4

**17.** Two wires of different densities but same area of cross section are soldered together at one end and are stretched to a tension $T$. The velocity of a transverse

wave in the first wire is double of that in the second wire. Find the ratio of the density of the first wire to that of the second wire.

**18.** A transverse wave described by

$$y = (0{\cdot}02 \text{ m}) \sin[(1{\cdot}0 \text{ m}^{-1})\, x + (30 \text{ s}^{-1})t]$$

propagates on a stretched string having a linear mass density of $1{\cdot}2 \times 10^{-4}$ kg m$^{-1}$. Find the tension in the string.

**19.** A travelling wave is produced on a long horizontal string by vibrating an end up and down sinusoidally. The amplitude of vibration is 1·0 cm and the displacement becomes zero 200 times per second. The linear mass density of the string is 0·10 kg m$^{-1}$ and it is kept under a tension of 90 N. (a) Find the speed and the wavelength of the wave. (b) Assume that the wave moves in the positive $x$-direction and at $t = 0$, the end $x = 0$ is at its positive extreme position. Write the wave equation. (c) Find the velocity and acceleration of the particle at $x$ = 50 cm at time $t = 10$ ms.

**20.** A string of length 40 cm and weighing 10 g is attached to a spring at one end and to a fixed wall at the other end. The spring has a spring constant of 160 N m$^{-1}$ and is stretched by 1·0 cm. If a wave pulse is produced on the string near the wall, how much time will it take to reach the spring ?

**21.** Two blocks each having a mass of 3·2 kg are connected by a wire $CD$ and the system is suspended from the ceiling by another wire $AB$ (figure 15-E5). The linear mass density of the wire $AB$ is 10 g m$^{-1}$ and that of $CD$ is 8 g m$^{-1}$. Find the speed of a transverse wave pulse produced in $AB$ and in $CD$.

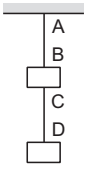



Figure 15-E5

**22.** In the arrangement shown in figure (15-E6), the string has a mass of 4·5 g. How much time will it take for a transverse disturbance produced at the floor to reach the pulley ? Take $g$ = 10 m s$^{-2}$.

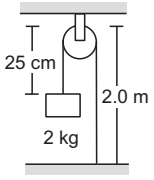



Figure 15-E6

**23.** A 4·0 kg block is suspended from the ceiling of an elevator through a string having a linear mass density of $19{\cdot}2 \times 10^{-3}$ kg m$^{-1}$. Find the speed (with respect to the string) with which a wave pulse can proceed on the string if the elevator accelerates up at the rate of 2·0 m s$^{-2}$. Take $g = 10$ m s$^{-2}$.

**24.** A heavy ball is suspended from the ceiling of a motor car through a light string. A transverse pulse travels at a speed of 60 cm s$^{-1}$ on the string when the car is at rest and 62 cm s$^{-1}$ when the car accelerates on a horizontal road. Find the acceleration of the car. Take $g = 10$ m s$^{-2}$.

**25.** A circular loop of string rotates about its axis on a frictionless horizontal plane at a uniform rate so that the tangential speed of any particle of the string is $v$. If a small transverse disturbance is produced at a point of the loop, with what speed (relative to the string) will this disturbance travel on the string ?

**26.** A heavy but uniform rope of length $L$ is suspended from a ceiling. (a) Write the velocity of a transverse wave travelling on the string as a function of the distance from the lower end. (b) If the rope is given a sudden sideways jerk at the bottom, how long will it take for the pulse to reach the ceiling ? (c) A particle is dropped from the ceiling at the instant the bottom end is given the jerk. Where will the particle meet the pulse ?

**27.** Two long strings $A$ and $B$, each having linear mass density $1{\cdot}2 \times 10^{-2}$ kg m$^{-1}$, are stretched by different tensions 4·8 N and 7·5 N respectively and are kept parallel to each other with their left ends at $x = 0$. Wave pulses are produced on the strings at the left ends at $t = 0$ on string $A$ and at $t = 20$ ms on string $B$. When and where will the pulse on $B$ overtake that on $A$ ?

**28.** A transverse wave of amplitude 0·50 mm and frequency 100 Hz is produced on a wire stretched to a tension of 100 N. If the wave speed is 100 m s$^{-1}$, what average power is the source transmitting to the wire ?

**29.** A 200 Hz wave with amplitude 1 mm travels on a long string of linear mass density 6 g m$^{-1}$ kept under a tension of 60 N. (a) Find the average power transmitted across a given point on the string. (b) Find the total energy associated with the wave in a 2·0 m long portion of the sring.

**30.** A tuning fork of frequency 440 Hz is attached to a long string of linear mass density 0·01 kg m$^{-1}$ kept under a tension of 49 N. The fork produces transverse waves of amplitude 0·50 mm on the string. (a) Find the wave speed and the wavelength of the waves. (b) Find the maximum speed and acceleration of a particle of the string. (c) At what average rate is the tuning fork transmitting energy to the string ?

**31.** Two waves, travelling in the same direction through the same region, have equal frequencies, wavelengths and amplitudes. If the amplitude of each wave is 4 mm and the phase difference between the waves is 90°, what is the resultant amplitude ?

**32.** Figure (15-E7) shows two wave pulses at $t = 0$ travelling on a string in opposite directions with the same wave

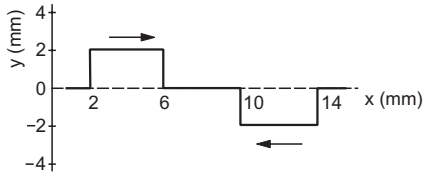



Figure 15-E7

speed 50 cm $s^{-1}$. Sketch the shape of the string at $t = 4$ ms, 6 ms, 8 ms, and 12 ms.

**33.** Two waves, each having a frequency of 100 Hz and a wavelength of 2·0 cm, are travelling in the same direction on a string. What is the phase difference between the waves (a) if the second wave was produced 0·015 s later than the first one at the same place, (b) if the two waves were produced at the same instant but the first one was produced a distance 4·0 cm behind the second one ? (c) If each of the waves has an amplitude of 2·0 mm, what would be the amplitudes of the resultant waves in part (a) and (b) ?

**34.** If the speed of a transverse wave on a stretched string of length 1 m is 60 m $s^{-1}$, what is the fundamental frequency of vibration ?

**35.** A wire of length 2·00 m is stretched to a tension of 160 N. If the fundamental frequency of vibration is 100 Hz, find its linear mass density.

**36.** A steel wire of mass 4·0 g and length 80 cm is fixed at the two ends. The tension in the wire is 50 N. Find the frequency and wavelength of the fourth harmonic of the fundamental.

**37.** A piano wire weighing 6·00 g and having a length of 90·0 cm emits a fundamental frequency corresponding to the "Middle C" ($\nu = 261{\cdot}63$ Hz). Find the tension in the wire.

**38.** A sonometer wire having a length of 1·50 m between the bridges vibrates in its second harmonic in resonance with a tuning fork of frequency 256 Hz. What is the speed of the transverse wave on the wire ?

**39.** The length of the wire shown in figure (15-E8) between the pulley is 1·5 m and its mass is 12·0 g. Find the frequency of vibration with which the wire vibrates in two loops leaving the middle point of the wire between the pulleys at rest.

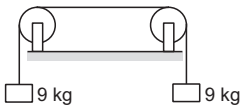



Figure 15-E8

**40.** A one-metre long stretched string having a mass of 40 g is attached to a tuning fork. The fork vibrates at 128 Hz in a direction perpendicular to the string. What should be the tension in the string if it is to vibrate in four loops ?

**41.** A wire, fixed at both ends is seen to vibrate at a resonant frequency of 240 Hz and also at 320 Hz. (a) What could be the maximum value of the fundamental frequency ? (b) If transverse waves can travel on this string at a speed of 40 m $s^{-1}$, what is its length ?

**42.** A string, fixed at both ends, vibrates in a resonant mode with a separation of 2·0 cm between the consecutive nodes. For the next higher resonant frequency, this separation is reduced to 1·6 cm. Find the length of the string.

**43.** A 660 Hz tuning fork sets up vibration in a string clamped at both ends. The wave speed for a transverse wave on this string is 220 m $s^{-1}$ and the string vibrates in three loops. (a) Find the length of the string. (b) If the maximum amplitude of a particle is 0·5 cm, write a suitable equation describing the motion.

**44.** A particular guitar wire is 30·0 cm long and vibrates at a frequency of 196 Hz when no finger is placed on it. The next higher notes on the scale are 220 Hz, 247 Hz, 262 Hz and 294 Hz. How far from the end of the string must the finger be placed to play these notes ?

**45.** A steel wire fixed at both ends has a fundamental frequency of 200 Hz. A person can hear sound of maximum frequency 14 kHz. What is the highest harmonic that can be played on this string which is audible to the person ?

**46.** Three resonant frequencies of a string are 90, 150 and 210 Hz. (a) Find the highest possible fundamental frequency of vibration of this string. (b) Which harmonics of the fundamental are the given frequencies ? (c) Which overtones are these frequencies ? (d) If the length of the string is 80 cm, what would be the speed of a transverse wave on this string ?

**47.** Two wires are kept tight between the same pair of supports. The tensions in the wires are in the ratio 2 : 1, the radii are in the ratio 3 : 1 and the densities are in the ratio 1 : 2. Find the ratio of their fundamental frequencies.

**48.** A uniform horizontal rod of length 40 cm and mass 1·2 kg is supported by two identical wires as shown in figure (15-E9). Where should a mass of 4·8 kg be placed on the rod so that the same tuning fork may excite the wire on left into its fundamental vibrations and that on right into its first overtone ? Take $g = 10$ m $s^{-2}$.

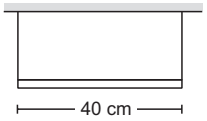



Figure 15-E9

**49.** Figure (15-E10) shows an aluminium wire of length 60 cm joined to a steel wire of length 80 cm and stretched between two fixed supports. The tension produced is 40 N. The cross-sectional area of the steel wire is 1·0 $mm^2$ and that of the aluminium wire is 3·0 $mm^2$. What could be the minimum frequency of a tuning fork which can produce standing waves in the system with the joint as a node ? The density of aluminium is 2·6 g $cm^{-3}$ and that of steel is 7·8 g $cm^{-3}$.

Figure 15-E10

**50.** A string of length $L$ fixed at both ends vibrates in its fundamental mode at a frequency $\nu$ and a maximum amplitude $A$. (a) Find the wavelength and the wave number $k$. (b) Take the origin at one end of the string and the $X$-axis along the string. Take the $Y$-axis along

the direction of the displacement. Take $t = 0$ at the instant when the middle point of the string passes through its mean position and is going towards the positive $y$-direction. Write the equation describing the standing wave.

**51.** A 2 m-long string fixed at both ends is set into vibrations in its first overtone. The wave speed on the string is 200 m $s^{-1}$ and the amplitude is 0·5 cm. (a) Find the wavelength and the frequency. (b) Write the equation giving the displacement of different points as a function of time. Choose the $X$-axis along the string with the origin at one end and $t = 0$ at the instant when the point $x = 50$ cm has reached its maximum displacement.

**52.** The equation for the vibration of a string, fixed at both ends vibrating in its third harmonic, is given by

$$y = (0{\cdot}4 \text{ cm}) \sin[(0{\cdot}314 \text{ cm}^{-1})\, x] \cos[(600\pi \text{ s}^{-1})t].$$

(a) What is the frequency of vibration ? (b) What are the positions of the nodes ? (c) What is the length of the string ? (d) What is the wavelength and the speed of two travelling waves that can interfere to give this vibration ?

**53.** The equation of a standing wave, produced on a string fixed at both ends, is

$$y = (0{\cdot}4 \text{ cm}) \sin[(0{\cdot}314 \text{ cm}^{-1})\, x] \cos[(600\pi \text{ s}^{-1})t].$$

What could be the smallest length of the string ?

**54.** A 40 cm wire having a mass of 3·2 g is stretched between two fixed supports 40·05 cm apart. In its fundamental mode, the wire vibrates at 220 Hz. If the area of cross section of the wire is 1·0 $mm^2$, find its Young modulus.

**55.** Figure (15-E11) shows a string stretched by a block going over a pulley. The string vibrates in its tenth harmonic in unison with a particular tuning fork. When a beaker containing water is brought under the block so that the block is completely dipped into the beaker, the string vibrates in its eleventh harmonic. Find the density of the material of the block.

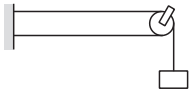


Figure 15-E11

**56.** A 2·00 m-long rope, having a mass of 80 g, is fixed at one end and is tied to a light string at the other end. The tension in the string is 256 N. (a) Find the frequencies of the fundamental and the first two overtones. (b) Find the wavelength in the fundamental and the first two overtones.

**57.** A heavy string is tied at one end to a movable support and to a light thread at the other end as shown in figure (15-E12). The thread goes over a fixed pulley and supports a weight to produce a tension. The lowest frequency with which the heavy string resonates is 120 Hz. If the movable support is pushed to the right by 10 cm so that the joint is placed on the pulley, what will be the minimum frequency at which the heavy string can resonate ?

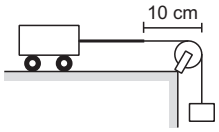



Figure 15-E12

□

## ANSWERS

### OBJECTIVE I

| | | | | | |
|---|---|---|---|---|---|
| 1. (c) | 2. (c) | 3. (a) | 4. (b) | 5. (d) | 6. (a) |
| 7. (a) | 8. (d) | 9. (d) | 10. (c) | 11. (b) | 12. (d) |
| 13. (b) | 14. (d) | 15. (b) | 16. (d) | 17. (a) | 18. (a) |
| 19. (b) | 20. (b) | 21. (a) | 22. (d) | | |

### OBJECTIVE II

| | | |
|---|---|---|
| 1. (c), (d) | 2. (d) | 3. (d) |
| 4. (b), (d) | 5. (a) | 6. (b), (c), (d) |
| 7. (c), (d) | 8. (a) | 9. (b) |
| 10. (c), (d) | | |

### EXERCISES

1. At $x = -2$ m

2. (a) $L$, $L$, $T$ (b) $a/T$
   (c) negative $x$-direction (d) $x = -a$ and $x = -2a$

5. $f(x, t) = A \sin\left(\frac{t}{T} - \frac{x}{vT}\right)$

6. (a) $L$, $L$ (b) $f(x, t) = A \sin \frac{x - vt}{a}$

7. $f(x, t) = A \sin \frac{x - v(t - t_0)}{a}$

8. (a) negative $x$-direction (b) 10 m $s^{-1}$, 20 cm, 50 Hz
   (c) 0·10 mm, 3·14 cm $s^{-1}$

9. (a) $y = (0{\cdot}20 \text{ cm}) \sin[(\pi \text{ cm}^{-1})\, x - (2\pi \times 10^3 \text{ s}^{-1})t]$
   (b) zero, $4\pi$ m $s^{-1}$

10. (a) 20 ms, 4·0 cm (b) zero
    (c) zero (d) 9·7 cm $s^{-1}$, 18 cm $s^{-1}$, 25 cm $s^{-1}$
11. 50 Hz, 4·0 cm, 2·0 m $s^{-1}$
12. (a) 1·0 mm (b) 4 cm (c) 1·6 $cm^{-1}$ (d) 5 Hz
13. (a) 20 cm (b) − 1·5 mm
14. 32 m $s^{-1}$
15. (a) 0·02 s
16. (a) 2 s (b) 3 s (c) $2 \times 10^{-3}$ N
17. 0·25
18. 0·108 N
19. (a) 30 m $s^{-1}$, 30 cm
    (b) $y = (1{\cdot}0 \text{ cm}) \cos 2\pi \left[\frac{x}{30 \text{ cm}} - \frac{t}{0{\cdot}01 \text{ s}}\right]$
    (c) − 5·4 m $s^{-1}$, 2·0 km $s^{-2}$
20. 0·05 s
21. 79 m $s^{-1}$ and 63 m $s^{-1}$
22. 0·02 s
23. 50 m $s^{-1}$
24. 3·7 m $s^{-2}$
25. $v$
26. (a) $\sqrt{gx}$ (b) $\sqrt{4L/g}$
    (c) at a distance $\frac{L}{3}$ from the bottom
27. at $t = 100$ ms at $x = 2{\cdot}0$ m
28. 49 mW
29. (a) 0·47 W (b) 9·4 mJ
30. (a) 70 m $s^{-1}$, 16 cm (b) 1·4 m $s^{-1}$, 3·8 km $s^{-2}$ (c) 0·67 W
31. $4\sqrt{2}$ mm
33. (a) $3\pi$ (b) $4\pi$ (c) zero, 4·0 mm
34. 30 Hz
35. 1·00 g $m^{-1}$
36. 250 Hz, 40 cm
37. 1480 N
38. 384 m $s^{-1}$
39. 70 Hz
40. 164 N
41. (a) 80 Hz (b) 25 cm
42. 8·0 cm
43. (a) 50 cm
    (b) (0·5 cm) $\sin[(0{\cdot}06\pi \text{ cm}^{-1})x] \times \cos[(1320\pi \text{ s}^{-1})t]$
44. 26·7 cm, 23·8 cm, 22·4 cm and 20·0 cm
45. 70
46. (a) 30 Hz (b) 3rd, 5th and 7th
    (c) 2nd, 4th and 6th (d) 48 m $s^{-1}$
47. 2 : 3
48. 5 cm from the left end
49. 180 Hz
50. (a) $2L$, $\pi/L$ (b) $y = A \sin(\pi x/L) \sin(2\pi\nu t)$
51. (a) 2 m, 100 Hz
    (b) (0·5 cm) $\sin[(\pi \text{m}^{-1})\, x] \cos[(200\pi \text{ s}^{-1})t]$
52. (a) 300 Hz (b) 0, 10 cm, 20 cm, 30 cm
    (c) 30 cm (d) 20 cm, 60 m/s
53. 10 cm
54. $1{\cdot}98 \times 10^{11}$ N $m^{-2}$
55. $5{\cdot}8 \times 10^{3}$ kg $m^{-3}$
56. (a) 10 Hz, 30 Hz, 50 Hz (b) 8·00 m, 2·67 m, 1·60 m
57. 240 Hz

□

## CHAPTER 16

# SOUND WAVES

### 16.1 THE NATURE AND PROPAGATION OF SOUND WAVES

Sound is produced in a material medium by a vibrating source. As the vibrating source moves forward, it compresses the medium past it, increasing the density locally. This part of the medium compresses the layer next to it by collisions. The compression travels in the medium at a speed which depends on the elastic and inertia properties of the medium. As the source moves back, it drags the medium and produces a rarefaction in the layer. The layer next to it is then dragged back and, thus, the rarefaction pulse passes forward. In this way, compression and rarefaction pulses are produced which travel in the medium.

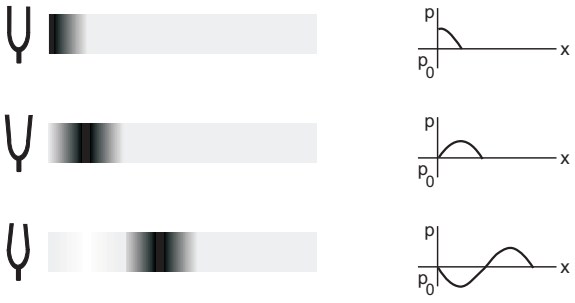



Figure 16.1

Figure (16.1) describes a typical case of propagation of sound waves. A tuning fork is vibrated in air. The prongs vibrate in simple harmonic motion. When the fork moves towards right, the layer next to it is compressed and consequently, the density is increased. The increase in density and hence, in pressure, is related to the velocity of the prong. The compression so produced travels in air towards right at the wave speed $v$. The velocity of the prong changes during the forward motion, being maximum at the mean position and zero at the extreme end. A compression wave pulse of length $vT/2$ is thus produced during the half period $T/2$ of forward motion. The prong now returns towards left and drags the air with it. The density and the pressure of the layer close to it go below the normal level, a rarefaction pulse is thus produced. During this half period of backward motion of the prong, a rarefaction pulse of length $vT/2$ is produced. As the prong continues executing its simple harmonic motion, a series of alternate compression and rarefaction pulses are produced which travel down the air.

As the prong vibrates in simple harmonic motion, the pressure variations in the layer close to the prong also change in a simple harmonic fashion. The increase in pressure above its normal value may, therefore, be written as

$$\delta P = P - P_0 = \delta P_0 \sin \omega t,$$

where $\delta P_0$ is the maximum increase in pressure above its normal value. As this disturbance travels towards right with the speed $v$ (the wave speed and not the particle speed), the equation for the excess pressure at any point $x$ at any time $t$ is given by

$$\delta P = \delta P_0 \sin \omega (t - x/v).$$

This is the equation of a wave travelling in $x$-direction with velocity $v$. The excess pressure oscillates between $+\delta P_0$ and $-\delta P_0$. The frequency of this wave is $\nu = \omega/(2\pi)$ and is equal to the frequency of vibration of the source. Henceforth, we shall use the symbol $p$ for the excess pressure developed above the equilibrium pressure and $p_0$ for the maximum change in pressure. The wave equation is then

$$p = p_0 \sin \omega (t - x/v). \qquad \ldots \quad (16.1)$$

Sound waves constitute alternate compression and rarefaction pulses travelling in the medium. However, sound is audible only if the frequency of alteration of pressure is between 20 Hz to 20,000 Hz. These limits are subjective and may vary slightly from person to person. An average human ear is not able to detect disturbance in the medium if the frequency is outside this range. Electronic detectors can detect waves of lower and higher frequencies as well. A dog can hear sound of frequency up to about 50 kHz and a bat up to about 100 kHz. The waves with frequency below the

audible range are called *infrasonic* waves and the waves with frequency above the audible range are called *ultrasonic.*

***Example 16.1***

*A wave of wavelength* 0·60 cm *is produced in air and it travels at a speed of* 300 m $s^{-1}$. *Will it be audible* ?

***Solution*** : From the relation $v = \nu\lambda$, the frequency of the wave is

$$\nu = \frac{v}{\lambda} = \frac{300 \text{ m s}^{-1}}{0{\cdot}60 \times 10^{-2} \text{ m}} = 50000 \text{ Hz}.$$

This is much above the audible range. It is an ultrasonic wave and will not be audible.

The disturbance produced by a source of sound is not always a sine wave. A pure sine wave has a unique frequency but a disturbance of other waveform may have many frequency components in it. For example, when we clap our hands, a pulse of disturbance is created which travels in the air. This pulse does not have the shape of a sine wave. However, it can be obtained by superposition of a large number of sine waves of different frequencies and amplitudes. We then say that the clapping sound has all these frequency components in it.

The compression and rarefaction in a sound wave is caused due to the back and forth motion of the particles of the medium. This motion is along the direction of propagation of sound and hence the sound waves are longitudinal.

All directions, perpendicular to the direction of propagation, are equivalent and hence, a sound wave cannot be polarized. If we make a slit on a cardboard and place it in the path of the sound, rotating the cardboard in its plane will produce no effect on the intensity of sound on the other side.

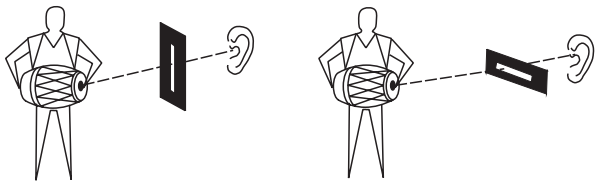


Figure 16.2

**Wavefront**

The sound produced at some point by a vibrating source travels in all directions in the medium if the medium is extended. The sound waves are, in general, three dimensional waves. For a small source, we have spherical layers of the medium on which the pressure at various elements have the same phase at a given instant. The compression, produced by the source at say $t = 0$, reaches the spherical surface of radius $r = vt$ at time $t$ and the pressure at all the points on this sphere is maximum at this instant. A half time-period later, the pressure at all the points on this sphere is reduced to minimum. The surface through the points, having the same phase of disturbance, is called a *wavefront.* For a homogeneous and isotropic medium, the wavefronts are normal to the direction of propagation.

For a point source placed in a homogeneous and isotropic medium, the wavefronts are spherical and the wave is called a *spherical wave.* If sound is produced by vibrating a large plane sheet, the disturbance produced in front of the sheet will have the same phase on a plane parallel to the sheet. The wavefronts are then planes (neglecting the end effects) and the direction of propagation is perpendicular to these planes. Such waves are called *plane waves.* The wavefront can have several other shapes. In this chapter, we shall mostly consider sound waves travelling in a fixed direction, i.e., plane waves. However, most of the results will be applicable to other waves also.

## 16.2 DISPLACEMENT WAVE AND PRESSURE WAVE

A longitudinal wave in a fluid (liquid or gas) can be described either in terms of the longitudinal displacement suffered by the particles of the medium or in terms of the excess pressure generated due to the compression or rarefaction. Let us see how the two representations are related to each other.

Consider a wave going in the $x$-direction in a fluid. Suppose that at a time $t$, the particle at the undisturbed position $x$ suffers a displacement $s$ in the $x$-direction. The wave can then be described by the equation

$$s = s_0 \sin \omega(t - x/v). \qquad \ldots \text{ (i)}$$

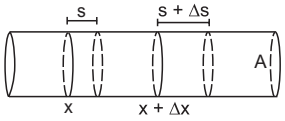



Figure 16.3

Consider the element of the material which is contained within $x$ and $x + \Delta x$ (figure 16.3) in the undisturbed state. Considering a cross-sectional area $A$, the volume of the element in the undisturbed state is $A\,\Delta x$ and its mass is $\rho A\,\Delta x$. As the wave passes, the ends at $x$ and $x + \Delta x$ are displaced by amounts $s$ and $s + \Delta s$ according to equation (i) above. The increase in volume of this element at time $t$ is

$$\Delta V = A\,\Delta s$$
$$= A\, s_0(-\omega/v) \cos \omega(t - x/v)\Delta x,$$

where $\Delta s$ has been obtained by differentiating equation (i) with respect to $x$. The element is, therefore, under a volume strain.

$$\frac{\Delta V}{V} = \frac{-A\, s_0 \omega \cos \omega(t - x/v)\Delta x}{vA\,\Delta x}$$

$$= \frac{-s_0\omega}{v} \cos \omega(t - x/v).$$

The corresponding stress, i.e., the excess pressure developed in the element at $x$ at time $t$ is,

$$p = B\left(\frac{-\Delta V}{V}\right),$$

where $B$ is the bulk modulus of the material. Thus,

$$p = B\,\frac{s_0\omega}{v} \cos \omega(t - x/v). \qquad \ldots \text{ (ii)}$$

Comparing with (i), we see that the pressure amplitude $p_0$ and the displacement amplitude $s_0$ are related as

$$p_0 = \frac{B\,\omega}{v}\, s_0 = Bks_0, \qquad \ldots \text{ (16.2)}$$

where $k$ is the wave number. Also, we see from (i) and (ii) that the pressure wave differs in phase by $\pi/2$ from the displacement wave. The pressure maxima occur where the displacement is zero and displacement maxima occur where the pressure is at its normal level.

The fact that, displacement is zero where the pressure change is maximum and vice versa, puts the two descriptions on different footings. The human ear or an electronic detector responds to the change in pressure and not to the displacement in a straight forward way. Suppose two audio speakers are driven by the same amplifier and are placed facing each other (figure 16.4). A detector is placed midway between them.

Figure 16.4

The displacement of the air particles near the detector will be zero as the two sources drive these particles in opposite directions. However, both send compression waves and rarefaction waves together. As a result, pressure increases at the detector simultaneously due to both sources. Accordingly, the pressure amplitude will be doubled, although the displacement remains zero here. A detector detects maximum intensity in such a condition. Thus, the description in terms of pressure wave is more appropriate than the description in terms of the displacement wave as far as sound properties are concerned.

***Example 16.2***

*A sound wave of wavelength* 40 cm *travels in air. If the difference between the maximum and minimum pressures at a given point is* $1{\cdot}0 \times 10^{-3}$ N m$^{-2}$, *find the amplitude of vibration of the particles of the medium. The bulk modulus of air is* $1{\cdot}4 \times 10^{5}$ N m$^{-2}$.

***Solution*** : The pressure amplitude is

$$p_0 = \frac{1{\cdot}0 \times 10^{-3}\ \text{N m}^{-2}}{2} = 0{\cdot}5 \times 10^{-3}\ \text{N m}^{-2}.$$

The displacement amplitude $s_0$ is given by

$$p_0 = B\,k\,s_0$$

or,
$$s_0 = \frac{p_0}{B\,k} = \frac{p_0\,\lambda}{2\,\pi\,B}$$

$$= \frac{0{\cdot}5 \times 10^{-3}\ \text{N m}^{-2} \times (40 \times 10^{-2}\ \text{m})}{2 \times 3{\cdot}14 \times 1{\cdot}4 \times 10^{5}\ \text{N m}^{-2}}$$

$$= 2{\cdot}2 \times 10^{-10}\ \text{m}.$$

## 16.3 SPEED OF A SOUND WAVE IN A MATERIAL MEDIUM

Consider again a sound wave going in $x$-direction in a fluid whose particles are displaced according to the equation

$$s = s_0 \sin \omega(t - x/v). \qquad \ldots \text{ (i)}$$

The pressure varies according to the equation

$$p = \frac{B\, s_0\omega}{v} \cos \omega(t - x/v). \qquad \ldots \text{ (ii)}$$

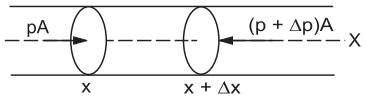



Figure 16.5

Consider the element of the fluid which is contained between the positions $x$ and $x + \Delta x$ in the undisturbed state (figure 16.5). The excess pressure at time $t$ at the end $x$ is $p$ and at $x + \Delta x$ it is $p + \Delta p$. Taking a cross-sectional area $A$, the force on the element from the left is $pA$ and from the right it is $(p + \Delta p)A$. The resultant force on the element at time $t$ is

$$\Delta F = Ap - A(p + \Delta p) = -A\,\Delta p$$

$$= -A\,\frac{B\,s_0\,\omega}{v}\,(\omega/v)\,\Delta x \sin \omega(t - x/v)$$

$$= -A\,\frac{B\,s_0\,\omega^{2}}{v^{2}} \sin \omega(t - x/v)\,\Delta x.$$

The change in pressure $\Delta p$ between $x$ and $x + \Delta x$ is obtained by differentiating equation (ii) with respect to $x$. If $\rho$ is the normal density of the fluid, the mass of the element considered is $\rho A \Delta x$. Using Newton's second law of motion, the acceleration of the element is given by

$$a = \frac{\Delta F}{\rho A \Delta x} = -\frac{B s_0 \omega^2}{\rho v^2} \sin \omega(t - x/v). \quad \ldots \text{(iii)}$$

However, the acceleration can also be obtained from equation (i). It is

$$a = \frac{\partial^2 s}{\partial t^2}$$

or, $$a = -\omega^2 s_0 \sin \omega(t - x/v). \quad \ldots \text{(iv)}$$

Comparing (iii) and (iv)

$$\frac{B s_0 \omega^2}{\rho v^2} = \omega^2 s_0$$

or, $$v = \sqrt{B/\rho}. \quad \ldots \text{(16.3)}$$

We see that the velocity of a longitudinal wave in a medium depends on its elastic properties and inertia properties as was the case with the waves on a string.

**Sound Waves in Solids**

Sound waves can travel in solids just like they can travel in fluids. The speed of longitudinal sound waves in a solid rod can be shown to be

$$v = \sqrt{Y/\rho} \quad \ldots \text{(16.4)}$$

where $Y$ is the Young modulus of the solid and $\rho$ its density. For extended solids, the speed is a more complicated function of bulk modulus and shear modulus. Table (16.1) gives the speed of sound in some common materials.

**Table 16.1**

| Medium | Speed m s$^{-1}$ | Medium | Speed m s$^{-1}$ |
|---|---|---|---|
| Air (dry 0°C) | 332 | Copper | 3810 |
| Hydrogen | 1330 | Aluminium | 5000 |
| Water | 1486 | Steel | 5200 |

## 16.4 SPEED OF SOUND IN A GAS : NEWTON'S FORMULA AND LAPLACE'S CORRECTION

The speed of sound in a gas can be expressed in terms of its pressure and density. The derivation uses some of the properties of gases that we shall study in another chapter. We summarise these properties below.

(a) For a given mass of an ideal gas, the pressure, volume and the temperature are related as $\frac{PV}{T} = \text{constant}$. If the temperature remains constant (called an isothermal process), the pressure and the volume of a given mass of a gas satisfy $PV$ = constant.

Here $T$ is the absolute temperature of the gas. This is known as *Boyle's law*.

(b) If no heat is supplied to a given mass of a gas (called an adiabatic process), its pressure and volume satisfy

$$PV^\gamma = \text{constant};$$

where $\gamma$ is a constant for the given gas. It is, in fact, the ratio $C_p / C_V$ of two specific heat capacities of the gas.

Newton suggested a theoretical expression for the velocity of sound wave in a gaseous medium. He assumed that when a sound wave propagates through a gas, the temperature variations in the layers of compression and rarefaction are negligible. The logic perhaps was that the layers are in contact with wider mass of the gas so that by exchanging heat with the surrounding the temperature of the layer will remain equal to that of the surrounding. Hence, the conditions are isothermal and Boyle's law will be applicable.

Thus, $$PV = \text{constant}$$

or, $$P\Delta V + V\Delta P = 0$$

or, $$B = -\frac{\Delta P}{\Delta V/V} = P. \quad \ldots \text{(i)}$$

Using this result in equation (16.3), the speed of sound in the gas is given by

$$v = \sqrt{P/\rho}. \quad \ldots \text{(16.5)}$$

The density of air at temperature 0°C and pressure 76 cm of mercury column is $\rho = 1{\cdot}293$ kg m$^{-3}$. This temperature and pressure is called standard temperature and pressure and is written as STP. According to equation (16.5), the speed of sound in air at this temperature and pressure should be 280 m s$^{-1}$. This value is somewhat smaller than the measured speed of sound which is about 332 m s$^{-1}$.

Laplace suggested that the compression or rarefaction takes place too rapidly and the gas element being compressed or rarefied does not get enough time to exchange heat with the surroundings. Thus, it is an adiabatic process and one should use the equation

$$PV^\gamma = \text{constant}.$$

Taking logarithms,

$$\ln P + \gamma \ln V = \text{constant}.$$

Taking differentials,

$$\frac{\Delta P}{P} + \gamma \frac{\Delta V}{V} = 0$$

or, $$B = -\frac{\Delta P}{\Delta V/V} = \gamma P$$

Thus, the speed of sound is $v = \sqrt{\frac{\gamma P}{\rho}}$ . … (16.6)

For air, $\gamma = 1{\cdot}4$ and putting values of $P$ and $\rho$ as before, equation (16.6) gives the speed of sound in air at STP to be 332 m $s^{-1}$ which is quite close to the observed value.

## 16.5 EFFECT OF PRESSURE, TEMPERATURE AND HUMIDITY ON THE SPEED OF SOUND IN AIR

We have stated that for an ideal gas, the pressure, volume and temperature of a given mass satisfy

$$\frac{PV}{T} = \text{constant.}$$

As the density of a given mass is inversely proportional to its volume, the above equation may also be written as

$$\frac{P}{\rho} = cT,$$

where $c$ is a constant. The speed of sound is

$$v = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\gamma cT}. \qquad \ldots \quad (16.7)$$

Thus, if pressure is changed but the temperature is kept constant, the density varies proportionally and $P/\rho$ remains constant. The speed of sound is not affected by the change in pressure provided the temperature is kept constant.

If the temperature of air is changed then the speed of sound is also changed.

From equation (16.7),

$$v \propto \sqrt{T}.$$

At STP, the temperature is 0°C or 273 K. If the speed of sound at 0°C is $v_0$, its value at the room temperature $T$ (in kelvin) will satisfy

$$\frac{v}{v_0} = \sqrt{\frac{T}{273}} = \sqrt{\frac{273 + t}{273}},$$

where $t$ is the temperature in °C. This may be approximated as

$$\frac{v}{v_0} = \left(1 + \frac{t}{273}\right)^{\frac{1}{2}} \cong 1 + \frac{t}{546}$$

or, $$v = v_0\left(1 + \frac{t}{546}\right).$$

The density of water vapour is less than dry air at the same pressure. Thus, the density of moist air is less than that of dry air. As a result, the speed of sound increases with increasing humidity.

## 16.6 INTENSITY OF SOUND WAVES

As a wave travels in a medium, energy is transported from one part of the space to another part. The intensity of a sound wave is defined as the average energy crossing a unit cross-sectional area perpendicular to the direction of propagation of the wave in unit time. It may also be stated as the average power transmitted across a unit cross-sectional area perpendicular to the direction of propagation.

The loudness of sound that we feel is mainly related to the intensity of sound. It also depends on the frequency to some extent.

Consider again a sound wave travelling along the $x$-direction. Let the equations for the displacement of the particles and the excess pressure developed by the wave be given by

$$\left.\begin{aligned} s &= s_0 \sin \omega(t - x/v) \\ \text{and} \quad p &= p_0 \cos \omega(t - x/v) \end{aligned}\right| \qquad \ldots \text{ (i)}$$

where $p_0 = \frac{B\,\omega\, s_0}{v}$ .

Consider a cross section of area $A$ perpendicular to the $x$-direction. The medium to the left to it exerts a force $pA$ on the medium to the right along the $X$-axis. The points of application of this force move longitudinally, that is along the force, with a speed $\frac{\partial s}{\partial t}$ . Thus, the power $W$, transmitted by the wave from left to right across the cross section considered, is

$$W = (pA)\,\frac{\partial s}{\partial t}.$$

By (i),

$$W = A\,p_0 \cos \omega(t - x/v)\; \omega s_0 \cos \omega(t - x/v)$$

$$= \frac{A\,\omega^2\, s_0^{\,2}\, B}{v} \cos^2 \omega(t - x/v).$$

The average of $\cos^2 \omega(t - x/v)$ over a complete cycle or over a long time is 1/2. The intensity $I$, which is equal to the average power transmitted across unit cross-sectional area is thus,

$$I = \frac{1}{2}\,\frac{\omega^2\, s_0^{\,2}\, B}{v} = \frac{2\pi^2\, B}{v}\, s_0^{\,2}\, \nu^2.$$

Using equation (16.2),

$$I = \frac{p_0^{\,2}\, v}{2\,B}. \qquad \ldots (16.8)$$

As $B = \rho v^2$, the intensity can also be written as

$$I = \frac{v}{2\rho\, v^2}\, p_0^{\,2} = \frac{p_0^{\,2}}{2\rho v}. \qquad \ldots (16.9)$$

We see that the intensity is proportional to the square of the pressure amplitude $p_0$ .

***Example 16.3***

*The pressure amplitude in a sound wave from a radio receiver is* $2{\cdot}0 \times 10^{-2}\ \text{N m}^{-2}$ *and the intensity at a point is* $5{\cdot}0 \times 10^{-7}\ \text{W m}^{-2}$. *If by turning the "volume" knob the pressure amplitude is increased to* $2{\cdot}5 \times 10^{-2}\ \text{N m}^{-2}$, *evaluate the intensity.*

***Solution*** : The intensity is proportional to the square of the pressure amplitude.

Thus, $\dfrac{I'}{I} = \left(\dfrac{p'_0}{p_0}\right)^2$

or, $I' = \left(\dfrac{p'_0}{p_0}\right)^2 I = \left(\dfrac{2{\cdot}5}{2{\cdot}0}\right)^2 \times 5{\cdot}0 \times 10^{-7}\ \text{W m}^{-2}$

$= 7{\cdot}8 \times 10^{-7}\ \text{W m}^{-2}$.

## 16.7 APPEARANCE OF SOUND TO HUMAN EAR

The appearance of sound to a human ear is characterised by three parameters (a) *pitch* (b) *loudness* and (c) *quality*. All the three are subjective description of sound though they are related to objectively defined quantities. Pitch is related to frequency, loudness is related to intensity and quality is related to the waveform of the sound wave.

### Pitch and Frequency

Pitch of a sound is that sensation by which we differentiate a buffallo voice, a male voice and a female voice. We say that a buffallo voice is of low pitch, a male voice has higher pitch and a female voice has (generally) still higher pitch. This sensation primarily depends on the dominant frequency present in the sound. Higher the frequency, higher will be the pitch and vice versa. The dominant frequency of a buffallo voice is smaller than that of a male voice which in turn is smaller than that of a female voice.

### Loudness and Intensity

The loudness that we sense is related to the intensity of sound though it is not directly proportional to it. Our perception of loudness is better correlated with the *sound level* measured in decibels (abbreviated as dB) and defined as follows.

$$\beta = 10 \log_{10}\left(\frac{I}{I_0}\right), \qquad \ldots \ (16.10)$$

where $I$ is the intensity of the sound and $I_0$ is a constant reference intensity $10^{-12}\ \text{W m}^{-2}$. The reference intensity represents roughly the minimum intensity that is just audible at intermediate frequencies. For $I = I_0$, the sound level $\beta = 0$. Table (16.2) shows the approximate sound levels of some of the sounds commonly encountered.

**Table 16.2 : *Sound Levels***

| | |
|---|---|
| Minimum audible sound | ≈ 0 dB |
| Whispering (at 1 m) | 10 dB |
| Normal talk (at 1 m) | 60 dB |
| Maximum tolerable sound | 120 dB |

***Example 16.4***

*If the intensity is increased by a factor of* 20, *by how many decibels is the sound level increased* ?

***Solution*** : Let the initial intensity be $I$ and the sound level be $\beta_1$. When the intensity is increased to 20 $I$, the level increases to $\beta_2$.

Then $\beta_1 = 10 \log (I/I_0)$

and $\beta_2 = 10 \log (20\, I/I_0)$.

Thus, $\beta_2 - \beta_1 = 10 \log (20\, I/I)$

$= 10 \log 20$

$= 13$ dB.

### Quality and Waveform

A sound generated by a source may contain a number of frequency components in it. Different frequency components have different amplitudes and superposition of them results in the actual waveform. The appearance of sound depends on this waveform apart from the dominant frequency and intensity. Figure (16.6) shows waveforms for a tuning fork, a clarinet and a cornet playing the same note (fundamental frequency = 440 Hz) with equal loudness.

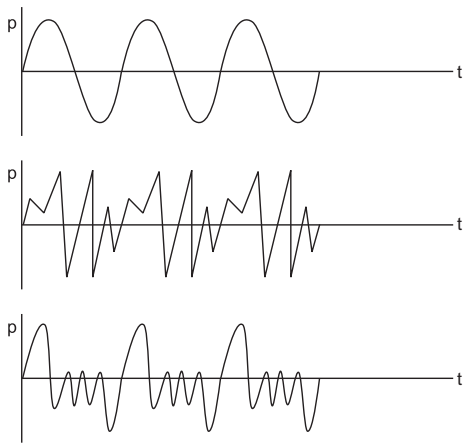



Figure 16.6

We differentiate between the sound from a tabla and that from a mridang by saying that they have different *quality*. A musical sound has certain well-defined frequencies which have considerable amplitude. These frequencies are generally harmonics of a fundamental frequency. Such a sound is particularly pleasant to the ear. On the other hand, a

noise has frequencies that do not bear any well-defined relationship among themselves.

## 16.8 INTERFERENCE OF SOUND WAVES

The principle of superposition introduced in the previous chapter is valid for sound waves as well. If two or more waves pass through the same region of a medium, the resultant disturbance is equal to the sum of the disturbances produced by individual waves. Depending on the phase difference, the waves can interfere constructively or destructively leading to a corresponding increase or decrease in the resultant intensity. While discussing the interference of two sound waves, it is advised that the waves be expressed in terms of pressure change. The resultant change in pressure is the algebraic sum of the changes in pressure due to the individual waves. Thus, one should not add the displacement vectors so as to obtain the resultant displacement wave.

Figure (16.7) shows two tuning forks $S_1$ and $S_2$, placed side by side, which vibrate with equal frequency and equal amplitude. The point $P$ is situated at a distance $x$ from $S_1$ and $x + \Delta x$ from $S_2$ .

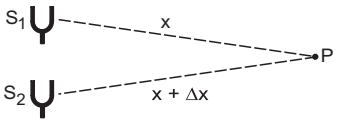



Figure 16.7

The forks may be set into vibration with a phase difference $\delta_0$. In case of tuning forks, the phase difference $\delta_0$ remains constant in time. Two sources whose phase difference remains constant in time are called *coherent sources.* If there were two drum beaters beating the drums independently, the sources would have been *incoherent.*

Suppose the two forks are vibrating in phase so that $\delta_0 = 0$. Also, let $p_{01}$ and $p_{02}$ be the amplitudes of the waves from $S_1$ and $S_2$ respectively. Let us examine the resultant change in pressure at a point $P$. The pressure change at $P$ due to the two waves are described by

$$p_1 = p_{01} \sin(kx - \omega t)$$
$$p_2 = p_{02} \sin[k(x + \Delta x) - \omega t]$$
$$= p_{02} \sin[(kx - \omega t) + \delta],$$

where $$\delta = k\,\Delta x = \frac{2\pi\,\Delta x}{\lambda} \qquad \ldots (16.11)$$

is the phase difference between the two waves reaching $P$. These equations are identical to those discussed in chapter 15, section 15.7. The resultant wave at $P$ is given by

$$p = p_0 \sin[(kx - \omega t) + \varepsilon],$$

where $$p_0^2 = p_{01}^2 + p_{02}^2 + 2\,p_{01}\,p_{02}\cos\delta,$$

and $$\tan\varepsilon = \frac{p_{02}\sin\delta}{p_{01} + p_{02}\cos\delta}.$$

The resultant amplitude is maximum when $\delta = 2n\pi$ and is minimum when $\delta = (2n + 1)\,\pi$ where $n$ is an integer. These are correspondingly the conditions for constructive and destructive interference

$$\left.\begin{array}{ll}\delta = 2n\pi & \text{constructive interference}\\ \delta = (2n+1)\,\pi & \text{destructive interference.}\end{array}\right| \ldots (16.12)$$

Using equation (16.11), i.e., $\delta = \frac{2\pi}{\lambda}\Delta x$, these conditions may be written in terms of the path difference as

$$\left.\begin{array}{ll}\Delta x = n\lambda & \text{(constructive)}\\ \Delta x = (n + 1/2)\,\lambda & \text{(destructive).}\end{array}\right| \quad \ldots (16.13)$$

At constructive interference,

$$p_0 = p_{01} + p_{02}$$

and at destructive interference,

$$p_0 = |\,p_{01} - p_{02}\,|.$$

Suppose $p_{01} = p_{02}$ and $\Delta x = \lambda/2$. The resultant pressure amplitude of the disturbance is zero and no sound is detected at such a point $P$. If $\Delta x = \lambda$, the amplitude is doubled. The intensity of a wave is proportional to the square of the amplitude and hence, at the points of constructive interference, the resultant intensity of sound is four times the intensity due to an individual source. This is a characteristic property of wave motion. Two sources can cancel the effects of each other or they can reinforce the effect.

If the sources have an initial phase difference $\delta_0$ between them, the waves reaching $P$ at time $t$ are represented by

$$p = p_{01} \sin[kx - \omega t]$$

and $$p = p_{02} \sin[k(x + \Delta x) - \omega t + \delta_0].$$

The phase difference between these waves is

$$\delta = \delta_0 + k\,\Delta x = \delta_0 + \frac{2\pi\,\Delta x}{\lambda}.$$

The interference maxima and minima occur according to equation (16.12).

For incoherent sources, $\delta_0$ is not constant and varies rapidly and randomly with time. At any point $P$, sometimes constructive and sometimes destructive interference takes place. If the intensity due to each source is $I$, the resultant intensity rapidly and randomly changes between zero and $4I$ so that the average observable intensity is $2I$. No interference effect is, therefore, observed. For observable interference, the sources must be coherent.

One way to obtain a pair of coherent sources is to obtain two sound waves from the same source by

dividing the original wave along two different paths and then combining them. The two waves then differ in phase only because of different paths travelled.

A popular demonstration of interference of sound is given by the Quinke's apparatus figure (16.8).

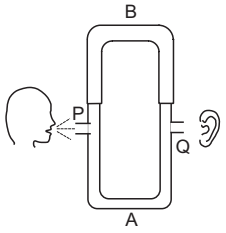



Figure 16.8

Sound produced near the end $P$ travels down in $P$ and is divided in two parts, one going through the tube $A$ and the other through $B$. They recombine and the resultant sound moves along $Q$ to reach the listener (which may be an electronic detector to do a quantitative analysis). The tube $B$ can be slid in and out to change the path length of one of the waves. If the sound is produced at a unique frequency (a tuning fork may be used for it), the wavelength $\lambda\ (= v/\nu)$ has a single value. The intensity at $Q$ will be a maximum or a minimum depending on whether the difference in path lengths is an integral multiple of $\lambda$ or a half integral multiple. Thus, when the tube $B$ is slowly pulled out the intensity at $Q$ oscillates.

**Phase Difference and Path Difference**

From equation (16.11) we see that if a wave travels an extra distance $\Delta x$ with respect to the other wave, a phase difference

$$\delta = \frac{\omega}{v}\Delta x = \frac{2\pi\,\Delta x}{\lambda}$$

is introduced between them.

Note carefully that this is the phase difference introduced due to the different path lengths covered by the waves from their origin. Any initial difference of phase that may exist between the sources must be added to it so as to get the actual phase difference.

***Example 16.5***

*Two sound waves, originating from the same source, travel along different paths in air and then meet at a point. If the source vibrates at a frequency of* 1·0 kHz *and one path is* 83 cm *longer than the other, what will be the nature of interference ? The speed of sound in air is* 332 m $s^{-1}$.

***Solution*** : The wavelength of sound wave is $\lambda = \frac{v}{\nu}$

$$= \frac{332\ \text{m s}^{-1}}{1{\cdot}0 \times 10^{3}\ \text{Hz}} = 0{\cdot}332\ \text{m}.$$

The phase difference between the waves arriving at the point of observation is

$$\delta = \frac{2\pi}{\lambda}\Delta x = 2\pi \times \frac{0{\cdot}83\ \text{m}}{0{\cdot}332\ \text{m}} = 2\pi \times 2{\cdot}5 = 5\pi.$$

As this is an odd multiple of $\pi$, the waves interfere destructively.

**Reflection of Sound Waves**

When there exists a discontinuity in the medium, the wave gets reflected. When a sound wave gets reflected from a rigid boundary, the particles at the boundary are unable to vibrate. Thus, a reflected wave is generated which interferes with the oncoming wave to produce zero displacement at the rigid boundary. At these points (zero displacement), the pressure variation is maximum. Thus, a reflected pressure wave has the same phase as the incident wave. That means, a compression pulse reflects as a compression pulse and a rarefaction pulse reflects as a rarefaction pulse.

A sound wave is also reflected if it encounters a low pressure region. A practical example is when a sound wave travels in a narrow open tube. When the wave reaches an open end, it gets reflected. The force on the particles there due to the outside air is quite small and hence, the particles vibrate with increased amplitude. As a result, the pressure there remains at the average value. Thus, the reflected pressure wave interferes destructively with the oncoming wave. There is a phase change of $\pi$ in the pressure wave when it is reflected by an open end. That means, a compression pulse reflects as a rarefaction pulse and vice versa.

## 16.9 STANDING LONGITUDINAL WAVES AND VIBRATIONS OF AIR COLUMNS

If two longitudinal waves of the same frequency and amplitude travel through a medium in the opposite directions, standing waves are produced. If the equations of the two waves are written as

$$p_1 = p_0 \sin \omega(t - x/v)$$

and $$p_2 = p_0 \sin \omega(t + x/v),$$

the resultant wave is by the principle of superposition,

$$\begin{aligned} p &= p_1 + p_2 \\ &= 2\,p_0 \cos(\omega x/v) \sin \omega t \\ &= 2\,p_0 \cos kx \sin \omega t. \end{aligned}$$

This equation is similar to the equation obtained in chapter 15 for standing waves on a string. Hence, all the characteristics of standing waves on a string

are also present in longitudinal standing waves. At different points in the medium, the pressure amplitudes have different magnitudes. In particular, at certain points the pressure remains permanently at its average value, they are called the *pressure nodes* and midway between the nodes, there are *pressure antinodes* where the amplitude is maximum. The separation between two consecutive nodes or between two consecutive antinodes is $\lambda/2$. It may be noted that a pressure node is a displacement antinode and a pressure antinode is a displacement node.

Standing waves can be produced in air columns trapped in tubes of cylindrical shape. Organ pipes are such vibrating air columns.

**(A) Closed Organ Pipe**

A closed organ pipe is a cylindrical tube having an air column with one end closed. Sound waves are sent in by a source vibrating near the open end. An ingoing pressure wave gets reflected from the fixed end. This inverted wave is again reflected at the open end. After two reflections, it moves towards the fixed end and interferes with the new wave sent by the source in that direction. The twice reflected wave has travelled an extra distance of $2l$ causing a phase advance of $\frac{2\pi}{\lambda} \cdot 2l = \frac{4\pi l}{\lambda}$ with respect to the new wave sent in by the source. Also, the twice reflected wave suffered a phase change of $\pi$ at the open end. The phase difference between the two waves is then $\delta = \frac{4\pi l}{\lambda} + \pi$. The waves interfere constructively if $\delta = 2n\pi$

or, $$\frac{4\pi l}{\lambda} + \pi = 2n\pi$$

or, $$l = (2n - 1)\frac{\lambda}{4}$$

where $n = 1, 2, 3, \ldots$ . This may also be written as

$$l = (2n + 1)\frac{\lambda}{4}, \qquad \ldots \ (16.14)$$

where $n = 0, 1, 2, \ldots$ .

In such a case, the amplitude goes on increasing until the energy dissipated through various damping effects equals the fresh input of energy from the source. Such waves exist in both the directions and they interfere to give standing waves of large amplitudes in the tube. The fixed end is always a pressure antinode (or displacement node) and the open end is a pressure node (or displacement antinode). In fact, the pressure node is not exactly at the open end because the air outside does exert some force on the air in the tube. We shall neglect this end correction for the time being.

The condition for having standing waves in a closed organ pipe is given by equation (16.14),

$$l = (2n + 1)\frac{\lambda}{4}.$$

The frequency $\nu$ is thus given by

$$\nu = \frac{v}{\lambda} = (2n + 1)\frac{v}{4l}. \qquad \ldots \ (16.15)$$

We see that there are certain discrete frequencies with which standing waves can be set up in a closed organ pipe. These frequencies are called *natural frequencies, normal frequencies or resonant frequencies.*

Figure (16.9) shows the variation of excess pressure and displacement of particles in a closed organ pipe for the first three resonant frequencies.

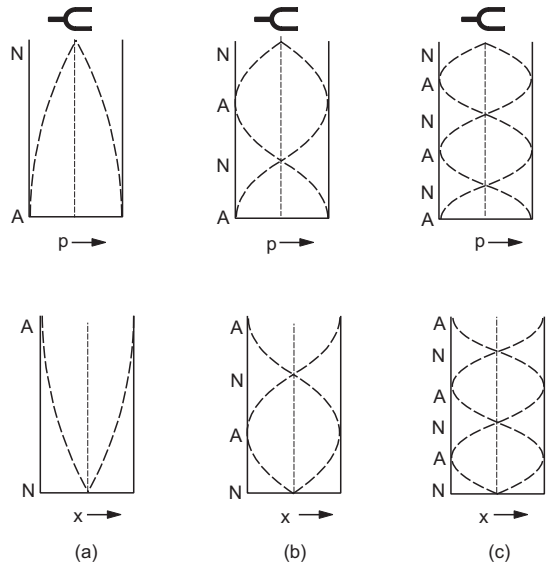



Figure 16.9

The minimum allowed frequency is obtained by putting $n = 0$ in equation (16.15). This is called the fundamental frequency $\nu_0$ of the tube. We have

$$\nu_0 = \frac{v}{4l}. \qquad \ldots \ (16.16)$$

By equation (16.14), $l = \lambda/4$ in this case. A pressure antinode is formed at the closed end and a node is formed at the open end. There are no other nodes or antinodes in between. The air column is said to vibrate in its fundamental mode (figure 16.9a).

Putting $n = 1$ in equation (16.15), we get the first overtone frequency

$$\nu_1 = \frac{3v}{4l} = 3\,\nu_0.$$

By equation (16.14), the length of the tube in this mode is

$$l = 3\lambda/4.$$

In the first overtone mode of vibration (figure 16.9b), there is an antinode and a node in the air column apart from the ends.

In the second overtone, there are two nodes and two antinodes in the column apart from the ends (figure 16.10c). In this mode, $l = 5\lambda/4$ and the frequency is

$$\nu_2 = \frac{5v}{4l} = 5\nu_0 .$$

The higher overtones can be described in a similar way.

Thus, an air column of length $l$ fixed at one end can vibrate with frequencies $(2n + 1)\,\nu_0$ where $\nu_0 = \frac{v}{4l}$ and $n$ is an integer. We see that all the overtone frequencies are harmonics (i.e., integral multiple) of the fundamental frequency but all the harmonics of fundamental frequency are not the allowed frequencies. Only the odd harmonics of the fundamental are the allowed frequencies.

**(B) Open Organ Pipe**

An open organ pipe is again a cylindrical tube containing an air column open at both ends. A source of sound near one of the ends sends the waves in the pipe (figure 16.10). The wave is reflected by the other open end and travels towards the source. It suffers second reflection at the open end near the source and then interferes with the new wave sent by the source. The twice reflected wave is ahead of the new wave coming in by a path difference $2l$. The phase difference is $\delta = \frac{2\pi}{\lambda} 2l = \frac{4\pi l}{\lambda}$. Constructive interference takes place if

$$\delta = 2n\pi$$
$$l = n\lambda/2 \qquad \ldots \quad (16.17)$$

where $n$ = 1, 2, 3, … .

The amplitude then keeps on growing. Waves moving in both the directions with large amplitudes are established and finally standing waves are set up. As already discussed, in this case the energy lost by various damping effects equals the energy input from the source. The frequencies with which a standing wave can be set up in an open organ pipe are

$$\nu = \frac{v}{\lambda} = \frac{nv}{2l} . \qquad \ldots (16.18)$$

Figure (16.10) shows the variation of excess pressure and displacement in the open organ pipe for the first three resonant frequencies. The minimum frequency $\nu_0$ is obtained by putting $n = 1$ in equation (16.18). Thus,

$$\nu_0 = \frac{v}{2l} . \qquad \ldots (16.19)$$

This corresponds to the fundamental mode of vibration. By equation (16.18), $l = \lambda/2$ in the fundamental mode of vibration (figure 16.10a).

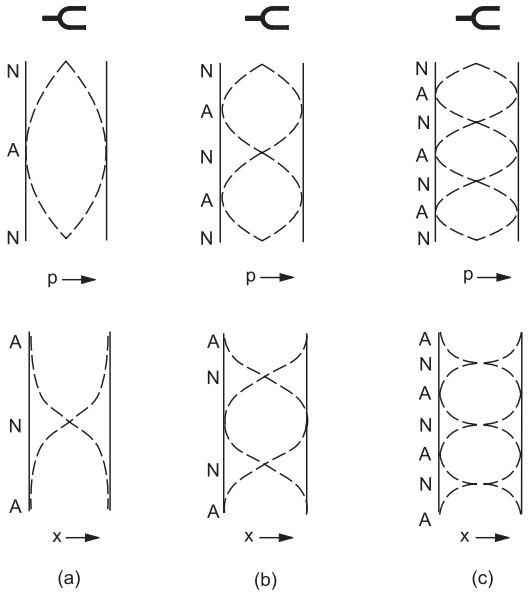



Figure 16.10

As the air at both ends is free to vibrate (neglecting the effect of the pressure of the air outside the pipe), pressure nodes are formed at these points. There is an antinode in between these nodes.

If the source vibrates at the frequency $\nu_0 = \frac{v}{2l}$, it will set up the column in the fundamental mode of vibration.

Putting $n = 2$ in equation (16.18), we get the frequency of the first overtone mode as

$$\nu_1 = 2\,\frac{v}{2l} = 2\nu_0 .$$

The length of the tube is, by equation (16.17), $l = \lambda$. There are two pressure antinodes and one node apart from the nodes at the two ends. In the $n$th overtone, there are $(n + 1)$ pressure antinodes. The frequency of the $n$th overtone is given by

$$\nu_n = \frac{nv}{2l} = n\nu_0 .$$

The overtone frequencies are $\frac{2v}{2l}$, $\frac{3v}{2l}$, ⋯, etc. All the overtone frequencies are harmonics of the fundamental and all the harmonics of the fundamental are allowed overtone frequencies. There are no missing harmonics in an open organ pipe. The quality of sound from an open organ pipe is, therefore, richer than that from a closed organ pipe in which all the even harmonics of the fundamental are missing.

***Example 16.6***

*An air column is constructed by fitting a movable piston in a long cylindrical tube. Longitudinal waves are sent in the tube by a tuning fork of frequency* 416 Hz. *How*

*far from the open end should the piston be so that the air column in the tube may vibrate in its first overtone ? Speed of sound in air is* 333 m $s^{-1}$.

***Solution :*** The piston provides the closed end of the column and an antinode of pressure is formed there. At the open end, a pressure node is formed. In the first overtone there is one more node and an antinode in the column as shown in figure (16.11). The length of the tube should then be $3\lambda/4$.

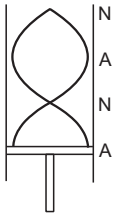



Figure 16.11

The wavelength is $\lambda = \frac{v}{\nu} = \frac{333 \text{ m s}^{-1}}{416 \text{ s}^{-1}} = 0{\cdot}800 \text{ m}$

Thus, the length of the tube is

$$\frac{3\lambda}{4} = \frac{3 \times 0{\cdot}800 \text{ m}}{4} = 60{\cdot}0 \text{ cm}.$$

## 16.10 DETERMINATION OF SPEED OF SOUND IN AIR

### (a) Resonance Column Method

Figure (16.12) shows schematically the diagram of a simple apparatus used in laboratories to measure the speed of sound in air. A long cylindrical glass tube (say about 1 m) is fixed on a vertical wooden frame. It is also called a resonance tube. A rubber tube connects the lower end of this glass tube to a vessel which can slide vertically on the same wooden frame. A meter scale is fitted parallel to and close to the glass tube.

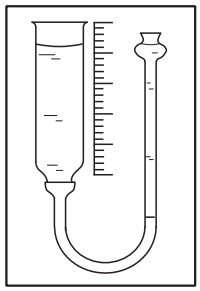


Figure 16.12

The vessel contains water which also goes in the resonance tube through the rubber tube. The level of water in the resonance tube is same as that in the vessel. Thus, by sliding the vessel up and down, one can change the water level in the resonance tube.

A tuning fork (frequency 256 Hz if the tube is 1 m long) is vibrated by hitting it on a rubber pad and is held near the open end of the tube in such a way that the prongs vibrate parallel to the length of the tube. Longitudinal waves are then sent in the tube.

The water level in the tube is initially kept high. The tuning fork is vibrated and kept close to the open end, and the loudness of sound coming from the tube is estimated. The vessel is brought down a little to decrease the water level in the resonance tube. The tuning fork is again vibrated, kept close to the open end and the loudness of the sound coming from the tube is estimated. The process is repeated until the water level corresponding to the maximum loudness is located. Fine adjustments of water level are made to locate accurately the level corresponding to the maximum loudness. The length of the air column is read on the scale attached. In this case, the air column vibrates in resonance with the tuning fork. The minimum length of the air column for which the resonance takes place corresponds to the fundamental mode of vibration. A pressure antinode is formed at the water surface (which is the closed end of the air column) and a pressure node is formed near the open end. In fact, the node is formed slightly above the open end (end correction) because of the air-pressure from outside.

Thus, for the first resonance the length $l_1$ of the air column in the resonance tube is given by

$$l_1 + d = \frac{\lambda}{4}, \qquad \ldots \text{ (i)}$$

where $d$ is the end correction.

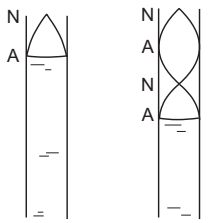



Figure 16.13

The length of the air column is increased to a little less than three times of $l$. The water level is adjusted so that the loudness of the sound coming from the tube becomes maximum again. The length of the air column is noted on the scale. In this second resonance the air column vibrates in the first overtone. There is one node and one antinode in between the ends of the column. The length $l_2$ of the column is given by

$$l_2 + d = 3\lambda/4. \qquad \ldots \text{ (ii)}$$

By (i) and (ii),

$$(l_2 - l_1) = \lambda/2, \quad \text{or,} \quad \lambda = 2(l_2 - l_1).$$

The frequency of the wave is same as the frequency of the tuning fork. Thus, the speed of sound in the air of the laboratory is

$$v = \nu\lambda = 2(l_2 - l_1)\,\nu. \qquad \ldots \quad (16.20)$$

**(b) Kundt's Tube Method of Determining the Speed of Sound in a Gas**

The resonance column method described above can be used to find the speed of sound in air only, as the tube is open at the end to the atmosphere. In the Kundt's method, a gas is enclosed in a long cylindrical tube closed at both ends, one by a disc $D$ and the other by a movable piston (figure 16.14). A metal rod is welded with the disc and is clamped exactly at the middle point. The length of the tube can be varied by moving the movable piston. Some powder is sprinkled inside the tube along its length.

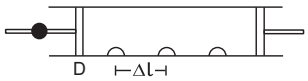



Figure 16.14

The rod is set into longitudinal vibrations electronically or by rubbing it with some rosined cloth or otherwise. If the length of the gas column is such that one of its resonant frequencies is equal to the frequency of the longitudinal vibration of the rod, standing waves are formed in the gas. The powder particles at the displacement antinodes fly apart due to the violent disturbance there, whereas the powder at the displacement nodes remain undisturbed because the particles here do not vibrate. Thus, the powder which was initially dispersed along the whole length of the tube collects in heaps at the displacement nodes. By measuring the separation $\Delta l$ between the successive heaps, one can find the wavelength of sound in the enclosed gas

$$\lambda = 2\Delta l.$$

The length of the gas column is adjusted by moving the movable piston such that the gas resonates with the disc and the wavelength $\lambda$ is obtained. If the frequency of the longitudinal vibration of the rod is $\nu$, the speed of sound in the gas is given by

$$v = \nu\lambda = 2\,\Delta l\,\nu. \qquad \ldots \text{ (i)}$$

If the frequency of the longitudinal vibrations in the rod is not known, the experiment is repeated with air filled in the tube. The length between the heaps of the powder, $\Delta l'$ is measured. The speed of sound in air is then

$$v' = 2\,\Delta l'\,\nu. \qquad \ldots \text{ (ii)}$$

By (i) and (ii),

$$\frac{v}{v'} = \frac{\Delta l}{\Delta l'}$$

or,
$$v = v'\,\frac{\Delta l}{\Delta l'}.$$

Knowing the speed $v'$ of sound in air, one can find the speed in the experimental gas.

Kundt's tube method can also be used to measure the speed of sound in a solid. Air at normal pressure is filled in the tube. The speed of sound in air is supposed to be known. The rod attached to the disc is clamped at the middle and is set into longitudinal vibration. The rod behaves like an open organ pipe as the two ends are free to vibrate. Assuming that it vibrates in its fundamental mode of vibration, the clamped point is a pressure antinode and the two ends of the rod are pressure nodes. Thus, the wavelength of sound in the rod is $\lambda = 2l$. If the powder piles up at successive distances $\Delta l$ and the speed of sound in air is $v_a$, then $v_a = 2\,\Delta l\,\nu$. Also, if $v$ be the speed of sound in the rod, $v = \nu\lambda = 2\,\nu l$.

Thus,
$$\frac{v}{v_a} = \frac{l}{\Delta l} \quad \text{or,} \quad v = \frac{l}{\Delta l}\,v_a.$$

As the speed $v_a$ of sound in air is known, measurements of $l$ and $\Delta l$ give the speed of sound in the material of the rod.

## 16.11 BEATS

So far we have considered superposition of two sound waves of equal frequency. Let us now consider two sound waves having equal amplitudes and travelling in a medium in the same direction but having slightly different frequencies. The equations of the two waves are given by

$$p_1 = p_0 \sin \omega_1(t - x/v)$$
$$p_2 = p_0 \sin \omega_2(t - x/v),$$

where we have chosen the two waves to be in phase at $x = 0, t = 0$. The speed of sound wave does not depend on its frequency and hence, same wave speed $v$ is used for both the equations. The angular frequencies $\omega_1$ and $\omega_2$ only slightly differ so that the difference $|\omega_1 - \omega_2|$ is small as compared to $\omega_1$ or $\omega_2$. By the principle of superposition, the resultant change in pressure is

$$p = p_1 + p_2$$
$$= p_0[\sin \omega_1(t - x/v) + \sin \omega_2(t - x/v)]$$
$$= 2\,p_0 \cos\left[\frac{\omega_1 - \omega_2}{2}(t - x/v)\right] \sin\left[\frac{\omega_1 + \omega_2}{2}(t - x/v)\right].$$

Writing $|\omega_1 - \omega_2| = \Delta\omega$, and $\dfrac{\omega_1 + \omega_2}{2} = \omega$, $\quad \ldots$ (i)

the resultant change in pressure is

$$p = 2\,p_0 \cos\frac{\Delta\omega}{2}(t - x/v)\,\sin\omega(t - x/v) \quad \ldots \quad (16.21)$$

$$= A\,\sin\omega(t - x/v),$$

where $A = 2\,p_0 \cos\frac{\Delta\omega}{2}(t - x/v)$. $\ldots$ (16.22)

As $\omega >> \frac{\Delta\omega}{2}$, the term $A$ varies slowly with time as compared to $\sin\omega(t - x/v)$. Thus, we can interpret this equation by saying that the resultant disturbance is a wave of angular frequency $\omega = (\omega_1 + \omega_2)/2$ whose amplitude varies with time and is given by equation (16.22). As a negative value of amplitude has no meaning, the amplitude is, in fact, $| A |$.

Let us concentrate our attention to a particular position $x$ and look for the pressure change there as a function of time. Equations (16.21) and (16.22) tell that the pressure oscillates back and forth with a frequency equal to the average frequency $\omega$ of the two waves but the amplitude of pressure variation itself changes periodically between $2p_0$ and zero.

Figure (16.15) shows the plots of

$$A = 2\,p_0 \cos\frac{\Delta\omega}{2}(t - x/v),\; B = \sin\omega(t - x/v)$$

and their product

$$p = 2\,p_0 \cos\frac{\Delta\omega}{2}(t - x/v)\,\sin\omega(t - x/v)$$

as a function of time for a fixed $x$.

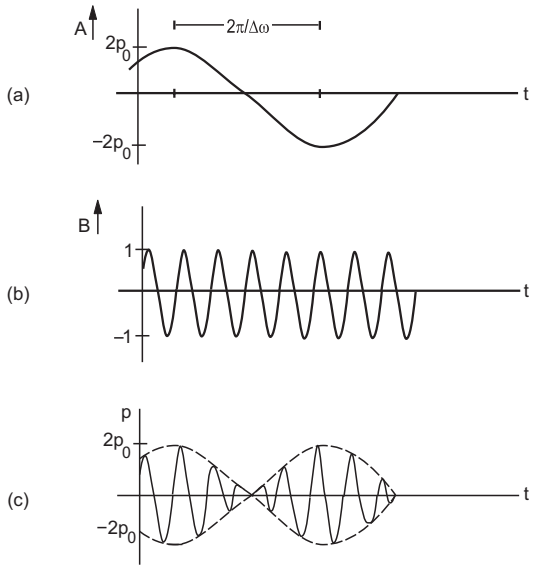



Figure 16.15

**Variation of Intensity at a Point**

The amplitude $| A |$ oscillates between 0 to $2\,p_0$ with a frequency which is double of the frequency of $A$. This is because as $A$ oscillates from $2\,p_0$ to $-2\,p_0$ (i.e., half the oscillation), the amplitude $| A |$ covers full oscillation from $2\,p_0$ to zero to $2\,p_0$ (figure 16.16).

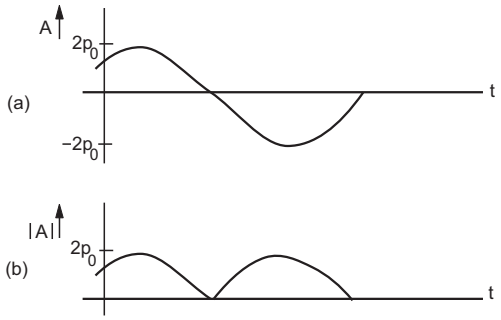



Figure 16.16

By equation (16.22), the frequency of variation of $A$ is $\frac{\Delta\omega/2}{2\pi} = \frac{\Delta\omega}{4\pi}$. Thus, the frequency of amplitude variation is

$$\nu' = 2 \times \frac{\Delta\omega}{4\pi} = \frac{|\,\omega_1 - \omega_2\,|}{2\pi} = |\,\nu_1 - \nu_2\,|,$$

where $\nu_1$ and $\nu_2$ are the frequencies of the original waves. Notice that we have written this frequency as $|\,\nu_1 - \nu_2\,|$. It is the difference between $\nu_1$ and $\nu_2$ with which the amplitude oscillates. The intensity is proportional to the square of the pressure amplitude and it also varies periodically with frequency $|\,\nu_1 - \nu_2\,|$. This phenomenon of periodic variation of intensity of sound when two sound waves of slightly different frequencies interfere, is called *beats*. Specifically, one cycle of maximum intensity and minimum intensity is counted as one beat, so that the frequency of beats is $|\,\nu_1 - \nu_2\,|$.

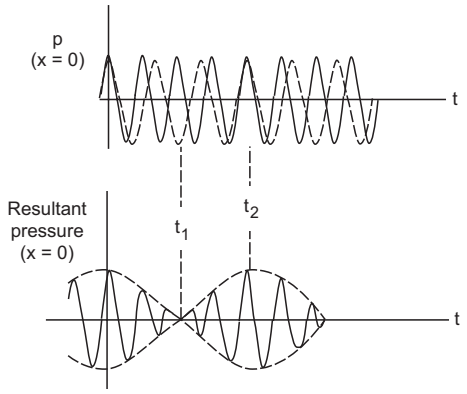



Figure 16.17

In figure (16.17), we explain the formation of beats graphically. The plots in part (a) show the pressure variation due to individual waves at a fixed position $x = 0$ as a function of time. At $t = 0$, the two waves produce pressure changes in phase. But as the frequencies and hence the time periods are slightly

different, the phase difference grows and at a time $t_1$, the two pressure changes become out of phase. The resultant becomes zero at this time. The phase difference increases further and at time $t_2$, it becomes $2\pi$. The pressure changes are again in phase. The superposition of the two waves (figure 16.17) gives the resultant pressure change as a function of time which is identical in shape to the part (c) of figure (16.15).

The phenomenon of beats can be observed by taking two tuning forks of the same frequency and putting some wax on the prongs of one of the forks. Loading with wax decreases the frequency of a tuning fork a little. When these two forks are vibrated and kept side by side, the listener can recognise the periodic variation of loudness of the resulting sound. The number of beats per second equals the difference in frequency.

For beats to be audible, the frequency $|\nu_1 - \nu_2|$ should not be very large. An average human ear cannot distinguish the variation of intensity if the variation is more than 16 times per second. Thus, the difference of the component frequencies should be less than 16 Hz for the beats to be heard.

***Example 16.7***

*A tuning fork A of frequency* 384 Hz *gives* 6 beats *in* 2 seconds *when sounded with another tuning fork B. What could be the frequency of B ?*

***Solution*** : The frequency of beats is $|\nu_1 - \nu_2|$, which is 3 Hz according to the problem. The frequency of the tuning fork $B$ is either 3 Hz more or 3 Hz less than the frequency of $A$. Thus, it could be either 381 Hz or 387 Hz.

## 16.12 DIFFRACTION

When waves are originated by a vibrating source, they spread in the medium. If the medium is homogeneous and isotropic, the waves from a point source have spherical wavefronts, the rays going in all directions. Far from the source, the wavefronts are nearly planes. The shape of the wavefront is changed when the wave meets an obstacle or an opening in its path. This leads to bending of the wave around the edges. For example, if a small cardboard is placed between a source of sound and a listener, the sound beyond the cardboard is not completely stopped, rather the waves bend at the edges of the cardboard to reach the listener. If a plane wave is passed through a small hole (an opening in a large obstacle), spherical waves are obtained on the other side as if the hole itself is a source sending waves in all directions (it is not a real source, no backward spherical waves are observed). Such bending of waves from an obstacle or an opening is called *diffraction.*

Diffraction is a characteristic property of wave motion and all kinds of waves exhibit diffraction.

The diffraction effects are appreciable when the dimensions of openings or the obstacles are comparable or smaller than the wavelength of the wave. If the opening or the obstacle is large compared to the wavelength, the diffraction effects are almost negligible.

The frequency of audible sound ranges from about 20 Hz to 20 kHz. Velocity of sound in air is around 332 m $s^{-1}$. The wavelength ($\lambda = v/\nu$) of audible sound in air thus ranges from 16 m to 1·6 cm. Quite often, the wavelength of sound is much larger than the obstacles or openings and diffraction is prominently displayed.

## 16.13 DOPPLER EFFECT

When a tuning fork is vibrated in air, sound waves travel from the fork. An observer stationed at some distance $x$ from the fork receives the sound as the wave disturbs the air near his ear and the pressure varies between a maximum and a minimum. The pitch of the sound heard by the observer depends on how many times the pressure near the ear oscillates per unit time. Each time the fork moves forward it sends a compression pulse and each time it moves backward it sends a rarefaction pulse. Suppose the source and the observer are both at rest with respect to the medium. Each compression or rarefaction pulse, sent by the tuning fork, takes same time to reach the air near the ear. Thus, the pressure near the ear oscillates as many times as the fork oscillates in a given interval. The frequency observed is then equal to the frequency of the source.

However, if the source or the observer or both, move with respect to the medium, the frequency observed may be different from the frequency of the source. This apparent change in frequency of the wave due to motion of the source or the observer is called the *Doppler effect*.

### Observer Stationary and Source Moving

Now suppose the observer is at rest with respect to the medium and the source moves towards the observer at a speed $u$ which is less than the wave speed $v$.

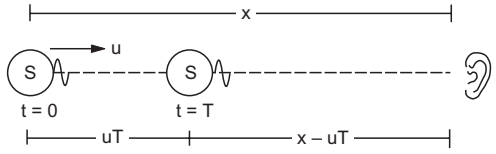



Figure 16.18

If the frequency of vibration of the source is $\nu_0$, it sends compression pulses at a regular interval of $T = 1/\nu_0$. Suppose the separation between the source and the observer is $x$ (figure 16.20) when a compression pulse is emitted at $t = 0$. The next compression pulse will be emitted after a time $T$. The source will travel a distance $uT$ in this time and hence this second compression wave is emitted from a place which is at a distance $x - uT$ from the observer. The first pulse takes a time $x/v$ to reach the observer whereas the next one takes $\frac{x - uT}{v}$.

Thus, the first compression wave reaches the observer at $t_1 = x/v$ and the next compression wave reaches at $t_2 = T + \frac{x - uT}{v}$. The time interval between the consecutive compression pulses detected by the observer is, therefore,

$$T' = t_2 - t_1$$

$$= T + \frac{x - uT}{v} - \frac{x}{v} = \left(1 - \frac{u}{v}\right) T = \frac{v - u}{v} T.$$

The apparent frequency of the sound as experienced by the observer is

$$\nu' = \frac{1}{T'},$$

or, $$\nu' = \frac{v}{v - u} \nu_0. \quad \ldots (16.23)$$

Similarly, if the source recedes from the observer at a speed $u$, the apparent frequency will be

$$\nu' = \frac{v}{v + u} \nu_0. \quad \ldots (16.24)$$

**Source Stationary and Observer Moving**

Next, consider the case when the source remains stationary with respect to the medium and the observer approaches the source with a speed $u$.

As the source remains stationary in the medium, compression pulses are emitted at regular interval $T$ from the same point in the medium. These pulses travel down the medium with a speed $v$ and at any instant the separation between two consecutive compression pulses is $\lambda = vT$ (figure 16.19)

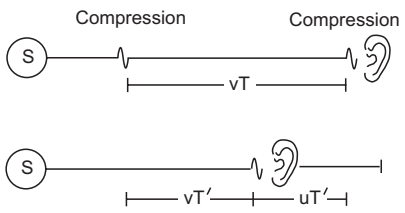



Figure 16.19

When the observer receives a compression pulse, the next compression pulse (towards the source) is a distance $vT$ away from it. This second compression pulse moves towards the observer at a speed $v$ and the observer moves towards it at a speed $u$. As a result, the observer will receive this second compression wave a time $T'$ after receiving the first one where

$$T' = \frac{vT}{v + u}.$$

The apparent frequency of sound experienced by the observer is then $\nu' = \frac{1}{T'}$

or, $$\nu' = \frac{v + u}{v} \nu_0. \quad \ldots (16.25)$$

Note that, in this case, it is not the same part of air that gives the sensation of pressure variation to the ear at frequency $\nu'$. The pressure in any part of the air still oscillates with a frequency $\nu_0$ but the observer moves in the medium to detect the pressure of some other part which reaches its maximum a little earlier. Similarly, if the source is stationary in the medium and the observer recedes from it at a speed $u$, the apparent frequency will be

$$\nu' = \frac{v - u}{v} \nu. \quad \ldots (16.26)$$

The equation (16.23) through (16.26) may be generalised as

$$\nu = \frac{v + u_o}{v - u_s} \nu_0 \quad \ldots (16.27)$$

where,

$v$ = speed of sound in the medium.

$u_o$ = speed of the observer with respect to the medium, considered positive when it moves towards the source and negative when it moves away from the source

and $u_s$ = speed of the source with respect to the medium, considered positive when it moves towards the observer and negative when it moves away from the observer.

It should be carefully noted that the speeds $u_s$ and $u_o$ are to be written with respect to the medium carrying the sound. If the medium itself is moving with respect to the given frame of reference, appropriate calculations must be made to obtain the speeds of the source and the observer with respect to the medium.

***Example 16.8***

*A sound detector is placed on a railway platform. A train, approaching the platform at a speed of* 36 km/h, *sounds its whistle. The detector detects* 12·0 kHz *as the most dominant frequency in the whistle. If the train stops at*

*the platform and sounds the whistle, what would be the most dominant frequency detected ? The speed of sound in air is* 340 m $s^{-1}$.

***Solution*** **:** Here the observer (detector) is at rest with respect to the medium (air). Suppose the dominant frequency as emitted by the train is $\nu_0$. When the train is at rest at the platform, the detector will detect the dominant frequency as $\nu_0$. When this same train was approaching the observer, the frequency detected was,

$$\nu' = \frac{v}{v - u_s}\nu_0$$

or, $$\nu_0 = \frac{v - u_s}{v}\nu' = \left(1 - \frac{u_s}{v}\right)\nu'.$$

The speed of the source is

$$u_s = 36 \text{ km h}^{-1} = \frac{36 \times 10^3 \text{ m}}{3600 \text{ s}} = 10 \text{ m s}^{-1}.$$

Thus, $$\nu_0 = \left(1 - \frac{10}{340}\right) \times 12{\cdot}0 \text{ kHz}$$

$$= 11{\cdot}6 \text{ kHz}.$$

We have derived the equations for Doppler shift in frequency assuming that the motion of the source or the observer is along the line joining the two. If the motion is along some other direction, the component of the velocity along the line joining the source and the observer should be used for $u_s$ and $u_o$.

It is helpful to remember that the apparent frequency is larger than the actual, if the separation between the source and the observer is decreasing and is smaller if the separation is increasing. If you are standing on a platform and a whistling train passes by, you can easily notice the change in the pitch of the whistle. When the train is approaching you, the pitch is higher. As it passes through, there is a sudden fall in the pitch.

**Change in Wavelength**

If the source moves with respect to the medium, the wavelength becomes different from the wavelength observed when there is no relative motion between the source and the medium. The formula for apparent wavelength may be derived immediately from the relation $\lambda = v/\nu$ and equation (16.23). It is

$$\lambda' = \frac{v - u}{v}\lambda. \qquad \ldots \ (16.28)$$

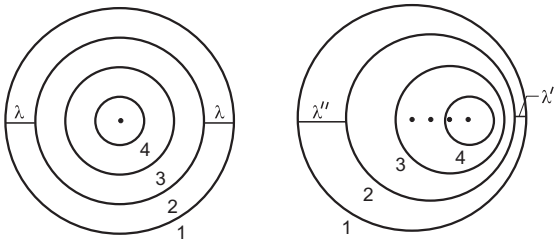



Figure 16.20

Figure 16.20 shows qualitatively the change in wavelength as the source moves through the medium. The circles represent the region of space where the pressure is maximum. They are wavefronts separated by a wavelength. We can say that these wavefronts representing the pressure maxima originate from the source and spread in all directions with a speed $v$.

Labels 1, 2, 3, 4 show wavefronts emitted successively at regular interval $T = 1/\nu$ from the source. Each wavefront will have its centre at the position where the source was situated while emitting the wavefront. The radius of a preceding wavefront will exceed the next one by an amount $\lambda = vt$.

When the source stays stationary, all the wavefronts are originated from the same position of the source and the separation between the successive wavefronts is equal to the difference in their radii. However, when the source moves, the centres of the wavefronts differ in position, so that the spacing between the successive wavefronts decreases along the direction of motion and increases on the opposite side. This separation being the new wavelength, the wavelength changes due to the motion of the source.

## 16.14 SONIC BOOMS

In the discussion of the Doppler effect, we considered only subsonic velocities for the source and the observer, that is, $u_s < v$ and $u_o < v$. What happens when the source moves through the medium at a speed $u_s$ greater than the wave speed $v$ ? A supersonic plane travels in air with a speed greater than the speed of sound in air. It sends a cracking sound called sonic boom which can break glass dishes, window panes and even cause damage to buildings. Let us extend figure (16.20) for the case where a tiny source moves through air with a speed $u_s > v$. The wavefronts are drawn for the pressure maxima. The spherical wavefronts intersect over the surface of a cone with the apex at the source. Because of constructive interference of a large number of waves arriving at the same instant on the surface of the cone, pressure waves of very large amplitude are sent with the conical wavefront. Such waves are one variety of shock waves.

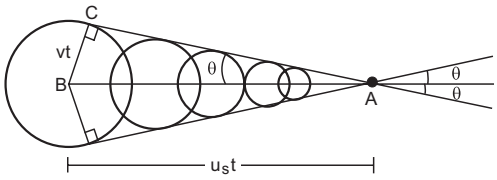



Figure 16.21

From the triangle *ABC*, the semivertical angle θ of the cone is given by

$$\sin\theta = \frac{vt}{u_s t} = \frac{v}{u_s}.$$

The ratio $\frac{u_s}{v}$ is called the "Mach Number".

As the tiny source moves, it drags the cone with it. When an observer on ground is intercepted by the cone surface, the boom is heard. There is a common misconception that the boom is produced at the instant the speed of the plane crosses the speed of sound and once it achieves the supersonic speed it sends no further shock wave. The sonic boom is not a one time affair that occurs when the speed just exceeds the speed of sound. As long as the plane moves with a supersonic speed, it continues to send the boom.

## 16.15 MUSICAL SCALE

A musical scale is a sequence of frequencies which have a particularly pleasing effect on the human ear. A widely used musical scale, called diatonic scale, has eight frequencies covering an octave. Each frequency is called a *note*. Table (16.3) gives these frequencies together with their Indian and Western names with the lowest of the octave at 256 Hz.

**Table 16.3**

| Symbol | Indian name | Western name | Frequency Hz |
|---|---|---|---|
| C | Sa | Do | 256 |
| D | Re | Re | 288 |
| E | Ga | Mi | 320 |
| F | Ma | Fa | $341\frac{1}{3}$ |
| G | Pa | Sol | 384 |
| A | Dha | La | $426\frac{2}{3}$ |
| B | Ni | Ti | 480 |
| $C_1$ | Sa | Do | 512 |

## 16.16 ACOUSTICS OF BUILDINGS

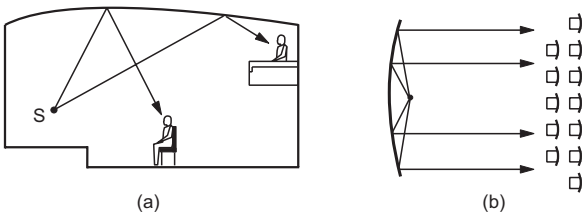



Figure 16.22

While designing an auditorium for speech or musical concerts, one has to take proper care for the absorption and reflection of sound. If these factors are poorly considered, a listener in the auditorium will not be able to clearly hear the sound. To have the intensity of sound almost uniform in the hall, the walls and the ceilings may be curved in proper fashion. Figure (16.22) shows a curved ceiling used to make sound audible uniformly at the balcony seats and the seats on the floor. Figure (16.22b) shows the use of a parabolic wall to make sound uniform across the width of the hall. Reflection of sound is helpful in maintaining a good loudness level throughout the hall. However, it also has several unwanted effects. Sound can reach a listener directly from the source as well as after reflection from a wall or the ceiling. This leads to *echo* which is heard after an interval of hearing the first sound. This echo interferes with the next sound signal affecting the clarity.

Another effect of multiple reflection is the *reverberation*. A listener hears the direct sound, sound coming after one reflection, after two reflections and so on. The time interval between the successive arrival of the same sound signal keeps on decreasing. Also, the intensity of the signal gradually decreases.

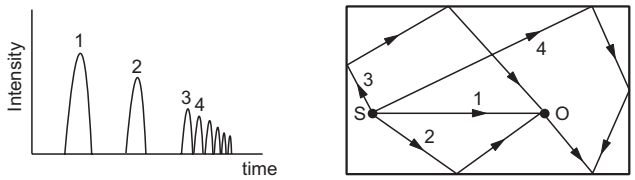



Figure 16.23

Figure (16.23) shows a typical situation. After sometime the signals coming from multiple reflections are so close that they form an almost continuous sound of decreasing intensity. This part of the sound is called *reverberant sound*. The time for which the reverberation persists is a major factor in the acoustics of halls. Quantitatively, the time taken by the reverberant sound to decrease its intensity by a factor of $10^6$ is called the *reverberation time*. If the reverberation time is too large, it disturbs the listener. A sound signal from the source may not be clearly heard due to the presence of reverberation of the previous signal. There are certain materials which absorb sound very effectively. Reverberation may be decreased by fixing such materials on the walls, ceiling, floor, furnitures, etc. However, this process decreases the overall intensity level in the hall. Also in musical concerts, some amount of reverberation adds to the quality of music. An auditorium with a very small reverberation time is called *acoustically dead*. Thus, one has to make a compromise. The reverberation time should not be very large, otherwise unpleasant echos will seriously affect the clarity. On the other hand it should not be too small, otherwise intensity and quality will be seriously affected.

Electrical amplifying systems are often used in large auditorium. If a loudspeaker is kept near the back or at the side walls, a listener may hear the sound from the speaker earlier than the sound from the stage. Loudspeakers are, therefore, placed with proper inclinations and electronic delays are installed so that sound from the stage and from the loudspeaker reach a listener almost simultaneously. Another problem with electrical amplifying systems is *ringing*. The amplifying system may pick up sound from the loudspeaker to again amplify it. This gives a very unpleasant whistling sound.

One also has to avoid any noise coming from outside the auditorium or from different equipment inside the auditorium. Sound of fans, exhausts, airconditioners, etc., often create annoyance to the listener.

### *Worked Out Examples*

**1.** *An ultrasound signal of frequency* 50 kHz *is sent vertically into sea water. The signal gets reflected from the ocean bed and returns to the surface* 0·80 s *after it was emitted. The speed of sound in sea water is* 1500 m $s^{-1}$. *(a) Find the depth of the sea. (b) What is the wavelength of this signal in water* ?

***Solution*** **:** (a) Let the depth of the sea be $d$. The total distance travelled by the signal is $2d$. By the question,

$$2d = (1500 \text{ m s}^{-1})\,(0{\cdot}8 \text{ s}) = 1200 \text{ m}$$

or, $$d = 600 \text{ m}.$$

(b) Using the equation $v = \nu\lambda$,

$$\lambda = \frac{v}{\nu} = \frac{1500 \text{ m s}^{-1}}{50 \times 10^3 \text{ s}^{-1}} = 3{\cdot}0 \text{ cm}.$$

**2.** *An aeroplane is going towards east at a speed of* 510 km $h^{-1}$ *at a height of* 2000 m. *At a certain instant, the sound of the plane heard by a ground observer appears to come from a point vertically above him. Where is the plane at this instant* ? *Speed of sound in air* = 340 m $s^{-1}$.

***Solution*** **:**

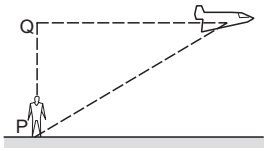



Figure 16-W1

The sound reaching the ground observer $P$, was emitted by the plane when it was at the point $Q$ vertically above his head. The time taken by the sound to reach the observer is

$$t = \frac{2000 \text{ m}}{340 \text{ m s}^{-1}} = \frac{100}{17} \text{ s}.$$

The distance moved by the plane during this period is

$$d = (510 \text{ km h}^{-1})\left(\frac{100}{17} \text{ s}\right)$$

$$= \frac{30 \times 10^5}{3600} \text{ m} = 833 \text{ m}.$$

Thus, the plane will be 833 m ahead of the observer on its line of motion when he hears the sound coming vertically to him.

**3.** *The equation of a sound wave in air is given by*

$$p = (0{\cdot}01 \text{ N m}^{-2}) \sin[(1000 \text{ s}^{-1})\,t - (3{\cdot}0 \text{ m}^{-1})\,x]$$

*(a) Find the frequency, wavelength and the speed of sound wave in air. (b) If the equilibrium pressure of air is* $1{\cdot}0 \times 10^5$ N $m^{-2}$, *what are the maximum and minimum pressures at a point as the wave passes through that point.*

***Solution*** **:** (a) Comparing with the standard form of a travelling wave

$$p = p_0 \sin[\omega(t - x/v)]$$

we see that $\omega = 1000 \text{ s}^{-1}$. The frequency is

$$\nu = \frac{\omega}{2\pi} = \frac{1000}{2\pi} \text{ Hz} = 160 \text{ Hz}.$$

Also from the same comparison, $\omega/v = 3{\cdot}0 \text{ m}^{-1}$

or, $$v = \frac{\omega}{3{\cdot}0 \text{ m}^{-1}} = \frac{1000 \text{ s}^{-1}}{3{\cdot}0 \text{ m}^{-1}}$$

$$\approx 330 \text{ m s}^{-1}.$$

The wavelength is $\lambda = \frac{v}{\nu} = \frac{330 \text{ m s}^{-1}}{160 \text{ Hz}} = 2{\cdot}1 \text{ m}.$

(b) The pressure amplitude is $p_0 = 0{\cdot}01$ N $m^{-2}$. Hence, the maximum and minimum pressures at a point in the wave motion will be $(1{\cdot}01 \times 10^5 \pm 0{\cdot}01)$ N $m^{-2}$.

**4.** *A sound wave of frequency* 10 kHz *is travelling in air with a speed of* 340 m $s^{-1}$. *Find the minimum separation between two points where the phase difference is* 60°.

***Solution*** **:** The wavelength of the wave is

$$\lambda = \frac{v}{\nu} = \frac{340 \text{ m s}^{-1}}{10 \times 10^3 \text{ s}^{-1}} = 3{\cdot}4 \text{ cm}.$$

The wave number is $k = \frac{2\pi}{\lambda} = \frac{2\pi}{3{\cdot}4} \text{ cm}^{-1}$.

The phase of the wave is $(kx - \omega t)$. At any given instant, the phase difference between two points at a separation $d$ is $kd$. If this phase difference is 60°, i.e., $\pi/3$ radian;

$$\frac{\pi}{3} = \left(\frac{2\pi}{3{\cdot}4} \text{ cm}^{-1}\right) d \text{ or } d = \frac{3{\cdot}4}{6} \text{ cm} = 0{\cdot}57 \text{ cm}.$$

**5.** *On a winter day sound travels* 336 meters *in one second. Find the atmospheric temperature. Speed of sound at* $0°C = 332\ m\ s^{-1}$.

***Solution*** **:** The speed of sound is proportional to the square root of the absolute temperature.

The speed of sound at 0°C or 273 K is 332 m $s^{-1}$. If the atmospheric temperature is $T$,

$$\frac{336\ m\ s^{-1}}{332\ m\ s^{-1}} = \sqrt{\frac{T}{273\ K}}$$

or, $$T = \left(\frac{336}{332}\right)^2 \times 273\ K = 280\ K$$

or, $$t = 7°C.$$

**6.** *The constant* $\gamma$ *for oxygen as well as for hydrogen is* 1·40. *If the speed of sound in oxygen is* 470 m $s^{-1}$, *what will be the speed in hydrogen at the same temperature and pressure* ?

***Solution*** **:** The speed of sound in a gas is given by $v = \sqrt{\frac{\gamma P}{\rho}}$. At STP, 22·4 litres of oxygen has a mass of 32 g whereas the same volume of hydrogen has a mass of 2 g. Thus, the density of oxygen is 16 times the density of hydrogen at the same temperature and pressure. As $\gamma$ is same for both the gases,

$$\frac{v\ (\text{hydrogen})}{v\ (\text{oxygen})} = \sqrt{\frac{\rho(\text{oxygen})}{\rho\ (\text{hydrogen})}}$$

or, $$v(\text{hydrogen}) = 4\ v\ (\text{oxygen})$$
$$= 4 \times 470\ m\ s^{-1} = 1880\ m\ s^{-1}.$$

**7.** *A microphone of cross-sectional area* $0{\cdot}80\ cm^2$ *is placed in front of a small speaker emitting* 3·0 W *of sound output. If the distance between the microphone and the speaker is* 2·0 m, *how much energy falls on the microphone in* 5·0 s?

***Solution*** **:** The energy emitted by the speaker in one second is 3·0 J. Let us consider a sphere of radius 2·0 m centred at the speaker. The energy 3·0 J falls normally on the total surface of this sphere in one second. The energy falling on the area $0{\cdot}8\ cm^2$ of the microphone in one second

$$= \frac{0{\cdot}8\ cm^2}{4\pi\ (2{\cdot}0\ m)^2} \times 3{\cdot}0\ J = 4{\cdot}8 \times 10^{-6}\ J.$$

The energy falling on the microphone in 5·0 s is

$$4{\cdot}8 \times 10^{-6}\ J \times 5 = 24\ \mu J.$$

**8.** *Find the amplitude of vibration of the particles of air through which a sound wave of intensity* $2{\cdot}0 \times 10^{-6}\ W\ m^{-2}$ *and frequency* 1·0 kHz *is passing. Density of air* $= 1{\cdot}2\ kg\ m^{-3}$ *and speed of sound in air* $= 330\ m\ s^{-1}$.

***Solution*** **:** The relation between the intensity of sound and the displacement amplitude is

$$I = 2\pi^2 s_0^2 \nu^2 \rho_0 v$$

or, $$s_0^2 = \frac{I}{2\pi^2 \nu^2 \rho_0 v}$$

$$= \frac{2{\cdot}0 \times 10^{-6}\ W\ m^{-2}}{2\pi^2 \times (1{\cdot}0 \times 10^6\ s^{-2}) \times (1{\cdot}2\ kg\ m^{-3}) \times (330\ m\ s^{-1})}$$

$$= 2{\cdot}53 \times 10^{-16}\ m^2$$

or, $$s_0 = 1{\cdot}6 \times 10^{-8}\ m.$$

**9.** *The sound level at a point is increased by* 30 dB. *By what factor is the pressure amplitude increased* ?

***Solution*** **:** The sound level in $dB$ is

$$\beta = 10 \log_{10}\left(\frac{I}{I_0}\right).$$

If $\beta_1$ and $\beta_2$ are the sound levels and $I_1$ and $I_2$ are the intensities in the two cases,

$$\beta_2 - \beta_1 = 10\left[\log_{10}\left(\frac{I_2}{I_0}\right) - \log_{10}\left(\frac{I_1}{I_0}\right)\right]$$

or, $$30 = 10 \log_{10}\left(\frac{I_2}{I_1}\right)$$

or, $$\frac{I_2}{I_1} = 10^3.$$

As the intensity is proportional to the square of the pressure amplitude, we have $\frac{p_2}{p_1} = \sqrt{\frac{I_2}{I_1}} = \sqrt{1000} \approx 32$.

**10.** *Figure (16-W2) shows a tube structure in which a sound signal is sent from one end and is received at the other end. The semicircular part has a radius of* 20·0 cm. *The frequency of the sound source can be varied electronically between* 1000 *and* 4000 Hz. *Find the frequencies at which maxima of intensity are detected. The speed of sound in air* $= 340\ m\ s^{-1}$.

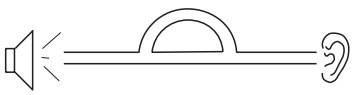


Figure 16-W2

***Solution*** **:** The sound wave bifurcates at the junction of the straight and the semicircular parts. The wave through the straight part travels a distance $l_1 = 2 \times 20$ cm and the wave through the curved part travels a distance $l_2 = \pi\ 20$ cm = 62·8 cm before they meet again and travel to the receiver. The path difference between the two waves received is, therefore,

$$\Delta l = l_2 - l_1 = 62{\cdot}8\ cm - 40\ cm = 22{\cdot}8\ cm = 0{\cdot}228\ m.$$

The wavelength of either wave is $\frac{v}{\nu} = \frac{340 \text{ m s}^{-1}}{\nu}$. For constructive interference, $\Delta l = n\,\lambda$, where $n$ is an integer.

or, $0{\cdot}228 \text{ m} = n\,\frac{340 \text{ m s}^{-1}}{\nu}$

or, $\nu = n\,\frac{340 \text{ m s}^{-1}}{0{\cdot}228 \text{ m}} = n\ 1491{\cdot}2 \text{ Hz} \approx n\ 1490 \text{ Hz}.$

Thus, the frequencies within the specified range which cause maxima of intensity are 1490 Hz and 2980 Hz.

**11.** *A source emitting sound of frequency* 180 Hz *is placed in front of a wall at a distance of* 2 m *from it. A detector is also placed in front of the wall at the same distance from it. Find the minimum distance between the source and the detector for which the detector detects a maximum of sound. Speed of sound in air* $= 360 \text{ m s}^{-1}$.

***Solution*** **:**

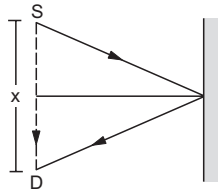



Figure 16-W3

The situation is shown in figure (16-W3). Suppose the detector is placed at a distance of $x$ meter from the source. The direct wave received from the source travels a distance of $x$ meter. The wave reaching the detector after reflection from the wall has travelled a distance of $2[(2)^2 + x^2/4]^{1/2}$ metre. The path difference between the two waves is

$$\Delta = \left\{2\left[(2)^2 + \frac{x^2}{4}\right]^{1/2} - x\right\} \text{ metre.}$$

Constructive interference will take place when $\Delta = \lambda, 2\lambda, \ldots$ . The minimum distance $x$ for a maximum corresponds to

$$\Delta = \lambda. \qquad \ldots \text{ (i)}$$

The wavelength is $\lambda = \frac{v}{\nu} = \frac{360 \text{ m s}^{-1}}{180 \text{ s}^{-1}} = 2 \text{ m}.$

Thus, by (i), $2\left[(2)^2 + \frac{x^2}{4}\right]^{1/2} - x = 2$

or, $\left[4 + \frac{x^2}{4}\right]^{1/2} = 1 + \frac{x}{2}$

or, $4 + \frac{x^2}{4} = 1 + \frac{x^2}{4} + x$

or, $3 = x.$

Thus, the detector should be placed at a distance of 3 m from the source. Note that there is no abrupt phase change.

**12.** *A tuning fork vibrates at* 264 Hz. *Find the length of the shortest closed organ pipe that will resonate with the tuning fork. Speed of sound in air is* $350 \text{ m s}^{-1}$.

***Solution*** **:** The resonant frequency of a closed organ pipe of length $l$ is $\frac{nv}{4l}$, where $n$ is a positive odd integer and $v$ is the speed of sound in air. To resonate with the given tuning fork,

$$\frac{nv}{4l} = 264 \text{ s}^{-1}$$

or, $$l = \frac{n \times 350 \text{ m s}^{-1}}{4 \times 264 \text{ s}^{-1}}.$$

For $l$ to be minimum, $n = 1$ so that

$$l_{\min} = \frac{350}{4 \times 264} \text{ m} = 33 \text{ cm}.$$

**13.** *The fundamental frequency of a closed organ pipe is equal to the first overtone frequency of an open organ pipe. If the length of the open pipe is* 60 cm, *what is the length of the closed pipe* ?

***Solution*** **:** The fundamental frequency of a closed organ pipe is $\frac{v}{4l_1}$. For an open pipe, the fundamental frequency is $\frac{v}{2l_2}$ and the first overtone is $\frac{2v}{2l_2} = \frac{v}{l_2}$. Here $l_1$ is the length of the closed pipe and $l_2 = 60$ cm is the length of the open pipe. We have,

$$\frac{v}{4l_1} = \frac{v}{60 \text{ cm}}$$

$$l_1 = \frac{1}{4} \times 60 \text{ cm} = 15 \text{ cm}.$$

**14.** *A tuning fork vibrating at frequency* 800 Hz *produces resonance in a resonance column tube. The upper end is open and the lower end is closed by the water surface which can be varied. Successive resonances are observed at lengths* 9·75 cm, 31·25 cm *and* 52·75 cm. *Calculate the speed of sound in air from these data.*

***Solution*** **:** For the tube open at one end, the resonance frequencies are $\frac{nv}{4l}$, where $n$ is a positive odd integer. If the tuning fork has a frequency $\nu$ and $l_1, l_2, l_3$ are the successive lengths of the tube in resonance with it, we have

$$\frac{nv}{4\,l_1} = \nu$$

$$\frac{(n+2)\,v}{4\,l_2} = \nu$$

$$\frac{(n+4)\,v}{4\,l_3} = \nu$$

giving $$l_3 - l_2 = l_2 - l_1 = \frac{2v}{4\nu} = \frac{v}{2\nu}.$$

By the question, $l_3 - l_2 = (52{\cdot}75 - 31{\cdot}25)$ cm $= 21{\cdot}50$ cm

and $l_2 - l_1 = (31{\cdot}25 - 9{\cdot}75)$ cm $= 21{\cdot}50$ cm.

Thus, $$\frac{v}{2\nu} = 21{\cdot}50 \text{ cm}$$

or, $v = 2\nu \times 21{\cdot}50$ cm $= 2 \times 800$ s$^{-1}$ $\times 21{\cdot}50$ cm $= 344$ m s$^{-1}$.

**15.** *A certain organ pipe resonates in its fundamental mode at a frequency of* 500 Hz *in air. What will be the fundamental frequency if the air is replaced by hydrogen at the same temperature? The density of air is* 1·20 kg m$^{-3}$ *and that of hydrogen is* 0·089 kg m$^{-3}$.

***Solution*** : Suppose the speed of sound in hydrogen is $v_h$ and that in air is $v_a$. The fundamental frequency of an organ pipe is proportional to the speed of sound in the gas contained in it. If the fundamental frequency with hydrogen in the tube is $\nu$, we have

$$\frac{\nu}{500 \text{ Hz}} = \frac{v_h}{v_a} = \sqrt{\frac{\rho_a}{\rho_h}} = \sqrt{\frac{1{\cdot}2}{0{\cdot}089}} = 3{\cdot}67$$

or, $$\nu = 3{\cdot}67 \times 500 \text{ Hz} \approx 1840 \text{ Hz}.$$

**16.** *An aluminium rod having a length of* 90·0 cm *is clamped at its middle point and is set into longitudinal vibrations by stroking it with a rosined cloth. Assume that the rod vibrates in its fundamental mode of vibration. The density of aluminium is* 2600 kg m$^{-3}$ *and its Young's modulus is* $7{\cdot}80 \times 10^{10}$ N m$^{-2}$. *Find (a) the speed of sound in aluminium, (b) the wavelength of sound waves produced in the rod, (c) the frequency of the sound produced and (d) the wavelength of the sound produced in air. Take the speed of sound in air to be* 340 m s$^{-1}$.

***Solution*** : (a) The speed of sound in aluminium is

$$v = \sqrt{\frac{Y}{\rho}} = \sqrt{\frac{7{\cdot}80 \times 10^{10} \text{ N m}^{-2}}{2600 \text{ kg m}^{-3}}}$$

$$= 5480 \text{ m s}^{-1}.$$

(b) Since the rod is clamped at the middle, the middle point is a pressure antinode. The free ends of the rod are the nodes. As the rod vibrates in its fundamental mode, there are no other nodes or antinodes. The length of the rod, which is also the distance between the successive nodes, is, therefore, equal to half the wavelength. Thus, the wavelength of sound in the aluminium rod is

$$\lambda = 2l = 180 \text{ cm}.$$

(c) The frequency of the sound produced which is also equal to the frequency of vibration of the rod is

$$\nu = \frac{v}{\lambda} = \frac{5480 \text{ m s}^{-1}}{180 \text{ cm}} = 3050 \text{ Hz}.$$

(d) The wavelength of sound in air is

$$\lambda = \frac{v}{\nu} = \frac{340 \text{ m s}^{-1}}{3050 \text{ Hz}} = 11{\cdot}1 \text{ cm}.$$

**17.** *The string of a violin emits a note of* 440 Hz *at its correct tension. The string is bit taut and produces* 4 beats *per second with a tuning fork of frequency* 440 Hz. *Find the frequency of the note emitted by this taut string.*

***Solution*** : The frequency of vibration of a string increases with increase in the tension. Thus, the note emitted by the string will be a little more than 440 Hz. As it produces 4 beats per second with the 440 Hz tuning fork, the frequency will be 444 Hz.

**18.** *A siren is fitted on a car going towards a vertical wall at a speed of* 36 km/h. *It produces sound of frequency* 500 Hz. *A person standing on the ground, behind the car, listens to the siren sound coming directly from the source as well as that coming after reflection from the wall. Calculate the apparent frequency of the wave (a) coming directly from the siren to the person and (b) coming after reflection. Take the speed of sound to be* 340 m s$^{-1}$.

***Solution*** :

Figure 16-W4

The speed of the car is 36 km h$^{-1}$ = 10 m s$^{-1}$.

(a) Here the observer is at rest with respect to the medium and the source is going away from the observer. The apparent frequency heard by the observer is, therefore,

$$\nu' = \frac{v}{v + u_s}\nu$$

$$= \frac{340}{340 + 10} \times 500 \text{ Hz} = 486 \text{ Hz}.$$

(b) The frequency received by the wall is

$$\nu'' = \frac{v}{v - u_s}\nu = \frac{340}{340 - 10} \times 500 \text{ Hz} = 515 \text{ Hz}.$$

The wall reflects this sound without changing the frequency. Thus, the frequency of the reflected wave as heard by the ground observer is 515 Hz.

**19.** *Two trains are moving towards each other at speeds of* 72 km h$^{-1}$ *and* 54 km h$^{-1}$ *relative to the ground. The first train sounds a whistle of frequency* 600 Hz. *Find the frequency of the whistle as heard by a passenger in the second train (a) before the trains meet and (b) after the trains have crossed each other. The speed of sound in air is* 340 m s$^{-1}$.

***Solution*** : The speed of the first train = 72 km h$^{-1}$ = 20 m s$^{-1}$ and that of the second = 54 km h$^{-1}$ = 15 m s$^{-1}$.

(a) Here both the source and the observer move with respect to the medium. Before the trains meet, the source is going towards the observer and the observer

is also going towards the source. The apparent frequency heard by the observer will be

$$\nu' = \frac{v + u_o}{v - u_s}\nu$$
$$= \frac{340 + 15}{340 - 20} \times 600 \text{ Hz} = 666 \text{ Hz}.$$

(b) After the trains have crossed each other, the source goes away from the observer and the observer goes away from the source. The frequency heard by the observer is, therefore,

$$\nu'' = \frac{v - u_o}{v + u_s}\nu$$
$$= \frac{340 - 15}{340 + 20} \times 600 \text{ Hz} = 542 \text{ Hz}.$$

**20.** *A person going away from a factory on his scooter at a speed of* 36 km h$^{-1}$ *listens to the siren of the factory. If the main frequency of the siren is* 600 Hz *and a wind is blowing along the direction of the scooter at* 36 km h$^{-1}$*, find the main frequency as heard by the person.*

***Solution*** **:** The speed of sound in still air is 340 m s$^{-1}$. Let us work from the frame of reference of the air. As both the observer and the wind are moving at the same speed along the same direction with respect to the ground, the observer is at rest with respect to the medium. The source (the siren) is moving with respect to the wind at a speed of 36 km h$^{-1}$, i.e., 10 m s$^{-1}$. As the source is going away from the observer who is at rest with respect to the medium, the frequency heard is

$$\nu' = \frac{v}{v + u_s}\nu = \frac{340}{340 + 10} \times 600 \text{ Hz} = 583 \text{ Hz}.$$

**21.** *A source and a detector move away from each other, each with a speed of* 10 m s$^{-1}$ *with respect to the ground with no wind. If the detector detects a frequency* 1950 Hz *of the sound coming from the source, what is the original frequency of the source ? Speed of sound in air* = 340 m s$^{-1}$.

***Solution*** **:** If the original frequency of the source is $\nu$, the apparent frequency heard by the observer is

$$\nu' = \frac{v - u_o}{v + u_s}\nu,$$

where $u_o$ is the speed of the observer going away from the source and $u_s$ is the speed of the source going away from the observer. By the question,

$$1950 \text{ Hz} = \frac{340 - 10}{340 + 10}\nu$$

or,
$$\nu = \frac{35}{33} \times 1950 \text{ Hz} = 2070 \text{ Hz}.$$

**22.** *The driver of a car approaching a vertical wall notices that the frequency of his car's horn changes from* 440 Hz *to* 480 Hz *when it gets reflected from the wall. Find the speed of the car if that of the sound is* 330 m s$^{-1}$.

***Solution*** :

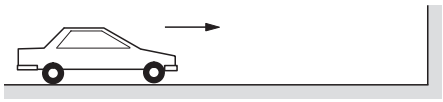


Figure 16-W5

Suppose the car is going towards the wall at a speed $u$. The wall is stationary with respect to the air and the horn is going towards it. If the frequency of the horn is $\nu$, that received by the wall is

$$\nu' = \frac{v}{v - u}\nu.$$

The wall reflects this sound without changing the frequency. Thus, the wall becomes the source of frequency $\nu'$ and the car-driver is the listener. The wall (which acts as the source now) is at rest with respect to the air and the car (which is the observer now) is going towards the wall at speed $u$. The frequency heard by the car-driver for this reflected wave is, therefore,

$$\nu'' = \frac{v + u}{v}\nu'$$
$$= \frac{v + u}{v} \cdot \frac{v}{v - u}\nu$$
$$= \frac{v + u}{v - u}\nu.$$

Putting the values,

$$480 = \frac{v + u}{v - u}440$$

or,
$$\frac{v + u}{v - u} = \frac{48}{44}$$

or,
$$\frac{u}{v} = \frac{4}{92}$$

or,
$$u = \frac{4}{92} \times 330 \text{ m s}^{-1} = 14{\cdot}3 \text{ m s}^{-1} \approx 52 \text{ km h}^{-1}.$$

**23.** *A train approaching a railway crossing at a speed of* 120 km h$^{-1}$ *sounds a short whistle at frequency* 640 Hz *when it is* 300 m *away from the crossing. The speed of sound in air is* 340 m s$^{-1}$*. What will be the frequency heard by a person standing on a road perpendicular to the track through the crossing at a distance of* 400 m *from the crossing ?*

***Solution*** **:**

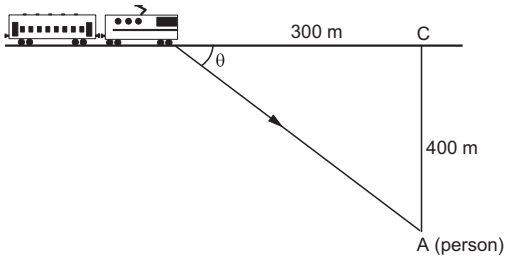



Figure 16-W6

The observer $A$ is at rest with respect to the air and the source is travelling at a velocity of 120 km h$^{-1}$ i.e., $\frac{100}{3}$ m s$^{-1}$. As is clear from the figure, the person receives the sound of the whistle in a direction $BA$ making an angle $\theta$ with the track where $\cos\theta = 300/500 = 3/5$. The component of the velocity of the source (i.e., of the train) along this direction $AB$ is $\frac{100}{3} \times \frac{3}{5}$ m s$^{-1}$ = 20 m s$^{-1}$. As the source is approaching the person with this component, the frequency heard by the observer is

$$\nu' = \frac{v}{v - u\cos\theta}\nu = \frac{340}{340 - 20} \times 640 \text{ Hz} = 680 \text{ Hz}.$$

□

## QUESTIONS FOR SHORT ANSWER

**1.** If you are walking on the moon, can you hear the sound of stones cracking behind you ? Can you hear the sound of your own footsteps ?

**2.** Can you hear your own words if you are standing in a perfect vacuum ? Can you hear your friend in the same conditions ?

**3.** A vertical rod is hit at one end. What kind of wave propagates in the rod if (a) the hit is made vertically (b) the hit is made horizontally ?

**4.** Two loudspeakers are arranged facing each other at some distance. Will a person standing behind one of the loudspeakers clearly hear the sound of the other loudspeaker or the clarity will be seriously damaged because of the 'collision' of the two sounds in between ?

**5.** The voice of a person, who has inhaled helium, has a remarkably high pitch. Explain on the basis of resonant vibration of vocal cord filled with air and with helium.

**6.** Draw a diagram to show the standing pressure wave and standing displacement wave for the 3rd overtone mode of vibration of an open organ pipe.

**7.** Two tuning forks vibrate with the same amplitude but the frequency of the first is double the frequency of the second. Which fork produces more intense sound in air ?

**8.** In discussing Doppler effect, we use the word "apparent frequency". Does it mean that the frequency of the sound is still that of the source and it is some physiological phenomenon in the listener's ear that gives rise to Doppler effect ? Think for the observer approaching the source and for the source approaching the observer.

## OBJECTIVE I

**1**. Consider the following statements about sound passing through a gas.
(A) The pressure of the gas at a point oscillates in time.
(B) The position of a small layer of the gas oscillates in time.
(a) Both $A$ and $B$ are correct.
(b) $A$ is correct but $B$ is wrong.
(c) $B$ is correct but $A$ is wrong.
(d) Both $A$ and $B$ are wrong.

**2.** When we clap our hands, the sound produced is best described by
(a) $p = p_0 \sin(kx - \omega t)$ (b) $p = p_0 \sin kx \cos \omega t$
(c) $p = p_0 \cos kx \sin \omega t$ (d) $p = \Sigma\, p_{0n} \sin(k_n x - \omega_n t)$.
Here $p$ denotes the change in pressure from the equilibrium value.

**3.** The bulk modulus and the density of water are greater than those of air. With this much of information, we can say that velocity of sound in air
(a) is larger than its value in water
(b) is smaller than its value in water
(c) is equal to its value in water
(d) cannot be compared with its value in water.

**4**. A tuning fork sends sound waves in air. If the temperature of the air increases, which of the following parameters will change ?
(a) Displacement amplitude (b) Frequency
(c) Wavelength (d) Time period

**5.** When sound wave is refracted from air to water, which of the following will remain unchanged ?
(a) Wave number (b) Wavelength
(c) Wave velocity (d) Frequency

**6.** The speed of sound in a medium depends on
(a) the elastic property but not on the inertia property
(b) the inertia property but not on the elastic property
(c) the elastic property as well as the inertia property
(d) neither the elastic property nor the inertia property.

**7.** Two sound waves move in the same direction in the same medium. The pressure amplitudes of the waves are equal but the wavelength of the first wave is double the second. Let the average power transmitted across a cross section by the first wave be $P_1$ and that by the second wave be $P_2$. Then
(a) $P_1 = P_2$ (b) $P_1 = 4P_2$
(c) $P_2 = 2P_1$ (d) $P_2 = 4P_1$ .

**8.** When two waves with same frequency and constant phase difference interfere,
(a) there is a gain of energy
(b) there is a loss of energy

(c) the energy is redistributed and the distribution changes with time
(d) the energy is redistributed and the distribution remains constant in time.

**9.** An open organ pipe of length $L$ vibrates in its fundamental mode. The pressure variation is maximum
(a) at the two ends
(b) at the middle of the pipe
(c) at distances $L/4$ inside the ends
(d) at distances $L/8$ inside the ends.

**10.** An organ pipe, open at both ends, contains
(a) longitudinal stationary waves
(b) longitudinal travelling waves
(c) transverse stationary waves
(d) transverse travelling waves.

**11.** A cylindrical tube, open at both ends, has a fundamental frequency $\nu$. The tube is dipped vertically in water so that half of its length is inside the water. The new fundamental frequency is
(a) $\nu/4$ (b) $\nu/2$ (c) $\nu$ (d) $2\nu$.

**12.** The phenomenon of beats can take place
(a) for longitudinal waves only
(b) for transverse waves only
(c) for both longitudinal and transverse waves
(d) for sound waves only.

**13.** A tuning fork of frequency 512 Hz is vibrated with a sonometer wire and 6 beats per second are heard. The beat frequency reduces if the tension in the string is slightly increased. The original frequency of vibration of the string is
(a) 506 Hz (b) 512 Hz (c) 518 Hz (d) 524 Hz.

**14.** The engine of a train sounds a whistle at frequency $\nu$. The frequency heard by a passenger is
(a) $> \nu$ (b) $< \nu$ (c) $= \frac{1}{\nu}$ (d) $= \nu$.

**15.** The change in frequency due to Doppler effect does not depend on
(a) the speed of the source
(b) the speed of the observer
(c) the frequency of the source
(d) separation between the source and the observer.

**16.** A small source of sound moves on a circle as shown in figure (16-Q1) and an observer is sitting at $O$. Let $\nu_1, \nu_2, \nu_3$ be the frequencies heard when the source is at $A$, $B$ and $C$ respectively.
(a) $\nu_1 > \nu_2 > \nu_3$ (b) $\nu_1 = \nu_2 > \nu_3$
(c) $\nu_2 > \nu_3 > \nu_1$ (d) $\nu_1 > \nu_3 > \nu_2$

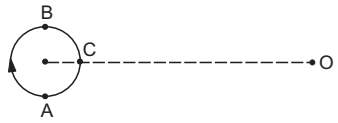


Figure 16-Q1

## OBJECTIVE II

**1.** When you speak to your friend, which of the following parameters have a unique value in the sound produced ?
(a) Frequency (b) Wavelength
(c) Amplitude (d) Wave velocity

**2.** An electrically maintained tuning fork vibrates with constant frequency and constant amplitude. If the temperature of the surrounding air increases but pressure remains constant, the sound produced will have
(a) larger wavelength (b) larger frequency
(c) larger velocity (d) larger time period.

**3.** The fundamental frequency of a vibrating organ pipe is 200 Hz.
(a) The first overtone is 400 Hz.
(b) The first overtone may be 400 Hz.
(c) The first overtone may be 600 Hz.
(d) 600 Hz is an overtone.

**4.** A source of sound moves towards an observer.
(a) The frequency of the source is increased.
(b) The velocity of sound in the medium is increased.
(c) The wavelength of sound in the medium towards the observer is decreased.
(d) The amplitude of vibration of the particles is increased.

**5.** A listener is at rest with respect to the source of sound. A wind starts blowing along the line joining the source and the observer. Which of the following quantities do not change ?
(a) Frequency (b) Velocity of sound
(c) Wavelength (d) Time period

## EXERCISES

**1.** A steel tube of length 1·00 m is struck at one end. A person with his ear close to the other end hears the sound of the blow twice, one travelling through the body of the tube and the other through the air in the tube. Find the time gap between the two hearings. Use the table in the text for speeds of sound in various substances.

**2.** At a prayer meeting, the disciples sing *JAI-RAM JAI-RAM.* The sound amplified by a loudspeaker comes back after reflection from a building at a distance of 80 m from the meeting. What maximum time interval can be kept between one *JAI-RAM* and the next *JAI-RAM* so that the echo does not disturb a listener sitting in the meeting. Speed of sound in air is 320 m $s^{-1}$.

**3.** A man stands before a large wall at a distance of 50·0 m and claps his hands at regular intervals. Initially, the interval is large. He gradually reduces the interval and fixes it at a value when the echo of a clap merges with the next clap. If he has to clap 10 times during every 3 seconds, find the velocity of sound in air.

**4.** A person can hear sound waves in the frequency range 20 Hz to 20 kHz. Find the minimum and the maximum wavelengths of sound that is audible to the person. The speed of sound is 360 m $s^{-1}$.

**5.** Find the minimum and maximum wavelengths of sound in water that is in the audible range (20–20000 Hz) for an average human ear. Speed of sound in water = 1450 m $s^{-1}$.

**6.** Sound waves from a loudspeaker spread nearly uniformly in all directions if the wavelength of the sound is much larger than the diameter of the loudspeaker. (a) Calculate the frequency for which the wavelength of sound in air is ten times the diameter of the speaker if the diameter is 20 cm. (b) Sound is essentially transmitted in the forward direction if the wavelength is much shorter than the diameter of the speaker. Calculate the frequency at which the wavelength of the sound is one tenth of the diameter of the speaker described above. Take the speed of sound to be 340 m $s^{-1}$.

**7.** Ultrasonic waves of frequency 4·5 MHz are used to detect tumour in soft tissues. The speed of sound in tissue is 1·5 km $s^{-1}$ and that in air is 340 m $s^{-1}$. Find the wavelength of this ultrasonic wave in air and in tissue.

**8.** The equation of a travelling sound wave is $y = 6{\cdot}0 \sin (600\, t - 1{\cdot}8\, x)$ where $y$ is measured in $10^{-5}$ m, $t$ in second and $x$ in metre. (a) Find the ratio of the displacement amplitude of the particles to the wavelength of the wave. (b) Find the ratio of the velocity amplitude of the particles to the wave speed.

**9.** A sound wave of frequency 100 Hz is travelling in air. The speed of sound in air is 350 m $s^{-1}$. (a) By how much is the phase changed at a given point in 2·5 ms ? (b) What is the phase difference at a given instant between two points separated by a distance of 10·0 cm along the direction of propagation ?

**10.** Two point sources of sound are kept at a separation of 10 cm. They vibrate in phase to produce waves of wavelength 5·0 cm. What would be the phase difference between the two waves arriving at a point 20 cm from one source (a) on the line joining the sources and (b) on the perpendicular bisector of the line joining the sources ?

**11.** Calculate the speed of sound in oxygen from the following data. The mass of 22·4 litre of oxygen at STP ($T = 273$ K and $p = 1{\cdot}0 \times 10^5$ N $m^{-2}$) is 32 g, the molar heat capacity of oxygen at constant volume is $C_V = 2{\cdot}5\, R$ and that at constant pressure is $C_p = 3{\cdot}5\, R$.

**12.** The speed of sound as measured by a student in the laboratory on a winter day is 340 m $s^{-1}$ when the room temperature is 17°C. What speed will be measured by another student repeating the experiment on a day when the room temperature is 32°C ?

**13.** At what temperature will the speed of sound be double of its value at 0°C ?

**14.** The absolute temperature of air in a region linearly increases from $T_1$ to $T_2$ in a space of width $d$. Find the time taken by a sound wave to go through the region in terms of $T_1, T_2, d$ and the speed $v$ of sound at 273 K. Evaluate this time for $T_1 = 280$ K, $T_2 = 310$ K, $d = 33$ m and $v = 330$ m $s^{-1}$.

**15.** Find the change in the volume of 1·0 litre kerosene when it is subjected to an extra pressure of $2{\cdot}0 \times 10^5$ N $m^{-2}$ from the following data. Density of kerosene = 800 kg $m^{-3}$ and speed of sound in kerosene = 1330 m $s^{-1}$.

**16.** Calculate the bulk modulus of air from the following data about a sound wave of wavelength 35 cm travelling in air. The pressure at a point varies between $(1{\cdot}0 \times 10^5 \pm 14)$ Pa and the particles of the air vibrate in simple harmonic motion of amplitude $5{\cdot}5 \times 10^{-6}$ m.

**17.** A source of sound operates at 2·0 kHz, 20 W emitting sound uniformly in all directions. The speed of sound in air is 340 m $s^{-1}$ and the density of air is 1·2 kg $m^{-3}$. (a) What is the intensity at a distance of 6·0 m from the source ? (b) What will be the pressure amplitude at this point ? (c) What will be the displacement amplitude at this point ?

**18.** The intensity of sound from a point source is $1{\cdot}0 \times 10^{-8}$ W $m^{-2}$ at a distance of 5·0 m from the source. What will be the intensity at a distance of 25 m from the source ?

**19.** The sound level at a point 5·0 m away from a point source is 40 dB. What will be the level at a point 50 m away from the source ?

**20.** If the intensity of sound is doubled, by how many decibels does the sound level increase ?

**21.** Sound with intensity larger than 120 dB appears painful to a person. A small speaker delivers 2·0 W of audio output. How close can the person get to the speaker without hurting his ears ?

**22.** If the sound level in a room is increased from 50 dB to 60 dB, by what factor is the pressure amplitude increased ?

**23.** The noise level in a classroom in absence of the teacher is 50 dB when 50 students are present. Assuming that on the average each student outputs same sound energy per second, what will be the noise level if the number of students is increased to 100 ?

**24.** In Quincke's experiment the sound detected is changed from a maximum to a minimum when the sliding tube is moved through a distance of 2·50 cm. Find the frequency of sound if the speed of sound in air is 340 m $s^{-1}$.

**25.** In Quincke's experiment, the sound intensity has a minimum value $I$ at a particular position. As the sliding tube is pulled out by a distance of 16·5 mm, the intensity

increases to a maximum of 9 $I$. Take the speed of sound in air to be 330 m $s^{-1}$. (a) Find the frequency of the sound source. (b) Find the ratio of the amplitudes of the two waves arriving at the detector assuming that it does not change much between the positions of minimum intensity and maximum intensity.

**26.** Two audio speakers are kept some distance apart and are driven by the same amplifier system. A person is sitting at a place 6·0 m from one of the speakers and 6·4 m from the other. If the sound signal is continuously varied from 500 Hz to 5000 Hz, what are the frequencies for which there is a destructive interference at the place of the listener? Speed of sound in air = 320 m $s^{-1}$.

**27.** A source of sound $S$ and a detector $D$ are placed at some distance from one another. A big cardboard is placed near the detector and perpendicular to the line $SD$ as shown in figure (16-E1). It is gradually moved away and it is found that the intensity changes from a maximum to a minimum as the board is moved through a distance of 20 cm. Find the frequency of the sound emitted. Velocity of sound in air is 336 m $s^{-1}$.

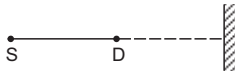



Figure 16-E1

**28.** A source $S$ and a detector $D$ are placed at a distance $d$ apart. A big cardboard is placed at a distance $\sqrt{2}d$ from the source and the detector as shown in figure (16-E2). The source emits a wave of wavelength = $d/2$ which is received by the detector after reflection from the cardboard. It is found to be in phase with the direct wave received from the source. By what minimum distance should the cardboard be shifted away so that the reflected wave becomes out of phase with the direct wave ?

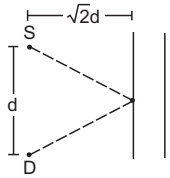



Figure 16-E2

**29.** Two stereo speakers are separated by a distance of 2·40 m. A person stands at a distance of 3·20 m directly in front of one of the speakers as shown in figure (16-E3). Find the frequencies in the audible range (20–2000 Hz) for which the listener will hear a minimum sound intensity. Speed of sound in air = 320 m $s^{-1}$.

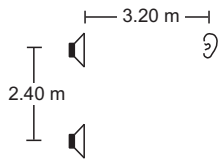



Figure 16-E3

**30.** Two sources of sound, $S_1$ and $S_2$, emitting waves of equal wavelength 20·0 cm, are placed with a separation of 20·0 cm between them. A detector can be moved on a line parallel to $S_1S_2$ and at a distance of 20·0 cm from it. Initially, the detector is equidistant from the two sources. Assuming that the waves emitted by the sources are in phase, find the minimum distance through which the detector should be shifted to detect a minimum of sound.

**31.** Two speakers $S_1$ and $S_2$, driven by the same amplifier, are placed at $y$ = 1·0 m and $y$ = –1·0 m (figure 16-E4). The speakers vibrate in phase at 600 Hz. A man stands at a point on the $X$-axis at a very large distance from the origin and starts moving parallel to the $Y$-axis. The speed of sound in air is 330 m $s^{-1}$. (a) At what angle θ will the intensity of sound drop to a minimum for the first time ? (b) At what angle will he hear a maximum of sound intensity for the first time ? (c) If he continues to walk along the line, how many more maxima can he hear ?

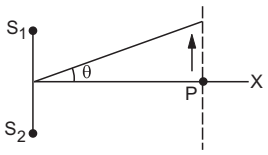



Figure 16-E4

**32.** Three sources of sound $S_1, S_2$ and $S_3$ of equal intensity are placed in a straight line with $S_1S_2 = S_2S_3$ (figure 16-E5). At a point $P$, far away from the sources, the wave coming from $S_2$ is 120° ahead in phase of that from $S_1$. Also, the wave coming from $S_3$ is 120° ahead of that from $S_2$. What would be the resultant intensity of sound at $P$ ?

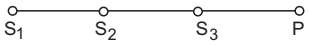



Figure 16-E5

**33.** Two coherent narrow slits emitting sound of wavelength λ in the same phase are placed parallel to each other at a small separation of 2λ. The sound is detected by moving a detector on the screen Σ at a distance $D(>> \lambda)$ from the slit $S_1$ as shown in figure (16-E6). Find the distance $x$ such that the intensity at $P$ is equal to the intensity at $O$.

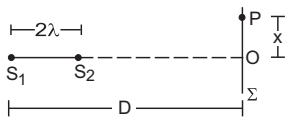



Figure 16-E6

**34.** Figure (16-E7) shows two coherent sources $S_1$ and $S_2$ which emit sound of wavelength λ in phase. The separation between the sources is 3λ. A circular wire of large radius is placed in such a way that $S_1S_2$ lies in its plane and the middle point of $S_1S_2$ is at the centre of

the wire. Find the angular positions θ on the wire for which constructive interference takes place.

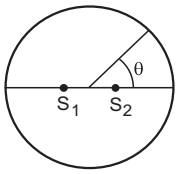



Figure 16-E7

**35.** Two sources of sound $S_1$ and $S_2$ vibrate at same frequency and are in phase (figure 16-E8). The intensity of sound detected at a point $P$ as shown in the figure is $I_0$. (a) If θ equals 45°, what will be the intensity of sound detected at this point if one of the sources is switched off ? (b) What will be the answer of the previous part if θ = 60° ?

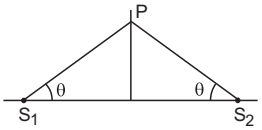



Figure 16-E8

**36.** Find the fundamental, first overtone and second overtone frequencies of an open organ pipe of length 20 cm. Speed of sound in air is 340 m $s^{-1}$.

**37.** A closed organ pipe can vibrate at a minimum frequency of 500 Hz. Find the length of the tube. Speed of sound in air = 340 m $s^{-1}$.

**38.** In a standing wave pattern in a vibrating air column, nodes are formed at a distance of 4·0 cm. If the speed of sound in air is 328 m $s^{-1}$, what is the frequency of the source ?

**39.** The separation between a node and the next antinode in a vibrating air column is 25 cm. If the speed of sound in air is 340 m $s^{-1}$, find the frequency of vibration of the air column.

**40.** A cylindrical metal tube has a length of 50 cm and is open at both ends. Find the frequencies between 1000 Hz and 2000 Hz at which the air column in the tube can resonate. Speed of sound in air is 340 m $s^{-1}$.

**41.** In a resonance column experiment, a tuning fork of frequency 400 Hz is used. The first resonance is observed when the air column has a length of 20·0 cm and the second resonance is observed when the air column has a length of 62·0 cm. (a) Find the speed of sound in air. (b) How much distance above the open end does the pressure node form ?

**42.** The first overtone frequency of a closed organ pipe $P_1$ is equal to the fundamental frequency of an open organ pipe $P_2$. If the length of the pipe $P_1$ is 30 cm, what will be the length of $P_2$ ?

**43.** A copper rod of length 1·0 m is clamped at its middle point. Find the frequencies between 20 Hz and 20,000 Hz at which standing longitudinal waves can be set up in the rod. The speed of sound in copper is 3·8 km $s^{-1}$.

**44.** Find the greatest length of an organ pipe open at both ends that will have its fundamental frequency in the normal hearing range (20–20,000 Hz). Speed of sound in air = 340 m $s^{-1}$.

**45.** An open organ pipe has a length of 5 cm. (a) Find the fundamental frequency of vibration of this pipe. (b) What is the highest harmonic of such a tube that is in the audible range ? Speed of sound in air is 340 m $s^{-1}$ and the audible range is 20–20,000 Hz.

**46.** An electronically driven loudspeaker is placed near the open end of a resonance column apparatus. The length of air column in the tube is 80 cm. The frequency of the loudspeaker can be varied between 20 Hz and 2 kHz. Find the frequencies at which the column will resonate. Speed of sound in air = 320 m $s^{-1}$.

**47.** Two successive resonance frequencies in an open organ pipe are 1620 Hz and 2268 Hz. Find the length of the tube. The speed of sound in air is 324 m $s^{-1}$.

**48.** A piston is fitted in a cylindrical tube of small cross section with the other end of the tube open. The tube resonates with a tuning fork of frequency 512 Hz. The piston is gradually pulled out of the tube and it is found that a second resonance occurs when the piston is pulled out through a distance of 32·0 cm. Calculate the speed of sound in the air of the tube.

**49**. A U-tube having unequal arm-lengths has water in it. A tuning fork of frequency 440 Hz can set up the air in the shorter arm in its fundamental mode of vibration and the same tuning fork can set up the air in the longer arm in its first overtone vibration. Find the length of the air columns. Neglect any end effect and assume that the speed of sound in air = 330 m $s^{-1}$.

**50.** Consider the situation shown in figure (16-E9). The wire which has a mass of 4·00 g oscillates in its second harmonic and sets the air column in the tube into vibrations in its fundamental mode. Assuming that the speed of sound in air is 340 m $s^{-1}$, find the tension in the wire.

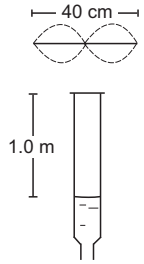



Figure 16-E9

**51.** A 30·0-cm-long wire having a mass of 10·0 g is fixed at the two ends and is vibrated in its fundamental mode. A 50·0-cm-long closed organ pipe, placed with its open end near the wire, is set up into resonance in its fundamental mode by the vibrating wire. Find the tension in the wire. Speed of sound in air = 340 m $s^{-1}$.

**52.** Show that if the room temperature changes by a small amount from $T$ to $T + \Delta T$, the fundamental frequency of an organ pipe changes from $\nu$ to $\nu + \Delta\nu$, where

$$\frac{\Delta v}{v} = \frac{1}{2}\frac{\Delta T}{T}.$$

**53.** The fundamental frequency of a closed pipe is 293 Hz when the air in it is at a temperature of 20°C. What will be its fundamental frequency when the temperature changes to 22°C ?

**54.** A Kundt's tube apparatus has a copper rod of length 1·0 m clamped at 25 cm from one of the ends. The tube contains air in which the speed of sound is 340 m/s. The powder collects in heaps separated by a distance of 5·0 cm. Find the speed of sound waves in copper.

**55.** A Kundt's tube apparatus has a steel rod of length 1·0 m clamped at the centre. It is vibrated in its fundamental mode at a frequency of 2600 Hz. The lycopodium powder dispersed in the tube collects into heaps separated by 6·5 cm. Calculate the speed of sound in steel and in air.

**56.** A source of sound with adjustable frequency produces 2 beats per second with a tuning fork when its frequency is either 476 Hz or 480 Hz. What is the frequency of the tuning fork ?

**57.** A tuning fork produces 4 beats per second with another tuning fork of frequency 256 Hz. The first one is now loaded with a little wax and the beat frequency is found to increase to 6 per second. What was the original frequency of the tuning fork ?

**58.** Calculate the frequency of beats produced in air when two sources of sound are activated, one emitting a wavelength of 32 cm and the other of 32·2 cm. The speed of sound in air is 350 m/s.

**59.** A tuning fork of unknown frequency makes 5 beats per second with another tuning fork which can cause a closed organ pipe of length 40 cm to vibrate in its fundamental mode. The beat frequency decreases when the first tuning fork is slightly loaded with wax. Find its original frequency. The speed of sound in air is 320 m/s.

**60.** A piano wire $A$ vibrates at a fundamental frequency of 600 Hz. A second identical wire $B$ produces 6 beats per second with it when the tension in $A$ is slightly increased. Find the ratio of the tension in $A$ to the tension in $B$.

**61.** A tuning fork of frequency 256 Hz produces 4 beats per second with a wire of length 25 cm vibrating in its fundamental mode. The beat frequency decreases when the length is slightly shortened. What could be the minimum length by which the wire be shortened so that it produces no beats with the tuning fork ?

**62.** A traffic policeman standing on a road sounds a whistle emitting the main frequency of 2·00 kHz. What could be the appparent frequency heard by a scooter-driver approaching the policeman at a speed of 36·0 km/h ? Speed of sound in air = 340 m/s.

**63**. The horn of a car emits sound with a dominant frequency of 2400 Hz. What will be the apparent dominant frequency heard by a person standing on the road in front of the car if the car is approaching at 18·0 km/h ? Speed of sound in air = 340 m/s.

**64.** A person riding a car moving at 72 km/h sounds a whistle emitting a wave of frequency 1250 Hz. What frequency will be heard by another person standing on the road (a) in front of the car (b) behind the car ? Speed of sound in air = 340 m/s.

**65.** A train approaching a platform at a speed of 54 km/h sounds a whistle. An observer on the platform finds its frequency to be 1620 Hz. The train passes the platform keeping the whistle on and without slowing down. What frequency will the observer hear after the train has crossed the platfrom ? The speed of sound in air = 332 m/s.

**66.** A bat emitting an ultrasonic wave of frequency $4{\cdot}5 \times 10^4$ Hz flies at a speed of 6 m/s between two parallel walls. Find the two frequencies heard by the bat and the beat frequency between the two. The speed of sound is 330 m/s.

**67.** A bullet passes past a person at a speed of 220 m/s. Find the fractional change in the frequency of the whistling sound heard by the person as the bullet crosses the person. Speed of sound in air = 330 m/s.

**68.** Two electric trains run at the same speed of 72 km/h along the same track and in the same direction with a separation of 2·4 km between them. The two trains simultaneously sound brief whistles. A person is situated at a perpendicular distance of 500 m from the track and is equidistant from the two trains at the instant of the whistling. If both the whistles were at 500 Hz and the speed of sound in air is 340 m/s, find the frequencies heard by the person.

**69.** A violin player riding on a slow train plays a 440 Hz note. Another violin player standing near the track plays the same note. When the two are close by and the train approaches the person on the ground, he hears 4·0 beats per second. The speed of sound in air = 340 m/s. (a) Calculate the speed of the train. (b) What beat frequency is heard by the player in the train ?

**70.** Two identical tuning forks vibrating at the same frequency 256 Hz are kept fixed at some distance apart. A listener runs between the forks at a speed of 3·0 m/s so that he approaches one tuning fork and recedes from the other (figure 16-E10). Find the beat frequency observed by the listener. Speed of sound in air = 332 m/s.

Figure 16-E10

**71.** Figure (16-E11) shows a person standing somewhere in between two identical tuning forks, each vibrating at

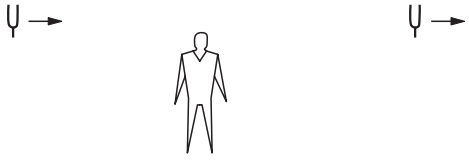


Figure 16-E11

512 Hz. If both the tuning forks move towards right at a speed of 5·5 m $s^{-1}$, find the number of beats heard by the listener. Speed of sound in air = 330 m $s^{-1}$.

**72.** A small source of sound vibrating at frequency 500 Hz is rotated in a circle of radius 100/π cm at a constant angular speed of 5·0 revolutions per second. A listener situates himself in the plane of the circle. Find the minimum and the maximum frequency of the sound observed. Speed of sound in air = 332 m $s^{-1}$.

**73.** Two trains are travelling towards each other both at a speed of 90 km $h^{-1}$. If one of the trains sounds a whistle at 500 Hz, what will be the apparent frequency heard in the other train ? Speed of sound in air = 350 m $s^{-1}$.

**74.** A traffic policeman sounds a whistle to stop a car-driver approaching towards him. The car-driver does not stop and takes the plea in court that because of the Doppler shift, the frequency of the whistle reaching him might have gone beyond the audible limit of 20 kHz and he did not hear it. Experiments showed that the whistle emits a sound with frequency close to 16 kHz. Assuming that the claim of the driver is true, how fast was he driving the car ? Take the speed of sound in air to be 330 m $s^{-1}$. Is this speed practical with today's technology ?

**75.** A car moving at 108 km $h^{-1}$ finds another car in front of it going in the same direction at 72 km $h^{-1}$. The first car sounds a horn that has a dominant frequency of 800 Hz. What will be the apparent frequency heard by the driver in the front car ? Speed of sound in air = 330 m $s^{-1}$.

**76.** Two submarines are approaching each other in a calm sea. The first submarine travels at a speed of 36 km $h^{-1}$ and the other at 54 km $h^{-1}$ relative to the water. The first submarine sends a sound signal (sound waves in water are also called *sonar*) at a frequency of 2000 Hz. (a) At what frequency is this signal received by the second submarine ? (b) The signal is reflected from the second submarine. At what frequency is this signal received by the first submarine. Take the speed of the sound wave in water to be 1500 m $s^{-1}$.

**77.** A small source of sound oscillates in simple harmonic motion with an amplitude of 17 cm. A detector is placed along the line of motion of the source. The source emits a sound of frequency 800 Hz which travels at a speed of 340 m $s^{-1}$. If the width of the frequency band detected by the detector is 8 Hz, find the time period of the source.

**78.** A boy riding on his bike is going towards east at a speed of $4\sqrt{2}$ m $s^{-1}$. At a certain point he produces a sound pulse of frequency 1650 Hz that travels in air at a speed of 334 m $s^{-1}$. A second boy stands on the ground 45° south of east from him. Find the frequency of the pulse as received by the second boy.

**79.** A sound source, fixed at the origin, is continuously emitting sound at a frequency of 660 Hz. The sound travels in air at a speed of 330 m $s^{-1}$. A listener is moving along the line $x = 336$ m at a constant speed of 26 m $s^{-1}$. Find the frequency of the sound as observed by the listener when he is (a) at $y = -140$ m, (b) at $y = 0$ and (c) at $y = 140$ m.

**80.** A train running at 108 km $h^{-1}$ towards east whistles at a dominant frequency of 500 Hz. Speed of sound in air is 340 m/s. (a) What frequency will a passenger sitting near the open window hear ? (b) What frequency will a person standing near the track hear whom the train has just passed ? (c) A wind starts blowing towards east at a speed of 36 km $h^{-1}$. Calculate the frequencies heard by the passenger in the train and by the person standing near the track.

**81.** A boy riding on a bicycle going at 12 km $h^{-1}$ towards a vertical wall whistles at his dog on the ground. If the frequency of the whistle is 1600 Hz and the speed of sound in air is 330 m $s^{-1}$, find (a) the frequency of the whistle as received by the wall (b) the frequency of the reflected whistle as received by the boy.

**82.** A person standing on a road sends a sound signal to the driver of a car going away from him at a speed of 72 km $h^{-1}$. The signal travelling at 330 m $s^{-1}$ in air and having a frequency of 1600 Hz gets reflected from the body of the car and returns. Find the frequency of the reflected signal as heard by the person.

**83.** A car moves with a speed of 54 km $h^{-1}$ towards a cliff. The horn of the car emits sound of frequency 400 Hz at a speed of 335 m $s^{-1}$. (a) Find the wavelength of the sound emitted by the horn in front of the car. (b) Find the wavelength of the wave reflected from the cliff. (c) What frequency does a person sitting in the car hear for the reflected sound wave ? (d) How many beats does he hear in 10 seconds between the sound coming directly from the horn and that coming after the reflection ?

**84.** An operator sitting in his base camp sends a sound signal of frequency 400 Hz. The signal is reflected back from a car moving towards him. The frequency of the reflected sound is found to be 410 Hz. Find the speed of the car. Speed of sound in air = 324 m $s^{-1}$.

**85.** Figure (16-E12) shows a source of sound moving along the *X*-axis at a speed of 22 m $s^{-1}$ continuously emitting a sound of frequency 2·0 kHz which travels in air at a speed of 330 m $s^{-1}$. A listener *Q* stands on the *Y*-axis at a distance of 330 m from the origin. At $t = 0$, the source crosses the origin *P*. (a) When does the sound emitted from the source at *P* reach the listener *Q* ? (b) What will be the frequency heard by the listener at this instant ? (c) Where will the source be at this instant ?

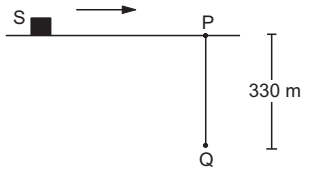



Figure 16-E12

**86.** A source emitting sound at frequency 4000 Hz, is moving along the *Y*-axis with a speed of 22 m $s^{-1}$. A listener is situated on the ground at the position (660 m, 0). Find the frequency of the sound received by the listener at the instant the source crosses the origin. Speed of sound in air = 330 m $s^{-1}$.

**87.** A source of sound emitting a 1200 Hz note travels along a straight line at a speed of 170 m $s^{-1}$. A detector is

placed at a distance of 200 m from the line of motion of the source. (a) Find the frequency of sound received by the detector at the instant when the source gets closest to it. (b) Find the distance between the source and the detector at the instant it detects the frequency 1200 Hz. Velocity of sound in air = 340 m $s^{-1}$.

**88.** A small source of sound $S$ of frequency 500 Hz is attached to the end of a light string and is whirled in a vertical circle of radius 1·6 m. The string just remains tight when the source is at the highest point. (a) An observer is located in the same vertical plane at a large distance and at the same height as the centre of the circle (figure 16-E13). The speed of sound in air = 330 m $s^{-1}$ and $g = 10$ m $s^{-2}$. Find the maximum frequency heard by the observer. (b) An observer is situated at a large distance vertically above the centre of the circle. Find the frequencies heard by the observer corresponding to the sound emitted by the source when it is at the same height as the centre.

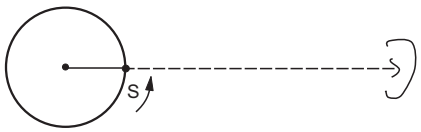



Figure 16-E13

**89.** A source emitting a sound of frequency ν is placed at a large distance from an observer. The source starts moving towards the observer with a uniform acceleration $a$. Find the frequency heard by the observer corresponding to the wave emitted just after the source starts. The speed of sound in the medium is $v$.

□

## ANSWERS

### OBJECTIVE I

1. (a) 2. (d) 3. (d) 4. (c) 5. (d) 6. (c)
7. (a) 8. (d) 9. (b) 10. (a) 11. (c) 12. (c)
13. (a) 14. (d) 15. (d) 16. (c)

### OBJECTIVE II

1. (d) 2. (a), (c) 3. (b), (c), (d)
4. (c) 5. (a), (d)

### EXERCISES

1. 2·75 m s
2. 0·5 s
3. 333 m $s^{-1}$
4. 18 mm, 18 m
5. 7·25 cm, 72·5 m
6. (a) 170 Hz (b) 17 kHz
7. $7{\cdot}6 \times 10^{-5}$ m, $3{\cdot}3 \times 10^{-4}$ m
8. (a) $1{\cdot}7 \times 10^{-5}$ (b) $1{\cdot}1 \times 10^{-4}$
9. (a) $\pi/2$ (b) $2\pi/35$
10. (a) zero (b) zero
11. 310 m $s^{-1}$
12. 349 m $s^{-1}$
13. 819°C
14. $\frac{2d}{v} \cdot \frac{\sqrt{273}}{\sqrt{T_1} + \sqrt{T_2}}$, 96 m $s^{-1}$
15. 0·14 $cm^3$
16. $1{\cdot}4 \times 10^5$ N $m^{-2}$
17. (a) 44 mW $m^{-2}$ (b) 6·0 Pa (c) $1{\cdot}2 \times 10^{-6}$ m
18. $4{\cdot}0 \times 10^{-10}$ W $m^{-2}$
19. 20 dB
20. 3 dB
21. 40 cm
22. $\sqrt{10}$
23. 53 dB
24. 3·4 kHz
25. (a) 5·0 kHz (b) 2
26. 1200 Hz, 2000 Hz, 2800 Hz, 3600 Hz and 4400 Hz
27. 420 Hz
28. 0·13 $d$
29. 200 $(2n + 1)$ Hz, where $n = 0, 1, 2, \ldots, 49$
30. 12·6 cm
31. (a) 7·9° (b) 16° (c) two
32. zero
33. $\sqrt{3}\,D$
34. 0°, 48·2°, 70·5°, 90° and similar points in other quadrants
35. (a) $I_0/4$ (b) $I_0/4$
36. 850 Hz, 1700 Hz and 2550 Hz
37. 17 cm
38. 4·1 kHz
39. 340 Hz
40. 1020 Hz, 1360 Hz and 1700 Hz
41. (a) 336 m $s^{-1}$ (b) 1 cm
42. 20 cm
43. 1·9$n$ kHz, where $n = 1, 2, 3, \ldots, 10$
44. 8·5 m
45. (a) 3·4 kHz (b) 5
46. $100(2n + 1)$ Hz, where $n = 0, 1, 2, 3, \ldots, 9$
47. 25 cm

48. 328 m/s
49. 18·8 cm, 56·3 cm
50. 11·6 N
51. 347 N
53. 294 Hz
54. 3400 m/s
55. 5200 m/s, 338 m/s
56. 478 Hz
57. 252 Hz
58. 7 per second
59. 205 Hz
60. 1·02
61. 0·39 cm
62. 2·06 kHz
63. 2436 Hz
64. (a) 1328 Hz (b) 1181 Hz
65. 1480 Hz
66. $4{\cdot}67 \times 10^4$ Hz, $4{\cdot}34 \times 10^4$ Hz, 3270 Hz
67. 0·8
68. 529 Hz, 474 Hz
69. (a) 11 km/h (b) a little less than 4 beats/s
70. 4·6 Hz
71. 17·5 Hz, may not be able to distinguish
72. 485 Hz and 515 Hz
73. 577 Hz
74. 300 km/h
75. 827 Hz
76. (a) 2034 Hz (b) 2068 Hz
77. 0·63 s
78. 1670 Hz
79. (a) 680 Hz (b) 660 Hz (c) 640 Hz
80. (a) 500 Hz (b) 459 Hz
(c) 500 Hz by the passenger and 458 by the person near the track
81. (a) 1616 Hz (b) 1632 Hz
82. 1417 Hz
83. (a) 80 cm (b) 80 cm (c) 437 Hz
(d) No beat may be heard
84. 4 m/s
85. (a) $t$ = 1 second (b) 2·0 kHz (c) at $x = 22$ m
86. 4018 Hz
87. (a) 1600 Hz (b) 224 m
88. (a) 514 Hz (b) 490 Hz and 511 Hz
89. $\dfrac{2v\nu^2}{2v\nu - a}$

□

## CHAPTER 17

# LIGHT WAVES

### 17.1 WAVES OR PARTICLES

The question whether light is a wave or a particle has a very interesting and a long history. The investigations about the nature of light has unfolded a huge treasure of knowledge and understanding. This question has great contributions to the development of the theory of quantum mechanics which presents an altogether different picture of the world in which there are no particles, no positions, no momenta in an ordinary sense.

Newton, the greatest among the great, believed that light is a collection of particles. He believed that a light source emits tiny corpuscles of light and these corpuscles travel in straight lines when not acted upon by external forces. The fact that light seems to travel in straight lines and cast shadows behind the obstacles was perhaps the strongest evidence of the particle nature of light. Newton could explain the laws of reflection of light on the basis of elastic collisions of the particles of light with the surface it is incident upon. The laws of refraction were explained by assuming that the particles of denser medium, such as glass or water, strongly attract the particles of light causing a bending at the surface. Newton had performed a number of experiments to study the behaviour of light. The book *Optiks* written by him, gives a classic account of these experiments.

The Dutch physicist Christian Huygens (1629–1695), who was a contemporary of Newton, suggested that light may be a wave phenomenon. The apparent rectilinear propagation of light may be due to the fact that the wavelength of light may be much smaller than the dimensions of these openings and obstacles.

Huygens' proposal remained in a dump for almost about a century. The scientific community by and large had great faith in Newton's writings and the particle theory remained in chair for a long time when it was seriously challenged by the double-slit experiment of Thomas Young (1773–1829) in 1801. This experiment clearly established that light coming from two coherent sources interferes and produces maxima and minima depending on the path difference. A series of experiments on diffraction of light conducted by the French physicist Augustin Jean Fresnel (1788–1827), measurement of velocity of light in water by Foucault in 1850, development of theory of electromagnetic waves by Maxwell in 1860 which correctly predicted the speed of light, were parts of a long activity which put the corpuscle theory of light to an end and convincingly established that light is a wave phenomenon.

But the drama was not yet over. The climax came when the wave theory of light failed to explain Hallwachs and Lenard's observation in 1900 that when light falls on a metal surface, electrons are ejected and that the kinetic energy of the emitted electrons does not depend on the intensity of the light used. Hertz possibly had the first observation of this phenomenon in early 1880's. This observation is known as *photoelectric effect* and we shall study it in detail in a later chapter. Photoelectric effect was explained by another giant, Albert Einstein, in 1905 on a particle model of light only. The old question "waves or particles" was reopened and an amicable understanding was reached in accepting that light has dual character. It can behave as particles as well as waves depending on its interaction with the surrounding. Later, it was found that even the well-established particles such as electrons also have a dual character and can show interference and diffraction under suitable situations. We shall study the wave particle duality in a later chapter. In this chapter, we shall study the wave aspect of light.

### 17.2 THE NATURE OF LIGHT WAVES

In a wave motion, there is some quantity which changes its value with time and space. In the wave on a string, it is the transverse displacement of the particles that changes with time and is different for different particles at the same instant. In the case of

sound waves, it is the pressure at a point in the medium that oscillates as time passes and has different values at different points at the same instant. We also know that it is the elastic properties of the medium that is responsible for the propagation of disturbance in a medium. The speed of a wave is determined by the elastic as well as the inertia properties of the medium.

The case with light waves is a bit different. The light waves need no material medium to travel. They can propagate in vacuum. Light is a nonmechanical wave. It was very difficult for the earlier physicists to conceive a wave propagating without a medium. Once the interference and diffraction experiments established the wave character of light, the search began for the medium responsible for the propagation of light waves. Light comes from the sun to the earth crossing millions of kilometers where apparently there is no material medium. What transmits the wave through this region ? Physicists assumed that a very dilute and highly elastic material medium is present everywhere in space and they named it "ether". Ether was never discovered and today we understand that light wave can propagate in vacuum.

The quantity that changes with space and time, in terms of which the wave equation should be written, is the electric field existing in space where light travels. We shall define and study electric field in later chapters, here we need only to know that (a) the electric field is a vector quantity and (b) the electric field is transverse to the direction of propagation of light (there are exceptions but we shall not discuss them).

Because light waves are transverse, they can be polarized. If a plane wave of light is travelling along the $x$-direction, the electric field may be along the $y$-direction or along the $z$-direction or in any other direction in the $y$-$z$ plane. The equation of such a light wave may be written as

$$E = E_0 \sin \omega(t - x/v), \qquad \ldots \quad (17.1)$$

where $E_0$ is the magnitude of the electric field at point $x$ at time $t$. The speed of light is itself an interesting quantity. The speed of light in vacuum with respect to any observer is always the same and is very nearly equal to $3 \times 10^8$ m s$^{-1}$. This speed is a fixed universal constant irrespective of the state of motion of the observer. This needs a basic revision of our concepts about space and time and is the basis of special theory of relativity. The speed of light in vacuum is generally denoted by the letter $c$. When a light wave travels in a transparent material, the speed is decreased by a factor $\mu$, called the *refractive index* of the material.

$$\mu = \frac{\text{speed of light in vacuum}}{\text{speed of light in the material}}. \quad \ldots \quad (17.2)$$

For a spherical wave originating from a point source, the equation of the wave is of the form

$$E = \frac{aE_0}{r} \sin \omega(t - r/v),$$

where $a$ is a constant.

The amplitude is proportional to the inverse of the distance and thus the intensity is proportional to the square of the inverse distance.

***Example 17.1***

*The refractive index of glass is* 1·5. *Find the speed of light in glass.*

***Solution*** : We have

$$\mu = \frac{\text{speed of light in vacuum}}{\text{speed of light in the material}}$$

Thus, speed of light in glass

$$= \frac{\text{speed of light in vacuum}}{\mu}$$

$$= \frac{3{\cdot}0 \times 10^8 \text{ m s}^{-1}}{1{\cdot}5} = 2{\cdot}0 \times 10^8 \text{ m s}^{-1}.$$

The frequency of visible light varies from about $3800 \times 10^{11}$ Hz to about $7800 \times 10^{11}$ Hz. The corresponding wavelengths (obtained from $\lambda = c/\nu$) are 380 nm to 780 nm. The colour sensation to a human eye is related to the wavelength of the light. Light of wavelength close to 780 nm appears red, and that close to 380 nm appears violet. Table (17.1) shows a rough relationship between the colour sensed and the wavelength of light.

**Table 17.1**

| Colour | Wavelength (order) |
|---|---|
| Red | 620–780 nm |
| Orange | 590–620 nm |
| Yellow | 570–590 nm |
| Green | 500–570 nm |
| Blue | 450–500 nm |
| Violet | 380–450 nm |

Light of single wavelength is called *monochromatic light*. Equation (17.1) represents a monochromatic light wave. Often, the light emitted by a source is a mixture of light corresponding to different wavelengths. Depending on the composition of the mixture, a human eye senses a large number of colours. White light itself is a mixture of light of all wavelengths from about 380 nm to about 780 nm in appropriate proportion.

In fact, a strictly monochromatic light is not possible to obtain. There is always a spread in wavelength. The best monochromatic light are LASERs in which the spread in wavelength is very small but not zero. We shall use the word "monochromatic light" to mean that the light contains a dominant wavelength with only a little spread.

We discussed in the previous chapter that if a wave is obstructed during its propagation by an obstacle or an opening, it gets diffracted. A plane wave going through a small opening becomes more like a spherical wave on the other side. Thus, the wave bends at the edges. Also, if the dimensions of the obstacle or the opening is much larger than the wavelength, the diffraction is negligible and the rays go along straight lines.

In the case of light, the wavelength is around 380–780 nm. The obstacles or openings encountered in normal situations are generally of the order of millimeters or even larger. Thus, the wavelength is several thousands times smaller than the usual obstacles or openings. The diffraction is almost negligible and the light waves propagate in straight lines and cast shadows of the obstacles. The light can then be treated as light rays which are straight lines drawn from the source and which terminate at an opaque surface and which pass through an opening undeflected. This is known as the *Geometrical optics* approximation and majority of the phenomena in normal life may be discussed in this approximation. The three major rules governing geometrical optics are the following.

1. *Rectilinear propagation of light* : Light travels in straight lines unless it is reflected by a polished surface or the medium of propagation is changed.

2. *Reflection of light* : The angle of incidence and the angle of reflection (i.e., the angles made by the incident and the reflected rays with the normal to the surface) are equal. Also, the incident ray, the reflected ray and the normal to the reflecting surface are coplanar.

3. *Refraction of light* : When light travelling in one medium enters another medium, the angle of incidence $i$ and the angle of refraction $r$ (angle made by the refracted ray with the normal) satisfy

$$\frac{\sin i}{\sin r} = \frac{v_1}{v_2},$$

where $v_1$ and $v_2$ are the speeds of light in the first medium and the second medium respectively. Also, the incident ray, the refracted ray and the normal to the separating surface are coplanar.

The rectilinear propagation of light is explained on the basis of wave theory by observing that the wavelength of light is much smaller than the obstacles or openings usually encountered. The laws of reflection and refraction can also be explained by wave theory. The rigorous derivation involves somewhat complicated mathematics, but things can be fairly well-understood by a geometrical method proposed by Huygens. This method tells us how to construct the shape of a wavefront of light wave from the given shape at an earlier instant. We refer again from the previous chapters that (a) a surface on which the wave disturbance is in same phase at all points is called a wavefront, (b) the direction of propagation of a wave at a point is perpendicular to the wavefront through that point, (c) the wavefronts of a wave originating from a point source are spherical and (d) the wavefronts for a wave going along a fixed direction are planes perpendicular to that direction.

### 17.3 HUYGENS' PRINCIPLE

The first proposer of the wave theory of light, Huygens, considered light to be a mechanical wave moving in a hypothetical medium which was named as ether. If we consider a surface σ enclosing a light source $S$, the optical disturbance at any point beyond σ must reach after crossing σ. The particles of the surface σ vibrate as the wave from $S$ reaches there and these vibrations cause the layer beyond to vibrate. We can thus assume that the particles on σ act as new sources of light waves emitting spherical waves and the disturbance at a point $A$ (figure 17.1) beyond σ, is caused by the superposition of all these spherical waves coming from different points of σ. Huygens called the particles spreading the vibration beyond them as *secondary sources* and the spherical wavefronts emitted from these secondary sources as the *secondary wavelets.*

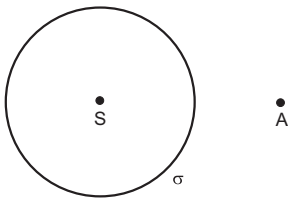



Figure 17.1

Huygens' principle may be stated in its most general form as follows :

*Various points of an arbitrary surface, when reached by a wavefront, become secondary sources of light emitting secondary wavelets. The disturbance beyond the surface results from the superposition of these secondary wavelets.*

Consider a spherical surface σ with its centre at a point source $S$ emitting a pulse of light (figure 17.2).

The optical disturbance reaches the particles on σ at time $t = 0$ and lasts for a short interval in which the positive and negative disturbances are produced. These particles on σ then send spherical wavelets which spread beyond σ. At time $t$, each of these wavelets has a radius $vt$. In figure (17.2), the solid lines represent positive optical disturbance and the dashed lines represent negative optical disturbance. The sphere Σ is the geometrical envelope of all the secondary wavelets which were emitted at time $t = 0$ from the primary wavefront σ.

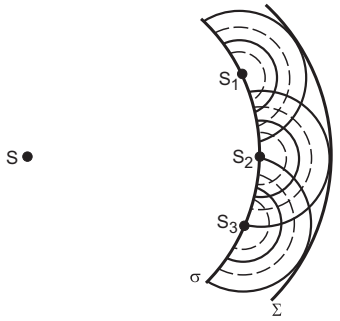



Figure 17.2

It is clear that at the points just inside Σ, only the positive disturbances of various secondary wavelets are meeting. The wavelets, therefore, interfere constructively at these points and produce finite disturbance. For points well inside Σ, some of the wavelets contribute positive disturbance and some others, centred at a nearby point of σ, produce negative disturbance. Thus, the resultant disturbance is zero at these points. The disturbance which was situated at σ at time $t = 0$ is, therefore, confined to a surface Σ at time $t$. Hence, the secondary wavelets from σ superpose in such a way that they produce a new wavefront at the geometrical envelope of the secondary wavelets.

This allows us to state the method of Huygens construction as follows :

**Huygens Construction**

*Various points of an arbitrary surface, as they are reached by a wavefront, become the sources of secondary wavelets. The geometrical envelope of these wavelets at any given later instant represents the new position of the wavefront at that instant.*

The method is quite general and although it was developed on the notion of mechanical waves it is valid for light waves. The surface used in the Huygens construction may have any arbitrary shape, not necessarily a wavefront itself. If the medium is homogeneous, (i.e., the optical properties of the medium are same everywhere) light moves forward and does not reflect back. We assume, therefore, that the secondary wavelets are emitted only in the forward direction and the geometrical envelope of the wavelets is to be taken in the direction of advancement of the wave. If there is a change of medium, the wave may be reflected from the discontinuity just as a wave on a string is reflected from a fixed end or a free end. In that case, secondary wavelets on the backward side should also be considered.

**Reflection of Light**

Let us suppose that a parallel light beam is incident upon a reflecting plane surface σ such as a plane mirror. The wavefronts of the incident wave will be planes perpendicular to the direction of incidence. After reflection, the light returns in the same medium. Consider a particular wavefront $AB$ of the incident light wave at $t = 0$ (figure 17.3). We shall construct the position of this wavefront at time $t$.

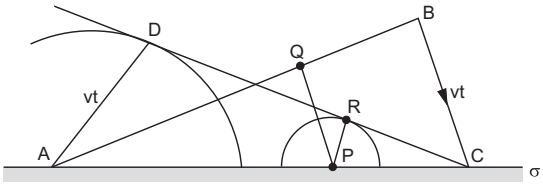



Figure 17.3

To apply Huygens construction, we use the reflecting surface σ for the sources of secondary wavelets. As the various points of σ are reached by the wavefront $AB$, they become sources of secondary wavelets. Because of the change of medium, the wavelets are emitted both in forward and backward directions. To study reflection, the wavelets emitted in the backward directions are to be considered.

Suppose, the point $A$ of σ is reached by the wavefront $AB$ at time $t = 0$. This point then emits a secondary wavelet. At time $t$, this wavelet becomes a hemispherical surface of radius $vt$ centred at $A$. Here $v$ is the speed of light. Let $C$ be the point which is just reached by the wavefront at time $t$ and hence the wavelet is a point at $C$ itself. Draw the tangent plane $CD$ from $C$ to the hemispherical wavelet originated from $A$. Consider an arbitrary point $P$ on the surface and let $AP/AC = x$. Let $PQ$ be the perpendicular from $P$ to $AB$ and let $PR$ be the perpendicular from $P$ to $CD$. By the figure,

$$\frac{PR}{AD} = \frac{PC}{AC} = \frac{AC - AP}{AC} = 1 - x$$

or, $$PR = AD\,(1 - x) = vt(1 - x). \quad \ldots \text{(i)}$$

Also, $$\frac{QP}{BC} = \frac{AP}{AC} = x$$

or, $QP = x\,BC = xvt.$

The time taken by the wavefront to reach the point $P$ is, therefore,

$$t_1 = \frac{QP}{v} = xt.$$

The point $P$ becomes a source of secondary wavelets at time $t_1$. The radius of the wavelet at time $t$, originated from $P$ is, therefore,

$$a = v(t - t_1) = v(t - xt) = vt(1 - x). \quad \ldots \text{ (ii)}$$

By (i) and (ii), we see that $PR$ is the radius of the secondary wavelet at time $t$ coming from $P$. As $CD$ is perpendicular to $PR$, $CD$ touches this wavelet. As $P$ is an arbitrary point on σ, all the wavelets originated from different points of σ, touch $CD$ at time $t$. Thus, $CD$ is the envelope of all these wavelets at time $t$. It is, therefore, the new position of the wavefront $AB$. The reflected rays are perpendicular to this wavefront $CD$.

In triangles $ABC$ and $ADC$ :

$$AD = BC = vt,$$

$AC$ is common,

and $\angle ADC = \angle ABC = 90°.$

Thus, the triangles are congruent and

$$\angle BAC = \angle DCA. \quad \ldots \text{ (iii)}$$

Now, the incident ray is perpendicular to $AB$ and the normal is perpendicular to $AC$. The angle between the incident ray and the normal is, therefore, equal to the angle between $AB$ and $AC$. Thus, $\angle BAC$ is equal to the angle of incidence.

Similarly, $\angle DCA$ represents the angle of reflection and we have proved in (iii) that the angle of incidence equals the angle of reflection. From the geometry, it is clear that the incident ray, the reflected ray and the normal to the surface $AC$ lie in the plane of drawing and hence, are coplanar.

**Refraction of Light**

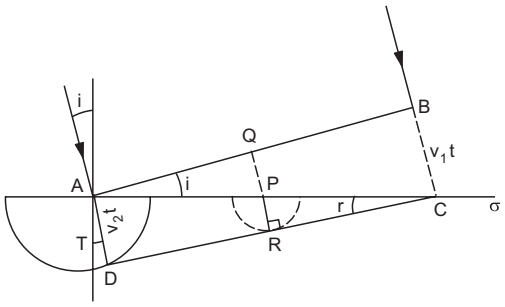



Figure 17.4

Suppose σ represents the surface separating two transparent media, medium 1 and medium 2 in which the speeds of light are $v_1$ and $v_2$ respectively. A parallel beam of light moving in medium 1 is incident on the surface and enters medium 2. In figure (17.4), we show the incident wavefront $AB$ in medium 1 at $t = 0$. The incident rays are perpendicular to this wavefront. To find the position of this wavefront after refraction, we apply the method of Huygens construction to the surface σ. The point $A$ of the surface is reached by the wavefront $AB$ at $t = 0$. This point becomes the source of secondary wavelet which expands in medium 2 at velocity $v_2$. At time $t$, this takes the shape of a hemisphere of radius $v_2 t$ centred at $A$. The point $C$ of the surface is just reached by the wavefront at time $t$ and hence, the wavelet is a point at $C$ itself. Draw the tangent plane $CD$ from $C$ to the wavelet originating from $A$.

Consider an arbitrary point $P$ on the surface σ and let $AP/AC = x$. Let $PQ$ and $PR$ be the perpendiculars from this arbitrary point $P$ to the planes $AB$ and $CD$ respectively. By the figure,

$$\frac{PR}{AD} = \frac{PC}{AC} = \frac{AC - AP}{AC} = 1 - x$$

or, $PR = AD(1 - x) = v_2\,t(1 - x).$ … (i)

Also, $\frac{QP}{BC} = \frac{AP}{AC} = x.$

Thus, $QP = x\,BC = xv_1\,t.$

The time, at which the wavefront arrives at $P$, is

$$t_1 = \frac{QP}{v_1} = xt.$$

The radius of the wavelet originated from $P$ and going into the second medium is, therefore,

$$a = v_2(t - t_1) = v_2\,t(\,1 - x). \quad \ldots \text{ (ii)}$$

By (i) and (ii), we see that $PR$ is the radius of the wavelet originating from $P$. As $CD$ is perpendicular to $PR$, $CD$ touches this wavelet. As $P$ is an arbitrary point on σ, all the wavelets which originated from different points on σ touch $CD$ at time $t$. The plane $CD$ is, therefore, the geometrical envelope of all the secondary wavelets at time $t$. It is, therefore, the position of the wavefront $AB$ at time $t$. The refracted rays are perpendicular to $CD$.

The angle $BAC$ is also equal to the angle between the incident ray (which is perpendicular to $AB$) and the normal to the surface and hence, it is equal to the angle of incidence $i$. Similarly, $\angle ACD$ is equal to the angle of refraction $r$.

We have $\sin i = \frac{BC}{AC}$

and $\sin r = \frac{AD}{AC}$

so that

$$\frac{\sin i}{\sin r} = \frac{BC}{AD} = \frac{v_1 t}{v_2 t}$$

or, $$\frac{\sin i}{\sin r} = \frac{v_1}{v_2}$$

which is called the *Snell's law*. The ratio $v_1/v_2$ is called the refractive index of medium 2 with respect to medium 1 and is denoted by $\mu_{21}$. If the medium 1 is vacuum, $\mu_{21}$ is simply the refractive index of the medium 2 and is denoted by $\mu$.

Also, $$\mu_{21} = \frac{v_1}{v_2} = \frac{c/v_2}{c/v_1} = \frac{\mu_2}{\mu_1}.$$

From the figure, it is clear that the incident ray, the refracted ray and the normal to the surface $\sigma$ are all in the plane of the drawing, i.e., they are coplanar.

Suppose light from air is incident on water. It bends towards the normal giving $i > r$. From Snell's law proved above, $v_1 > v_2$. Thus, according to the wave theory the speed of light should be greater in air than in water. This is opposite to the prediction of Newton's corpuscle theory. If light bends due to the attraction of the particles of a medium then speed of light should be greater in the medium. Later, experiments on measurement of speed of light confirmed wave theory.

Thus, the basic rules of geometrical optics could be understood in terms of the wave theory of light using Huygens' principle. In the rest of this chapter, we shall study the wave behaviour, such as the interference, diffraction and polarization of light.

## 17.4 YOUNG'S DOUBLE HOLE EXPERIMENT

Thomas Young in 1801 reported his experiment on the interference of light. He made a pinhole in a cardboard and allowed sunlight to pass through. This light was then allowed to fall upon another cardboard having two pinholes side by side placed symmetrically. The emergent light was received on a plane screen placed at some distance. At a given point on the screen, the waves from the two holes had different phases. These waves interfered to give a pattern of bright and dark areas. The variation of intensity on the screen demonstrated the interference taking place between the light waves reaching the screen from the two pinholes.

The pattern of bright and dark areas are sharply defined only if light of a single wavelength is used. Young's original experiments were performed with white light and he deduced from the experiments that the wavelength of extreme red light was around 1/36000 inch and that of the extreme violet was around 1/60000 inch. These results are quite close to their accurate measurements done with modern instruments.

## 17.5 YOUNG'S DOUBLE SLIT EXPERIMENT

In the double slit experiment, we use two long parallel slits as the sources of light in place of pin holes. The light coming out of the two slits is intercepted on a screen placed parallel to the plane of the slits. The slits are illuminated by a parallel beam of a monochromatic (of nearly a single wavelength) light. A series of dark and bright strips, called *fringes*, are observed on the screen. The arrangement of the experiment is schematically shown in figure (17.5a). Figure (17.5b) shows a cross section of the arrangement of the Young's double slit experiment.

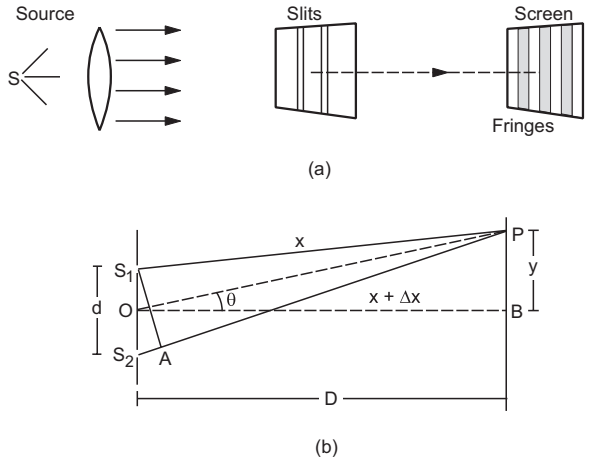



Figure 17.5

The two waves interfering at $P$ have covered different distances $S_1P = x$ and $S_2P = x + \Delta x$. The electric fields at $P$ due to the two waves may be written as

$$E_1 = E_{01} \sin(kx - \omega t)$$

and $$E_2 = E_{02} \sin[k(x + \Delta x) - \omega t]$$
$$= E_{02} \sin[kx - \omega t + \delta],$$

where $$\delta = \frac{\omega}{v}\Delta x = \frac{2\pi}{\lambda}\Delta x. \quad \ldots (17.3)$$

The situation is mathematically identical to that discussed in chapter 15, section 15.7. The resultant field at the point $P$ is

$$E = E_0 \sin(kx - \omega t + \varepsilon),$$

where $$E_0^2 = E_{01}^2 + E_{02}^2 + 2E_{01}E_{02}\cos\delta \quad \ldots (17.4)$$

and $$\tan\varepsilon = \frac{E_{02}\sin\delta}{E_{01} + E_{02}\cos\delta}.$$

The conditions for constructive (bright fringe) and destructive (dark fringe) interferences are,

$$\left.\begin{array}{ll} \delta = 2n\pi & \text{for bright fringes} \\ \text{and} \quad \delta = (2n+1)\pi & \text{for dark fringes} \end{array}\right| \ldots (17.5)$$

where $n$ is an integer.

Using, (17.3) these conditions may also be written as

$$\left.\begin{array}{ll} \Delta x = n\lambda & \text{for bright fringes} \\ \text{and} \quad \Delta x = \left(n + \frac{1}{2}\right)\lambda & \text{for dark fringes.} \end{array}\right| \ldots \quad (17.6)$$

At the point $B$ in figure (17.5b), $\Delta x = 0$ as $S_1B = S_2B$. This point is the centre of the bright fringe corresponding to $n = 0$.

**Intensity Variation**

If the two slits are identical, $E_{01} = E_{02} = E_0'$ and from equation (17.4),

$$E_0^2 = 2E_0'^2(1 + \cos\delta).$$

As the intensity is proportional to the square of the amplitude, we get

$$I = 2I'(1 + \cos\delta) = 4I'\cos^2\frac{\delta}{2}, \qquad \ldots \quad (17.7)$$

where $I$ is the resultant intensity and $I'$ is the intensity due to a single slit.

The equation gives intensity as a function of $\delta$. At the centre of a bright fringe, $\delta = 2n\pi$ and $I = 4I'$. At the centre of a dark fringe, $\delta = (2n + 1)\pi$ and $I = 0$. At other points, the intensity is in between 0 and $4I'$ as given by equation (17.7).

**Fringe-width and Determination of Wavelength**

The separation on the screen between the centres of two consecutive bright fringes or two consecutive dark fringes is called the *fringe-width*. Suppose $S_1A$ is the perpendicular from $S_1$ to $S_2P$ (figure 17.5b). Suppose,

$D = OB$ = separation between the slits and the screen,

$d$ = separation between the slits

and $D >> d$.

Under the above approximation ($D >> d$), $S_1P$ and $S_2P$ are nearly parallel and hence $S_1A$ is very nearly perpendicular to $S_1P, S_2P$ and $OP$. As $S_1S_2$ is perpendicular to $OB$ and $S_1A$ is perpendicular (nearly) to $OP$, we have

$$\angle S_2S_1A = \angle POB = \theta.$$

This is a small angle as $D >> d$.

The path difference is

$$\Delta x = PS_2 - PS_1 \approx PS_2 - PA$$
$$= S_2A = d\sin\theta$$
$$\approx d\tan\theta = d\frac{y}{D}.$$

The centres of the bright fringes are obtained at distances $y$ from the point $B$, where

$$\Delta x = d\frac{y}{D} = n\lambda \quad (\text{where } n \text{ is an integer})$$

or, $$y = \frac{nD\lambda}{d},$$

i.e., at $y = 0, \pm\frac{D\lambda}{d}, \pm\frac{2D\lambda}{d}, \pm\frac{3D\lambda}{d}, \ldots,$ etc.

The centres of dark fringes will be obtained, where

$$\Delta x = d\frac{y}{D} = \left(n + \frac{1}{2}\right)\lambda$$

or, $$y = \left(n + \frac{1}{2}\right)\frac{D\lambda}{d},$$

i.e., at $y = \pm\frac{D\lambda}{2d}, \pm\frac{3D\lambda}{2d}, \pm\frac{5D\lambda}{2d}, \ldots$

The fringe-width is, therefore,

$$w = \frac{D\lambda}{d}. \qquad \ldots \quad (17.8)$$

By measuring $D$, $d$ and $W$ in an experiment, one can calculate the wavelength of the light used. We see from equation (17.8) that as the separation $d$ between the slits is increased, the fringe-width is decreased. If $d$ becomes much larger than $\lambda$, the fringe-width will be very small. The maxima and minima, in this case, will be so closely spaced that it will look like a uniform intensity pattern. This is an example of the general result that the wave effects are difficult to observe, if the wavelength is small compared to the dimensions of the obstructions or openings to the incident wavefront.

***Example 17.2***

*In a Young's double slit experiment, the separation between the slits is* 0·10 mm, *the wavelength of light used is* 600 nm *and the interference pattern is observed on a screen* 1·0 m *away. Find the separation between the successive bright fringes.*

***Solution*** : The separation between the successive bright fringes is

$$w = \frac{D\lambda}{d} = \frac{1{\cdot}0\text{ m} \times 600 \times 10^{-9}\text{ m}}{0{\cdot}10 \times 10^{-3}\text{ m}}$$
$$= 6{\cdot}0 \times 10^{-3}\text{ m} = 6{\cdot}0\text{ mm}.$$

**17.6 OPTICAL PATH**

Consider a light wave travelling in a medium of refractive index $\mu$. Its equation may be written as

$$E = E_0 \sin\omega(t - x/v) = E_0 \sin\omega(t - \mu x/c).$$

If the light wave travels a distance $\Delta x$, the phase changes by

$$\delta_1 = \mu\frac{\omega}{c}\Delta x. \qquad \ldots \quad (i)$$

Instead, if the light wave travels in vacuum, its equation will be

$$E = E_0 \sin \omega(t - x/c).$$

If the light travels through a distance $\mu \Delta x$, the phase changes by

$$\delta_2 = \frac{\omega}{c}(\mu \Delta x) = \mu \frac{\omega}{c} \Delta x. \quad \ldots \text{(ii)}$$

By (i) and (ii), we see that a wave travelling through a distance $\Delta x$ in a medium of refractive index $\mu$ suffers the same phase change as when it travels a distance $\mu\Delta x$ in vacuum. In other words, a path length of $\Delta x$ in a medium of refractive index $\mu$ is equivalent to a path length of $\mu\Delta x$ in vacuum. The quantity $\mu\Delta x$ is called the *optical path* of the light. In dealing with interference of two waves, we need the difference between the optical paths travelled by the waves. The geometrical path and the optical path are equal only when light travels in vacuum or in air where the refractive index is close to 1.

The concept of optical path may also be introduced in terms of the change in wavelength as the wave changes its medium. The frequency of a wave is determined by the frequency of the source and is not changed when the wave enters a new medium. If the wavelength of light in vacuum is $\lambda_0$ and that in the medium is $\lambda_n$, then

$$\lambda_0 = \frac{c}{\nu}$$

and

$$\lambda_n = \frac{v}{\nu} = \frac{c}{\mu\nu}$$

so that

$$\lambda_n = \frac{\lambda_0}{\mu}.$$

At any given instant, the points differing by one wavelength have same phase of vibration. Thus, the points at separation $\lambda_n$ in the medium have same phase of vibration. On the other hand, in vacuum, points at separation $\lambda_0$ will have same phase of vibration. Thus, a path $\lambda_n$ in a medium is equivalent to a path $\lambda_0 = \mu\lambda_n$ in vacuum. In general, a path $\Delta x$, in a medium of refractive index $\mu$, is equivalent to a path $\mu\Delta x$ in vacuum which is called the optical path.

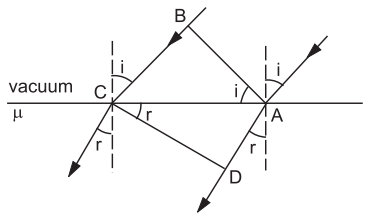



Figure 17.6

We can also understand the idea of optical path with the help of figure (17.6). Suppose a parallel beam of light travelling in vacuum is incident on the surface $AC$ of a medium of refractive index $\mu$. $AB$ is perpendicular to the incident rays and hence represents a wavefront of the incident light. Similarly $CD$ is perpendicular to the refracted rays and represents a wavefront of the refracted light. Now phase of the wave has a constant value at different points of a wavefront. Thus phase at $A$ = phase at $B$ and phase at $C$ = phase at $D$.

Thus, the phase difference between $A$ and $D$ = Phase difference between $B$ and $C$. From the figure,

$$\mu = \frac{\sin i}{\sin r} = \frac{BC}{AC} \times \frac{AC}{AD} = \frac{BC}{AD}$$

or, $BC = \mu\, AD$.

The phase of light wave changes by equal amount whether it covers a distance $BC = \mu\, AD$ in vacuum or $AD$ in the medium. Thus, a path $AD$ in a medium of refractive index $\mu$ is equivalent to a path $\mu(AD)$ in vacuum which we call optical path.

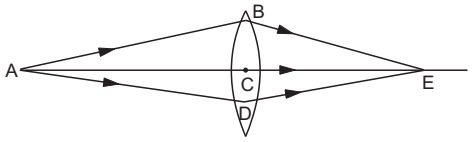



Figure 17.7

Consider the situation in figure (17.7). The geometrical paths $ABE$, $ACE$ and $ADE$ are different, but the optical paths are equal. This is because each path leads to the same phase difference, phase at $E$-phase at $A$. Note that the ray having longer geometrical path covers less distance in the lens as compared to the ray having shorter geometrical path.

***Example 17.3***

*The wavelength of light coming from a sodium source is* 589 nm. *What will be its wavelength in water ? Refractive index of water* = 1·33.

***Solution*** : The wavelength in water is $\lambda = \lambda_0 / \mu$, where $\lambda_0$ is the wavelength in vacuum and $\mu$ is the refractive index of water. Thus,

$$\lambda = \frac{589 \text{ nm}}{1{\cdot}33} = 443 \text{ nm}.$$

## 17.7 INTERFERENCE FROM THIN FILMS

When oil floating on water is viewed in sunlight, beautiful colours appear. These colours appear because of interference between the light waves sent by the film as explained below.

Consider a thin film made of a transparent material with plane parallel faces separated by a distance $d$. Suppose a parallel beam of light is incident

on the film at an angle $i$ as shown in figure (17.8). The wave is divided into two parts at the upper surface, one is reflected and the other is refracted. The refracted part, which enters into the film, again gets divided at the lower surface into two parts; one is transmitted out of the film and the other is reflected back. Multiple reflections and refractions take place and a number of reflected waves as well as transmitted waves are sent by the film. The film may be viewed by the reflected light (more usual case) or by the transmitted light. We shall discuss the transmitted light first.

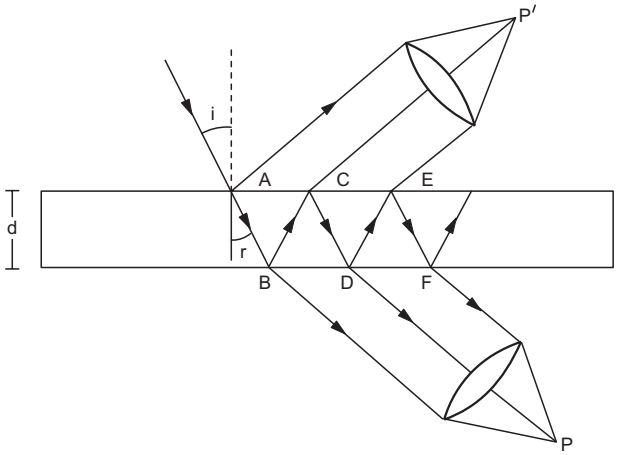



Figure 17.8

In figure (17.8), we have collected the parallel rays transmitted by the film by a converging lens at a point $P$. The amplitude of the individual transmitted waves is different for different waves; it gradually decreases as more reflections are involved. The wave *BP, DP, FP,* etc., interfere at $P$ to produce a resultant intensity. Let us consider the phase difference between the two waves *BP* and *DP*. The two waves moved together and hence, remained in phase up to $B$ where splitting occured and one wave followed the path *BP* and the other *BCDP*.

Let us discuss the special case for normal incidence when the angle of incidence $i = 0$. Then, the points $B$ and $D$ coincide. The path *BP* equals *DP* and the only extra distance travelled by the wave along *DP* is $BC + CD = 2d$. As this extra path is traversed in a medium of refractive index $\mu$, the optical path difference between the waves *BP* and *DP* interfering at $P$ is

$$\Delta x = 2\mu d.$$

The phase difference is

$$\delta = 2\pi \frac{\Delta x}{\lambda} = 2\pi \frac{2\mu d}{\lambda}.$$

This is also the phase difference between the waves *DP* and *FP* or in fact, between any consecutively transmitted waves. All these waves are in phase if

$$\delta = 2n\pi$$

or, $$2\mu d = n\lambda, \quad \ldots (17.9)$$

where $n$ is an integer.

If this condition is satisfied, constructive interference takes place and the film is seen illuminated. On the other hand, if

$$2\mu d = \left(n + \frac{1}{2}\right)\lambda, \quad \ldots (17.10)$$

$\delta = (2n + 1)\pi$ and the consecutive waves are out of phase. The waves cancel each other although complete cancellation does not take place because the interfering waves do not have equal amplitude. Still, the illumination will be comparatively less.

If white light is used, the film's thickness $d$ will satisfy condition (17.9) for certain wavelengths and these colours will be strongly transmitted due to constructive interference. The colours corresponding to the wavelengths for which (17.10) is satisfied will be poorly transmitted due to destructive interference. This gives coloured appearance of the film.

Next, let us consider the case when the film is viewed by the light reflected by it. The reflected light consists of waves from *A, C, E,* ...., etc., (figure 17.8) which may be brought to a focus at a point $P'$ by a converging lens. The optical path difference between the consecutively reflected waves reaching at $P'$ is again $2\mu d$ in the limit of normal incidence $i = 0$. Experimental arrangement may be a bit difficult for an exact normal incidence and then collection of reflected light along the same direction. However, we can suppose that it is viewed by light falling very nearly normal to it. We may expect that the condition of maximum illumination and minimum illumination will be same as equations (17.9) and (17.10). But a simple argument conflicts the case. If for a given thickness and wavelength, destructive interference takes place both in reflection as well as in transmission; where does the light go then ? What happens to the energy incident on the film ? Similarly, if the intensity is enhanced both in transmission and reflection, where does this extra energy come from ? It seems logical that if the intensity in transmission is increased, it should be at the cost of reflection and vice versa. So the conditions for maximum and minimum illumination in reflection should be opposite to that in transmission. We should have

$2\mu d = n\,\lambda$ for minimum illumination in reflection

$\ldots (17.11)$

and

$$2\mu d = \left(n + \frac{1}{2}\right)\lambda \quad \text{for maximum illumination in reflection.} \quad \ldots \ (17.12)$$

This comes out to be true experimentally. To explain why destructive interference takes place even when the optical path difference is an integral multiple of wavelength, let us recall our discussion of reflection and transmission of waves in chapter 15. If a composite string is prepared by joining a light string to a heavier one and if a wave pulse is sent from the lighter one towards the heavier one, a part is reflected from the junction and other is transmitted into the heavier string (figure 17.9a). The reflected pulse is inverted with respect to the incident pulse.

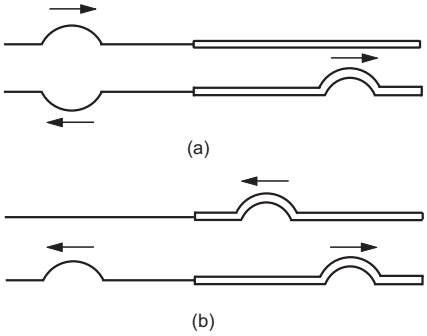



Figure 17.9

There is a sudden phase-change of π when a wave is reflected from a denser string. No such sudden phase change takes place if the wave is reflected from a rarer string (figure 17.9b). Same is true for light waves. The medium with higher refractive index is optically denser. When light is incident from air to a film, the reflected wave suffers a sudden phase change of π. The next wave, with which it interferes, suffers no such sudden phase change. If $2\mu d$ is equal to λ or its integral multiple, the second wave is out of phase with the first because the first has suffered a phase change of π. This explains the conditions (17.11) and (17.12).

***Example 17.4***

*Find the minimum thickness of a film which will strongly reflect the light of wavelength* 589 nm. *The refractive index of the material of the film is* 1·25.

***Solution*** : For strong reflection, the least optical path difference introduced by the film should be λ/2. The optical path difference between the waves reflected from the two surfaces of the film is $2\mu d$. Thus, for strong reflection,

$$2\mu d = \lambda/2$$

or,
$$d = \frac{\lambda}{4\mu} = \frac{589 \text{ nm}}{4 \times 1{\cdot}25} = 118 \text{ nm}.$$

## 17.8 FRESNEL BIPRISM

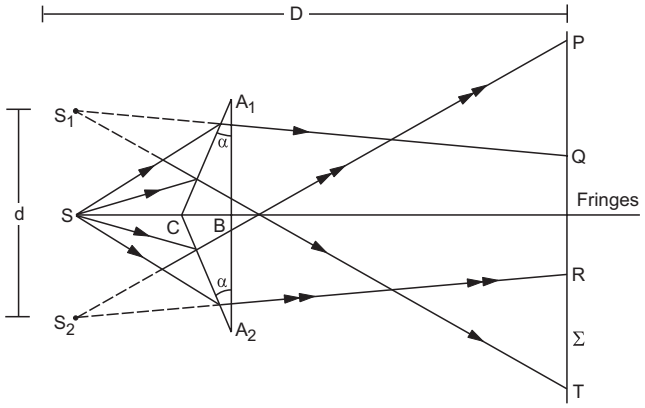



Figure 17.10

Figure (17.10) shows a schematic diagram of Fresnel biprism and interference of light using it. Two thin prisms $A_1BC$ and $A_2BC$ are joined at the bases to form a biprism. The refracting angles $A_1$ and $A_2$ (denoted by α in the figure) are of the order of half a degree each. In fact, it is a simple prism whose base angles are extremely small. A narrow slit $S$, allowing monochromatic light, is placed parallel to the refracting edge $C$. The light going through the prism $A_1BC$ appears in a cone $S_1QT$ and the light going through $A_2BC$ appears in a cone $S_2PR$. Here $S_1$ and $S_2$ are the virtual images of $S$ as formed by the prisms $A_1BC$ and $A_2BC$. A screen Σ is placed to intercept the transmitted light. Interference fringes are formed on the portion $QR$ of the screen where the two cones overlap.

One can treat the points $S_1$ and $S_2$ as two coherent sources sending light to the screen. The arrangement is then equivalent to a Young's double slit experiment with $S_1$ and $S_2$ acting as the two slits. Suppose the separation between $S_1$ and $S_2$ is $d$ and the separation between the plane of $S_1S_2$ and Σ is $D$. The fringe-width obtained on the screen is

$$w = \frac{D\lambda}{d}.$$

## 17.9 COHERENT AND INCOHERENT SOURCES

Two sources of light waves are said to be *coherent* if the initial phase difference $\delta_0$ between the waves emitted by the sources remains constant in time. If $\delta_0$ changes randomly with time, the sources are called *incoherent*. Two waves produce interference pattern only if they originate from coherent sources. This condition is same as discussed for sound waves in the previous chapter. The process of light emission from ordinary sources such as the sun, a candle, an electric

bulb, is such that one has to use special techniques to get coherent sources. In an ordinary source, light is emitted by atoms in discrete steps. An atom after emitting a short light pulse becomes inactive for some time. It again gains energy by some interaction, becomes active and emits another pulse of light. Thus, at a particular time a particular group of atoms is active and the rest are inactive. The active time is of the order of $10^{-8}$ s and during this period, a wavetrain of several meters is emitted.

We can picture the light coming from an ordinary source as a collection of several wavetrains, each several meters long and having no fixed phase relation with each other. Such a source is incoherent in itself. Different wavetrains are emitted by different groups of atoms and these groups act independently of each other, hence the phase varies randomly from train to train. If two lamps are substituted in place of the slits $S_1$ and $S_2$ in a Young's interference experiment, no fringe will be seen. This is because each source keeps on changing its phase randomly and hence, the phase difference between the two sources also changes randomly. That is why, a narrow aperture $S_0$ is used to select a particular wavetrain which is incident on the two slits together. This ensures that the initial phase difference of the wavelets originating from $S_1$ and $S_2$ does not change with time. When a new wavetrain is emitted by the lamp, the phase is randomly changed but that change is simultaneously communicated to both $S_1$ and $S_2$ and the phase difference remains unchanged. In order to obtain a fairly distinct interference pattern, the path difference between the two waves originating from coherent sources should be kept small. This is so because the wavetrains are finite in length and hence with large difference in path, the waves do not overlap at the same instant in the same region of space. The second wavetrain arrives well after the first train has already passed and hence, no interference takes place. In practice, the path difference should not exceed a few centimeters to observe a good interference pattern.

Because of the incoherent nature of the basic process of light emission in ordinary sources, these sources cannot emit highly monochromatic light. A strictly monochromatic light, having a well-defined single frequency or wavelength, must be a sine wave which has an infinite extension. A wavetrain of finite length may be described by the superposition of a number of sine waves of different wavelengths. Thus, the light emitted by an ordinary source always has a spread in wavelength. An ordinary sodium vapour lamp emits light of wavelength 589·0 nm and 589·6 nm with a spread of about ± 0·01 nm in each line. Shorter the length of the wavetrain, larger is the spread in wavelength.

It has been made possible to produce light sources which emit very long wavetrains, of the order of several hundred metres. The spread in wavelength is accordingly very small. These sources are called *laser* sources. The atoms behave in a cooperative manner in such a source and hence the light is coherent. Two independent laser sources can produce interference fringes and the path difference may be several metres long.

## 17.10 DIFFRACTION OF LIGHT

When a wave is obstructed by an obstacle, the rays bend round the corner. This phenomenon is known as *diffraction*. We can explain the effect using Huygens' principle. When a wavefront is partially obstructed, only the wavelets from the exposed parts superpose and the resulting wavefront has a different shape. This allows for the bending round the edges. In case of light waves, beautiful fringe patterns comprising maximum and minimum intensity are formed due to diffraction.

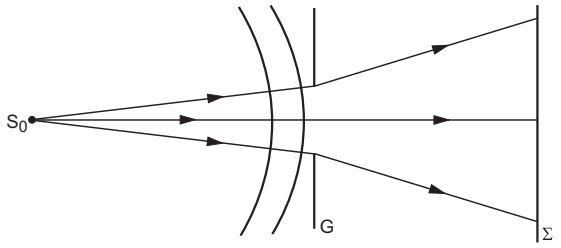



Figure 17.11

Figure (17.11) shows the basic arrangement for observing diffraction effects in light waves. It consists of a narrow source of light $S_0$, a diffracting element $G$ (an obstacle or an opening) and a screen Σ. The wavefronts emitted by the source $S_0$ are partially obstructed by the element $G$. The secondary wavelets originating from different points of the unobstructed part interfere on the screen Σ and produce the diffraction pattern of varying intensity. A special case of diffraction, which is very important in practice and which is simpler to analyse mathematically, arises when the source $S_0$ and the screen Σ are far away from the diffracting element $G$. Plane waves are incident on $G$ and the waves interfering at a particular point come parallel to each other. This special class of diffraction is called *Fraunhofer diffraction* after the physicist Joseph von Fraunhofer (1787–1826) who investigated such diffraction cases in great detail. Fraunhofer diffraction can be observed in a laboratory by placing converging lenses before and after $G$ and keeping the source $S_0$ and the screen Σ in their focal planes

respectively (figure 17.12). The source and the screen are effectively at infinite distance from the diffracting element.

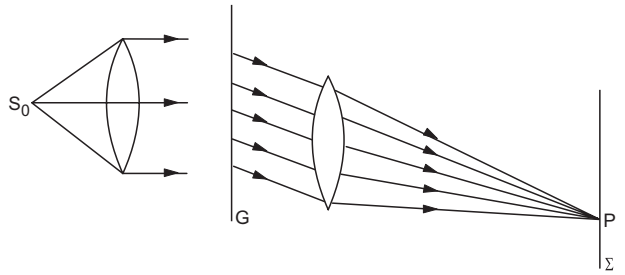



Figure 17.12

If the source or the screen is at a finite distance from the diffracting element $G$, it is called *Fresnel diffraction* after the physicist Augustin Jean Fresnel (1788–1827).

It is a coincidence that the two great physicists, Fraunhofer and Fresnel, investigating diffraction phenomenon lived a short life of equal number of years.

We shall now discuss some of the important cases of diffraction.

## 17.11 FRAUNHOFER DIFFRACTION BY A SINGLE SLIT

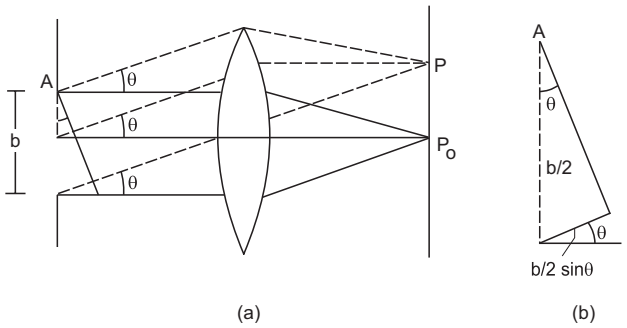



Figure 17.13

Suppose a parallel beam of light is incident normally on a slit of width $b$ (figure 17.13). According to Huygens' principle, each and every point of the exposed part of the plane wavefront (i.e., every point of the slit) acts as a source of secondary wavelets spreading in all directions. The light is received by a screen placed at a large distance. In practice, this condition is achieved by placing the screen at the focal plane of a converging lens placed just after the slit. $A$ particular point $P$ on the screen receives waves from all the secondary sources. All these waves start parallel to each other from different points of the slit and interfere at $P$ to give the resultant intensity.

At the point $P_0$ which is at the bisector plane of the slit, all the waves reach after travelling equal optical path and hence, are in phase. The waves, thus, interfere constructively with each other and maximum intensity is observed. As we move away from $P_0$, the waves arrive with different phases and the intensity is changed.

Let us consider a point $P$ which collects the waves originating from different points of the slit at an angle θ. Figure (17.13) shows the perpendicular from the point $A$ to the parallel rays. This perpendicular also represents the wavefront of the parallel beam diffracted at an angle θ. The optical paths from any point on this wavefront to the point $P$ are equal. The optical path difference between the waves sent by the upper edge $A$ of the slit and the wave sent by the centre of the slit is $\frac{b}{2}\sin\theta$. This is shown in expanded view in figure (17.13b). Consider the angle for which $\frac{b}{2}\sin\theta = \lambda/2$. The above mentioned two waves will have a phase difference

$$\delta = \frac{2\pi}{\lambda}\cdot\frac{\lambda}{2} = \pi.$$

The two waves will cancel each other. The wave from any point in the upper half of the slit is exactly cancelled by the wave from the point $b/2$ distance below it. The whole slit can be divided into such pairs and hence, the intensity at $P$ will be zero. This is the condition of the first minimum, i.e., the first dark fringe.

So, $$\frac{b}{2}\sin\theta = \frac{\lambda}{2}$$

or, $$b\sin\theta = \lambda \text{ (first minimum).}$$

Similar arguments show that other minima (zero intensity) are located at points corresponding to $b\sin\theta = 2\lambda, 3\lambda, \ldots$

or, $$b\sin\theta = n\lambda \text{ (dark fringe).} \quad \ldots (17.13)$$

The points of the maximum intensity lie nearly midway between the successive minima. A detailed mathematical analysis shows that the amplitude $E_0'$ of the electric field at a general point $P$ is,

$$E_0' = E_0\frac{\sin\beta}{\beta}, \quad \ldots (17.14)$$

where $$\beta = \frac{1}{2}\frac{\omega}{v}b\sin\theta = \frac{\pi}{\lambda}b\sin\theta \quad \ldots (17.15)$$

and $E_0$ is the amplitude at the point $P_0$ which corresponds at $\theta = 0$.

The intensity is proportional to the square of the amplitude. If $I_0$ represents the intensity at $P_0$, its value at $P$ is

$$I = I_0 \frac{\sin^2\beta}{\beta^2} \cdot \qquad \ldots \quad (17.16)$$

We draw, in figure (17.14), variation of the intensity as a function of sinθ.

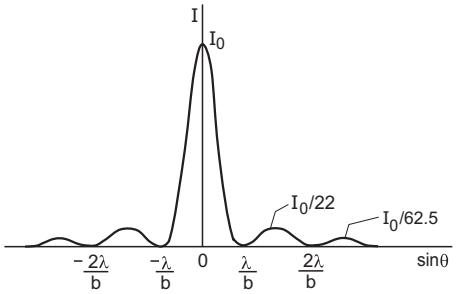



Figure 17.14

Most of the diffracted light is distributed between $\sin\theta = -\frac{\lambda}{b}$ and $+\frac{\lambda}{b}$. The intensity at the first maximum after the central one is only 1/22 of the intensity of the central maximum. This divergence in θ is inversely proportional to the width $b$ of the slit. If the slit-width $b$ is decreased, the divergence is increased and the light is diffracted in a wider cone. On the other hand, if the slit-width is large compared to the wavelength, $\lambda/b \approx 0$ and the light continues undiffracted in the direction $\theta = 0$. This clearly indicates that diffraction effects are observable only when the obstacle or the opening has dimensions comparable to the wavelength of the wave.

***Example 17.5***

*A parallel beam of monochromatic light of wavelength* 450 nm *passes through a long slit of width* 0·2 mm. *Find the angular divergence in which most of the light is diffracted.*

***Solution*** : Most of the light is diffracted between the two first order minima. These minima occur at angles given by $b \sin\theta = \pm\lambda$

or, $\sin\theta = \pm\lambda/b$

$$= \pm \frac{450 \times 10^{-9}\ \text{m}}{0{\cdot}2 \times 10^{-3}\ \text{m}} = \pm 2{\cdot}25 \times 10^{-3}.$$

or, $\theta \approx \pm 2{\cdot}25 \times 10^{-3}$ rad.

The angular divergence $= 4{\cdot}5 \times 10^{-3}$ rad.

## 17.12 FRAUNHOFER DIFFRACTION BY A CIRCULAR APERTURE

When a parallel beam of light is passed through an opaque board with a circular hole in it, the light is diffracted by the hole. If received on a screen at a large distance, the pattern is a bright disc surrounded by alternate dark and bright rings of decreasing intensity as shown in figure (17.15). The wavefront is obstructed by the opaque board and only the points of the wavefront, that are exposed by the hole, send the secondary wavelets. The bright and dark rings are formed by the superposition of these wavelets. The mathematical analysis shows that the first dark ring is formed by the light diffracted from the hole at an angle θ with the axis, where

$$\sin\theta \approx 1{\cdot}22 \frac{\lambda}{b} \cdot \qquad \ldots \quad (17.17)$$

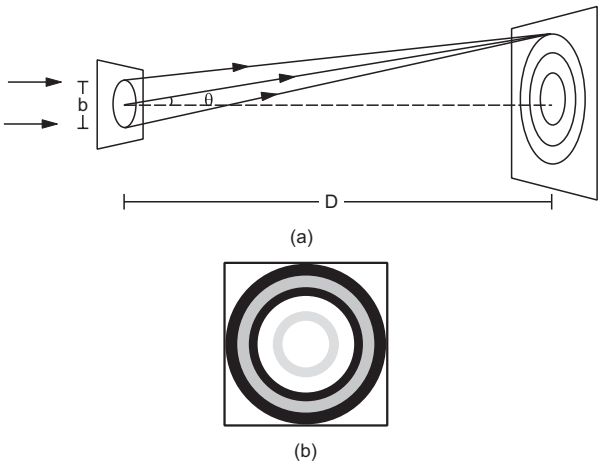



Figure 17.15

Here λ is the wavelength of the light used and $b$ is the diameter of the hole. If the screen is at a distance $D (D >> b)$ from the hole, the radius of the first dark ring is

$$R \approx 1{\cdot}22 \frac{\lambda D}{b} \cdot \qquad \ldots \quad (17.18)$$

If the light transmitted by the hole is converged by a converging lens at the screen placed at the focal plane of this lens, the radius of the first dark ring is

$$R = 1{\cdot}22 \frac{\lambda f}{b} \cdot \qquad \ldots \quad (17.19)$$

As most of the light coming from the hole is concentrated within the first dark ring, this radius is also called the *radius of the diffraction disc.*

Diffraction by a circular aperture is of great practical importance. In many of the optical instruments, lenses are used. When light passes through a lens, the wavefront is limited by its rim which is usually circular. If a parallel beam of light is incident on a converging lens, only the part intercepted by the lens gets transmitted into the converging beam. Thus, the light is diffracted by the lens. This lens itself works to converge the diffracted light in its focal plane and hence, we observe a bright disc, surrounded by

alternate dark and bright rings, as the image. The radius of the diffraction disc is given by equation (17.19) where, $b$ now stands for the diameter of the aperture of the lens.

The above discussion shows that a converging lens can never form a point image of a distant point source. In the best conditions, it produces a bright disc surrounded by dark and bright rings. If we assume that most of the light is concentrated within the central bright disc, we can say that the lens produces a disc image for a distant point source. This is not only true for a distant point source but also for any point source. The radius of the image disc is

$$R = 1{\cdot}22\,\frac{\lambda}{b}\,D,$$

where $D$ is the distance from the lens at which the light is focussed.

***Example 17.6***

*A beam of light of wavelength* 590 nm *is focussed by a converging lens of diameter* 10·0 cm *at a distance of* 20 cm *from it. Find the diameter of the disc image formed.*

***Solution*** **:** The angular radius of the central bright disc in a diffraction pattern from circular aperture is given by

$$\sin\theta \approx \frac{1{\cdot}22\,\lambda}{b}$$

$$= \frac{1{\cdot}22 \times 590 \times 10^{-9}\ \text{m}}{10{\cdot}0 \times 10^{-2}\ \text{m}} = 0{\cdot}7 \times 10^{-5}\ \text{rad.}$$

The radius of the bright disc is

$$0{\cdot}7 \times 10^{-5} \times 20\ \text{cm} = 1{\cdot}4 \times 10^{-4}\ \text{cm.}$$

The diameter of the disc image $= 2{\cdot}8 \times 10^{-4}$ cm.

## 17.13 FRESNEL DIFFRACTION AT A STRAIGHT EDGE

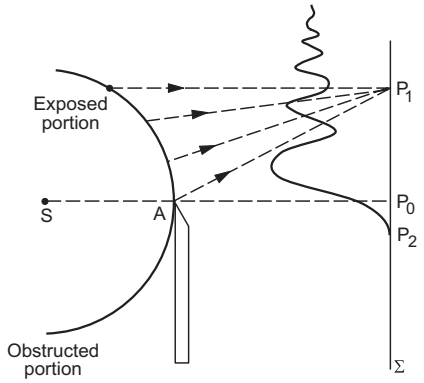



Figure 17.16

Consider the situation shown in figure (17.16). Let $S$ be a narrow slit sending monochromatic light. The light is obstructed by an opaque obstacle having a sharp edge $A$. The light is collected on a screen Σ. The portion of the screen below $P_0$ in the figure is the region of the geometrical shadow.

Cylindrical wavefronts emitted from the slit are obstructed by the obstacle. The points on the exposed portion of the wavefront emit secondary wavelets which interfere to produce varying intensity on the screen Σ.

The curve in the figure shows the variation of intensity of light on the screen. We see that the intensity gradually decreases as we go farther inside the region of geometrical shadow, i.e., below $P_0$. As we go above $P_0$, the intensity alternately increases and decreases. The difference of the maximum intensity and minimum intensity goes on decreasing as we go farther away from $P_0$ and finally we get uniform illumination.

## 17.14 LIMIT OF RESOLUTION

The fact that a lens forms a disc image of a point source, puts a limit on resolving two neighbouring points imaged by a lens.

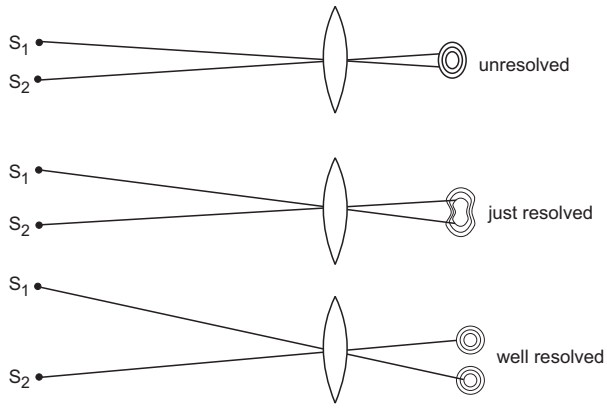



Figure 17.17

Suppose $S_1$ and $S_2$ are two point sources placed before a converging lens (figure 17.17). If the separation between the centres of the image-discs is small in comparison to the radii of the discs, the discs will largely overlap on one another and it will appear like a single disc. The two points are then not resolved. If $S_1$ and $S_2$ are moved apart, the centres of their image-discs also move apart. For a sufficient separation, one can distinguish the presence of two discs in the pattern. In this case, we say that the points are just resolved.

The angular radius θ of the diffraction disc is given by $\sin\theta = \frac{1{\cdot}22\,\lambda}{b}$, where $b$ is the radius of the lens. Thus, increasing the radius of the lens improves the resolution. This is the reason why objective lenses of powerful microscopes and telescopes are kept large in size.

The human eye is itself a converging lens which forms the image of the points (we see) on the retina. The above discussion then shows that two points very close to each other cannot be seen as two distinct points by the human eye.

**Rayleigh Criterion**

Whether two disc images of nearby points are resolved or not may depend on the person viewing the images. Rayleigh suggested a quantitative criterion for resolution. Two images are called just resolved in this criterion if the centre of one bright disc falls on the periphery of the second. This means, the radius of each bright disc should be equal to the separation between them. In this case, the resultant intensity has a minimum between the centres of the images. Figure (17.18) shows the variation of intensity when the two images are just resolved.

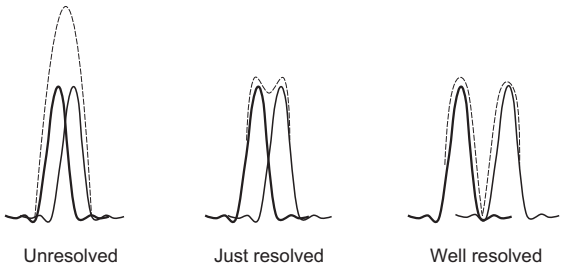



Figure 17.18

## 17.15 SCATTERING OF LIGHT

When a parallel beam of light passes through a gas, a part of it appears in directions other than the incident direction. This phenomenon is called *scattering of light*. The basic process in scattering is absorption of light by the molecules followed by its re-radiation in different directions. The strength of scattering can be measured by the loss of energy in the light beam as it passes through the gas. It should be distinguished from the absorption of light as it passes through a medium. In absorption, the light energy is converted into internal energy of the medium whereas in scattering, the light energy is radiated in other directions. The strength of scattering depends on the wavelength of the light beside the size of the particles which cause scattering. If these particles are smaller than the wavelength, the scattering is proportional to $1/\lambda^4$. This is known as *Rayleigh's law of scattering*. Thus, red light is scattered the least and violet is scattered the most. This is why, red signals are used to indicate dangers. Such a signal goes to large distances without an appreciable loss due to scattering.

The blue appearance of sky is due to scattering of sunlight from the atmosphere. When you look at the sky, it is the scattered light that enters the eyes. Among the shorter wavelengths, the colour blue is present in larger proportion in sunlight. Light of short wavelengths are strongly scattered by the air molecules and reach the observer. This explains the blue colour of sky. Another natural phenomenon related to the scattering of light is the red appearance of sun at the sunset and at the sunrise. At these times, the sunlight has to travel a large distance through the atmosphere. The blue and neighbouring colours are scattered away in the path and the light reaching the observer is predominantly red.

If the earth had no atmosphere, the sky would appear black and stars could be seen during day hours. In fact if you go about 20 km up, where the atmosphere becomes quite thin, the sky does appear black and stars are visible during day hours as astronauts have found.

Besides air molecules, water particles, dust, etc., also scatter light. The appearance of sky is affected by the presence of these scattering centres. On a humid day before rains, the sky appears light blue whereas, on a clear day it appears deep blue. The change in the quality of colour of sky results from the fact that the water droplets and the dust particles may have size greater than the wavelength of light. Rayleigh's law of scattering does not operate in this case and colours other than blue may be scattered in larger proportion. The appearance of sky in large industrial cities is also different from villages. An automobile engine typically ejects about $10^{11}$ particles per second, similarly for other machines. Such particles remain suspended in air for quite long time unless rain or wind clears them. Often the sky looks hazy with a greyish tinge in such areas.

## 17.16 POLARIZATION OF LIGHT

In writing equation (17.1) for light wave, we assumed that the direction of electric field is fixed and the magnitude varies sinusoidally with space and time. The electric field in a light wave propagating in free space is perpendicular to the direction of propagation. However, there are infinite number of directions perpendicular to the direction of propagation and the electric field may be along any of these directions. For example, if the light propagates along the $X$-axis, the electric field may be along the $Y$-axis, or along the $Z$-axis or along any direction in the $Y$-$Z$ plane. If the electric field at a point always remains parallel to a fixed direction as the time passes, the light is called *linearly polarized* along that direction. For example, if the electric field at a point is always parallel to the

$Y$-axis, we say that the light is linearly polarized along the $Y$-axis. The same is also called *plane polarized* light. The plane containing the electric field and the direction of propagation is called the *plane of polarization.*

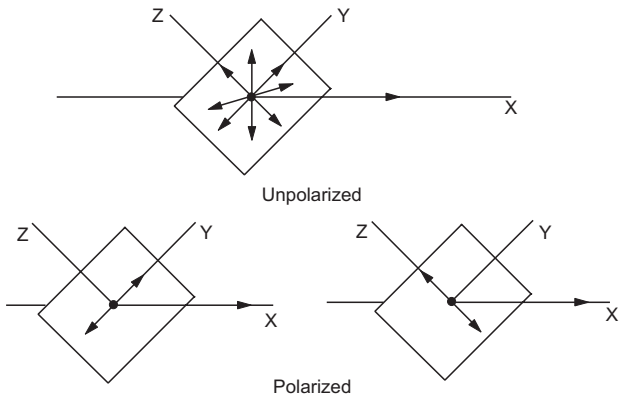



Figure 17.19

As we have mentioned earlier, light is emitted by atoms. The light pulse emitted by one atom in a single event has a fixed direction of electric field. However, the light pulses emitted by different atoms, in general, have electric fields in different directions. Hence, the resultant electric field at a point keeps on changing its direction randomly and rapidly. Such a light is called *unpolarized.* The light emitted by an ordinary source such as an electric lamp, a mercury tube, a candle, the sun, are unpolarized.

Suppose an unpolarized light wave travels along the $X$-axis. The electric field at any instant is in the $Y$-$Z$ plane, we can break the field into its components $E_y$ and $E_z$ along the $Y$-axis and the $Z$-axis respectively. The fact that the resultant electric field changes its direction randomly may be mathematically expressed by saying that $E_y$ and $E_z$ have a phase difference $\delta$ that changes randomly with time. Thus,

$$E_y = E_1 \sin(\omega t - kx + \delta)$$

$$E_z = E_2 \sin(\omega t - kx).$$

The resultant electric field makes an angle $\theta$ with the $Y$-axis, where

$$\tan\theta = \frac{E_z}{E_y} = \frac{E_2 \sin(\omega t - kx)}{E_1 \sin(\omega t - kx + \delta)}.$$

Since $\delta$ changes randomly with time, so does $\theta$ and the light is unpolarized.

If $\delta$ is zero, $\tan\theta = E_2/E_1 = \text{constant}$ and the electric field is always parallel to a fixed direction. The light is linearly polarized.

If $\delta = \pi$, $\tan\theta = -E_2/E_1$ and again the electric field is parallel to a fixed direction and the light is linearly polarized.

If $\delta = \pi/2$ and $E_1 = E_2$, then

$$\tan\theta = \frac{E_z}{E_y} = \frac{E_2 \sin(\omega t - kx)}{E_1 \sin(\omega t - kx + \pi/2)}$$

$$= \tan(\omega t - kx)$$

or, $\theta = \omega t - kx.$

At any point $x$, the angle $\theta$ increases at a uniform rate $\omega$. The electric field, therefore, rotates at a uniform angular speed $\omega$. Also,

$$E^2 = E_y^2 + E_z^2 = E_1^2 \cos^2(\omega t - kx) + E_1^2 \sin^2(\omega t - kx) = E_1^2,$$

i.e., the magnitude of the field remains constant. The tip of the electric field, thus, goes in a circle at a uniform angular speed. Such a light is called a *circularly polarized light.*

If $\delta = \pi/2$ but $E_1 \neq E_2$, the tip of the electric field traces out an ellipse. Such a light wave is called an *elliptically polarized light.*

**Polaroids**

There are several methods to produce polarized light from the unpolarized light. An instrument used to produce polarized light from unpolarized light is called a *polarizer.* Plane sheets in the shape of circular discs called *polaroids* are commercially available which transmit light with $E$-vector parallel to a special direction in the sheet. These polaroids have long chains of hydrocarbons which become conducting at optical frequencies. When light falls perpendicularly on the sheet, the electric field parallel to the chains is absorbed in setting up electric currents in the chains but the field perpendicular to the chains gets transmitted. The direction perpendicular to the chains is called the *transmission axis* of the polaroid. When light passes through the polaroid, the transmitted light becomes linearly polarized with $E$-vector parallel to the transmission axis.

If linearly polarized light is incident on a polaroid with the $E$-vector parallel to the transmission axis, the light is completely transmitted by the polaroid. If the $E$-vector is perpendicular to the transmission axis, the light is completely stopped by the polaroid. If the $E$-vector is at an angle $\theta$ with the transmission axis, light is partially transmitted. The intensity of the transmitted light is

$$I = I_0 \cos^2\theta, \qquad \ldots \ (17.20)$$

where $I_0$ is the intensity when the incident $E$-vector is parallel to the transmission axis. This is known as the *law of Malus.*

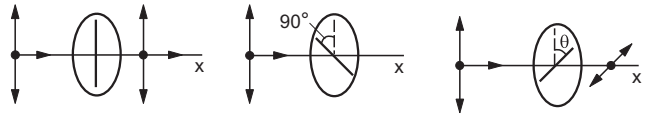



Figure 17.20

**Polarization by Reflection and Refraction**

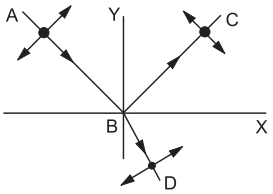



Figure 17.21

Consider a light beam from air incident on the surface of a transparent medium of refractive index μ. The incident ray, the reflected ray and the refracted ray are all in one plane. The plane is called the plane of incidence. In figure (17.21) we have shown this plane as the *X*-*Y* plane. Consider the incident light going along AB. The electric field $\vec{E}$ must be perpendicular to *AB*. If the incident light is unpolarized the electric field will randomly change its direction, remaining at all times in a plane perpendicular to *AB*. We can resolve the field in two components, one in the *X*-*Y* plane and the other along the *Z*-direction. In figure (17.21), the component in the *X*-*Y* plane is shown by the double-arrow perpendicular to *AB* and the component along the *Z*-direction by the solid dot.

Light with electric field along the *Z*-direction is more strongly reflected as compared to that in the *X*-*Y* plane. This is shown in the figure by reduced size of the double arrow. Similarly, the refracted light has a larger component of electric field in the *X*-*Y* plane shown in the figure by the reduced size of the solid dot.

If the light is incident on the surface with an angle of incidence *i* given by

$$\tan i = \mu, \qquad \ldots \quad (17.21)$$

the reflected light is completely polarized with the electric field along the *Z*-direction as suggested by figure (17.22). The refracted ray is never completely polarized. The angle *i* given by equation (17.21) is called the *Brewster angle* and equation (17.21) itself is known as the *Brewster's law*.

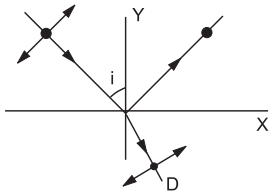



Figure 17.22

The fact that reflected light is polarized is used in preparing 'polarizing sunglasses' which reduce the glare from roads, snow, water surfaces, etc. The glasses are in fact polaroids with their transmission axis perpendicular to the electric field of the polarized reflected light. The reflected light, which is responsible for the glare, is thus largely absorbed. The direct light coming to the glasses is unpolarized and is less absorbed. In this respect the polarizing sunglasses are different from the ordinary dark-coloured sunglasses which absorb any light passing through them reducing the intensity to a large extent.

**Polarization by Scattering**

When unpolarized light is scattered by small particles, the scattered light is partially polarized. The blue light received from the sky is accordingly partially polarized. Though human eye does not distinguish between an unpolarized light and a polarized light, the eyes of a bee can detect the difference. Austrian Nobel Laureate Karl Von Frisch performed experiments for several years on bees and concluded that the bees cannot only distinguish unpolarized light from polarized light but can also determine the direction of polarization.

## *Worked Out Examples*

**1.** *White light is a mixture of light of wavelengths between* 400 nm *and* 700 nm. *If this light goes through water* (μ = 1·33) *what are the limits of the wavelength there* ?

***Solution*** **:** When a light having wavelength $\lambda_0$ in vacuum goes through a medium of refractive index μ, the wavelength in the medium becomes $\lambda = \lambda_0 / \mu$.

For $\lambda_0 = 400$ nm, $\lambda = \dfrac{400 \text{ nm}}{1{\cdot}33} = 300$ nm

and for $\lambda_0 = 700$ nm, $\lambda = \dfrac{700 \text{ nm}}{1{\cdot}33} = 525$ nm.

Thus, the limits are 300 nm and 525 nm.

**2.** *The optical path of a monochromatic light is the same if it goes through* 2·00 cm *of glass or* 2·25 cm *of water. If the refractive index of water is* 1·33, *what is the refractive index of glass* ?

***Solution*** : When light travels through a distance *x* in a medium of refractive index μ, its optical path is μ*x*. Thus, if μ is the refractive index of glass,

$$\mu(2{\cdot}00 \text{ cm}) = 1{\cdot}33 \times (2{\cdot}25 \text{ cm})$$

or, $$\mu = 1{\cdot}33 \times \frac{2{\cdot}25}{2{\cdot}00} = 1{\cdot}50.$$

**3.** *White light is passed through a double slit and interference pattern is observed on a screen* 2·5 m *away. The separation between the slits is* 0·5 mm. *The first violet and red fringes are formed* 2·0 mm *and* 3·5 mm *away from the central white fringe. Calculate the wavelengths of the violet and the red light.*

***Solution*** **:** For the first bright fringe, the distance from the centre is

$$y = \frac{D\lambda}{d}.$$

For violet light, $y = 2{\cdot}0$ mm. Thus,

$$2{\cdot}0 \text{ mm} = \frac{(2{\cdot}5 \text{ m})\lambda}{0{\cdot}5 \text{ mm}}$$

or, $$\lambda = \frac{(0{\cdot}5 \text{ mm})\,(2{\cdot}0 \text{ mm})}{2{\cdot}5 \text{ m}} = 400 \text{ nm}.$$

Similarly, for red light, $y = 3{\cdot}5$ mm. Thus,

$$3{\cdot}5 \text{ mm} = \frac{(2{\cdot}5 \text{ m})\lambda}{0{\cdot}5 \text{ mm}}$$

or, $$\lambda = 700 \text{ nm}.$$

**4.** *A double slit experiment is performed with sodium (yellow) light of wavelength* 589·3 nm *and the interference pattern is observed on a screen* 100 cm *away. The tenth bright fringe has its centre at a distance of* 12 mm *from the central maximum. Find the separation between the slits.*

***Solution*** **:** For the $n$th maximum fringe, the distance above the central line is

$$x = \frac{n\lambda D}{d}.$$

According to the data given,

$$x = 12 \text{ mm},\ n = 10,\ \lambda = 589{\cdot}3 \text{ nm},\ D = 100 \text{ cm}.$$

Thus, the separation between the slits is

$$d = \frac{n\lambda D}{x} = \frac{10 \times 589{\cdot}3 \times 10^{-9} \text{ m} \times 100 \times 10^{-2} \text{ m}}{12 \times 10^{-3} \text{ m}}$$

$$= 4{\cdot}9 \times 10^{-4} \text{ m} = 0{\cdot}49 \text{ mm}.$$

**5.** *The intensity of the light coming from one of the slits in a Young's double slit experiment is double the intensity from the other slit. Find the ratio of the maximum intensity to the minimum intensity in the interference fringe pattern observed.*

***Solution*** **:** The intensity of the light originating from the first slit is double the intensity from the second slit. The amplitudes of the two interfering waves are in the ratio $\sqrt{2} : 1$, say $\sqrt{2}\,A$ and $A$.

At the point of constructive interference, the resultant amplitude becomes $(\sqrt{2} + 1)\,A$. At the points of destructive interference, this amplitude is $(\sqrt{2} - 1)\,A$. The ratio of the resultant intensities at the maxima to that at the minima is

$$\frac{(\sqrt{2} + 1)^2 A^2}{(\sqrt{2} - 1)^2 A^2} = 34.$$

**6.** *The width of one of the two slits in a Young's double slit experiment is double of the other slit. Assuming that the amplitude of the light coming from a slit is proportional to the slit-width, find the ratio of the maximum to the minimum intensity in the interference pattern.*

***Solution*** **:** Suppose the amplitude of the light wave coming from the narrower slit is $A$ and that coming from the wider slit is $2A$. The maximum intensity occurs at a place where constructive interference takes place. Then the resultant amplitude is the sum of the individual amplitudes. Thus,

$$A_{max} = 2A + A = 3A.$$

The minimum intensity occurs at a place where destructive interference takes place. The resultant amplitude is then difference of the individual amplitudes. Thus,

$$A_{min} = 2A - A = A.$$

As the intensity is proportional to the square of the amplitude,

$$\frac{I_{max}}{I_{min}} = \frac{(A_{max})^2}{(A_{min})^2} = \frac{(3A)^2}{A^2} = 9.$$

**7.** *Two sources* $S_1$ and $S_2$ *emitting light of wavelength* 600 nm *are placed a distance* $1{\cdot}0 \times 10^{-2}$ cm *apart. A detector can be moved on the line* $S_1P$ *which is perpendicular to* $S_1S_2$. *(a) What would be the minimum and maximum path difference at the detector as it is moved along the line* $S_1P$ ? *(b) Locate the position of the farthest minimum detected.*

***Solution*** **:**

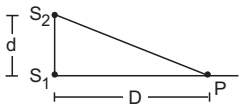



Figure 17-W1

(a) The situation is shown in figure (17-W1). The path difference is maximum when the detector is just at the position of $S_1$ and its value is equal to $d = 1{\cdot}0 \times 10^{-2}$ cm. The path difference is minimum when the detector is at a large distance from $S_1$. The path difference is then close to zero.

(b) The farthest minimum occurs at a point $P$ where the path difference is $\lambda/2$. If $S_1P = D$,

$$S_2P - S_1P = \frac{\lambda}{2}$$

or, $$\sqrt{D^2 + d^2} - D = \frac{\lambda}{2}$$

or, $$D^2 + d^2 = \left(D + \frac{\lambda}{2}\right)^2$$

or, $$d^2 = D\lambda + \frac{\lambda^2}{4}$$

or, $$D = \frac{d^2}{\lambda} - \frac{\lambda}{4}$$

$$= \frac{(1{\cdot}0 \times 10^{-4}\ \text{m})^2}{600 \times 10^{-9}\ \text{m}} - 150 \times 10^{-9}\ \text{m} = 1{\cdot}7\ \text{cm}.$$

**8.** *A beam of light consisting of two wavelengths,* 6500 Å *and* 5200 Å *is used to obtain interference fringes in a Young's double slit experiment* (1 Å = $10^{-10}$ m). *The distance between the slits is* 2·0 mm *and the distance between the plane of the slits and the screen is* 120 cm. *(a) Find the distance of the third bright fringe on the screen from the central maximum for the wavelength* 6500 Å. *(b) What is the least distance from the central maximum where the bright fringes due to both the wavelengths coincide* ?

***Solution*** **:** (a) The centre of the $n$th bright fringe is at a distance $y = \frac{n\lambda D}{d}$ from the central maximum. For the 3rd bright fringe of 6500 Å,

$$y = \frac{3 \times 6500 \times 10^{-10}\ \text{m} \times 1{\cdot}2\ \text{m}}{2 \times 10^{-3}\ \text{m}}$$

$$= 0{\cdot}117\ \text{cm} \approx 0{\cdot}12\ \text{cm}.$$

(b) Suppose the $m$th bright fringe of 6500 Å coincides with the $n$th bright fringe of 5200 Å.

Then, $$\frac{m \times 6500\ \text{Å} \times D}{d} = \frac{n \times 5200\ \text{Å} \times D}{d}$$

or, $$\frac{m}{n} = \frac{5200}{6500} = \frac{4}{5}.$$

The minimum values of $m$ and $n$ that satisfy this equation are 4 and 5 respectively. The distance of the 4th bright fringe of 6500 Å or the 5th bright fringe of 5200 Å from the central maximum is

$$y = \frac{4 \times 6500 \times 10^{-10}\ \text{m} \times 1{\cdot}2\ \text{m}}{2 \times 10^{-3}\ \text{m}}$$

$$= 0{\cdot}156\ \text{cm} \approx 0{\cdot}16\ \text{cm}.$$

**9.** *Monochromatic light of wavelength* 600 nm *is used in a Young's double slit experiment. One of the slits is covered by a transparent sheet of thickness* $1{\cdot}8 \times 10^{-5}$ m *made of a material of refractive index* 1·6. *How many fringes will shift due to the introduction of the sheet* ?

***Solution*** **:** When the light travels through a sheet of thickness $t$, the optical path travelled is $\mu t$, where $\mu$ is the refractive index. When one of the slits is covered by the sheet, air is replaced by the sheet and hence, the optical path changes by $(\mu - 1)t$. One fringe shifts when the optical path changes by one wavelength. Thus, the number of fringes shifted due to the introduction of the sheet is

$$\frac{(\mu - 1)t}{\lambda} = \frac{(1{\cdot}6 - 1) \times 1{\cdot}8 \times 10^{-5}\ \text{m}}{600 \times 10^{-9}\ \text{m}} = 18.$$

**10.** *White light is incident normally on a glass plate of thickness* $0{\cdot}50 \times 10^{-6}$ *and index of refraction* 1·50. *Which wavelengths in the visible region* (400 nm–700 nm) *are strongly reflected by the plate* ?

***Solution*** **:** The light of wavelength $\lambda$ is strongly reflected if

$$2\mu d = \left(n + \frac{1}{2}\right)\lambda, \qquad \ldots \text{(i)}$$

where $n$ is a nonnegative integer.

Here, $$2\mu d = 2 \times 1{\cdot}50 \times 0{\cdot}5 \times 10^{-6}\ \text{m}$$

$$= 1{\cdot}5 \times 10^{-6}\ \text{m}. \qquad \ldots \text{(ii)}$$

Putting $\lambda = 400$ nm in (i) and using (ii),

$$1{\cdot}5 \times 10^{-6}\ \text{m} = \left(n + \frac{1}{2}\right)(400 \times 10^{-9}\ \text{m})$$

or, $$n = 3{\cdot}25.$$

Putting $\lambda = 700$ nm in (i) and using (ii),

$$1{\cdot}5 \times 10^{-6}\ \text{m} = \left(n + \frac{1}{2}\right)(700 \times 10^{-9}\ \text{m})$$

or, $$n = 1{\cdot}66.$$

Thus, between 400 nm and 700 nm the integer $n$ can take the values 2 and 3. Putting these values of $n$ in (i), the wavelength become

$$\lambda = \frac{4\,\mu d}{2n + 1} = 600\ \text{nm and } 429\ \text{nm}.$$

Thus, light of wavelengths 429 nm and 600 nm are strongly reflected.

**11.** *A parallel beam of green light of wavelength* 546 nm *passes through a slit of width* 0·40 mm. *The transmitted light is collected on a screen* 40 cm *away. Find the distance between the two first order minima.*

***Solution*** **:** The minima occur at an angular deviation $\theta$ given by $b \sin\theta = n\lambda$, where $n$ is an integer. For the first order minima, $n = \pm 1$ so that $\sin\theta = \pm \frac{\lambda}{b}$. As the fringes are observed at a distance much larger than the width of the slit, the linear distances from the central maximum are given by

$$x = D\tan\theta$$

$$\approx D\sin\theta = \pm \frac{\lambda D}{b}.$$

Thus, the minima are formed at a distance $\frac{\lambda D}{b}$ from the central maximum on its two sides. The separation between the minima is

$$\frac{2\lambda D}{b} = \frac{2 \times 546 \times 10^{-9}\ \text{m} \times 40 \times 10^{-2}\ \text{m}}{0{\cdot}40 \times 10^{-3}\ \text{m}} = 1{\cdot}1\ \text{mm}.$$

## QUESTIONS FOR SHORT ANSWER

**1.** Is the colour of 620 nm light and 780 nm light same ? Is the colour of 620 nm light and 621 nm light same ? How many colours are there in white light ?

**2.** The wavelength of light in a medium is $\lambda = \lambda_0 / \mu$, where $\lambda$ is the wavelength in vacuum. A beam of red light ($\lambda_0 = 720$ nm) enters into water. The wavelength in water is $\lambda = \lambda_0 / \mu = 540$ nm. To a person under water does this light appear green ?

**3.** Whether the diffraction effects from a slit will be more clearly visible or less clearly, if the slit-width is increased ?

**4.** If we put a cardboard (say $20\ \text{cm} \times 20\ \text{cm}$) between a light source and our eyes, we can't see the light. But when we put the same cardboard between a sound source and our ear, we hear the sound almost clearly. Explain.

**5.** TV signals broadcast by Delhi studio cannot be directly received at Patna which is about 1000 km away. But the same signal goes some 36000 km away to a satellite, gets reflected and is then received at Patna. Explain.

**6.** Can we perform Young's double slit experiment with sound waves ? To get a reasonable "fringe pattern", what should be the order of separation between the slits ? How can the bright fringes and the dark fringes be detected in this case ?

**7.** Is it necessary to have two waves of equal intensity to study interference pattern ? Will there be an effect on clarity if the waves have unequal intenstity ?

**8.** Can we conclude from the interference phenomenon whether light is a transverse wave or a longitudinal wave ?

**9.** Why don't we have interference when two candles are placed close to each other and the intensity is seen at a distant screen ? What happens if the candles are replaced by laser sources ?

**10.** If the separation between the slits in a Young's double slit experiment is increased, what happens to the fringe-width ? If the separation is increased too much, will the fringe pattern remain detectable ?

**11.** Suppose white light falls on a double slit but one slit is covered by a violet filter (allowing $\lambda = 400$ nm). Describe the nature of the fringe pattern observed.

## OBJECTIVE I

**1.** Light is
(a) wave phenomenon (b) particle phenomenon
(c) both particle and wave phenomenon.

**2.** The speed of light depends
(a) on elasticity of the medium only
(b) on inertia of the medium only
(c) on elasticity as well as inertia
(d) neither on elasticity nor on inertia.

**3.** The equation of a light wave is written as $y = A \sin(kx - \omega t)$. Here, $y$ represents
(a) displacement of ether particles
(b) pressure in the medium
(c) density of the medium
(d) electric field.

**4.** Which of the following properties show that light is a transverse wave ?
(a) Reflection (b) Interference
(c) Diffraction (d) Polarization

**5.** When light is refracted into a medium,
(a) its wavelength and frequency both increase
(b) its wavelength increases but frequency remains unchanged
(c) its wavelength decreases but frequency remains unchanged
(d) its wavelength and frequency both decrease.

**6.** When light is refracted, which of the following does not change ?
(a) Wavelength (b) Frequency
(c) Velocity (d) Amplitude

**7.** The amplitude modulated (AM) radio wave bends appreciably round the corners of a $1\ \text{m} \times 1\ \text{m}$ board but the frequency modulated (FM) wave only negligibly bends. If the average wavelengths of AM and FM waves are $\lambda_a$ and $\lambda_f$,
(a) $\lambda_a > \lambda_f$ (b) $\lambda_a = \lambda_f$ (c) $\lambda_a < \lambda_f$
(d) we don't have sufficient information to decide about the relation of $\lambda_a$ and $\lambda_f$.

**8.** Which of the following sources gives best monochromatic light ?
(a) A candle (b) A bulb (c) A mercury tube (d) A laser

**9.** The wavefronts of a light wave travelling in vacuum are given by $x + y + z = c$. The angle made by the direction of propagation of light with the $X$-axis is
(a) 0° (b) 45° (c) 90° (d) $\cos^{-1}(1/\sqrt{3})$.

**10.** The wavefronts of light coming from a distant source of unknown shape are nearly
(a) plane (b) elliptical (c) cylindrical (d) spherical.

**11.** The inverse square law of intensity (i.e., the intensity $\propto \frac{1}{r^2}$) is valid for a
(a) point source (b) line source
(c) plane source (d) cylindrical source.

**12.** Two sources are called coherent if they produce waves
(a) of equal wavelength (b) of equal velocity
(c) having same shape of wavefront
(d) having a constant phase difference.

**13.** When a drop of oil is spread on a water surface, it displays beautiful colours in daylight because of

(a) disperson of light (b) reflection of light
(c) polarization of light (d) interference of light.

**14.** Two coherent sources of different intensities send waves which interfere. The ratio of maximum intensity to the minimum intensity is 25. The intensities of the sources are in the ratio
(a) 25 : 1 (b) 5 : 1 (c) 9 : 4 (d) 625 : 1.

**15.** The slits in a Young's double slit experiment have equal width and the source is placed symmetrically with respect to the slits. The intensity at the central fringe is $I_0$. If one of the slits is closed, the intensity at this point will be
(a) $I_0$ (b) $I_0/4$ (c) $I_0/2$ (d) $4I_0$.

**16.** A thin transparent sheet is placed in front of a Young's double slit. The fringe-width will
(a) increase (b) decrease
(c) remain same (d) become nonuniform.

**17.** If Young's double slit experiment is performed in water,
(a) the fringe width will decrease
(b) the fringe width will increase
(c) the fringe width will remain unchanged
(d) there will be no fringe.

## OBJECTIVE II

**1.** A light wave can travel
(a) in vacuum (b) in vacuum only
(c) in a material medium (d) in a material medium only.

**2.** Which of the following properties of light conclusively support wave theory of light ?
(a) Light obeys laws of reflection.
(b) Speed of light in water is smaller than the speed in vacuum.
(c) Light shows interference.
(d) Light shows photoelectric effect.

**3.** When light propagates in vacuum there is an electric field and a magnetic field. These fields
(a) are constant in time
(b) have zero average value
(c) are perpendicular to the direction of propagation of light.
(d) are mutually perpendicular.

**4.** Huygens' principle of secondary wavelets may be used to
(a) find the velocity of light in vacuum
(b) explain the particle behaviour of light
(c) find the new position of a wavefront
(d) explain Snell's law.

**5.** Three observers *A, B* and *C* measure the speed of light coming from a source to be $v_A$, $v_B$ and $v_C$. The observer *A* moves towards the source and *C* moves away from the source at the same speed. The observer *B* stays stationary. The surrounding space is vacuum everywhere.
(a) $v_A > v_B > v_C$ (b) $v_A < v_B < v_C$
(c) $v_A = v_B = v_C$ (d) $v_B = \frac{1}{2}(v_A + v_C)$.

**6.** Suppose the medium in the previous question is water. Select the correct option(s) from the list given in that question.

**7.** Light waves travel in vacuum along the *X*-axis. Which of the following may represent the wavefronts ?
(a) $x = c$ (b) $y = c$ (c) $z = c$ (d) $x + y + z = c$.

**8.** If the source of light used in a Young's double slit experiment is changed from red to violet,
(a) the fringes will become brighter
(b) consecutive fringes will come closer
(c) the intensity of minima will increase
(d) the central bright fringe will become a dark fringe.

**9.** A Young's double slit experiment is performed with white light.
(a) The central fringe will be white.
(b) There will not be a completely dark fringe.
(c) The fringe next to the central will be red.
(d) The fringe next to the central will be violet.

**10.** Four light waves are represented by
(i) $y = a_1 \sin \omega t$ (ii) $y = a_2 \sin(\omega t + \varepsilon)$
(iii) $y = a_1 \sin 2\omega t$ (iv) $y = a_2 \sin 2(\omega t + \varepsilon)$.
Interference fringes may be observed due to superposition of
(a) (i) and (ii) (b) (i) and (iii)
(c) (ii) and (iv) (d) (iii) and (iv).

## EXERCISES

**1.** Find the range of frequency of light that is visible to an average human being (400 nm < λ < 700 nm).

**2.** The wavelength of sodium light in air is 589 nm. (a) Find its frequency in air. (b) Find its wavelength in water (refractive index = 1·33). (c) Find its frequency in water. (d) Find its speed in water.

**3.** The index of refraction of fused quartz is 1·472 for light of wavelength 400 nm and is 1·452 for light of wavelength 760 nm. Find the speeds of light of these wavelengths in fused quartz.

**4.** The speed of the yellow light in a certain liquid is $2{\cdot}4 \times 10^8$ m s$^{-1}$. Find the refractive index of the liquid.

**5.** Two narrow slits emitting light in phase are separated by a distance of 1·0 cm. The wavelength of the light is $5{\cdot}0 \times 10^{-7}$ m. The interference pattern is observed on a screen placed at a distance of 1·0 m. (a) Find the separation between the consecutive maxima. Can you

expect to distinguish between these maxima ? (b) Find the separation between the sources which will give a separation of 1·0 mm between the consecutive maxima.

**6.** The separation between the consecutive dark fringes in a Young's double slit experiment is 1·0 mm. The screen is placed at a distance of 2·5 m from the slits and the separation between the slits is 1·0 mm. Calculate the wavelength of light used for the experiment.

**7.** In a double slit interference experiment, the separation between the slits is 1·0 mm, the wavelength of light used is $5{\cdot}0 \times 10^{-7}$ m and the distance of the screen from the slits is 1·0 m. (a) Find the distance of the centre of the first minimum from the centre of the central maximum. (b) How many bright fringes are formed in one centimeter width on the screen ?

**8.** In a Young's double slit experiment, two narrow vertical slits placed 0·800 mm apart are illuminated by the same source of yellow light of wavelength 589 nm. How far are the adjacent bright bands in the interference pattern observed on a screen 2·00 m away ?

**9.** Find the angular separation between the consecutive bright fringes in a Young's double slit experiment with blue-green light of wavelength 500 nm. The separation between the slits is $2{\cdot}0 \times 10^{-3}$ m.

**10.** A source emitting light of wavelengths 480 nm and 600 nm is used in a double slit interference experiment. The separation between the slits is 0·25 mm and the interference is observed on a screen placed at 150 cm from the slits. Find the linear separation between the first maximum (next to the central maximum) corresponding to the two wavelengths.

**11.** White light is used in a Young's double slit experiment. Find the minimum order of the violet fringe ($\lambda = 400$ nm) which overlaps with a red fringe ($\lambda = 700$ nm).

**12.** Find the thickness of a plate which will produce a change in optical path equal to half the wavelength $\lambda$ of the light passing through it normally. The refractive index of the plate is $\mu$.

**13.** A plate of thickness $t$ made of a material of refractive index $\mu$ is placed in front of one of the slits in a double slit experiment. (a) Find the change in the optical path due to introduction of the plate. (b) What should be the minimum thickness $t$ which will make the intensity at the centre of the fringe pattern zero ? Wavelength of the light used is $\lambda$. Neglect any absorption of light in the plate.

**14.** A transparent paper (refractive index = 1·45) of thickness 0·02 mm is pasted on one of the slits of a Young's double slit experiment which uses monochromatic light of wavelength 620 nm. How many fringes will cross through the centre if the paper is removed ?

**15.** In a Young's double slit experiment using monochromatic light, the fringe pattern shifts by a certain distance on the screen when a mica sheet of refractive index 1·6 and thickness 1·964 micron (1 micron $= 10^{-6}$ m) is introduced in the path of one of the interfering waves. The mica sheet is then removed and the distance between the screen and the slits is doubled. It is found that the distance between the successive maxima now is the same as the observed fringe-shift upon the introduction of the mica sheet. Calculate the wavelength of the monochromatic light used in the experiment.

**16.** A mica strip and a polysterene strip are fitted on the two slits of a double slit apparatus. The thickness of the strips is 0·50 mm and the separation between the slits is 0·12 cm. The refractive index of mica and polysterene are 1·58 and 1·55 respectively for the light of wavelength 590 nm which is used in the experiment. The interference is observed on a screen a distance one meter away. (a) What would be the fringe-width ? (b) At what distance from the centre will the first maximum be located ?

**17.** Two transparent slabs having equal thickness but different refractive indices $\mu_1$ and $\mu_2$ are pasted side by side to form a composite slab. This slab is placed just after the double slit in a Young's experiment so that the light from one slit goes through one material and the light from the other slit goes through the other material. What should be the minimum thickness of the slab so that there is a minimum at the point $P_0$ which is equidistant from the slits ?

**18.** A thin paper of thickness 0·02 mm having a refractive index 1·45 is pasted across one of the slits in a Young's double slit experiment. The paper transmits 4/9 of the light energy falling on it. (a) Find the ratio of the maximum intensity to the minimum intensity in the fringe pattern. (b) How many fringes will cross through the centre if an identical paper piece is pasted on the other slit also ? The wavelength of the light used is 600 nm.

**19.** A Young's double slit apparatus has slits separated by 0·28 mm and a screen 48 cm away from the slits. The whole apparatus is immersed in water and the slits are illuminated by the red light ($\lambda = 700$ nm in vacuum). Find the fringe-width of the pattern formed on the screen.

**20.** A parallel beam of monochromatic light is used in a Young's double slit experiment. The slits are separated by a distance $d$ and the screen is placed parallel to the plane of the slits. Show that if the incident beam makes an angle $\theta = \sin^{-1}\left(\frac{\lambda}{2d}\right)$ with the normal to the plane of the slits, there will be a dark fringe at the centre $P_0$ of the pattern.

**21.** A narrow slit $S$ transmitting light of wavelength $\lambda$ is placed a distance $d$ above a large plane mirror as shown in figure (17-E1). The light coming directly from the slit

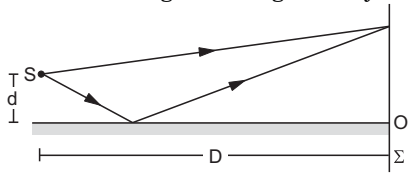



Figure 17-E1

and that coming after the reflection interfere at a screen Σ placed at a distance $D$ from the slit. (a) What will be the intensity at a point just above the mirror, i.e., just above $O$ ? (b) At what distance from $O$ does the first maximum occur ?

**22.** A long narrow horizontal slit is placed 1 mm above a horizontal plane mirror. The interference between the light coming directly from the slit and that after reflection is seen on a screen 1·0 m away from the slit. Find the fringe-width if the light used has a wavelength of 700 nm.

**23.** Consider the situation of the previous problem. If the mirror reflects only 64% of the light energy falling on it, what will be the ratio of the maximum to the minimum intensity in the interference pattern observed on the screen ?

**24.** A double slit $S_1 - S_2$ is illuminated by a coherent light of wavelength λ. The slits are separated by a distance $d$. A plane mirror is placed in front of the double slit at a distance $D_1$ from it and a screen Σ is placed behind the double slit at a distance $D_2$ from it (figure 17-E2). The screen Σ receives only the light reflected by the mirror. Find the fringe-width of the interference pattern on the screen.

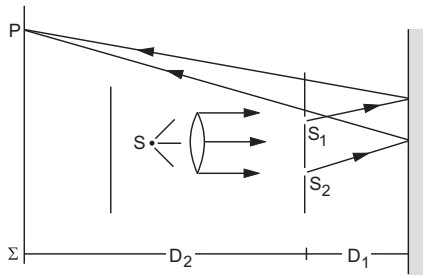



Figure 17-E2

**25.** White coherent light (400 nm–700 nm) is sent through the slits of a Young's double slit experiment (figure 17-E3). The separation between the slits is 0·5 mm and the screen is 50 cm away from the slits. There is a hole in the screen at a point 1·0 mm away (along the width of the fringes) from the central line. (a) Which wavelength(s) will be absent in the light coming from the hole ? (b) which wavelength(s) will have a strong intensity ?

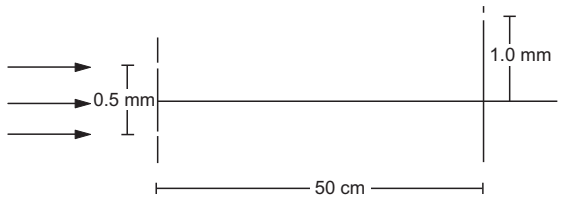



Figure 17-E3

**26.** Consider the arrangement shown in figure (17-E4). The distance $D$ is large compared to the separation $d$ between the slits. (a) Find the minimum value of $d$ so that there is a dark fringe at $O$. (b) Suppose $d$ has this value. Find the distance $x$ at which the next bright fringe is formed. (c) Find the fringe-width.

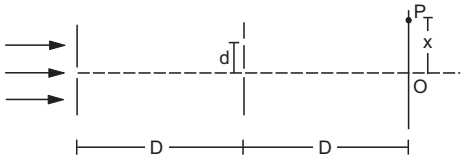



Figure 17-E4

**27.** Two coherent point sources $S_1$ and $S_2$ vibrating in phase emit light of wavelength λ. The separation between the sources is 2λ. Consider a line passing through $S_2$ and perpendicular to the line $S_1 S_2$. What is the smallest distance from $S_2$ where a minimum of intensity occurs ?

**28.** Figure (17-E5) shows three equidistant slits being illuminated by a monochromatic parallel beam of light. Let $BP_0 - AP_0 = \lambda/3$ and $D >> \lambda$. (a) Show that in this case $d = \sqrt{2\lambda D/3}$. (b) Show that the intensity at $P_0$ is three times the intensity due to any of the three slits individually.

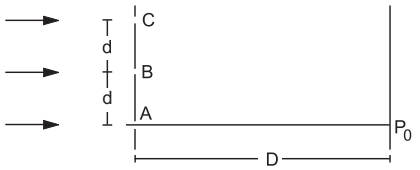



Figure 17-E5

**29.** In a Young's double slit experiment, the separation between the slits = 2·0 mm, the wavelength of the light = 600 nm and the distance of the screen from the slits = 2·0 m. If the intensity at the centre of the central maximum is 0·20 W $m^{-2}$, what will be the intensity at a point 0·5 cm away from this centre along the width of the fringes ?

**30.** In a Young's double slit interference experiment the fringe pattern is observed on a screen placed at a distance $D$ from the slits. The slits are separated by a distance $d$ and are illuminated by monochromatic light of wavelength λ. Find the distance from the central point where the intensity falls to (a) half the maximum, (b) one fourth of the maximum.

**31.** In a Young's double slit experiment $\lambda = 500$ nm, $d$ = 1·0 mm and $D$ = 1·0 m. Find the minimum distance from the central maximum for which the intensity is half of the maximum intensity.

**32.** The line-width of a bright fringe is sometimes defined as the separation between the points on the two sides of the central line where the intensity falls to half the maximum. Find the line-width of a bright fringe in a Young's double slit experiment in terms of λ, $d$ and $D$ where the symbols have their usual meanings.

**33.** Consider the situation shown in figure (17-E6). The two slits $S_1$ and $S_2$ placed symmetrically around the central line are illuminated by a monochromatic light of wavelength $\lambda$. The separation between the slits is $d$. The light transmitted by the slits falls on a screen $\Sigma_1$ placed at a distance $D$ from the slits. The slit $S_3$ is at the central line and the slit $S_4$ is at a distance $z$ from $S_3$. Another screen $\Sigma_2$ is placed a further distance $D$ away from $\Sigma_1$. Find the ratio of the maximum to minimum intensity observed on $\Sigma_2$ if $z$ is equal to

(a) $z = \frac{\lambda D}{2d}$, (b) $\frac{\lambda D}{d}$, (c) $\frac{\lambda D}{4d}$.

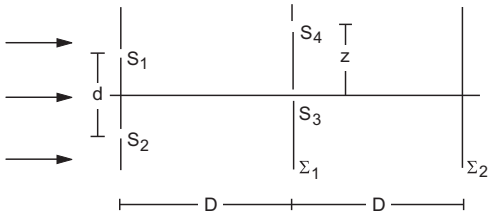



Figure 17-E6

**34.** Consider the arrangement shown in figure (17-E7). By some mechanism, the separation between the slits $S_3$ and $S_4$ can be changed. The intensity is measured at the point $P$ which is at the common perpendicular bisector of $S_1S_2$ and $S_3S_4$. When $z = \frac{D\lambda}{2d}$, the intensity measured at $P$ is $I$. Find this intensity when $z$ is equal to

(a) $\frac{D\lambda}{d}$, (b) $\frac{3D\lambda}{2d}$ and (c) $\frac{2D\lambda}{d}$.

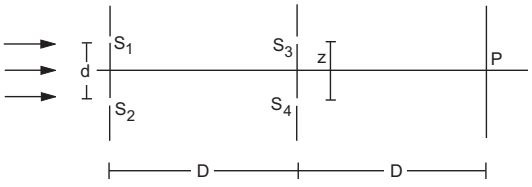



Figure 17-E7

**35.** A soap film of thickness 0·0011 mm appears dark when seen by the reflected light of wavelength 580 nm. What is the index of refraction of the soap solution, if it is known to be between 1·2 and 1·5 ?

**36.** A parallel beam of light of wavelength 560 nm falls on a thin film of oil (refractive index = 1·4). What should be the minimum thickness of the film so that it strongly reflects the light ?

**37.** A parallel beam of white light is incident normally on a water film $1{\cdot}0 \times 10^{-4}$ cm thick. Find the wavelength in the visible range (400 nm–700 nm) which are strongly transmitted by the film. Refractive index of water = 1·33.

**38.** A glass surface is coated by an oil film of uniform thickness $1{\cdot}00 \times 10^{-4}$ cm. The index of refraction of the oil is 1·25 and that of the glass is 1·50. Find the wavelengths of light in the visible region (400 nm–750 nm) which are completely transmitted by the oil film under normal incidence.

**39.** Plane microwaves are incident on a long slit having a width of 5·0 cm. Calculate the wavelength of the microwaves if the first diffraction minimum is formed at $\theta = 30°$.

**40.** Light of wavelength 560 nm goes through a pinhole of diameter 0·20 mm and falls on a wall at a distance of 2·00 m. What will be the diameter of the central bright spot formed on the wall ?

**41.** A convex lens of diameter 8·0 cm is used to focus a parallel beam of light of wavelength 620 nm. If the light be focused at a distance of 20 cm from the lens, what would be the diameter of the central bright spot formed ?

□

## ANSWERS

### OBJECTIVE I

| | | | | | |
|---|---|---|---|---|---|
| 1. (c) | 2. (d) | 3. (d) | 4. (d) | 5. (c) | 6. (b) |
| 7. (a) | 8. (d) | 9. (d) | 10. (a) | 11. (a) | 12. (d) |
| 13. (d) | 14. (c) | 15. (b) | 16. (c) | 17. (a) | |

### OBJECTIVE II

| | | |
|---|---|---|
| 1. (a), (c) | 2. (b), (c) | 3. (b), (c), (d) |
| 4. (c), (d) | 5. (c), (d) | 6. (a), (d) |
| 7. (a) | 8. (b) | 9. (a), (b), (d) |
| 10. (a), (d) | | |

### EXERCISES

1. $4{\cdot}3 \times 10^{14}$ Hz-$7{\cdot}5 \times 10^{14}$ Hz
2. (a) $5{\cdot}09 \times 10^{14}$ Hz (b) 443 nm
   (c) $5{\cdot}09 \times 10^{14}$ Hz (d) $2{\cdot}25 \times 10^{8}$ m $s^{-1}$
3. $2{\cdot}04 \times 10^{8}$ m $s^{-1}$, $2{\cdot}07 \times 10^{8}$ m $s^{-1}$
4. 1·25
5. (a) 0·05 mm (b) 0·50 mm
6. 400 nm
7. (a) 0·25 mm (b) 20
8. 1·47 mm

9. 0·014 degree
10. 0·72 mm
11. 7
12. $\frac{\lambda}{2(\mu - 1)}$
13. (a) $(\mu - 1)t$ (b) $\frac{\lambda}{2(\mu - 1)}$
14. 14·5
15. 590 nm
16. (a) $4{\cdot}9 \times 10^{-4}$ m
    (b) 0·021 cm on one side and 0·028 cm on the other side
17. $\frac{\lambda}{2|\mu_1 - \mu_2|}$
18. (a) 25 (b) 15
19. 0·90 mm
21. (a) zero (b) $\frac{D\lambda}{4d}$
22. 0·35 mm
23. 81 : 1
24. $\lambda(2D_1 + D_2)/d$
25. (a) 400 nm, 667 nm (b) 500 nm
26. (a) $\sqrt{\frac{\lambda D}{2}}$ (b) $d$ (c) $2d$
27. $7\lambda/12$
29. 0·05 W $m^{-2}$
30. (a) $\frac{D\lambda}{4d}$ (b) $\frac{D\lambda}{3d}$
31. $1{\cdot}25 \times 10^{-4}$ m
32. $\frac{D\lambda}{2d}$
33. (a) 1 (b) $\infty$ (c) 34
34. (a) zero (b) $I$ (c) $2I$
35. 1·32
36. 100 nm
37. 443 nm, 532 nm and 666 nm
38. 455 nm, 556 nm, 714 nm
39. 2·5 cm
40. 6·85 mm
41. $1{\cdot}9 \times 10^{-6}$ m

□

## CHAPTER 18

# GEOMETRICAL OPTICS

We have learnt that light in many cases behaves as a wave of short wavelength. A ray of light gives the direction of propagation of light. In absence of an obstacle, the rays advance in straight lines without changing directions. When light meets a surface separating two transparent media, reflection and refraction occur and the light rays bend. Light rays also bend round the edge of an obstacle limiting the wave front. But as the wavelength of light is usually much smaller than the size of the obstacles, this diffraction effect can usually be neglected. We then deal with *geometrical optics.*

### 18.1 REFLECTION AT SMOOTH SURFACES

A light ray is reflected by a smooth surface in accordance with the two laws of reflection :

(a) the angle of incidence is equal to the angle of reflection

(b) the incident ray, the reflected ray and the normal to the reflecting surface are coplanar.

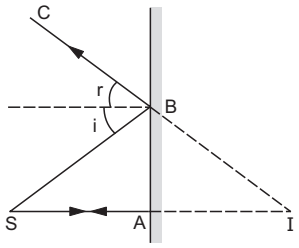



Figure 18.1

Figure (18.1) shows a point source $S$ placed before a plane mirror. Consider a ray $SA$ that falls normally on the mirror. This ray is reflected back along $AS$. Consider any other ray $SB$ making an angle $i$ with the normal. It is reflected along $BC$. Suppose $AS$ and $BC$ meet at a point $I$ when produced behind the mirror. It is simple to show that the triangles $SAB$ and $ABI$ are congruent and $SA = AI$. Thus, all the reflected rays meet at $I$ when produced behind the mirror. An eye receiving the reflected rays feels that the rays are diverging from the point $I$. The point $I$ is called the *image* of the object $S$.

The basic laws of reflection are same for plane and curved surfaces. A normal can be drawn from any point of the curved surface by first drawing the tangent plane from that point and then drawing the line perpendicular to that plane. Angles of incidence and reflection are defined from this normal (figure 18.2). The angle of incidence is equal to the angle of reflection. The incident ray, the normal and the reflected ray are in the same plane.

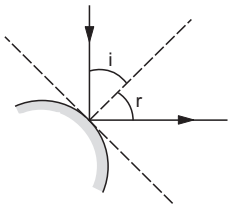



Figure 18.2

### 18.2 SPHERICAL MIRRORS

A spherical mirror is a part cut from a hollow sphere. Spherical mirrors are generally constructed from glass. One surface of the glass is silvered. The reflection takes place at the other surface. If reflection takes place at the convex surface, it is called a *convex mirror* and if reflection takes place at the concave surface, it is called a *concave mirror.*

Generally, a spherical mirror is constructed with a circular boundary. The centre of the sphere, of which the mirror is a part, is called the *centre of curvature* of the mirror. The radius of this sphere is called the *radius of curvature* of the mirror. The point on the mirror at the middle of the surface is called its *pole.* The line joining the pole and the centre of curvature is called the *principal axis.*

#### Focus

Suppose a light beam travelling in a direction parallel to the principal axis is incident on a concave

mirror. If the aperture of the beam is small so that the light falls "near" the pole only, all the reflected rays cross the principal axis at nearly the same point (figure 18.3a). This point where the reflected rays converge is called the *focus* of the mirror.

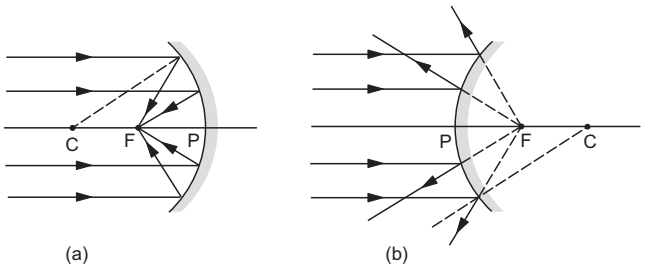



Figure 18.3

In case of a convex mirror, the reflected rays diverge after the reflection. Again, if the incident parallel beam has small aperture, the reflected rays appear to diverge from a point on the principal axis (figure 18.3b). This point is the focus of the convex mirror.

The plane through the focus and perpendicular to the principal axis is called the *focal plane*. The distance of the focus from the pole is called the *focal length* of the mirror.

**Paraxial Rays**

A ray close to the principal axis is called a paraxial ray. In this chapter, we shall consider only paraxial rays in image formation.

**Image Tracing**

When a point object is placed before a spherical mirror of small aperture, a point image is formed. To locate the position of the image, we draw two rays from the point object, make them incident on the mirror and trace the reflected rays. The line joining the point of incidence and the centre of curvature is the normal. A reflected ray is traced by applying the laws of reflection. If the reflected rays intersect, the point of intersection is the *real image*. If the rays diverge after reflection, a *virtual image* is formed at the point from where the rays seem to diverge. Figure (18.4) shows some examples.

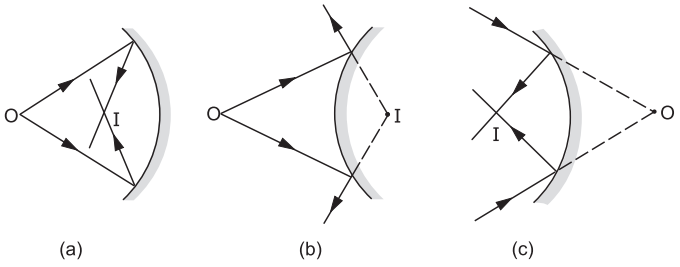



Figure 18.4

If the incident rays diverge from a point object, the object is called a *real object*. Sometimes the rays incident on the mirror do not diverge from a point, rather they converge towards the mirror (figure 18.4c). In this case, the point where these rays would meet if there were no mirror, is treated as the object. Such a point is called a *virtual object*.

Thus, the point of intersection of the incident rays is called the *object* and the point of intersection of the corresponding reflected rays is called its *image*.

**Sign Convention**

In image tracing, we come across the object distance, the image distance, the focal length and the radius of curvature. A system of signs for these quantities is necessary to derive relations connecting them which are consistent in all types of physical situations. We shall describe coordinate sign convention which is now widely used.

In this method, the pole is taken to be the origin and the principal axis as the $X$-axis. Usually, the positive of the axis is taken along the incident rays. The quantities $u$, $v$, $R$ and $f$ denote the $x$-coordinates of the object, the image, the centre of curvature and the focus respectively. Any of these quantities is positive if the corresponding point lies on the positive side of the origin and is negative if it is on the negative side. Figure (18.5) shows some typical situations. The signs of various quantities are tabulated below

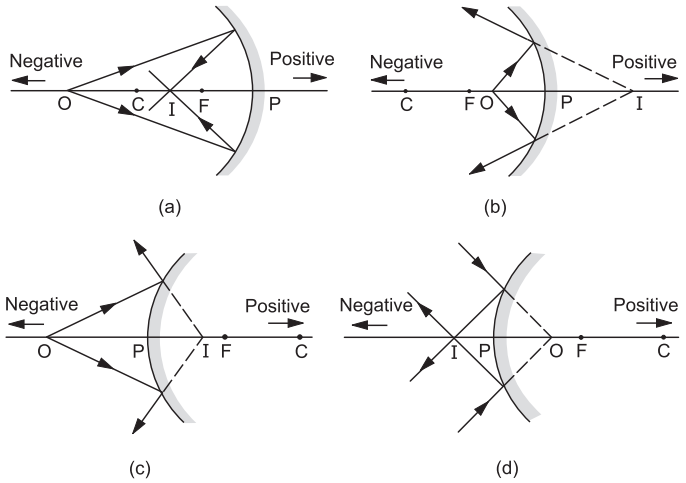



Figure 18.5

| Figure | $u$ | $v$ | $R$ | $f$ |
|---|---|---|---|---|
| 18.5a | – | – | – | – |
| 18.5b | – | + | – | – |
| 18.5c | – | + | + | + |
| 18.5d | + | – | + | + |

If the lengths perpendicular to the principal axis are needed, we should fix the positive direction of the $Y$-axis. Generally the upward is taken as positive of the $Y$-axis and downward as the negative of the $Y$-axis. Heights along the positive $Y$-axis are positive and heights along the negative $Y$-axis are negative. Quite often, we shall use the words "object-distance" and "image-distance" for $u$ and $v$.

### 18.3 RELATION BETWEEN $u$, $v$ AND $R$ FOR SPHERICAL MIRRORS

Consider the situation shown in figure (18.6). A point object is placed at the point $O$ of the principal axis of a concave mirror. A ray $OA$ is incident on the mirror at $A$. It is reflected in the direction $AI$. Another ray $OP$ travels along the principal axis. As $PO$ is normal to the mirror at $P$, the ray is reflected back along $PO$. The reflected rays $PO$ and $AI$ intersect at $I$ where the image is formed.

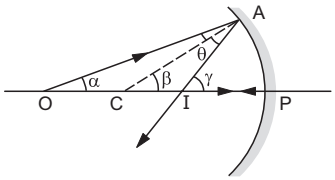



Figure 18.6

Let $C$ be the centre of curvature. The line $CA$ is the normal at $A$. Thus, by the laws of reflection, $\angle OAC = \angle CAI$. Let $\alpha$, $\beta$, $\gamma$ and $\theta$ denote the angles $AOP$, $ACP$, $AIP$ and $OAC$ respectively. As the exterior angle in a triangle equals the sum of the two opposite interior angles, we have,

$$\text{from triangle } OAC \qquad \beta = \alpha + \theta \qquad \ldots \text{ (i)}$$

$$\text{and} \quad \text{from triangle } OAI \qquad \gamma = \alpha + 2\theta. \qquad \ldots \text{ (ii)}$$

Eliminating $\theta$ from (i) and (ii),

$$2\beta = \alpha + \gamma. \qquad \ldots \text{ (iii)}$$

If the point $A$ is close to $P$, the angles $\alpha$, $\beta$ and $\gamma$ are small and we can write

$$\alpha \approx \frac{AP}{PO}, \quad \beta = \frac{AP}{PC} \quad \text{and} \quad \gamma \approx \frac{AP}{PI}.$$

As $C$ is the centre of curvature, the equation for $\beta$ is exact whereas the remaining two are approximate. Putting in (iii),

$$2\frac{AP}{PC} = \frac{AP}{PO} + \frac{AP}{PI}$$

$$\text{or,} \qquad \frac{1}{PO} + \frac{1}{PI} = \frac{2}{PC}. \qquad \ldots \text{ (iv)}$$

The pole $P$ is taken as the origin and the principal axis as the $X$-axis. The rays are incident from left to right. We take the direction from left to right as the positive $X$-direction. The point $O$, $I$ and $C$ are situated to the left of the origin $P$ in the figure. The quantities $u$, $v$ and $R$ are, therefore, negative. As the distances $PO$, $PI$ and $PC$ are positive, $PO = -u$, $PI = -v$ and $PC = -R$. Putting in (iv),

$$\frac{1}{-u} + \frac{1}{-v} = \frac{2}{-R}$$

$$\text{or,} \qquad \frac{1}{u} + \frac{1}{v} = \frac{2}{R}. \qquad \ldots \text{ (18.1)}$$

Although equation (18.1) is derived for a special situation shown in figure (18.6), it is also valid in all other situations with a spherical mirror. This is because we have taken proper care of the signs of $u$, $v$ and $R$ appearing in figure (18.6).

***Example 18.1***

*A convex mirror has its radius of curvature* 20 cm. *Find the position of the image of an object placed at a distance* of 12 cm *from the mirror.*

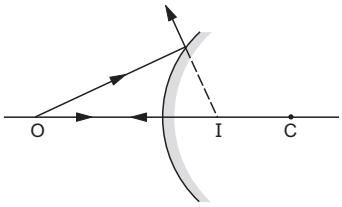



Figure 18.7

***Solution*** : The situation is shown in figure (18.7). Here $u = -12$ cm and $R = +20$ cm. We have,

$$\frac{1}{u} + \frac{1}{v} = \frac{2}{R}$$

$$\text{or,} \qquad \frac{1}{v} = \frac{2}{R} - \frac{1}{u}$$

$$= \frac{2}{20\text{ cm}} - \frac{1}{-12\text{ cm}} = \frac{11}{60\text{ cm}}$$

$$\text{or,} \qquad v = \frac{60}{11}\text{ cm}.$$

The positive sign of $v$ shows that the image is formed on the right side of the mirror. It is a virtual image.

#### Relation between the Focal Length and the Radius of Curvature

If the object $O$ in figure (18.6) is taken at a large distance, the rays coming from $O$ and incident on the mirror become almost parallel. The image is then formed close to the focus. Thus, if $u = \infty$, $v = f$. Putting in (18.1),

$$\frac{1}{\infty} + \frac{1}{f} = \frac{2}{R}$$

$$\text{or,} \qquad f = R/2. \qquad \ldots \text{ (18.2)}$$

Equation (18.1) may also be written as

$$\frac{1}{u}+\frac{1}{v}=\frac{1}{f}. \qquad \ldots (18.3)$$

The focus is midway between the pole and the centre of curvature.

## 18.4 EXTENDED OBJECTS AND MAGNIFICATION

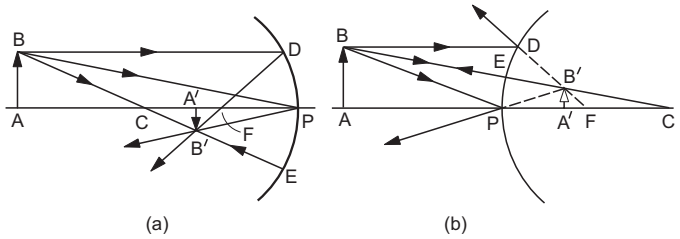



Figure 18.8

Suppose an object $AB$ is placed on the principal axis of a spherical mirror with the length $AB$ perpendicular to the principal axis (figure 18.8). Consider two rays $BD$ and $BE$, the first parallel to the principal axis and the other directed towards the centre of curvature. The ray $BD$ will go through the focus $F$ after reflection. The ray $BE$ will return along $EB$ as it hits the mirror normally. The image of $B$ is formed at the intersection of these two reflected rays. Thus, the image $B'$ of the point $B$ is traced. If we drop a perpendicular $B'A'$ on the principal axis, it can be shown that $A'$ is the image of $A$ and $A'B'$ is the image of $AB$. Figures (18.8a) and (18.8b) show the construction in two different situations.

### Lateral Magnification

The ratio $\dfrac{\text{height of the image}}{\text{height of the objective}}$ is called *lateral* or *transverse magnification*. The height of the object (placed perpendicular to the principal axis) is taken to be positive. If the image is also on the same side of the principal axis, its height is also positive. The image is then erect. Figure (18.8b) shows an example. If the image is inverted, its height is taken as negative.

Consider the ray $BP$ in figure (18.8) hitting the mirror at the pole $P$. The reflected ray passes through $B'$ as $B'$ is the image of $B$. The principal axis is the normal at $P$ to the mirror. By the laws of reflection,

$$\angle BPA = \angle APB'.$$

Thus, the right-angled triangles $ABP$ and $A'B'P$ are similar. Thus,

$$\frac{A'B'}{AB}=\frac{PA'}{PA}. \qquad \ldots (i)$$

In figure (18.8a); $A'B' = -h_2$, $AB = h_1$, $PA' = -v$ and $PA = -u$. Equation (i) gives

$$\frac{-h_2}{h_1}=\frac{-v}{-u}$$

or, $$m=\frac{h_2}{h_1}=-\frac{v}{u}. \qquad \ldots (18.4)$$

Since proper signs are used, this same relation is also valid in all other situations. For example, in figure 18.8(b), $A'B' = +h_2, AB = +h_1, PA' = +v$ and $PA = -u$. Equation (i) gives

$$\frac{h_2}{h_1}=\frac{v}{-u} \quad \text{or,} \quad m=-\frac{v}{u}.$$

***Example 18.2***

*An object of length* 2·5 cm *is placed at a distance of* 1·5 $f$ *from a concave mirror where* $f$ *is the magnitude of the focal length of the mirror. The length of the object is perpendicular to the principal axis. Find the length of the image. Is the image erect or inverted* ?

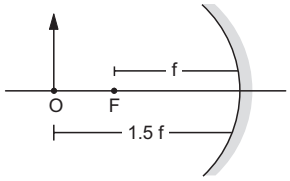



Figure 18.9

***Solution*** : The given situation is shown in figure (18.9). The focal length $F = -f$, and $u = -1{\cdot}5\ f$. We have,

$$\frac{1}{u}+\frac{1}{v}=\frac{1}{F} \quad \text{or,} \quad \frac{1}{-1{\cdot}5f}+\frac{1}{v}=\frac{1}{-f}$$

or, $$\frac{1}{v}=\frac{1}{1{\cdot}5f}-\frac{1}{f}=\frac{-1}{3f}$$

or, $$v=-3f.$$

Now $$m=-\frac{v}{u}=\frac{3f}{-1{\cdot}5f}=-2$$

or, $$\frac{h_2}{h_1}=-2 \quad \text{or,} \quad h_2=-2\,h_1=-5{\cdot}0 \text{ cm}.$$

The image is 5·0 cm long. The minus sign shows that it is inverted.

## 18.5 REFRACTION AT PLANE SURFACES

When a light ray is incident on a surface separating two transparent media, the ray bends at the time of changing the medium. The angle of incidence $i$ and the angle of refraction $r$ follow *Snell's law*

$$\frac{\sin i}{\sin r}=\frac{v_1}{v_2}=\frac{\mu_2}{\mu_1},$$

where $v_1$ and $v_2$ are the speeds of light in media 1 and 2 respectively and $\mu_1$ and $\mu_2$ are the refractive indices

of media 1 and 2 respectively. For vacuum the refractive index μ equals 1. For air also, it is very close to 1.

**Image due to Refraction at a Plane Surface**

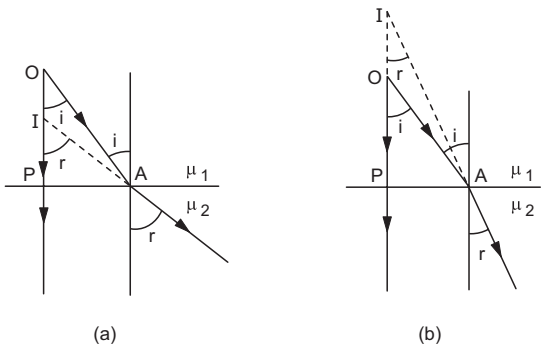



Figure 18.10

Consider the situation shown in figure (18.10). A point object $O$ is placed in a medium of refractive index $\mu_1$. Another medium of refractive index $\mu_2$ has its boundary at $PA$. Consider two rays $OP$ and $OA$ originating from $O$. Let $OP$ fall perpendicularly on $PA$ and $OA$ fall at $PA$ at a small angle $i$ with the normal. $OP$ enters the second medium undeviated and $OA$ enters making an angle $r$ with the normal. When produced backward, these rays meet at $I$ which is the virtual image of $O$. If $i$ and $r$ are small,

$$\sin i \approx \tan i = \frac{PA}{PO}$$

and $$\sin r \approx \tan r = \frac{PA}{PI}.$$

Thus, $$\frac{\mu_2}{\mu_1} = \frac{\sin i}{\sin r}$$

$$= \left(\frac{PA}{PO}\right)\cdot\left(\frac{PI}{PA}\right) = \frac{PI}{PO}. \qquad \ldots \text{(i)}$$

Suppose medium 2 is air and an observer looks at the image from this medium (figure 18.10a). The real depth of the object inside medium 1 is $PO$ whereas the depth as it appears to the observer is $PI$. Writing $\mu_2 = 1$ and $\mu_1 = \mu$, equation (i) gives

$$\frac{1}{\mu} = \frac{\text{apparent depth}}{\text{real depth}}$$

or, $$\mu = \frac{\text{real depth}}{\text{apparent depth}}. \qquad \ldots \text{(18.5)}$$

The image shifts closer to eye by an amount $OI = PO - PI$

$$= \left(\frac{PO - PI}{PO}\right)PO = \left(1 - \frac{PI}{PO}\right)PO$$

or, $$\Delta t = \left(1 - \frac{1}{\mu}\right)t, \qquad \ldots \text{(18.6)}$$

where $t$ is the thickness of the medium over the object and $\Delta t$ is the apparent shift in its position towards the observer. Note that $\Delta t$ is positive in figure (18.10a) and negative in figure (18.10b).

***Example 18.3***

*A printed page is kept pressed by a glass cube* ($\mu = 1{\cdot}5$) *of edge* 6·0 cm. *By what amount will the printed letters appear to be shifted when viewed from the top* ?

***Solution*** : The thickness of the cube = $t$ = 6·0 cm. The shift in the position of the printed letters is

$$\Delta t = \left(1 - \frac{1}{\mu}\right)t$$

$$= \left(1 - \frac{1}{1{\cdot}5}\right) \times 6{\cdot}0 \text{ cm} = 2{\cdot}0 \text{ cm}.$$

## 18.6 CRITICAL ANGLE

When a ray passes from an optically denser medium (larger μ) to an optically rarer medium (smaller μ), the angle of refraction $r$ is greater than the corresponding angle of incidence $i$. We have,

$$\frac{\sin i}{\sin r} = \frac{\mu_2}{\mu_1} < 1.$$

If we gradually increase $i$, the corresponding $r$ will also increase and at a certain stage $r$ will become 90°. Let the angle of incidence for this case be $\theta_c$. If $i$ is increased further, there is no $r$ which can satisfy Snell's law. Thus, the ray will not be refracted. Entire light is then reflected back into the first medium. This is called *total internal reflection*. The angle $\theta_c$ is called the *critical angle* for the given pair of media. Generally, critical angle of a medium is quoted for light going from the medium to the air. In this case, $\mu_2 = 1$. Writing $\mu_1 = \mu$, Snell's law gives

$$\frac{\sin\theta_c}{\sin 90^\circ} = \frac{1}{\mu}$$

or, $$\sin\theta_c = (1/\mu)$$

or, $$\theta_c = \sin^{-1}(1/\mu). \qquad \ldots \text{(18.7)}$$

***Example 18.4***

*The critical angle for water is* 48·2°. *Find its refractive index.*

***Solution*** : $\mu = \dfrac{1}{\sin\theta_c} = \dfrac{1}{\sin 48{\cdot}2^\circ} = 1{\cdot}34.$

## 18.7 OPTICAL FIBRE

Total internal reflection is the basic principle of a very useful branch of physics known as *fibre optics*. An *optical fibre* is a very thin fibre made of glass or

plastic having a radius of the order of a micrometer ($10^{-6}$ m). A bundle of such thin fibres forms a *light pipe*.

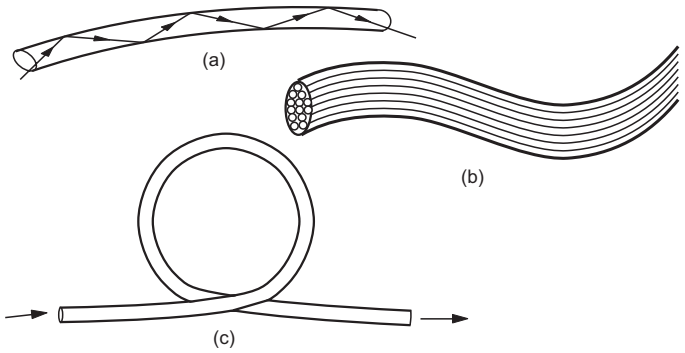



Figure 18.11

Figure (18.11a) shows the principle of light transmission by an optical fibre. Figure (18.11b) sketches a light pipe. Because of the small radius of the fibre, light going into it makes a nearly glancing incidence on the wall. The angle of incidence is greater than the critical angle and hence total internal reflection takes place. The light is thus transmitted along the fibre. Even if a light pipe is put in a complicated shape (figure 18.11c), the light is transmitted without any appreciable loss.

Light pipes using optical fibres may be used to see places which are difficult to reach such as inside of a human body. For example, a patient's stomach can be viewed by inserting one end of a light pipe into the stomach through the mouth. Light is sent down through one set of fibres in the pipe. This illuminates the inside of the stomach. The light from the inside travels back through another set of fibres in the pipe and the viewer gets the image at the outer end.

The other important application of fibre optics is to transmit communication signals through light pipes. For example, about 2000 telephone signals, appropriately mixed with light waves, may be simultaneously transmitted through a typical optical fibre. The clarity of the signals transmitted in this way is much better than other conventional methods.

The fibres in a light pipe must be optically insulated from each other. This is usually done by coating each fibre with a material having refractive index less than that of the fibre.

## 18.8 PRISM

Figure (18.12) shows the cross section of a prism. *AB* and *AC* represent the refracting surfaces. The angle *BAC* is the angle of the prism. Consider the prism to be placed in air. A ray *PQ*, incident on a refracting surface *AB,* gets refracted along *QR*. The angle of incidence and the angle of refraction are $i$ and $r$ respectively. The ray *QR* is incident on the surface *AC*. Here the light goes from an optically denser medium to an optically rarer medium. If the angle of incidence $r'$ is not greater than the critical angle, the ray is refracted in air along *RS*. The angle of refraction is $i'$. The angle $i'$ is also called the angle of emergence. If the prism were not present, the incident ray would have passed undeviated along *PQTU*. Because of the prism, the final ray goes along *RS*. The angle $UTS = \delta$ is called the *angle of deviation*. From triangle *TQR*,

$$\angle UTS = \angle TQR + \angle TRQ$$

or, $$\delta = (\angle TQV - \angle RQV) + (\angle TRV - \angle QRV)$$

$$= (i - r) + (i' - r')$$

$$= (i + i') - (r + r'). \quad \ldots \text{(i)}$$

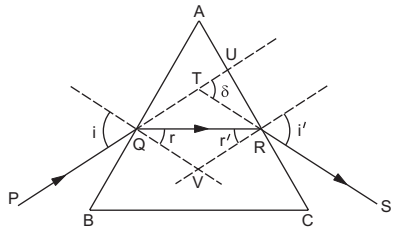



Figure 18.12

Now, the four angles of the quadrangle *AQVR* add to 360°. The angles *AQV* and *ARV* are 90° each. Thus,

$$A + \angle QVR = 180°.$$

Also, from the triangle *QRV,*

$$r + r' + \angle QVR = 180°.$$

So, $$r + r' = A. \quad \ldots \text{(18.8)}$$

Substituting in (i),

$$\delta = i + i' - A. \quad \ldots \text{(18.9)}$$

### Angle of Minimum Deviation

The angle $i'$ is determined by the angle of incidence $i$. Thus, the angle of deviation $\delta$ is also determined by $i$. For a particular value of angle of incidence, the angle of deviation is minimum. In this situation, the ray passes symmetrically through the prism, so that $i = i'$.

The above statement can be justified by assuming that there is a unique angle of minimum deviation. Suppose, when deviation is minimum, the angle of incidence is greater than the angle of emergence. Suppose, figure (18.13) shows the situation for minimum deviation. According to our assumption,

$$i > i'.$$

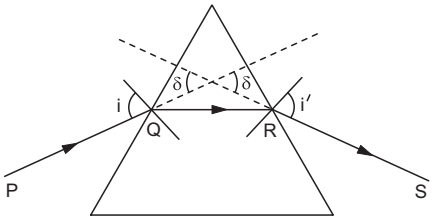



Figure 18.13

Now, if we send a ray along *SR*, it will retrace the path and will emerge along *QP*. Thus, the angle of deviation is same as before and hence, is minimum. According to our assumption, the angle of incidence is greater than the angle of emergence. Hence,

$$i' > i.$$

Thus, we get a contradiction. Similarly, if we assume that the angle of incidence is smaller than the angle of emergence for minimum deviation, we again get a contradiction. Hence, for minimum deviation, $i = i'$.

**Relation between Refractive Index and the Angle of Minimum Deviation**

Let the angle of minimum deviation be $\delta_m$. For minimum deviation, $i = i'$ and $r = r'$. We have,

$$\delta_m = i + i' - A$$

$$= 2\,i - A$$

or, $$i = \frac{A + \delta_m}{2}. \quad \ldots \text{(i)}$$

Also, $$r + r' = A$$

or, $$r = A/2. \quad \ldots \text{(ii)}$$

The refractive index is

$$\mu = \frac{\sin i}{\sin r}.$$

Using (i) and (ii),

$$\mu = \frac{\sin \frac{A + \delta_m}{2}}{\sin \frac{A}{2}}. \quad \ldots (18.10)$$

If the angle of prism $A$ is small, $\delta_m$ is also small. Equation (18.10) then becomes

$$\mu \approx \frac{\frac{A + \delta_m}{2}}{\frac{A}{2}}$$

or, $$\delta_m = (\mu - 1)\,A. \quad \ldots (18.11)$$

***Example 18.5***

*The angle of minimum deviation from a prism is* 37°. *If the angle of prism is* 53°, *find the refractive index of the material of the prism.*

***Solution*** : $$\mu = \frac{\sin \frac{A + \delta_m}{2}}{\sin \frac{A}{2}} = \frac{\sin \frac{53° + 37°}{2}}{\sin \frac{53°}{2}} = \frac{\sin 45°}{\sin 26{\cdot}5°}$$

$$= 1{\cdot}58.$$

## 18.9 REFRACTION AT SPHERICAL SURFACES

When two transparent media are separated by a spherical surface, light incident on the surface gets refracted into the medium on other side. Suppose two transparent media having refractive indices $\mu_1$ and $\mu_2$ are separated by a spherical surface *AB* (figure 18.14). Let *C* be the centre of curvature of *AB*. Consider a point object *O* in the medium 1. Suppose the line *OC* cuts the spherical surface at *P*.

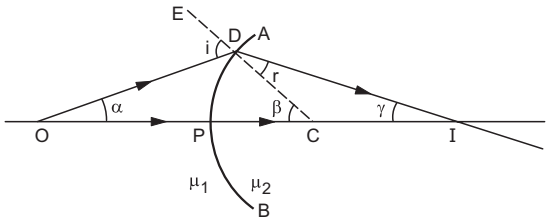



Figure 18.14

Several cases may arise. The surface may be concave towards the higher μ side or it may be convex. The object may be on the convex side or on the concave side. In figure (18.14), it is assumed that $\mu_2 > \mu_1$ and the object *O* is on the convex side of the surface.

**Image Tracing**

Consider two rays *OD* and *OP* originating from *O*. The ray *OP* falls normally on *AB*. It goes into the medium 2 undeviated. Suppose the ray *OD* makes a small angle α with the line *OPC* and falls on the surface *AB* at a point *D*. The normal to *AB* at the point *D* is *DC*. The angle $ODE = i$ is the angle of incidence. The ray is refracted along *DI*. The two refracted rays meet at the point *I* where the image is formed. The angle $CDI = r$ is the angle of refraction. If the refracted rays actually meet, a real image is formed. If the refracted rays diverge after refraction, a virtual image is formed at the point from where these rays seem to diverge.

**Sign Convention**

The sign convention for refraction at spherical surface is quite similar to that used for spherical mirrors.

The line joining the object and the centre is taken as the $X$-axis. The positive direction of the axis is generally chosen along the direction of the incident rays. The point of intersection of the spherical surface with the axis is taken as the origin. The quantities $u$, $v$ and $R$ denote the $x$-coordinates of the object, the image and the centre of curvature respectively. Any of these quantities is positive if the corresponding point lies on the positive side of the origin and is negative if it is on the negative side. Similarly for the lengths perpendicular to the $X$-axis.

**Relation between *u, v* and *R***

Refer to figure (18.14). Let $\angle DOP = \alpha$, $\angle DCP = \beta$ and $\angle DIC = \gamma$. For paraxial rays, $D$ is close to $P$ and $\alpha$, $i$, $r$, $\beta$ and $\gamma$ are all small. From the triangle $ODC$,

$$\alpha + \beta = i \qquad \ldots \text{(i)}$$

and from $DCI$,

$$r + \gamma = \beta. \qquad \ldots \text{(ii)}$$

Also, from Snell's law,

$$\frac{\sin i}{\sin r} = \frac{\mu_2}{\mu_1}.$$

We can write $\sin i \approx i$ and $\sin r \approx r$ so that the above equation becomes

$$\mu_1 i = \mu_2 r. \qquad \ldots \text{(iii)}$$

Putting $i$ and $r$ from (i) and (ii) into (iii),

$$\mu_1(\alpha + \beta) = \mu_2(\beta - \gamma)$$

or, $$\mu_1\alpha + \mu_2\gamma = (\mu_2 - \mu_1)\beta. \qquad \ldots \text{(iv)}$$

As $\alpha$, $\beta$ and $\gamma$ are small, from figure (18.14),

$$\alpha \approx \frac{DP}{PO},\ \beta = \frac{DP}{PC} \text{ and } \gamma \approx \frac{DP}{PI}.$$

The expression for $\beta$ is exact as $C$ is the centre of curvature. Putting in (iv),

$$\mu_1\left(\frac{DP}{PO}\right) + \mu_2\left(\frac{DP}{PI}\right) = (\mu_2 - \mu_1)\frac{DP}{PC}$$

or, $$\frac{\mu_1}{PO} + \frac{\mu_2}{PI} = \frac{\mu_2 - \mu_1}{PC}. \qquad \ldots \text{(v)}$$

At this stage, proper sign convention must be used so that the formula derived is also valid for situations other than that shown in the figure.

In figure (18.14), the point $P$ is the origin and $OPC$ is the axis. As the incident ray comes from left to right, we choose this direction as the positive direction of the axis. We see that $u$ is negative whereas $v$ and $R$ are positive. As the distances $PO$, $PI$ and $PC$ are positive, $PO = -u$, $PI = +v$ and $PC = +R$. From (v),

$$\frac{\mu_1}{-u} + \frac{\mu_2}{v} = \frac{\mu_2 - \mu_1}{R}$$

or, $$\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}. \qquad \ldots \text{(18.12)}$$

Although the formula (18.12) is derived for a particular situation of figure (18.14), it is valid for all other situations of refraction at a single spherical surface. This is because we have used the proper sign convention.

***Example 18.6***

*Locate the image of the point object O in the situation shown in figure (18.15). The point C denotes the centre of curvature of the separating surface.*

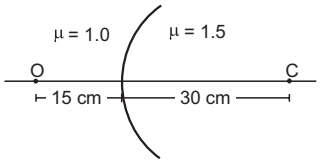



Figure 18.15

***Solution*** : Here $u = -15$ cm, $R = 30$ cm, $\mu_1 = 1$ and $\mu_2 = 1{\cdot}5$. We have,

$$\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$$

or, $$\frac{1{\cdot}5}{v} - \frac{1{\cdot}0}{-15\text{ cm}} = \frac{1{\cdot}5 - 1}{30\text{ cm}}$$

or, $$\frac{1{\cdot}5}{v} = \frac{0{\cdot}5}{30\text{ cm}} - \frac{1}{15\text{ cm}}$$

or, $$v = -30\text{ cm}.$$

The image is formed 30 cm left to the spherical surface and is virtual.

## 18.10 EXTENDED OBJECTS : LATERAL MAGNIFICATION

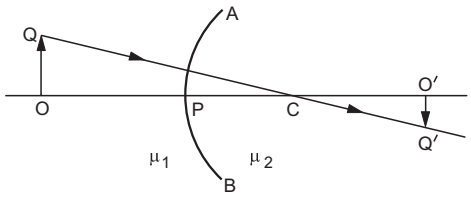



Figure 18.16

Consider the situation shown in figure (18.16). Let $OQ$ be an extended object placed perpendicular to the line $OPC$. Consider the ray originating from $Q$ and going towards $QC$. This ray is incident normally on the spherical surface $AB$. Thus, it goes undeviated in medium 2. The image of $Q$ must be on the line $QC$.

The image of $O$ will be formed on the line $OPC$. Let it be formed at $O'$. The position of $O'$ may be located by using equation (18.12). If we drop a perpendicular from $O'$ on $OPC$, the intersection $Q'$ of this perpendicular with $QC$ will be the image of $Q$. Thus, $O'Q'$ will be the image of $OQ$.

**Lateral Magnification**

The lateral or transverse magnification is defined as

$$m = \frac{h_2}{h_1},$$

where $h_2$ = height of the image and $h_1$ = height of the object. In figure (18.16), $OQ = +h_1$ and $O'Q' = -h_2$.

$$m = \frac{h_2}{h_1} = -\frac{O'Q'}{OQ}.$$

The triangle $OCQ$ and $O'CQ'$ are similar. So,

$$m = -\frac{O'Q'}{OQ} = -\frac{O'C}{OC}$$

$$= -\frac{PO' - PC}{PO + PC}. \quad \ldots \text{(i)}$$

In figure (18.16), $PO = -u$, $PC = +R$ and $PO' = +v$. Equation (i) gives

$$m = -\frac{v - R}{-u + R}$$

$$= \frac{R - v}{R - u}. \quad \ldots \text{(ii)}$$

Also,

$$\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$$

or, $$\frac{\mu_2 u - \mu_1 v}{uv} = \frac{\mu_2 - \mu_1}{R}$$

or, $$R = \frac{(\mu_2 - \mu_1)uv}{\mu_2 u - \mu_1 v}.$$

This gives

$$R - v = \frac{\mu_1 v(v - u)}{\mu_2 u - \mu_1 v}$$

and $$R - u = \frac{\mu_2 u(v - u)}{\mu_2 u - \mu_1 v}$$

Thus, by (ii),

$$m = \frac{\mu_1 v}{\mu_2 u}. \quad \ldots \text{(18.13)}$$

***Example 18.7***

*Find the size of the image formed in the situation shown in figure (18.17).*

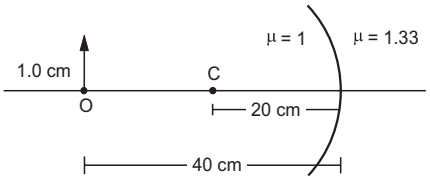



Figure 18.17

***Solution*** : Here $u = -40$ cm, $R = -20$ cm, $\mu_1 = 1$, $\mu_2 = 1{\cdot}33$. We have,

$$\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$$

or, $$\frac{1{\cdot}33}{v} - \frac{1}{-40 \text{ cm}} = \frac{1{\cdot}33 - 1}{-20 \text{ cm}}$$

or, $$\frac{1{\cdot}33}{v} = -\frac{1}{40 \text{ cm}} - \frac{0{\cdot}33}{20 \text{ cm}}$$

or, $$v = -32 \text{ cm}.$$

The magnification is

$$m = \frac{h_2}{h_1} = \frac{\mu_1 v}{\mu_2 u}$$

or, $$\frac{h_2}{1{\cdot}0 \text{ cm}} = \frac{-32 \text{ cm}}{1{\cdot}33 \times (-40 \text{ cm})}$$

or, $$h_2 = +0{\cdot}6 \text{ cm}.$$

The image is erect.

## 18.11 REFRACTION THROUGH THIN LENSES

A lens is one of the most familiar optical devices for a human being. We have lenses in our eyes and a good number of us supplement them with another set of lenses in our spectacles. A lens is made of a transparent material bounded by two spherical surfaces. The surfaces may be both convex, both concave or one convex and one concave. When the thickness of the lens is small compared to the other dimensions like object distance, we call it a *thin lens.* Figure (18.18) shows several lenses and paths of a number of rays going through them.

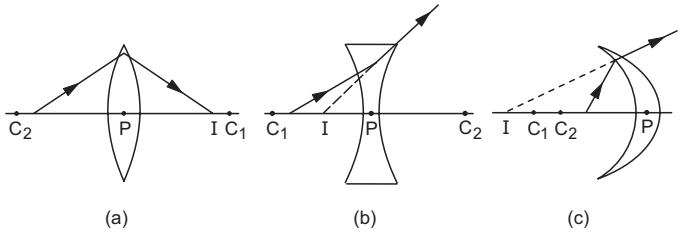



Figure 18.18

As there are two spherical surfaces, there are two centres of curvature $C_1$ and $C_2$ and correspondingly two radii of curvature $R_1$ and $R_2$. The line joining $C_1$ and

$C_2$ is called the *principal axis* of the lens. The centre $P$ of the thin lens which lies on the principal axis, is called the *optical centre*.

**Focus**

Suppose, a narrow beam of light travelling parallel to the principal axis is incident on the lens near the optical centre (figure 18.19a).

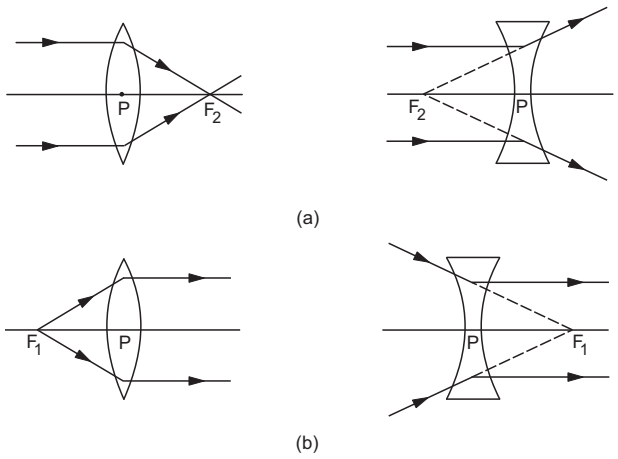



Figure 18.19

The rays are refracted twice and then come out of the lens. The emergent rays may converge at a point $F_2$ on the principal axis or they may seem to diverge from a point $F_2$ on the principal axis. In the first case, the lens is called a *convergent lens, converging lens or convex lens.* In the second case, it is called a *divergent lens, diverging lens* or *concave lens*. The point $F_2$ is called the *second focus* of the lens. The distance $PF_2$ from the optical centre is called the *second focal length.*

The *first focus* $F_1$ is defined as a point where an object should be placed to give emergent rays parallel to the principal axis (figure 18.19b). For a convergent lens, such an object is a real object and for a divergent lens, it is a virtual object. The distance $PF_1$ is the *first focal length.*

If the media on the two sides of a thin lens have same refractive index, the two focal lengths are equal. We shall be largely using the second focus $F_2$ in our discussions. Thus, when we write just *focus*, we shall mean the second focus and when we write just *focal length*, we shall mean second focal length.

**Sign Conventions**

The coordinate sign conventions for a lens are similar to those for mirrors or refraction at spherical surfaces. The optical centre is taken as the origin and the principal axis as the $X$-axis. The positive direction of the axis is generally taken along the incident rays. The quantities $u$, $v$, $f$, $R_1$ and $R_2$ represent the $x$-coordinates of the object, the image, the focus, first centre of curvature and second centre of curvature respectively. The table below shows the signs of $u$, $v$, $f$, $R_1$ and $R_2$ in certain cases shown in figure (18.20).

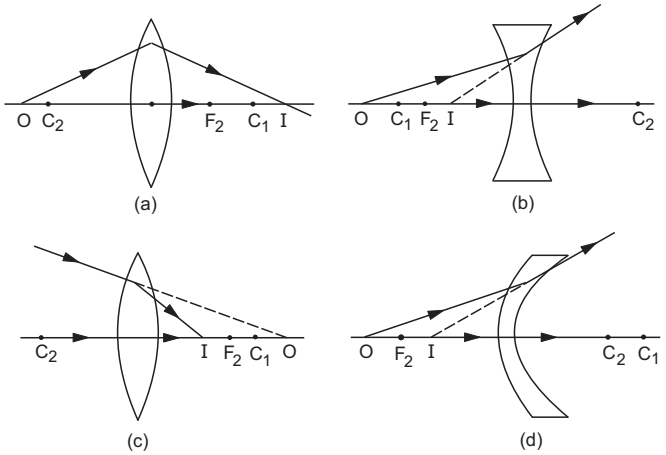



Figure 18.20

| Figure | $u$ | $v$ | $f$ | $R_1$ | $R_2$ |
|---|---|---|---|---|---|
| 18.20a | – | + | + | + | – |
| 18.20b | – | – | – | – | + |
| 18.20c | + | + | + | + | – |
| 18.20d | – | – | – | + | + |

Generally, the incident rays and hence the positive direction of the axis is taken from left to right. Heights measured upwards are taken to be positive and the heights measured downward are taken to be negative.

With the usual choice of axes, $f$ of a lens is positive for a converging lens and is negative for a diverging lens.

## 18.12 LENS MAKER'S FORMULA AND LENS FORMULA

Consider the situation shown in figure (18.21). $ADBE$ is a thin lens. An object $O$ is placed on its principal axis. The two spherical surfaces of the lens have their centres at $C_1$ and $C_2$ . The optical centre is at $P$ and the principal axis cuts the two spherical surfaces at $D$ and $E$.

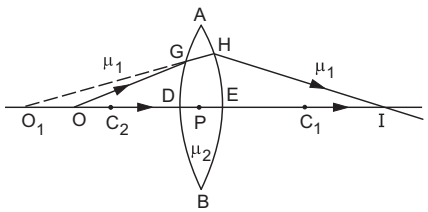



Figure 18.21

Let the refractive index of the material of the lens be $\mu_2$ and suppose it is placed in a medium of refractive index $\mu_1$. To trace the image of $O$, consider two rays $OP$ and $OG$ originating from $O$. The ray $OP$ falls on the spherical surfaces perpendicularly and hence, it goes undeviated through the lens. The ray $OG$ is refracted from a medium of refractive index $\mu_1$ to another medium of refractive index $\mu_2$. The centre of curvature of the surface $ADB$ is at $C_1$. The ray is refracted along $GH$ which meets the principal axis at $O_1$ (when produced backward in figure 18.21). Thus, due to this single refraction, the image of $O$ is formed at $O_1$. The ray $GH$ is incident on the spherical surface $AEB$. It is refracted from medium $\mu_2$ to medium $\mu_1$. The emergent ray $HI$ intersects the principal axis at $I$ where the final image is formed.

The general equation for refraction at a spherical surface is

$$\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}. \qquad \ldots \text{(i)}$$

To use this equation for the first refraction at $ADB$, we should take the origin at $D$, whereas, for the second refraction at $AEB$, we should take the origin at $E$. As the lens is thin, the points $D$, $P$ and $E$ are all close to each other and we may take the origin at $P$ for both these refractions.

For the first refraction, the object is at $O$, the image is at $O_1$ and the centre of curvature is at $C_1$. If $u$, $v_1$, and $R_1$, denote their $x$-coordinates,

$$\frac{\mu_2}{v_1} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R_1}. \qquad \ldots \text{(ii)}$$

For the second refraction at $AEB$, the incident rays $GH$ and $DE$ diverge from $O_1$. Thus, $O_1$ is the object for this refraction and its $x$-coordinate is $v_1$. The image is formed at $I$ and the centre of curvature is at $C_2$. Their $x$-coordinates are $v$ and $R_2$ respectively. The light goes from the medium $\mu_2$ to medium $\mu_1$. Applying equation (i),

$$\frac{\mu_1}{v} - \frac{\mu_2}{v_1} = \frac{\mu_1 - \mu_2}{R_2}. \qquad \ldots \text{(iii)}$$

Adding (ii) and (iii),

$$\mu_1\left(\frac{1}{v} - \frac{1}{u}\right) = \left(\mu_2 - \mu_1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

or,
$$\frac{1}{v} - \frac{1}{u} = \left(\frac{\mu_2}{\mu_1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right). \qquad \ldots \text{(18.14)}$$

If the object $O$ is taken far away from the lens, the image is formed close to the focus. Thus, for $u = \infty$, $v = f$. Putting in (18.14), we get,

$$\frac{1}{f} = \left(\frac{\mu_2}{\mu_1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right). \qquad \ldots \text{(18.15)}$$

If the refractive index of the material of the lens is $\mu$ and it is placed in air, $\mu_2 = \mu$ and $\mu_1 = 1$ so that (18.15) becomes

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right). \qquad \ldots \text{(18.16)}$$

This is called *lens maker's formula* because it tells what curvatures will be needed to make a lens of desired focal length. Combining (18.14) and (18.15),

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f} \qquad \ldots \text{(18.17)}$$

which is known as the *lens formula.*

***Example 18.8***

*A biconvex lens has radii of curvature* 20 cm *each. If the refractive index of the material of the lens is* 1·5*, what is its focal length* ?

***Solution*** **:** In a biconvex lens, centre of curvature of the first surface is on the positive side of the lens and that of the second surface is on the negative side. Thus, $R_1 = 20$ cm and $R_2 = -20$ cm.

We have,

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

or,
$$\frac{1}{f} = (1{\cdot}5 - 1)\left(\frac{1}{20\text{ cm}} - \frac{1}{-20\text{ cm}}\right)$$

or,
$$f = 20 \text{ cm}.$$

## 18.13 EXTENDED OBJECTS : LATERAL MAGNIFICATION

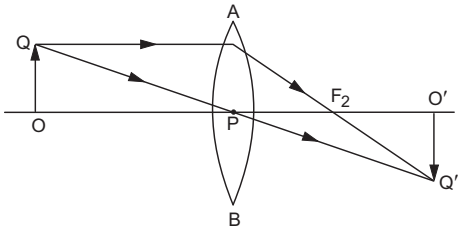



Figure 18.22

Consider the situation shown in figure (18.22). $OQ$ is an extended object placed on the principal axis with its height perpendicular to the principal axis. To locate the image of $Q$, consider two rays $QP$ and $QA$, the first one through the optical centre and the other parallel to the principal axis. The parts of the two surfaces at which the ray $QP$ is refracted, are nearly parallel to each other. The lens near the optical centre, therefore, behaves like a rectangular slab. Thus, the ray passing through this region does not bend. Also, the lateral displacement produced is negligible as the thickness of the lens is small. Thus, a ray passing through the optical centre goes undeviated. The ray $QP$ emerges in the same direction $PQ'$. The ray $QA$ parallel to the

principal axis must pass through the focus $F_2$. Thus, the emergent ray is along $AF_2Q'$. The image is formed where $QPQ'$ and $AF_2Q'$ intersect. Drop a perpendicular from $Q'$ on the principal axis. This perpendicular $O'Q'$ is the image of $OQ$. Figure (18.23) shows image formation in some other cases.

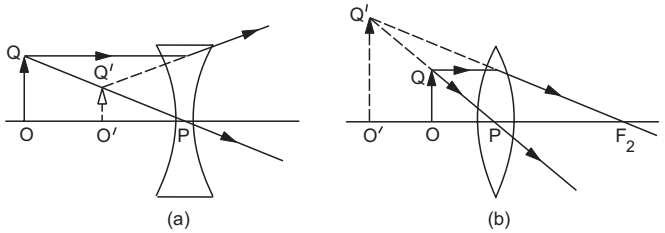



Figure 18.23

The lateral or transverse magnification is defined as

$$m = \frac{h_2}{h_1},$$

where $h_2$ = height of the image and $h_1$ = height of the object.

Referring to figure (18.22), the magnification is

$$m = \frac{h_2}{h_1} = \frac{-O'Q'}{OQ}.$$

From the similar triangles $OQP$ and $O'Q'P$,

$$\frac{O'Q'}{OQ} = \frac{PO'}{PO}$$

so that $$m = -\frac{PO'}{PO}. \quad \ldots \text{(i)}$$

But $PO = -u$ and $PO' = +v$

so that by (i), $$m = \frac{v}{u}. \quad \ldots (18.18)$$

As usual, negative $m$ indicates inverted image and positive $m$ indicates erect image.

Since proper sign conventions are used, equation (18.18) is valid in all situations with a single thin lens, although it is derived for the particular situation of figure (18.22).

***Example 18.9***

*An object of length* 2·0 cm *is placed perpendicular to the principal axis of a convex lens of focal length* 12 cm. *Find the size of the image if the object is at a distance of* 8·0 cm *from the lens.*

***Solution*** : We have $u = -8{\cdot}0$ cm, and $f = +12$ cm

Using $$\frac{1}{v} - \frac{1}{u} = \frac{1}{f},$$

$$\frac{1}{v} = \frac{1}{12 \text{ cm}} + \frac{1}{-8{\cdot}0 \text{ cm}}$$

or, $$v = -24 \text{ cm}.$$

Thus, $$m = \frac{v}{u} = \frac{-24 \text{ cm}}{-8{\cdot}0 \text{ cm}} = 3.$$

Thus, $h_2 = 3\, h_1 = 3 \times 2{\cdot}0$ cm $= 6{\cdot}0$ cm. The positive sign shows that the image is erect.

## 18.14 POWER OF A LENS

The power $P$ of a lens is defined as $P = 1/f$, where $f$ is the focal length. The SI unit of power of a lens is obviously $m^{-1}$. This is also known as *dioptre*. The focal length of a converging lens is positive and that of a diverging lens is negative. Thus, the power of a converging lens is positive and that of a diverging lens is negative.

## 18.15 THIN LENSES IN CONTACT

Figure (18.24) shows two lenses $L_1$ and $L_2$ placed in contact. The focal lengths of the lenses are $f_1$ and $f_2$ respectively.

Suppose, a point object is placed at a point $O$ on the common principal axis. The first lens would form its image at $O_1$. This point $O_1$ works as the object for the second lens and the final image is formed at $I$.

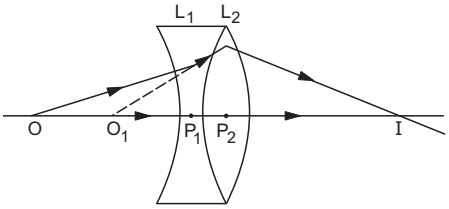



Figure 18.24

Let $u$ = object-distance for the first lens,

$v$ = final image-distance for the second lens,

$v_1$ = image-distance of the first image $O_1$ for the first lens. As the lenses are assumed to be thin, $v_1$ is also the object-distance for the second lens.

Then,

$$\frac{1}{v_1} - \frac{1}{u} = \frac{1}{f_1}$$

and $$\frac{1}{v} - \frac{1}{v_1} = \frac{1}{f_2}.$$

Adding these equations, we get

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f_1} + \frac{1}{f_2}. \quad \ldots \text{(i)}$$

If the combination is replaced by a single lens of focal length $F$ such that it forms the image of $O$ at the same position $I$,

$$\frac{1}{v}-\frac{1}{u}=\frac{1}{F}. \qquad \ldots \text{ (ii)}$$

Such a lens is called the equivalent lens for the combination.

Comparing (i) and (ii),

$$\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}. \qquad \ldots \text{ (18.19)}$$

This $F$ is the focal length of the equivalent lens for the combination. As the power of a lens is $P = 1/F$, equation (18.19) immediately gives

$$P = P_1 + P_2. \qquad \ldots \text{ (18.20)}$$

Though equation (18.19) is derived for the situation shown in figure (18.24), it is true for any situation involving two thin lenses in contact.

## 18.16 TWO THIN LENSES SEPARATED BY A DISTANCE

When two thin lenses are separated by a distance, it is *not* equivalent to a single thin lens. In fact, such a combination can only be equivalent to a thick lens which has a more complicated theory.

In a special case when the object is placed at infinity, the combination may be replaced by a single thin lens. We shall now derive the position and focal length of the equivalent lens in this special case. To start with, let us derive an expression for the angle of deviation of a ray when it passes through a lens.

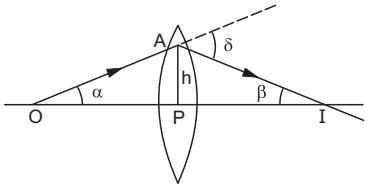



Figure 18.25

Let $O$ be a point object on the principal axis of a lens (figure 18.25). Let $OA$ be a ray incident on the lens at a point $A$, a height $h$ above the optical centre. It is deviated through an angle $\delta$ and comes out along $AI$. It strikes the principal axis at $I$ where the image is formed.

Let $\angle AOP = \alpha$ and $\angle AIP = \beta$. By triangle $OAI$,

$$\delta = \alpha + \beta.$$

If the height $h$ is small as compared to $PO$ and $PI$, the angles $\alpha$, $\beta$ are also small. Then,

$$\alpha \approx \tan\alpha = h/OP \text{ and } \beta \approx \tan\beta = h/PI.$$

Thus,
$$\delta = \frac{h}{PO}+\frac{h}{PI} \qquad \ldots \text{ (i)}$$

Now, $PO = -u$ and $PI = +v$

so that by (i), $\delta = h\left(\frac{1}{v}-\frac{1}{u}\right)$

or,
$$\delta = \frac{h}{f}. \qquad \ldots \text{ (18.21)}$$

Now, consider the situation shown in figure (18.26). Two thin lenses are placed coaxially at a separation $d$. The incident ray $AB$ and the emergent ray $CD$ intersect at $E$. The perpendicular from $E$ to the principal axis falls at $P$. The equivalent lens should be placed at this position $P$. A ray $ABE$ going parallel to the principal axis will go through the equivalent lens and emerge along $ECD$. The angle of deviation is $\delta = \delta_1 + \delta_2$ from triangle $BEC$. The focal length of the equivalent lens is $F = PD$.

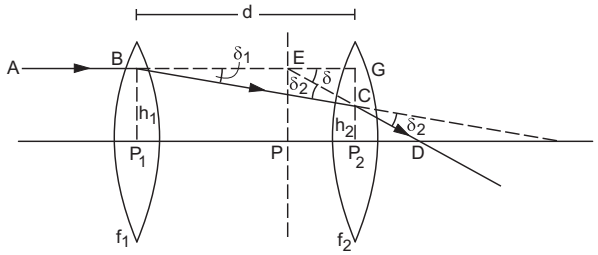



Figure 18.26

Using equation (18.21),

$$\delta_1 = \frac{h_1}{f_1},\ \delta_2 = \frac{h_2}{f_2} \text{ and } \delta = \frac{h_1}{F}.$$

As $\delta = \delta_1 + \delta_2$,

$$\frac{h_1}{F}=\frac{h_1}{f_1}+\frac{h_2}{f_2}. \qquad \ldots \text{ (ii)}$$

Now,

$$h_1 - h_2 = P_2G - P_2C = CG$$
$$= BG\tan\delta_1 \approx BG\,\delta_1$$

or,
$$h_1 - h_2 = d\frac{h_1}{f_1} \qquad \ldots \text{ (iii)}$$

or,
$$h_2 = h_1 - d\frac{h_1}{f_1}. \qquad \ldots \text{ (iv)}$$

Thus, by (ii),

$$\frac{h_1}{F}=\frac{h_1}{f_1}+\frac{h_1}{f_2}-\frac{d\,(h_1/f_1)}{f_2}.$$

or,
$$\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}-\frac{d}{f_1f_2}. \qquad \ldots \text{ (18.22)}$$

**Position of the Equivalent Lens**

We have,
$$PP_2 = EG$$
$$= GC\cot\delta$$
$$= \frac{h_1 - h_2}{\tan\delta} \approx \frac{h_1 - h_2}{\delta}$$

By (iii), $h_1 - h_2 = \dfrac{d\,h_1}{f_1}\cdot$ Also, $\delta = \dfrac{h_1}{F}$ so that

$$PP_2 = \left(\frac{d\,h_1}{f_1}\right)\left(\frac{F}{h_1}\right) = \frac{d\,F}{f_1}\cdot \qquad \ldots \quad (18.23)$$

Thus, the equivalent lens is to be placed at a distance $d\,F/f_1$ behind the second lens.

Equation (18.22) and (18.23) are true only for the special case of parallel incident beam. If the object is at a finite distance, one should not use the above equations. The image position should be worked out using the lens equations for the two lenses separately.

## 18.17 DEFECTS OF IMAGES

The simple theory of image formation developed for mirrors and lenses suffers from various approximations. As a result, the actual images formed contain several defects. These defects can be broadly divided in two categories, (a) *chromatic aberration* and (b) *monochromatic aberration*. The index of refraction of a transparent medium differs for different wavelengths of the light used. The defects arising from such a variation of the refractive index are termed as chromatic aberrations. The other defects, which arise even if light of a single colour is used, are called monochromatic aberrations. We shall first discuss this type of defects.

### A. Monochromatic Aberrations

#### (a) Spherical Aberration

All through the discussion of lenses and mirrors with spherical surfaces, it has been assumed that the aperture of the lens or the mirror is small and the light rays of interest make small angles with the principal axis. It is then possible to have a point image of a point object. However, this is only an approximation even if we neglect diffraction.

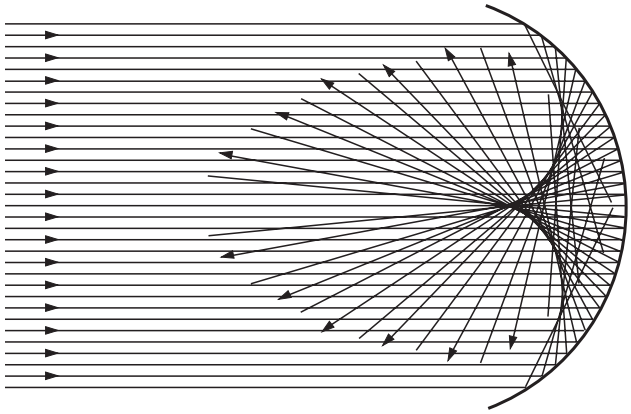


Figure 18.27

The rays reflect or refract from points at different distances from the principal axis. In general, they meet each other at different points. Thus, the image of a point object is a blurred surface. Such a defect is called *spherical aberration.* Figure (18.27) shows *spherical aberration* for a concave mirror for an object at infinity. The rays parallel to the principal axis are incident on the spherical surface of the concave mirror. The rays close to the principal axis (paraxial rays) are focused at the geometrical focus $F$ of the mirror as given by the mirror formula. The rays farthest from the principal axis are called the marginal rays and are focused at a point $F'$ somewhat closer to the mirror. The intermediate rays focus at different points between $F$ and $F'$. Also, the rays reflected from a small portion away from the pole meet at a point off the axis. Thus, a three-dimensional blurred image is formed. The intersection of this image with the plane of figure is shown blackened in figure (18.27) and is called the *caustic curve.* If a screen is placed perpendicular to the principal axis, a disc image is formed on the screen. As the screen is moved parallel to itself, the disc becomes smallest at one position. This disc is closest to the ideal image and its periphery is called the *circle of least confusion*. The magnitude of spherical aberration may be measured from the distance $FF'$ between the point where the paraxial rays converge and the point where the marginal rays converge.

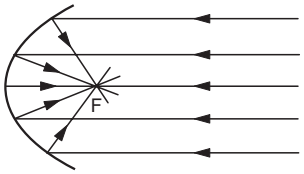



Figure 18.28

The parallel rays may be brought to focus at one point if a parabolic mirror is used. Also, if a point source is placed at the focus of a parabolic mirror, the reflected rays will be very nearly parallel. The reflectors in automobile headlights are made parabolic and the bulb is placed at the focus. The light beam is then nearly parallel and goes up to large distances.

Because of the finite aperture, a lens too produces a blurred disc type image of a point object. Figure (18.29) shows the situation for a convex and a concave

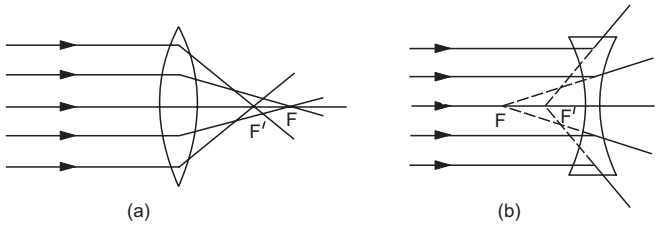



Figure 18.29

lens for the rays coming parallel to the principal axis. We see from the figure that the marginal rays are deviated a bit too strongly and hence, they meet at a point different from that given by geometrical optics formulae. Also, in the situation shown, the spherical aberration is opposite for convex and concave lens. The point $F'$, where the marginal rays meet, is to the left of the focus for convex lens and is to the right of the focus for the concave lens.

The magnitude of spherical aberration for a lens depends on the radii of curvature and the object distance. The spherical aberration for a particular object distance can be reduced by properly choosing the radii of curvature. However, it cannot be reduced to zero for a single lens which forms a real image of a real object. A simple method to reduce spherical aberration is to use a stop before and in front of the lens. A stop is an opaque sheet with a small circular opening in it. It only allows a narrow pencil of rays to go through the lens hence reducing the aberration. However, this method reduces the intensity of the image as most of the light is cut off.

The spherical aberration is less if the total deviation of the rays is distributed over the two surfaces of the lens. A striking example is a planoconvex lens forming the image of a distant object. If the plane surface faces the incident rays, the spherical aberration is much larger than that in the case when the curved surface faces the incident rays (figure 18.30). In the former case, the total deviation occurs at a single surface whereas it is distributed at both the surfaces in the latter case.

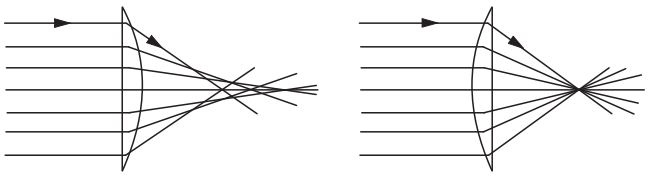


Figure 18.30

The spherical aberration can also be reduced by using a combination of convex and concave lenses. A suitable combination can reduce the spherical aberration by compensation of positive and negative aberrations.

**(b) Coma**

We have seen that if a point object is placed on the principal axis of a lens and the image is received on a screen perpendicular to the principal axis, the image has a shape of a disc because of spherical aberration. The basic reason of this aberration is that the rays passing through different regions of the lens meet the principal axis at different points. If the point object is placed away from the principal axis and the image is received on a screen perpendicular to the axis, the shape of the image is like a comet. This defect is called *coma*. The basic reason is again the same. The lens fails to converge all the rays passing at different distances from the axis at a single point. Figure (18.31) explains the formation of coma. The paraxial rays form an image of $P$ at $P'$. The rays passing through the shaded zone forms a circular image on the screen. The rays through outer zones of the lens form bigger circles with centres shifted. The image seen on the screen thus have a comet-like appearance.

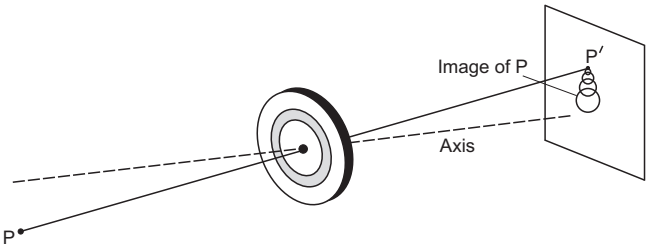



Figure 18.31

Coma can also be reduced by properly designing radii of curvature of the lens surfaces. It can also be reduced by appropriate stops placed at appropriate distances from the lens.

**(c) Astigmatism**

Spherical aberration and coma refer to the spreading of the image of a point object in a plane perpendicular to the principal axis. The image is also spread along the principal axis. Suppose, a point object is placed at a point off the axis of a converging lens. A screen is placed perpendicular to the axis and is moved along the axis. At a certain distance, an approximate line image is focussed. If the screen is moved further away, the shape of the image changes but it remains on the screen for quite a distance moved by the screen. The spreading of image along the principal axis is known as *astigmatism* (not to be confused with a defect of vision having the same name).

**(d) Curvature**

We have so far considered the image formed by a lens on a plane. However, it is not always true that the best image is formed along a plane. For a point object placed off the axis, the image is spread both along and perpendicular to the principal axis. The best image is, in general, obtained not on a plane but on a curved surface. This defect is known as *curvature*. It is intrinsically related to astigmatism. The astigmation or the curvature may be reduced by using proper stops placed at proper locations along the axis.

**(e) Distortion**

Distortion is the defect arising when extended objects are imaged. Different portions of the object are, in general, at different distances from the axis. The relation between the object distance and the image distance is not linear and hence, the magnification is not the same for all portions of the extended object. As a result, a line object is not imaged into a line but into a curve. Figure (18.32) shows some distorted images.

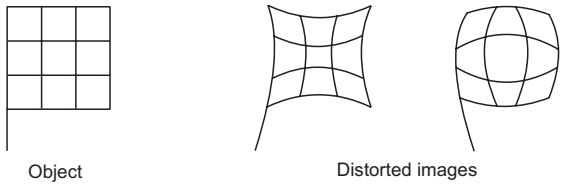



Figure 18.32

## B. Chromatic Aberrations

The refractive index of the material of a lens varies slightly with the wavelength and hence, the focal length is also different for different wavelengths. In the visible region, the focal length is maximum for red and minimum for violet. Thus, if white light is used, each colour forms a separate image of the object.

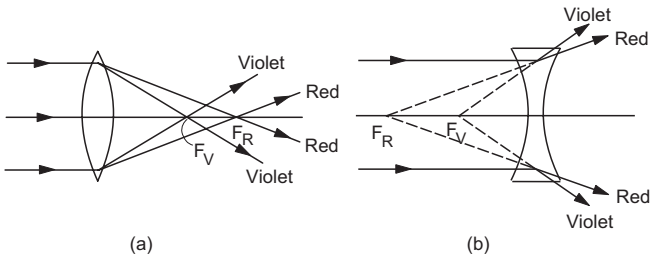



Figure 18.33

The violet rays are deviated more and hence, they form an image closer to the lens as compared to the image formed by the red rays. If light is incident on the lens from left to right, the violet image is to the left of the red image for convex lens and it is to the right of the red image for the concave lens. In the first case, the chromatic aberration is called positive and in the second case, it is negative. Thus, a proper combination of a convex and a concave lens may result in no chromatic aberration. Such a combination is called an *achromatic combination* for the pair of wavelengths. Also, the magnification $v/u$ depends on the focal length and hence, on the wavelength. For an extended object, the images formed by light of different colours are of different sizes. A typical situation is shown in figure (18.33). Monochromatic aberration are assumed to be absent.

The separation between the images formed by extreme wavelengths of the visible range is called the *axial chromatic aberration* or *longitudinal chromatic aberration*. The difference in the size of the images (perpendicular to the principal axis) formed by the extreme wavelengths of the range is called the *lateral chromatic aberration* (figure 18.34).

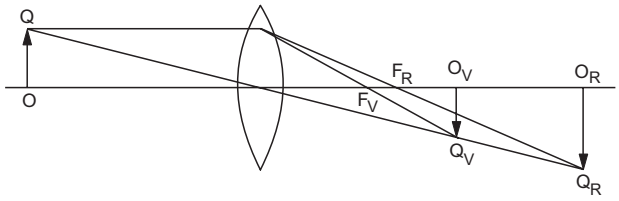



Figure 18.34

### *Worked Out Examples*

**1.** *An object is placed on the principal axis of a concave mirror of focal length* 10 cm *at a distance of* 8·0 cm *from the pole. Find the position and the nature of the image.*

***Solution*** **:** Here $u = -8{\cdot}0$ cm and $f = -10$ cm.

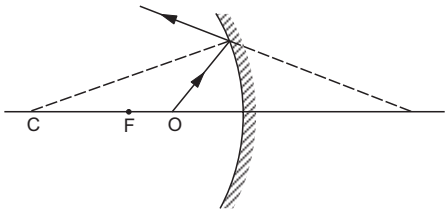



Figure 18-W1

We have, $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$

or, $\frac{1}{v} = \frac{1}{f} - \frac{1}{u}$

$$= \frac{1}{-10\text{ cm}} - \frac{1}{-8{\cdot}0\text{ cm}}$$

$$= \frac{1}{40\text{ cm}}$$

or, $v = 40$ cm.

The positive sign shows that the image is formed at 40 cm from the pole on the other side of the mirror (figure 18-W1). As the image is formed beyond the mirror, the reflected rays do not intersect, the image is thus virtual.

**2.** *A rod of length* 10 cm *lies along the principal axis of a concave mirror of focal length* 10 cm *in such a way that the end closer to the pole is* 20 cm *away from it. Find the length of the image.*

***Solution*** :

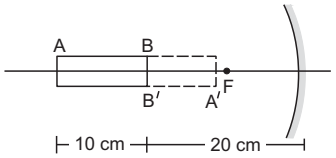



Figure 18-W2

The situation is shown in figure (18-W2). The radius of curvature of the mirror is $r = 2f = 20$ cm. Thus, the nearer end $B$ of the rod $AB$ is at the centre of the curvature and hence, its image will be formed at $B$ itself. We shall now locate the image of $A$.

Here $u = -30$ cm and $f = -10$ cm. We have

$$\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$$

or, $$\frac{1}{v}=\frac{1}{f}-\frac{1}{u}$$

$$=\frac{1}{-10\text{ cm}}-\frac{1}{-30\text{ cm}}$$

or, $$v = -15\text{ cm}.$$

Thus, the image of $A$ is formed at 15 cm from the pole. The length of the image is, therefore, 5·0 cm.

**3.** *At what distance from a convex mirror of focal length* 2·5 m *should a boy stand so that his image has a height equal to half the original height ? The principal axis is perpendicular to the height.*

***Solution*** : We have,

$$m = -\frac{v}{u}=\frac{1}{2}$$

or, $$v = -\frac{u}{2}.$$

Also, $$\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$$

or, $$\frac{1}{u}+\frac{1}{-u/2}=\frac{1}{2{\cdot}5\text{ m}}$$

or, $$-\frac{1}{u}=\frac{1}{2{\cdot}5\text{ m}}$$

or, $$u = -2{\cdot}5\text{ m}.$$

Thus, he should stand at a distance of 2·5 m from the mirror.

**4.** *A* 2·0 cm *high object is placed on the principal axis of a concave mirror at a distance of* 12 cm *from the pole. If the image is inverted, real and* 5·0 cm *high, find the location of the image and the focal length of the mirror.*

***Solution*** : The magnification is $m = -\frac{v}{u}$

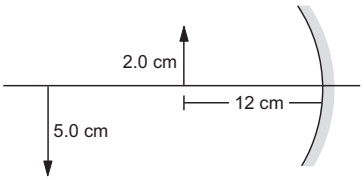



Figure 18-W3

or, $$\frac{-5{\cdot}0\text{ cm}}{2{\cdot}0\text{ cm}}=\frac{-v}{-12\text{ cm}}$$

or, $$v = -30\text{ cm}.$$

The image is formed at 30 cm from the pole on the side of the object. We have,

$$\frac{1}{f}=\frac{1}{v}+\frac{1}{u}$$

$$=\frac{1}{-30\text{ cm}}+\frac{1}{-12\text{ cm}}=-\frac{7}{60\text{ cm}}$$

or, $$f = -\frac{60\text{ cm}}{7}=-8{\cdot}6\text{ cm}.$$

The focal length of the mirror is 8·6 cm.

**5**. *Consider the situation shown in figure (18-W4). Find the maximum angle* θ *for which the light suffers total internal reflection at the vertical surface.*

***Solution*** :

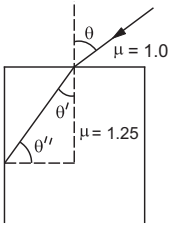



Figure 18-W4

The critical angle for this case is

$$\theta'' = \sin^{-1}\frac{1}{1{\cdot}25}=\sin^{-1}\frac{4}{5}$$

or, $$\sin\theta'' = \frac{4}{5}.$$

Since $\theta'' = \frac{\pi}{2}-\theta'$, we have $\sin\theta' = \cos\theta'' = 3/5$. From Snell's law,

$$\frac{\sin\theta}{\sin\theta'}=1{\cdot}25$$

or, $$\sin\theta = 1{\cdot}25\times\sin\theta'$$

$$=1{\cdot}25\times\frac{3}{5}=\frac{3}{4}$$

or, $\theta = \sin^{-1}\frac{3}{4}$.

If $\theta''$ is greater than the critical angle, $\theta$ will be smaller than this value. Thus, the maximum value of $\theta$, for which total reflection takes place at the vertical surface, is $\sin^{-1}(3/4)$.

**6.** *A right prism is to be made by selecting a proper material and the angles A and B ($B \le A$), as shown in figure (18-W5a). It is desired that a ray of light incident normally on AB emerges parallel to the incident direction after two total internal reflections. (a) What should be the minimum refractive index $\mu$ for this to be possible? (b) For $\mu = 5/3$, is it possible to achieve this with the angle A equal to* 60 degrees?

***Solution*** :

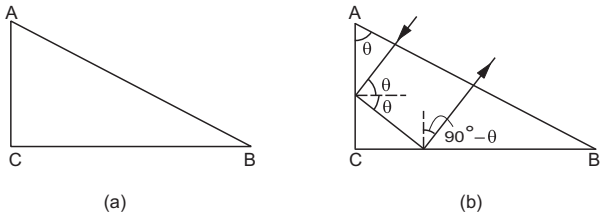



Figure 18-W5

(a) Consider the ray incident normally on *AB (figure 18-W5b)*. The angle of reflection at the surface *AC* is $\theta$. It is clear from the figure that the angle of incidence at the second surface *CB* is $90° - \theta$. The emergent ray will be parallel to the incident ray after two total internal reflections. The critical angle $\theta_c$ should be less than $\theta$ as well as $90° - \theta$. Thus, $\theta_c$ should be smaller than or equal to the smaller of $\theta$ and $90° - \theta$, i.e.,

$$\theta_c \le \min(\theta, 90° - \theta).$$

As $\min(\theta, 90° - \theta) \le 45°$, $\theta_c \le 45°$

or, $\sin\theta_c \le \frac{1}{\sqrt{2}}$ or, $\frac{1}{\mu} \le \frac{1}{\sqrt{2}}$

or, $\mu \ge \sqrt{2}$.

Thus, the refractive index of the material of the prism should be greater than or equal to $\sqrt{2}$. In this case the given ray can undergo two internal reflections for a suitable $\theta$.

(b) For $\mu = 5/3$, the critical angle $\theta_c$ is

$$\sin^{-1}(3/5) = 37°.$$

As the figure suggests, we consider the light incident normally on the face *AB*. The angle of incidence $\theta$ on the surface *AC* is equal to $\theta = 60°$. As this is larger than the critical angle 37°, total internal reflection takes place here. The angle of incidence at the surface *CB* is $90° - \theta = 30°$. As this is less than the critical angle, total internal reflection does not take place at this surface.

**7.** *A point object O is placed in front of a transparent slab at a distance x from its closer surface. It is seen from the other side of the slab by light incident nearly normally to the slab. The thickness of the slab is t and its refractive index is $\mu$. Show that the apparent shift in the position of the object is independent of x and find its value.*

***Solution*** :

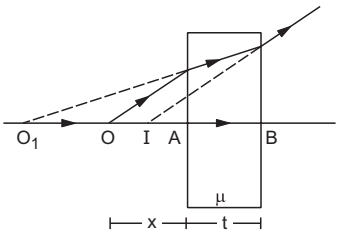



Figure 18-W6

The situation is shown in figure (18-W6). Because of the refraction at the first surface, the image of *O* is formed at $O_1$. For this refraction, the real depth is $AO = x$ and the apparent depth is $AO_1$. Also the first medium is air and the second is the slab. Thus,

$$\frac{x}{AO_1} = \frac{1}{\mu} \text{ or, } AO_1 = \mu x.$$

The point $O_1$ acts as the object for the refraction at the second surface. Due to this refraction the image of $O_1$ is formed at *I*. Thus,

$$\frac{BO_1}{BI} = \mu$$

or, $\frac{AB + AO_1}{BI} = \mu$ or, $\frac{t + \mu x}{BI} = \mu$

or, $BI = x + \frac{t}{\mu}$.

The net shift in $OI = OB - BI = (x + t) - \left(x + \frac{t}{\mu}\right)$

$$= t\left(1 - \frac{1}{\mu}\right),$$

which is independent of *x*.

**8.** *Consider the situation shown in figure (18-W7). A plane mirror is fixed at a height h above the bottom of a beaker containing water (refractive index $\mu$) up to a height d. Find the position of the image of the bottom formed by the mirror.*

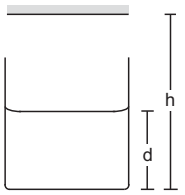



Figure 18-W7

***Solution*** : The bottom of the beaker appears to be shifted up by a distance

$$\Delta t = \left(1 - \frac{1}{\mu}\right) d.$$

Thus, the apparent distance of the bottom from the mirror is $h - \Delta t = h - \left(1 - \frac{1}{\mu}\right) d = h - d + \frac{d}{\mu}$. The image is formed behind the mirror at a distance $h - d + \frac{d}{\mu}$.

**9.** *A beaker contains water up to a height $h_1$ and K.oil above water up to another height $h_2$. Find the apparent shift in the position of the bottom of the beaker when viewed from above. Refractive index of water is $\mu_1$ and that of K.oil is $\mu_2$.*

***Solution*** **:** The apparent shift of the bottom due to the water is

$$\Delta t_1 = \left(1 - \frac{1}{\mu_1}\right) h_1$$

and due to the K.oil is

$$\Delta t_2 = \left(1 - \frac{1}{\mu_2}\right) h_2.$$

The total shift $= \Delta t_1 + \Delta t_2 = \left(1 - \frac{1}{\mu_1}\right) h_1 + \left(1 - \frac{1}{\mu_2}\right) h_2.$

**10.** *Monochromatic light is incident on the plane interface AB between two media of refractive indices $\mu_1$ and $\mu_2$ ($\mu_2 > \mu_1$) at an angle of incidence $\theta$ as shown in figure (18-W8). The angle $\theta$ is infinitesimally greater than the critical angle for the two media so that total internal reflection takes place. Now, if a transparent slab DEFG of uniform thickness and of refractive index $\mu_3$ is introduced on the interface (as shown in the figure), show that for any value of $\mu_3$ all light will ultimately be reflected back into medium II.*

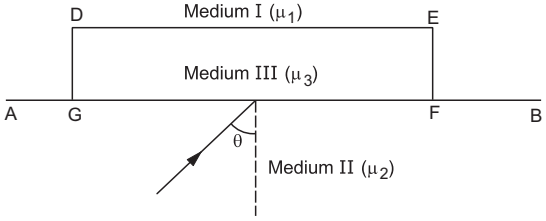



Figure 18-W8

***Solution*** **:** We shall use the symbol $\gtrsim$ to mean "infinitesimally greater than".

When the slab is not inserted, $\theta \gtrsim \theta_c = \sin^{-1}(\mu_1/\mu_2)$

or, $\sin\theta \gtrsim \mu_1/\mu_2.$

When the slab is inserted, we have two cases $\mu_3 \le \mu_1$ and $\mu_3 > \mu_1$.

**Case I** **:** $\mu_3 \le \mu_1$

We have $\sin\theta \gtrsim \mu_1/\mu_2 \ge \mu_3/\mu_2$.

Thus, the light is incident on *AB* at an angle greater than the critical angle $\sin^{-1}(\mu_3/\mu_2)$. It suffers total internal reflection and goes back to medium II.

**Case II** **:** $\mu_3 > \mu_1$

$\sin\theta \gtrsim \mu_1/\mu_2 < \mu_3/\mu_2$

Thus, the angle of incidence $\theta$ may be smaller than the critical angle $\sin^{-1}(\mu_3/\mu_2)$ and hence it may enter medium III. The angle of refraction $\theta'$ is given by (figure 18-W9)

$$\frac{\sin\theta}{\sin\theta'} = \frac{\mu_3}{\mu_2} \quad \ldots \text{(i)}$$

or, $$\sin\theta' = \frac{\mu_2}{\mu_3}\sin\theta$$

$$\gtrsim \frac{\mu_2}{\mu_3} \cdot \frac{\mu_1}{\mu_2}.$$

Thus, $$\sin\theta' \gtrsim \frac{\mu_1}{\mu_3}$$

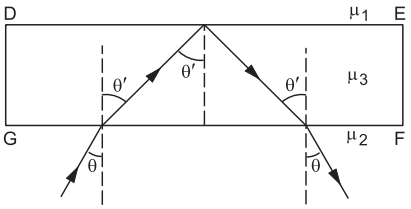



Figure 18-W9

or, $$\theta' \gtrsim \sin^{-1}\left(\frac{\mu_1}{\mu_3}\right). \quad \ldots \text{(ii)}$$

As the slab has parallel faces, the angle of refraction at the face *FG* is equal to the angle of incidence at the face *DE*. Equation (ii) shows that this angle is infinitesimally greater than the critical angle here. Hence, the light suffers total internal reflection and falls at the surface *FG* at an angle of incidence $\theta'$. At this face, it will refract into medium II and the angle of refraction will be $\theta$ as shown by equation (i). Thus, the total light energy is ultimately reflected back into medium II.

**11.** *A concave mirror of radius* 40 cm *lies on a horizontal table and water is filled in it up to a height of* 5·00 cm *(figure 18-W10). A small dust particle floats on the water surface at a point P vertically above the point of contact of the mirror with the table. Locate the image of the dust particle as seen from a point directly above it. The refractive index of water is* 1·33.

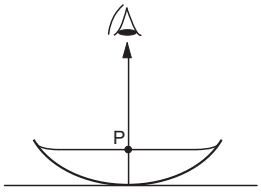



Figure 18-W10

***Solution*** :

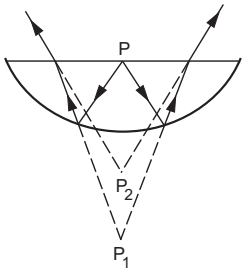



Figure 18-W11

The ray diagram is shown in figure (18-W11). Let us first locate the image formed by the concave mirror. Let us take vertically upward as the negative axis. Then $R = -40$ cm. The object distance is $u = -5$ cm. Using the mirror equation,

$$\frac{1}{u}+\frac{1}{v}=\frac{2}{R}$$

or, $$\frac{1}{v}=\frac{2}{R}-\frac{1}{u}=\frac{2}{-40\text{ cm}}-\frac{1}{-5\text{ cm}}=\frac{6}{40\text{ cm}}$$

or, $$v = 6{\cdot}67 \text{ cm}.$$

The positive sign shows that the image $P_1$ is formed below the mirror and hence, it is virtual. These reflected rays are refracted at the water surface and go to the observer. The depth of the point $P_1$ from the surface is 6·67 cm + 5·00 cm = 11·67 cm. Due to refraction at the water surface, the image $P_1$ will be shifted above by a distance

$$(11{\cdot}67\text{ cm})\left(1-\frac{1}{1{\cdot}33}\right)=2{\cdot}92\text{ cm}.$$

Thus, the final image is formed at a point (11·67 – 2·92) cm = 8·75 cm below the water surface.

**12.** *An object is placed* 21 cm *in front of a concave mirror of radius of curvature* 20 cm. *A glass slab of thickness* 3 cm *and refractive index* 1·5 *is placed close to the mirror in the space between the object and the mirror. Find the position of the final image formed. The distance of the nearer surface of the slab from the mirror is* 10 cm.

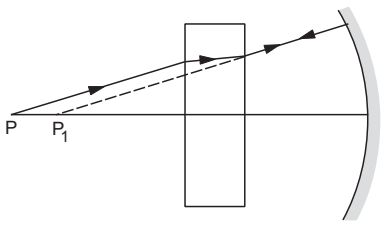



Figure 18-W12

***Solution*** :

The situation is shown in figure (18-W12). Because of the refraction at the two surfaces of the slab, the image of the object $P$ is formed at $P_1$, shifted towards the mirror by a distance

$$t\left(1-\frac{1}{\mu}\right)=(3\text{ cm})\left(1-\frac{1}{1{\cdot}5}\right)=1\text{ cm}.$$

Thus, the rays falling on the concave mirror are diverging from $P_1$ which is at 21 cm – 1 cm = 20 cm from the mirror. But the radius of curvature is also 20 cm, hence $P_1$ is at the centre. The rays, therefore, fall normally on the mirror and hence, retrace their path. The final image is formed at $P$ itself.

**13.** *The refractive indices of silicate flint glass for wavelengths* 400 nm *and* 700 nm *are* 1·66 *and* 1·61 *respectively. Find the minimum angles of deviation of an equilateral prism made of this glass for light of wavelength* 400 nm *and* 700 nm.

***Solution*** : The minimum angle of deviation $\delta_m$ is given by

$$\mu=\frac{\sin\frac{A+\delta_m}{2}}{\sin\frac{A}{2}}=\frac{\sin\left(30^\circ+\frac{\delta_m}{2}\right)}{\sin 30^\circ}$$

$$=2\sin\left(30^\circ+\frac{\delta_m}{2}\right).$$

For 400 nm light,

$$1{\cdot}66 = 2\sin(30^\circ+\delta_m/2)$$

or, $$\sin(30^\circ+\delta_m/2)=0{\cdot}83$$

or, $$(30^\circ+\delta_m/2)=56^\circ$$

or, $$\delta_m = 52^\circ.$$

For 700 nm light,

$$1{\cdot}61 = 2\sin(30^\circ+\delta_m/2).$$

This gives $\delta_m = 48^\circ$.

**14.** *Consider the situation shown in figure (18-W13). Light from a point source S is made parallel by a convex lens L. The beam travels horizontally and falls on an* 88°-88°-4° *prism (refractive index 1.5) as shown in the figure. It passes through the prism symmetrically. The transmitted light falls on a vertical mirror. Through what angle should the mirror be rotated so that an image of S is formed on S itself* ?

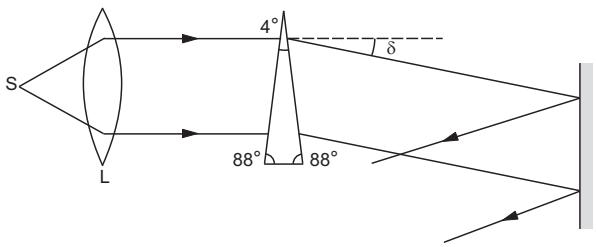



Figure 18-W13

***Solution*** **:** The parallel beam after going through the prism will be deviated by an angle δ. If the mirror is also rotated by this angle δ, the rays will fall normally on it. The rays will be reflected back along the same path and form the image of $S$ on itself.

As the prism is thin, the angle δ is given by

$$\delta = (\mu - 1)\, A$$

$$= (1{\cdot}5 - 1) \times 4^\circ = 2^\circ.$$

Thus, the mirror should be rotated by 2°.

**15.** *Locate the image formed by refraction in the situation shown in figure (18-W14). The point C is the centre of curvature.*

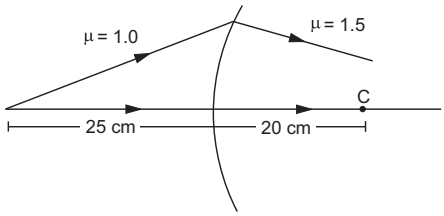



Figure 18-W14

***Solution*** **:** We have,

$$\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}. \quad \ldots \text{(i)}$$

Here $u = -25$ cm, $R = 20$ cm, $\mu_1 = 1{\cdot}0$ and $\mu_2 = 1{\cdot}5$.

Putting the values in (i),

$$\frac{1{\cdot}5}{v} + \frac{1{\cdot}0}{25\text{ cm}} = \frac{1{\cdot}5 - 1{\cdot}0}{20\text{ cm}}$$

or, $$\frac{1{\cdot}5}{v} = \frac{1}{40\text{ cm}} - \frac{1}{25\text{ cm}}$$

or, $$v = -100\text{ cm}.$$

As $v$ is negative, the image is formed to the left of the separating surface at a distance of 100 cm from it.

**16.** *One end of a horizontal cylindrical glass rod* ($\mu = 1{\cdot}5$) *of radius* 5·0 cm *is rounded in the shape of a hemisphere. An object* 0·5 mm *high is placed perpendicular to the axis of the rod at a distance* of 20·0 cm *from the rounded edge. Locate the image of the object and find its height.*

***Solution*** **:** Taking the origin at the vertex, $u = -20{\cdot}0$ cm *and* $R = 5{\cdot}0$ cm.

We have,

$$\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$$

or, $$\frac{1{\cdot}5}{v} = \frac{1}{-20{\cdot}0\text{ cm}} + \frac{0{\cdot}5}{5{\cdot}0\text{ cm}} = \frac{1}{20\text{ cm}}$$

or, $$v = 30\text{ cm}.$$

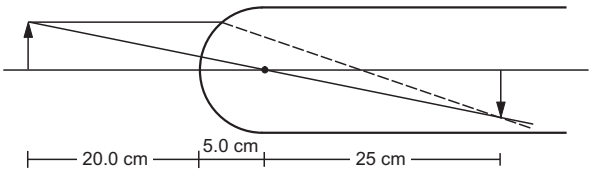



Figure 18-W15

The image is formed inside the rod at a distance of 30 cm from the vertex.

The magnification is $m = \dfrac{\mu_1 v}{\mu_2 u}$

$$= \frac{30\text{ cm}}{-1{\cdot}5 \times 20\text{ cm}} = -1.$$

Thus, the image will be of same height (0·5 mm) as the object but it will be inverted.

**17.** *There is a small air bubble inside a glass sphere* ($\mu = 1{\cdot}5$) *of radius* 10 cm. *The bubble is* 4·0 cm *below the surface and is viewed normally from the outside (figure 18-W16). Find the apparent depth of the bubble.*

***Solution*** **:**

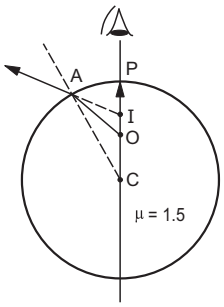



Figure 18-W16

The observer sees the image formed due to refraction at the spherical surface when the light from the bubble goes from the glass to the air.

Here $u = -4{\cdot}0$ cm, $R = -10$ cm, $\mu_1 = 1{\cdot}5$ and $\mu_2 = 1$.

We have,

$$\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$$

or, $$\frac{1}{v} - \frac{1{\cdot}5}{-4{\cdot}0\text{ cm}} = \frac{1 - 1{\cdot}5}{-10\text{ cm}}$$

or, $$\frac{1}{v} = \frac{0{\cdot}5}{10\text{ cm}} - \frac{1{\cdot}5}{4{\cdot}0\text{ cm}}$$

or, $$v = -3{\cdot}0\text{ cm}.$$

Thus, the bubble will apear 3·0 cm below the surface.

**18.** *A parallel beam of light travelling in water (refractive index = 4/3) is refracted by a spherical air bubble of radius* 2 mm *situated in water. Assuming the light rays to be paraxial, (i) find the position of the image due to*

*refraction at the first surface and the position of the final image, and (ii) draw a ray diagram showing the positions of both the images.*

***Solution :***

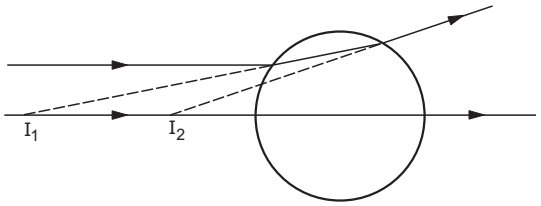



Figure 18-W17

The ray diagram is shown in figure 18-W17. The equation for refraction at a spherical surface is

$$\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}. \quad \ldots \text{(i)}$$

For the first refraction (water to air); $\mu_1 = 1{\cdot}33$, $\mu_2 = 1$, $u = \infty$, $R = +2$ mm.

Thus, $$\frac{1}{v} = \frac{1 - 1{\cdot}33}{2 \text{ mm}}$$

or, $$v = -6 \text{ mm}.$$

The negative sign shows that the image $I_1$ is virtual and forms at 6 mm from the surface of the bubble on the water side. The refracted rays (which seem to come from $I_1$) are incident on the farther surface of the bubble. For this refraction,

$$\mu_1 = 1, \mu_2 = 1{\cdot}33, R = -2 \text{ mm}.$$

The object distance is $u = -(6 \text{ mm} + 4 \text{ mm}) = -10$ mm. Using equation (i),

$$\frac{1{\cdot}33}{v} + \frac{1}{10 \text{ mm}} = \frac{1{\cdot}33 - 1}{-2 \text{ mm}}$$

or, $$\frac{1{\cdot}33}{v} = -\frac{0{\cdot}33}{2 \text{ mm}} - \frac{1}{10 \text{ mm}}$$

or, $$v = -5 \text{ mm}.$$

The minus sign shows that the image is formed on the air side at 5 mm from the refracting surface.

Measuring from the centre of the bubble, the first image is formed at 8·0 mm from the centre and the second image is formed at 3·0 mm from the centre. Both images are formed on the side from which the incident rays are coming.

**19.** *Calculate the focal length of the thin lens shown in figure (18-W18). The points $C_1$ and $C_2$ denote the centres of curvature and the refractive index is 1.5.*

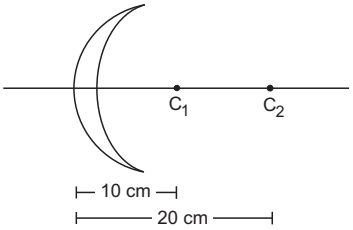



Figure 18-W18

***Solution :*** As is clear from the figure, both the radii of curvature are positive. Thus, $R_1 = +10$ cm and $R_2 = +20$ cm. The focal length is given by

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$= (1{\cdot}5 - 1)\left(\frac{1}{10 \text{ cm}} - \frac{1}{20 \text{ cm}}\right)$$

$$= 0{\cdot}5 \times \frac{1}{20 \text{ cm}} = \frac{1}{40 \text{ cm}}$$

or, $$f = 40 \text{ cm}.$$

**20.** *A point source S is placed at a distance of* 15 cm *from a converging lens of focal length* 10 cm *on its principal axis. Where should a diverging mirror of focal length* 12 cm *be placed so that a real image is formed on the source itself*?

***Solution :***

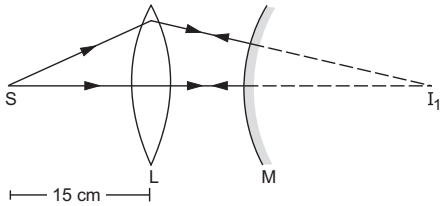



Figure 18-W19

The equation for the lens is

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}. \quad \ldots \text{(i)}$$

Here $u = -15$ cm and $f = +10$ cm.

Using equation (i),

$$\frac{1}{v} + \frac{1}{15 \text{ cm}} = \frac{1}{10 \text{ cm}}$$

$$\frac{1}{v} = \frac{1}{10 \text{ cm}} - \frac{1}{15 \text{ cm}} = \frac{1}{30 \text{ cm}}$$

or, $$v = 30 \text{ cm}.$$

The positive sign of $v$ shows that the image $I_1$ is formed to the right of the lens in the figure. The diverging mirror is to be placed to the right in such a way that the light rays fall on the mirror perpendicularly. Then only the rays will retrace their path and form the final image on the object. Thus, the image $I_1$ formed by the lens should be at the centre of curvature of the mirror.

We have, $LI_1 = 30$ cm,

$$MI_1 = R = 2f = 24 \text{ cm},$$

Hence, $LM = LI_1 - MI_1 = 6$ cm.

Thus, the mirror should be placed 6 cm to the right of the lens.

**21.** *A converging lens of focal length* 15 cm *and a converging mirror of focal length* 20 cm *are placed with their principal axes coinciding. A point source S is placed on the principal axis at a distance of* 12 cm *from the lens as shown in figure (18-W20). It is found that the final beam comes out parallel to the principal axis. Find the separation between the mirror and the lens.*

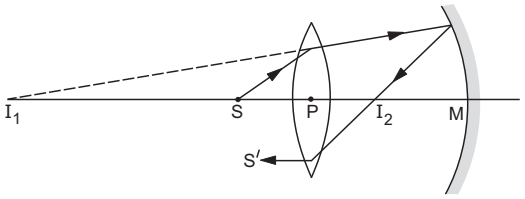



Figure 18-W20

***Solution*** **:** Let us first locate the image of $S$ formed by the lens. Here $u = -12$ cm and $f = 15$ cm. We have,

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

or,
$$\frac{1}{v} = \frac{1}{f} + \frac{1}{u}$$

$$= \frac{1}{15 \text{ cm}} - \frac{1}{12 \text{ cm}}$$

or,
$$v = -60 \text{ cm}.$$

The negative sign shows that the image is formed to the left of the lens as suggested in the figure. The image $I_1$ acts as the source for the mirror. The mirror forms an image $I_2$ of the source $I_1$. This image $I_2$ then acts as the source for the lens and the final beam comes out parallel to the principal axis. Clearly $I_2$ must be at the focus of the lens. We have,

$$I_1I_2 = I_1P + PI_2 = 60 \text{ cm} + 15 \text{ cm} = 75 \text{ cm}.$$

Suppose the distance of the mirror from $I_2$ is $x$ cm. For the reflection from the mirror,

$u = MI_1 = -(75 + x)$ cm, $v = -x$ cm and $f = -20$ cm.

Using
$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f},$$

$$\frac{1}{x} + \frac{1}{75 + x} = \frac{1}{20}$$

or,
$$\frac{75 + 2x}{(75 + x)\,x} = \frac{1}{20}$$

or,
$$x^2 + 35x - 1500 = 0$$

or,
$$x = \frac{-35 \pm \sqrt{35 \times 35 + 4 \times 1500}}{2}.$$

This gives $x = 25$ or $-60$.

As the negative sign has no physical meaning, only positive sign should be taken. Taking $x = 25$, the separation between the lens and the mirror is $(15 + 25)$ cm = 40 cm.

**22.** *A biconvex thin lens is prepared from glass* ($\mu = 1{\cdot}5$), *the two bounding surfaces having equal radii* of 25 cm *each. One of the surfaces is silvered from outside to make it reflecting. Where should an object be placed before this lens so that the image is formed on the object itself* ?

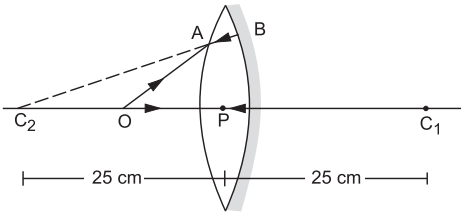



Figure 18-W21

***Solution*** **:**

Refer to figure (18-W21). The object is placed at $O$. A ray $OA$ starting from $O$ gets refracted into the glass at the first surface and hits the silvered surface along $AB$. To get the image at the object, the rays should retrace their path after reflection from the silvered surface. This will happen only if $AB$ falls normally on the silvered surface. Thus, $AB$ should appear to come from the centre $C_2$ of the second surface. Thus, due to the refraction at the first surface, a virtual image of $O$ is formed at $C_2$. For this case,

$v = PC_2 = -25$ cm, $R = PC_1 = +25$ cm, $\mu_1 = 1$, $\mu_2 = 1{\cdot}5$.

We have,

$$\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$$

or,
$$\frac{1{\cdot}5}{-25 \text{ cm}} - \frac{1}{u} = \frac{1{\cdot}5 - 1}{25 \text{ cm}}$$

or,
$$\frac{1}{u} = -\frac{1{\cdot}5}{25 \text{ cm}} - \frac{0{\cdot}5}{25 \text{ cm}}$$

or,
$$u = -12{\cdot}5 \text{ cm}.$$

Thus, the object should be placed at a distance of 12·5 cm from the lens.

**23.** *A concavo-convex (figure 18-W22) lens made of glass* ($\mu = 1{\cdot}5$) *has surfaces of radii* 20 cm *and* 60 cm. *(a) Locate the image of an object placed* 80 cm *to the left of the lens along the principal axis. (b) A similar lens is placed coaxially at a distance* of 160 cm *right to it. Locate the position of the image.*

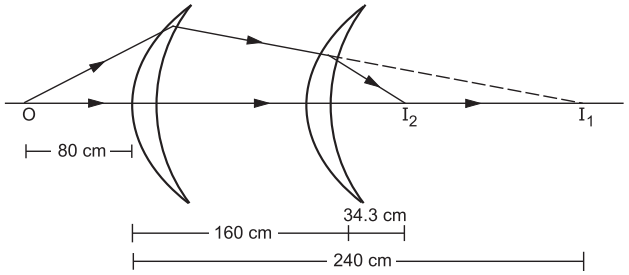



Figure 18-W22

***Solution :*** The focal length of the lens is given by

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$= (1{\cdot}5 - 1)\left(\frac{1}{20\text{ cm}} - \frac{1}{60\text{ cm}}\right) = \frac{1}{60\text{ cm}}$$

or, $f = 60$ cm.

(a) For the image formed by the first lens, $u = -80$ cm so that

$$\frac{1}{v} = \frac{1}{u} + \frac{1}{f}$$

$$= \frac{1}{-80\text{ cm}} + \frac{1}{60\text{ cm}} = \frac{1}{240\text{ cm}}$$

or, $v = 240$ cm.

The first image $I_1$ would form 240 cm to the right of the first lens.

(b) The second lens intercepts the converging beam as suggested by the figure. The image $I_1$ acts as a virtual source for the second lens. For the image formed by this lens, $u = 240\text{ cm} - 160\text{ cm} = +80$ cm so that

$$\frac{1}{v} = \frac{1}{u} + \frac{1}{f}$$

$$= \frac{1}{80\text{ cm}} + \frac{1}{60\text{ cm}} = \frac{7}{240\text{ cm}}$$

or, $v = 34{\cdot}3$ cm.

The final image is formed 34·3 cm to the right of the second lens.

**24.** *A thin lens* ($\mu = 1.5$) *of focal length* + 12 cm *is immersed in water* ($\mu = 1{\cdot}33$). *What is its new focal length* ?

***Solution :*** We have, $\frac{1}{f} = \left(\frac{\mu_2}{\mu_1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$.

When the lens is placed in air, $f = 12$ cm. Thus,

$$\frac{1}{12\text{ cm}} = (1{\cdot}5 - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

or, $$\frac{1}{R_1} - \frac{1}{R_2} = \frac{1}{6\text{ cm}}.$$

If the focal length becomes $f'$ when placed in water,

$$\frac{1}{f'} = \left(\frac{1{\cdot}5}{1{\cdot}33} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$= \frac{1}{8} \times \frac{1}{6\text{ cm}} = \frac{1}{48\text{ cm}} \quad \text{or, } f' = 48\text{ cm}.$$

**25.** *A long cylindrical tube containing water is closed by an equiconvex lens of focal length* 10 cm *in air. A point source is placed along the axis of the tube outside it at a distance* of 21 cm *from the lens. Locate the final image of the source. Refractive index of the material of the lens* = 1·5 *and that of water* = 1·33.

***Solution :***

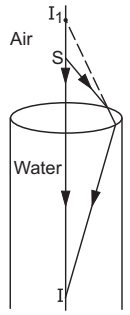



Figure 18-W23

The light from the source $S$ gets refracted at the air–glass interface and then at the glass–water interface. Referring to the figure (18-W23), let us take vertically downward as the positive direction of the axis.

If the image due to the refraction at the first surface is formed at a distance $v_1$ from the surface, we have,

$$\frac{1{\cdot}5}{v_1} - \frac{1}{u} = \frac{1{\cdot}5 - 1}{R}, \qquad \ldots \text{ (i)}$$

where $R$ is the radius of curvature of the surface. As the lens is equiconvex, the radius of curvature of the second surface is $-R$. Also, the image formed by the first surface acts as the object for the second surface. Thus,

$$\frac{1{\cdot}33}{v} - \frac{1{\cdot}5}{v_1} = \frac{1{\cdot}33 - 1{\cdot}5}{-R}. \qquad \ldots \text{ (ii)}$$

Adding (i) and (ii),

$$\frac{1{\cdot}33}{v} - \frac{1}{u} = \frac{1}{R}(0{\cdot}5 + 0{\cdot}17) = \frac{0{\cdot}67}{R}$$

or, $$\frac{1{\cdot}33}{v} - \frac{1}{-21\text{ cm}} = \frac{0{\cdot}67}{R}$$

or, $$\frac{4}{3v} + \frac{1}{21\text{ cm}} = \frac{2}{3R}$$

or, $$\frac{1}{v} = \frac{1}{2R} - \frac{1}{28\text{ cm}} \qquad \ldots \text{ (iii)}$$

The focal length of the lens in air is 10 cm. Using

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right),$$

$$\frac{1}{10\text{ cm}} = (1{\cdot}5 - 1)\left(\frac{1}{R} + \frac{1}{R}\right)$$

or, $R = 10$ cm.

Thus, by (iii),

$$\frac{1}{v} = \frac{1}{20\text{ cm}} - \frac{1}{28\text{ cm}}$$

or, $v = 70$ cm.

The image is formed 70 cm inside the tube.

**26.** *A slide projector produces* 500 times *enlarged image of a slide on a screen* 10 m *away. Assume that the projector consists of a single convex lens used for magnification. If the screen is moved* 2·0 m *closer, by what distance should the slide be moved towards or away from the lens so that the image remains focussed on the screen* ?

***Solution*** : In the first case, $v = 10$ m and $\frac{v}{u} = -500$.

Thus, $u = -\frac{v}{500} = -\frac{1}{50}\text{ m} = -2{\cdot}0\text{ cm}$. The focal length $f$ is given by

$$\frac{1}{f} = \frac{1}{v} - \frac{1}{u} = \frac{1}{10\text{ m}} + \frac{1}{2{\cdot}0\text{ cm}}.$$

If the screen is moved 2·0 m closer, $v = 8{\cdot}0$ m. The object distance $u'$ is given by

$$\frac{1}{v} - \frac{1}{u'} = \frac{1}{f}$$

or, $$\frac{1}{u'} = \frac{1}{v} - \frac{1}{f} = \frac{1}{8{\cdot}0\text{ m}} - \frac{1}{10\text{ m}} - \frac{1}{2{\cdot}0\text{ cm}}$$

$$= \frac{1}{40\text{ m}} - \frac{1}{2{\cdot}0\text{ cm}}$$

$$= -\frac{1}{2{\cdot}0\text{ cm}}\left(1 - \frac{1}{2000}\right)$$

or, $$u' = -2{\cdot}0\text{ cm}\left(1 - \frac{1}{2000}\right)^{-1}$$

$$\approx -2{\cdot}0\text{ cm}\left(1 + \frac{1}{2000}\right) = -2{\cdot}0\text{ cm} - \frac{1}{1000}\text{ cm}.$$

Thus, the slide should be taken $\frac{1}{1000}$ cm away from the lens.

**27.** *A convex lens focusses an object* 10 cm *from it on a screen placed* 10 cm *away from it. A glass plate* ($\mu = 1{\cdot}5$) *of thickness* 1·5 cm *is inserted between the lens and the screen. Where should the object be placed so that its image is again focussed on the screen* ?

***Solution*** :

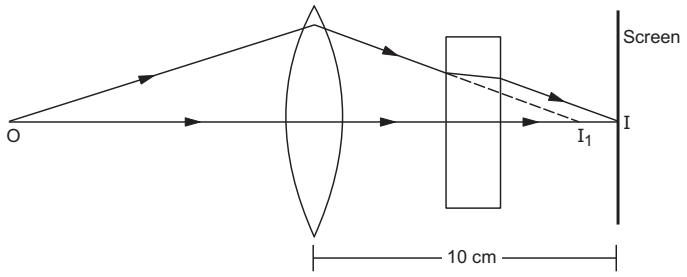



Figure 18-W24

$$\frac{1}{f} = \frac{1}{v} - \frac{1}{u} = \frac{1}{10\text{ cm}} - \left(\frac{1}{-10\text{ cm}}\right) = \frac{1}{5\text{ cm}}.$$

So, the focal length of the lens is 5 cm. The situation with the glass plate inserted is shown in figure (18-W24). The object is placed at $O$. The lens would form the image at $I_1$ but the glass plate intercepts the rays and forms the image $I$ on the screen.

The shift $I_1I = t\left(1 - \frac{1}{\mu}\right)$

$$= (1{\cdot}5\text{ cm})\left(1 - \frac{1}{1{\cdot}5}\right) = 0{\cdot}5\text{ cm}.$$

Thus, the lens forms the image at a distance 9·5 cm from itself. Using

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f},$$

$$\frac{1}{u} = \frac{1}{v} - \frac{1}{f} = \frac{1}{9{\cdot}5\text{ cm}} - \frac{1}{5\text{ cm}}$$

or, $u = -10{\cdot}6$ cm.

Thus, the object should be placed at a distance of 10·6 cm from the lens.

**28.** *Two convex lenses of focal length* 20 cm *each are placed coaxially with a separation of* 60 cm *between them. Find the image of a distant object formed by the combination by (a) using thin lens formula separately for the two lenses and (b) using the equivalent lens. Note that although the combination forms a real image of a distant object on the other side, it is equivalent to a diverging lens as far as the location of the final image is concerned.*

***Solution*** :

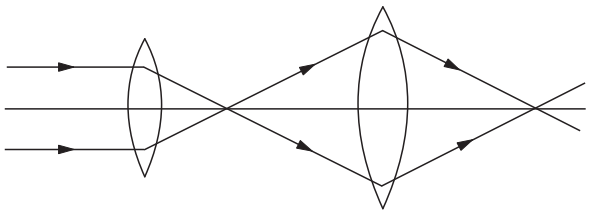


Figure 18-W25

(a) The first image is formed at the focus of the first lens. This is at 20 cm from the first lens and hence at $u = -40$ cm from the second. Using the lens formula for the second lens,

$$\frac{1}{v} = \frac{1}{u} + \frac{1}{f} = -\frac{1}{40\text{ cm}} + \frac{1}{20\text{ cm}}$$

or, $v = 40$ cm.

The final image is formed 40 cm to the right of the second lens.

(b) The equivalent focal length is

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$$

$$= \frac{1}{20 \text{ cm}} + \frac{1}{20 \text{ cm}} - \frac{60 \text{ cm}}{(20 \text{ cm})^2}$$

or, $F = -20$ cm.

It is a divergent lens. It should be kept at a distance $D = \frac{dF}{f_1}$ behind the second lens.

Here, $D = \frac{(60 \text{ cm})(-20 \text{ cm})}{20 \text{ cm}} = -60$ cm.

Thus, the equivalent divergent lens should be placed at a distance of 60 cm to the right of the second lens. The final image is formed at the focus of this divergent lens i.e., 20 cm to the left of it. It is, therefore, 40 cm to the right of the second lens.

□

## QUESTIONS FOR SHORT ANSWER

**1.** Is the formula "Real depth/Apparent depth = μ" valid if viewed from a position quite away from the normal ?

**2.** Can you ever have a situation in which a light ray goes undeviated through a prism ?

**3.** Why does a diamond shine more than a glass piece cut to the same shape ?

**4.** A narrow beam of light passes through a slab obliquely and is then received by an eye (figure 18-Q1). The index of refraction of the material in the slab fluctuates slowly with time. How will it appear to the eye? The twinkling of stars has a similar explanation.

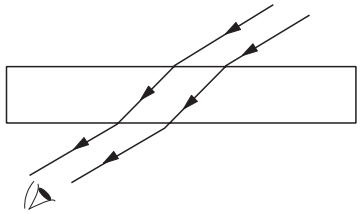


Figure 18-Q1

**5.** Can a plane mirror ever form a real image ?

**6.** If a piece of paper is placed at the position of a virtual image of a strong light source, will the paper burn after sufficient time ? What happens if the image is real? What happens if the image is real but the source is virtual ?

**7.** Can a virtual image be photographed by a camera ?

**8.** In motor vehicles, a convex mirror is attached near the driver's seat to give him the view of the traffic behind. What is the special function of this convex mirror which a plane mirror can not do ?

**9.** If an object far away from a convex mirror moves towards the mirror, the image also moves. Does it move faster, slower or at the same speed as compared to the object ?

**10.** Suppose you are inside the water in a swimming pool near an edge. A friend is standing on the edge. Do you find your friend taller or shorter than his usual height ?

**11.** The equation of refraction at a spherical surface is

$$\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}.$$

Taking $R = \infty$, show that this equation leads to the equation

$$\frac{\text{Real depth}}{\text{Apparent depth}} = \frac{\mu_2}{\mu_1}$$

for refraction at a plane surface.

**12.** A thin converging lens is formed with one surface convex and the other plane. Does the position of image depend on whether the convex surface or the plane surface faces the object ?

**13.** A single lens is mounted in a tube. A parallel beam enters the tube and emerges out of the tube as a divergent beam. Can you say with certainty that there is a diverging lens in the tube ?

**14.** An air bubble is formed inside water. Does it act as a converging lens or a diverging lens ?

**15.** Two converging lenses of unequal focal lengths can be used to reduce the aperture of a parallel beam of light without loosing the energy of the light. This increases the intensity. Describe how the converging lenses should be placed to do this.

**16.** If a spherical mirror is dipped in water, does its focal length change ?

**17.** If a thin lens is dipped in water, does its focal length change ?

**18.** Can mirrors give rise to chromatic aberration ?

**19.** A laser light is focussed by a converging lens. Will there be a significant chromatic aberration ?

## OBJECTIVE I

**1.** A point source of light is placed in front of a plane mirror.

(a) All the reflected rays meet at a point when produced backward.

(b) Only the reflected rays close to the normal meet at a point when produced backward.

(c) Only the reflected rays making a small angle with

the mirror meet at a point when produced backward.
(d) Light of different colours make different images.

**2**. Total internal reflection can take place only if
(a) light goes from optically rarer medium (smaller refractive index) to optically denser medium
(b) light goes from optically denser medium to rarer medium
(c) the refractive indices of the two media are close to each other
(d) the refractive indices of the two media are widely different.

**3**. In image formation from spherical mirrors, only paraxial rays are considered because they
(a) are easy to handle geometrically
(b) contain most of the intensity of the incident light
(c) form nearly a point image of a point source
(d) show minimum dispersion effect.

**4.** A point object is placed at a distance of 30 cm from a convex mirror of focal length 30 cm. The image will form at
(a) infinity (b) pole
(c) focus (d) 15 cm behind the mirror.

**5.** Figure (18-Q2) shows two rays $A$ and $B$ being reflected by a mirror and going as $A'$ and $B'$. The mirror
(a) is plane (b) is convex
(c) is concave (d) may be any spherical mirror.

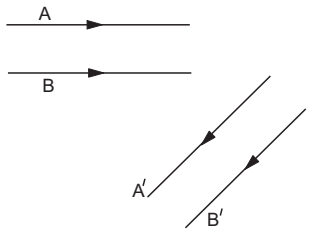



Figure 18-Q2

**6**. The image formed by a concave mirror
(a) is always real (b) is always virtual
(c) is certainly real if the object is virtual
(d) is certainly virtual if the object is real.

**7.** Figure (18-Q3) shows three transparent media of refractive indices $\mu_1$, $\mu_2$ and $\mu_3$. A point object $O$ is placed in the medium $\mu_2$. If the entire medium on the right of the spherical surface has refractive index $\mu_1$, the image forms at $O'$. If this entire medium has refractive index $\mu_3$, the image forms at $O''$. In the situation shown,
(a) the image forms between $O'$ and $O''$
(b) the image forms to the left of $O'$
(c) the image forms to the right of $O''$
(d) two images form, one at $O'$ and the other at $O''$.

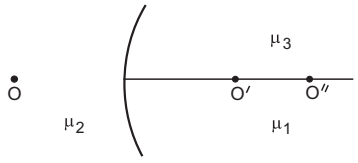



Figure 18-Q3

**8.** Four modifications are suggested in the lens formula to include the effect of the thickness $t$ of the lens. Which one is likely to be correct ?
(a) $\frac{1}{v}-\frac{1}{u}=\frac{t}{uf}$ (b) $\frac{t}{v^2}-\frac{1}{u}=\frac{1}{f}$
(c) $\frac{1}{v-t}-\frac{1}{u+t}=\frac{1}{f}$ (d) $\frac{1}{v}-\frac{1}{u}+\frac{t}{uv}=\frac{t}{f}$.

**9**. A double convex lens has two surfaces of equal radii $R$ and refractive index $m = 1{\cdot}5$. We have,
(a) $f = R/2$ (b) $f = R$ (c) $f = -R$ (d) $f = 2R$.

**10.** A point source of light is placed at a distance of $2f$ from a converging lens of focal length $f$. The intensity on the other side of the lens is maximum at a distance
(a) $f$ (b) between $f$ and $2f$ (c) $2f$ (d) more than $2f$.

**11**. A parallel beam of light is incident on a converging lens parallel to its principal axis. As one moves away from the lens on the other side on its principal axis, the intensity of light
(a) remains constant (b) continuously increases
(c) continuously decreases
(d) first increases then decreases.

**12.** A symmetric double convex lens is cut in two equal parts by a plane perpendicular to the principal axis. If the power of the original lens was 4 D, the power of a cut-lens will be
(a) 2 D (b) 3 D (c) 4 D (d) 5 D.

**13.** A symmetric double convex lens is cut in two equal parts by a plane containing the principal axis. If the power of the original lens was 4 D, the power of a divided lens will be
(a) 2 D (b) 3 D (c) 4 D (d) 5 D.

**14.** Two concave lenses $L_1$ and $L_2$ are kept in contact with each other. If the space between the two lenses is filled with a material of smaller refractive index, the magnitude of the focal length of the combination
(a) becomes undefined (b) remains unchanged
(c) increases (d) decreases.

**15.** A thin lens is made with a material having refractive index $\mu = 1{\cdot}5$. Both the sides are convex. It is dipped in water ($\mu = 1{\cdot}33$). It will behave like
(a) a convergent lens (b) a divergent lens
(c) a rectangular slab (d) a prism.

**16**. A convex lens is made of a material having refractive index 1·2. Both the surfaces of the lens are convex. If it is dipped into water ($\mu = 1{\cdot}33$), it will behave like
(a) a convergent lens (b) a divergent lens
(c) a rectangular slab (d) a prism.

**17.** A point object $O$ is placed on the principal axis of a convex lens of focal length $f = 20$ cm at a distance of 40 cm to the left of it. The diameter of the lens is 10 cm. An eye is placed 60 cm to right of the lens and a distance $h$ below the principal axis. The maximum value of $h$ to see the image is
(a) 0 (b) 2·5 cm (c) 5 cm (d) 10 cm.

**18.** The rays of different colours fail to converge at a point after going through a converging lens. This defect is called
(a) spherical aberration (b) distortion
(c) coma (d) chromatic aberration.

## OBJECTIVE II

**1.** If the light moving in a straight line bends by a small but fixed angle, it may be a case of
(a) reflection (b) refraction
(c) diffraction (d) dispersion.

**2.** Mark the correct options.
(a) If the incident rays are converging, we have a real object.
(b) If the final rays are converging, we have a real image.
(c) The image of a virtual object is called a virtual image.
(d) If the image is virtual, the corresponding object is called a virtual object.

**3.** Which of the following (referred to a spherical mirror) do (does) not depend on whether the rays are paraxial or not ?
(a) Pole (b) Focus
(c) Radius of curvature (d) Principal axis

**4.** The image of an extended object, placed perpendicular to the principal axis of a mirror, will be erect if
(a) the object and the image are both real
(b) the object and the image are both virtual
(c) the object is real but the image is virtual
(d) the object is virtual but the image is real.

**5.** A convex lens forms a real image of a point object placed on its principal axis. If the upper half of the lens is painted black,
(a) the image will be shifted downward
(b) the image will be shifted upward
(c) the image will not be shifted
(d) the intensity of the image will decrease.

**6.** Consider three converging lenses $L_1, L_2$ and $L_3$ having identical geometrical construction. The index of refraction of $L_1$ and $L_2$ are $\mu_1$ and $\mu_2$ respectively. The upper half of the lens $L_3$ has a refractive index $\mu_1$ and the lower half has $\mu_2$ (figure 18-Q3). A point object $O$ is imaged at $O_1$ by the lens $L_1$ and at $O_2$ by the lens $L_2$ placed in same position. If $L_3$ is placed at the same place,
(a) there will be an image at $O_1$
(b) there will be an image at $O_2$.
(c) the only image will form somewhere between $O_1$ and $O_2$
(d) the only image will form away from $O_2$.

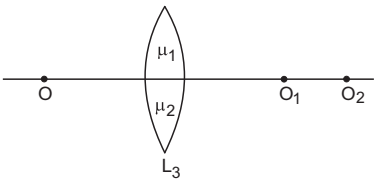



Figure 18-Q4

**7.** A screen is placed at a distance 40 cm away from an illuminated object. A converging lens is placed between the source and the screen and it is attempted to form the image of the source on the screen. If no position could be found, the focal length of the lens
(a) must be less than 10 cm
(b) must be greater than 10 cm
(c) must not be greater than 20 cm
(d) must not be less than 10 cm.

## EXERCISES

**1.** A concave mirror having a radius of curvature 40 cm is placed in front of an illuminated point source at a distance of 30 cm from it. Find the location of the image.

**2.** A concave mirror forms an image of 20 cm high object on a screen placed 5·0 m away from the mirror. The height of the image is 50 cm. Find the focal length of the mirror and the distance between the mirror and the object.

**3.** A concave mirror has a focal length of 20 cm. Find the position or positions of an object for which the image-size is double of the object-size.

**4.** A 1 cm object is placed perpendicular to the principal axis of a convex mirror of focal length 7·5 cm. Find its distance from the mirror if the image formed is 0·6 cm in size.

**5.** A candle flame 1·6 cm high is imaged in a ball bearing of diameter 0·4 cm. If the ball bearing is 20 cm away from the flame, find the location and the height of the image.

**6.** A 3 cm tall object is placed at a distance of 7·5 cm from a convex mirror of focal length 6 cm. Find the location, size and nature of the image.

**7.** A U-shaped wire is placed before a concave mirror having radius of curvature 20 cm as shown in figure (18-E1). Find the total length of the image.

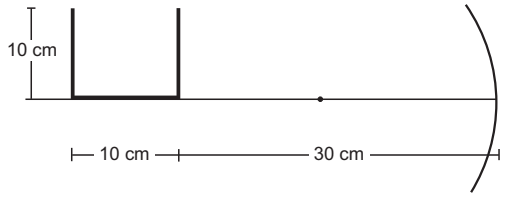



Figure 18-E1

**8.** A man uses a concave mirror for shaving. He keeps his face at a distance of 25 cm from the mirror and gets an image which is 1·4 times enlarged. Find the focal length of the mirror.

**9.** Find the diameter of the image of the moon formed by a spherical concave mirror of focal length 7·6 m. The diameter of the moon is 3450 km and the distance between the earth and the moon is $3{\cdot}8 \times 10^5$ km.

**10.** A particle goes in a circle of radius 2·0 cm. *A* concave mirror of focal length 20 cm is placed with its principal axis passing through the centre of the circle and perpendicular to its plane. The distance between the pole of the mirror and the centre of the circle is 30 cm. Calculate the radius of the circle formed by the image.

**11.** A concave mirror of radius $R$ is kept on a horizontal table (figure 18-E2). Water (refractive index = $\mu$) is poured into it up to a height $h$. Where should an object be placed so that its image is formed on itself ?

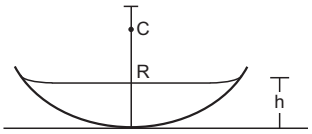



Figure 18-E2

**12.** A point source $S$ is placed midway between two converging mirrors having equal focal length $f$ as shown in figure (18-E3). Find the values of $d$ for which only one image is formed.

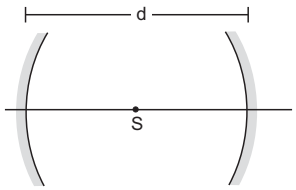



Figure 18-E3

**13.** A converging mirror $M_1$, a point source $S$ and a diverging mirror $M_2$ are arranged as shown in figure (18-E4). The source is placed at a distance of 30 cm from $M_1$. The focal length of each of the mirrors is 20 cm. Consider only the images formed by a maximum of two reflections. It is found that one image is formed on the source itself. (a) Find the distance between the two mirrors. (b) Find the location of the image formed by the single reflection from $M_2$ .

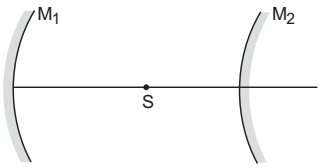



Figure 18-E4

**14.** A light ray falling at an angle of 45° with the surface of a clean slab of ice of thickness 1·00 m is refracted into it at an angle of 30°. Calculate the time taken by the light rays to cross the slab. Speed of light in vacuum = $3 \times 10^8$ m $s^{-1}$.

**15.** A pole of length 1·00 m stands half dipped in a swimming pool with water level 50·0 cm higher than the bed. The refractive index of water is 1·33 and sunlight is coming at an angle of 45° with the vertical. Find the length of the shadow of the pole on the bed.

**16.** A small piece of wood is floating on the surface of a 2·5 m deep lake. Where does the shadow form on the bottom when the sun is just setting? Refractive index of water = 4/3.

**17.** An object $P$ is focussed by a microscope $M$. *A* glass slab of thickness 2·1 cm is introduced between $P$ and $M$. If the refractive index of the slab is 1·5, by what distance should the microscope be shifted to focus the object again ?

**18.** A vessel contains water up to a height of 20 cm and above it an oil up to another 20 cm. The refractive indices of the water and the oil are 1·33 and 1·30 respectively. Find the apparent depth of the vessel when viewed from above.

**19.** Locate the image of the point $P$ as seen by the eye in the figure (18-E5).

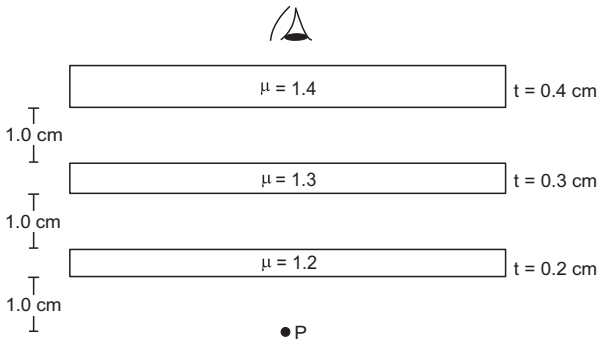



Figure 18-E5

**20.** $k$ transparent slabs are arranged one over another. The refractive indices of the slabs are $\mu_1, \mu_2, \mu_3, \ldots \mu_k$ and the thicknesses are $t_1, t_2, t_3, \ldots t_k$ . An object is seen through this combination with nearly perpendicular light. Find the equivalent refractive index of the system which will allow the image to be formed at the same place.

**21.** A cylindrical vessel of diameter 12 cm contains $800\pi$ $cm^3$ of water. A cylindrical glass piece of diameter 8·0 cm and height 8·0 cm is placed in the vessel. If the bottom of the vessel under the glass piece is seen by the paraxial rays (see figure 18-E6), locate its image. The index of refraction of glass is 1·50 and that of water is 1·33.

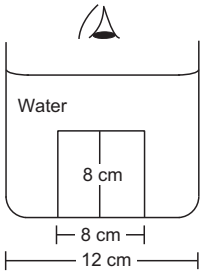



Figure 18-E6

**22.** Consider the situation in figure (18-E7). The bottom of the pot is a reflecting plane mirror, $S$ is a small fish and $T$ is a human eye. Refractive index of water is $\mu$. (a) At what distance(s) from itself will the fish see the image(s) of the eye ? (b) At what distance(s) from itself will the eye see the image(s) of the fish.

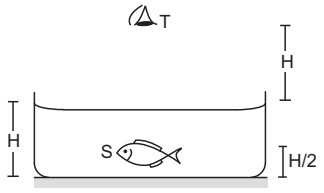



Figure 18-E7

**23**. A small object is placed at the centre of the bottom of a cylindrical vessel of radius 3 cm and height 4 cm filled completely with water. Consider the ray leaving the vessel through a corner. Suppose this ray and the ray along the axis of the vessel are used to trace the image. Find the apparent depth of the image and the ratio of real depth to the apparent depth under the assumptions taken. Refractive index of water = 1·33.

**24.** A cylindrical vessel, whose diameter and height both are equal to 30 cm, is placed on a horizontal surface and a small particle $P$ is placed in it at a distance of 5·0 cm from the centre. An eye is placed at a position such that the edge of the bottom is just visible (see figure 18-E8). The particle $P$ is in the plane of drawing. Up to what minimum height should water be poured in the vessel to make the particle $P$ visible ?

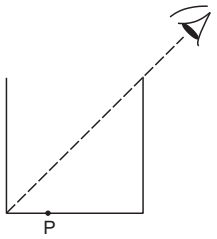



Figure 18-E8

**25**. A light ray is incident at an angle of 45° with the normal to a $\sqrt{2}$ cm thick plate ($\mu = 2{\cdot}0$). Find the shift in the path of the light as it emerges out from the plate.

**26.** An optical fibre ($\mu = 1{\cdot}72$) is surrounded by a glass coating ($\mu = 1{\cdot}50$). Find the critical angle for total internal reflection at the fibre-glass interface.

**27**. A light ray is incident normally on the face $AB$ of a right-angled prism $ABC$ ($\mu = 1{\cdot}50$) as shown in figure (18-E9). What is the largest angle $\phi$ for which the light ray is totally reflected at the surface $AC$ ?

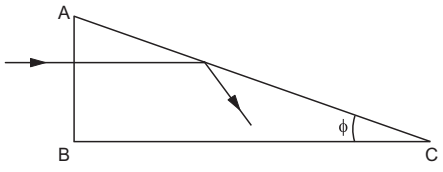



Figure 18-E9

**28.** Find the maximum angle of refraction when a light ray is refracted from glass ($\mu = 1{\cdot}50$) to air.

**29.** Light is incident from glass ($\mu = 1{\cdot}5$) to air. Sketch the variation of the angle of deviation $\delta$ with the angle of incident $i$ for $0 < i < 90°$.

**30**. Light is incident from glass ($\mu = 1{\cdot}50$) to water ($\mu = 1{\cdot}33$). Find the range of the angle of deviation for which there are two angles of incidence.

**31.** Light falls from glass ($\mu = 1{\cdot}5$) to air. Find the angle of incidence for which the angle of deviation is 90°.

**32**. A point source is placed at a depth $h$ below the surface of water (refractive index = $\mu$). (a) Show that light escapes through a circular area on the water surface with its centre directly above the point source. (b) Find the angle subtended by a radius of the area on the source.

**33.** A container contains water up to a height of 20 cm and there is a point source at the centre of the bottom of the container. $A$ rubber ring of radius $r$ floats centrally on the water. The ceiling of the room is 2·0 m above the water surface. (a) Find the radius of the shadow of the ring formed on the ceiling if $r = 15$ cm. (b) Find the maximum value of $r$ for which the shadow of the ring is formed on the ceiling. Refractive index of water = 4/3.

**34**. Find the angle of minimum deviation for an equilateral prism made of a material of refractive index 1·732. What is the angle of incidence for this deviation ?

**35.** Find the angle of deviation suffered by the light ray shown in figure (18-E10). The refractive index $\mu = 1{\cdot}5$ for the prism material.

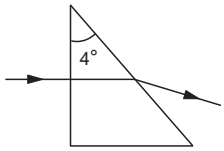



Figure 18-E10

**36.** A light ray, going through a prism with the angle of prism 60°, is found to deviate by 30°. What limit on the refractive index can be put from these data ?

**37**. Locate the image formed by refraction in the situation shown in figure (18-E11).

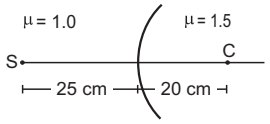



Figure 18-E11

**38.** A spherical surface of radius 30 cm separates two transparent media $A$ and $B$ with refractive indices 1·33 and 1·48 respectively. The medium $A$ is on the convex side of the surface. Where should a point object be placed in medium $A$ so that the paraxial rays become parallel after refraction at the surface ?

**39.** Figure (18-E12) shows a transparent hemisphere of radius 3·0 cm made of a material of refractive index 2·0. (a) A narrow beam of parallel rays is incident on the hemisphere as shown in the figure. Are the rays totally reflected at the plane surface ? (b) Find the image formed by the refraction at the first surface. (c) Find the image formed by the reflection or by the refraction at the plane surface. (d) Trace qualitatively the final rays as they come out of the hemisphere.

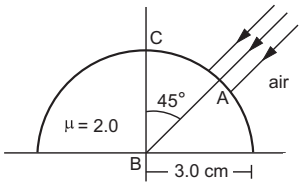



Figure 18-E12

**40.** A small object is embedded in a glass sphere ($\mu = 1{\cdot}5$) of radius 5·0 cm at a distance 1·5 cm left to the centre. Locate the image of the object as seen by an observer standing (a) to the left of the sphere and (b) to the right of the sphere.

**41.** A biconvex thick lens is constructed with glass ($\mu = 1{\cdot}50$). Each of the surfaces has a radius of 10 cm and the thickness at the middle is 5 cm. Locate the image of an object placed far away from the lens.

**42.** A narrow pencil of parallel light is incident normally on a solid transparent sphere of radius $r$. What should be the refractive index if the pencil is to be focussed (a) at the surface of the sphere, (b) at the centre of the sphere.

**43.** One end of a cylindrical glass rod ($\mu = 1{\cdot}5$) of radius 1·0 cm is rounded in the shape of a hemisphere. The rod is immersed in water ($\mu = 4/3$) and an object is placed in the water along the axis of the rod at a distance of 8·0 cm from the rounded edge. Locate the image of the object.

**44.** A paperweight ($\mu = 1{\cdot}5$) in the form of a hemisphere of radius 3·0 cm is used to hold down a printed page. An observer looks at the page vertically through the paperweight. At what height above the page will the printed letters near the centre appear to the observer ?

**45.** Solve the previous problem if the paperweight is inverted at its place so that the spherical surface touches the paper.

**46.** A hemispherical portion of the surface of a solid glass sphere ($\mu = 1{\cdot}5$) of radius $r$ is silvered to make the inner side reflecting. An object is placed on the axis of the hemisphere at a distance $3r$ from the centre of the sphere. The light from the object is refracted at the unsilvered part, then reflected from the silvered part and again refracted at the unsilvered part. Locate the final image formed.

**47.** The convex surface of a thin concavo-convex lens of glass of refractive index 1·5 has a radius of curvature 20 cm. The concave surface has a radius of curvature 60 cm. The convex side is silvered and placed on a horizontal surface as shown in figure (18-E13). (a) Where should a pin be placed on the axis so that its image is formed at the same place ? (b) If the concave part is filled with water ($\mu = 4/3$), find the distance through which the pin should be moved so that the image of the pin again coincides with the pin.

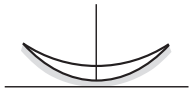


Figure 18-E13

**48.** A double convex lens has focal length 25 cm. The radius of curvature of one of the surfaces is double of the other. Find the radii, if the refractive index of the material of the lens is 1·5.

**49.** The radii of curvature of a lens are + 20 cm and + 30 cm. The material of the lens has a refracting index 1·6. Find the focal length of the lens (a) if it is placed in air, and (b) if it is placed in water ($\mu = 1{\cdot}33$).

**50.** Lenses are constructed by a material of refractive index 1·50. The magnitude of the radii of curvature are 20 cm and 30 cm. Find the focal lengths of the possible lenses with the above specifications.

**51.** A thin lens made of a material of refractive index $\mu_2$ has a medium of refractive index $\mu_1$ on one side and a medium of refractive index $\mu_3$ on the other side. The lens is biconvex and the two radii of curvature have equal magnitude $R$. A beam of light travelling parallel to the principal axis is incident on the lens. Where will the image be formed if the beam is incident from (a) the medium $\mu_1$ and (b) from the medium $\mu_3$ ?

**52.** A convex lens has a focal length of 10 cm. Find the location and nature of the image if a point object is placed on the principal axis at a distance of (a) 9·8 cm, (b) 10·2 cm from the lens.

**53.** A slide projector has to project a 35 mm slide (35 mm × 23 mm) on a 2 m × 2 m screen at a distance of 10 m from the lens. What should be the focal length of the lens in the projector ?

**54.** A particle executes a simple harmonic motion of amplitude 1·0 cm along the principal axis of a convex lens of focal length 12 cm. The mean position of oscillation is at 20 cm from the lens. Find the amplitude of oscillation of the image of the particle.

**55.** An extended object is placed at a distance of 5·0 cm from a convex lens of focal length 8·0 cm. (a) Draw the ray diagram (to the scale) to locate the image and from this, measure the distance of the image from the lens. (b) Find the position of the image from the lens formula and see how close the drawing is to the correct result.

**56.** A pin of length 2·00 cm is placed perpendicular to the principal axis of a converging lens. An inverted image of size 1·00 cm is formed at a distance of 40·0 cm from the pin. Find the focal length of the lens and its distance from the pin.

**57.** A convex lens produces a double size real image when an object is placed at a distance of 18 cm from it. Where should the object be placed to produce a triple size real image ?

**58.** A pin of length 2·0 cm lies along the principal axis of a converging lens, the centre being at a distance of 11 cm from the lens. The focal length of the lens is 6 cm. Find the size of the image.

**59.** The diameter of the sun is $1{\cdot}4 \times 10^{9}$ m and its distance from the earth is $1{\cdot}5 \times 10^{11}$ m. Find the radius of the image of the sun formed by a lens of focal length 20 cm.

**60.** A 5·0 diopter lens forms a virtual image which is 4 times the object placed perpendicularly on the principal axis of the lens. Find the distance of the object from the lens.

**61.** A diverging lens of focal length 20 cm and a converging mirror of focal length 10 cm are placed coaxially at a separation of 5 cm. Where should an object be placed so that a real image is formed at the object itself ?

**62**. A converging lens of focal length 12 cm and a diverging mirror of focal length 7·5 cm are placed 5·0 cm apart with their principal axes coinciding. Where should an object be placed so that its image falls on itself ?

**63.** A converging lens and a diverging mirror are placed at a separation of 15 cm. The focal length of the lens is 25 cm and that of the mirror is 40 cm. Where should a point source be placed between the lens and the mirror so that the light, after getting reflected by the mirror and then getting transmitted by the lens, comes out parallel to the principal axis ?

**64.** A converging lens of focal length 15 cm and a converging mirror of focal length 10 cm are placed 50 cm apart with common principal axis. A point source is placed in between the lens and the mirror at a distance of 40 cm from the lens. Find the locations of the two images formed.

**65**. Consider the situation described in the previous problem. Where should a point source be placed on the principal axis so that the two images form at the same place ?

**66**. A converging lens of focal length 15 cm and a converging mirror of focal length 10 cm are placed 50 cm apart. If a pin of length 2·0 cm is placed 30 cm from the lens farther away from the mirror, where will the final image form and what will be the size of the final image ?

**67**. A point object is placed on the principal axis of a convex lens ($f$ = 15 cm) at a distance of 30 cm from it. A glass plate ($\mu$ = 1·50) of thickness 1 cm is placed on the other side of the lens perpendicular to the axis. Locate the image of the point object.

**68**. A convex lens of focal length 20 cm and a concave lens of focal length 10 cm are placed 10 cm apart with their principal axes coinciding. A beam of light travelling parallel to the principal axis and having a beam diameter 5·0 mm, is incident on the combination. Show that the emergent beam is parallel to the incident one. Find the beam diameter of the emergent beam.

**69.** A diverging lens of focal length 20 cm and a converging lens of focal length 30 cm are placed 15 cm apart with their principal axes coinciding. Where should an object be placed on the principal axis so that its image is formed at infinity ?

**70.** A 5 mm high pin is placed at a distance of 15 cm from a convex lens of focal length 10 cm. A second lens of focal length 5 cm is placed 40 cm from the first lens and 55 cm from the pin. Find (a) the position of the final image, (b) its nature and (c) its size.

**71.** A point object is placed at a distance of 15 cm from a convex lens. The image is formed on the other side at a distance of 30 cm from the lens. When a concave lens is placed in contact with the convex lens, the image shifts away further by 30 cm. Calculate the focal lengths of the two lenses.

**72**. Two convex lenses, each of focal length 10 cm, are placed at a separation of 15 cm with their principal axes coinciding. (a) Show that a light beam coming parallel to the principal axis diverges as it comes out of the lens system. (b) Find the location of the virtual image formed by the lens system of an object placed far away. (c) Find the focal length of the equivalent lens. (Note that the sign of the focal length is positive although the lens system actually diverges a parallel beam incident on it.)

**73**. A ball is kept at a height $h$ above the surface of a heavy transparent sphere made of a material of refractive index $\mu$. The radius of the sphere is $R$. At $t = 0$, the ball is dropped to fall normally on the sphere. Find the speed of the image formed as a function of time for $t < \sqrt{\frac{2h}{g}}$. Consider only the image by a single refraction.

**74.** A particle is moving at a constant speed $V$ from a large distance towards a concave mirror of radius $R$ along its principal axis. Find the speed of the image formed by the mirror as a function of the distance $x$ of the particle from the mirror.

**75.** A small block of mass $m$ and a concave mirror of radius $R$ fitted with a stand lie on a smooth horizontal table with a separation $d$ between them. The mirror together with its stand has a mass $m$. The block is pushed at $t = 0$ towards the mirror so that it starts moving towards the mirror at a constant speed $V$ and collides with it. The collision is perfectly elastic. Find the velocity of the image (a) at a time $t < d/V$, (b) at a time $t > d/V$.

**76.** A gun of mass $M$ fires a bullet of mass $m$ with a horizontal speed $V$. The gun is fitted with a concave mirror of focal length $f$ facing towards the receding bullet. Find the speed of separation of the bullet and the image just after the gun was fired.

**77**. A mass $m$ = 50 g is dropped on a vertical spring of spring constant 500 N m$^{-1}$ from a height $h$ = 10 cm as shown in figure (18-E14). The mass sticks to the spring and

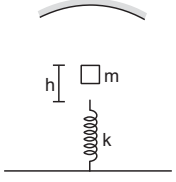



Figure 18-E14

executes simple harmonic oscillations after that. A concave mirror of focal length 12 cm facing the mass is fixed with its principal axis coinciding with the line of motion of the mass, its pole being at a distance of 30 cm from the free end of the spring. Find the length in which the image of the mass oscillates.

**78.** Two concave mirrors of equal radii of curvature $R$ are fixed on a stand facing opposite directions. The whole system has a mass $m$ and is kept on a frictionless horizontal table (figure 18-E15).

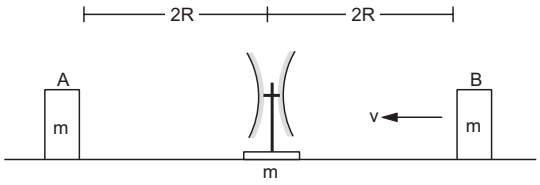


Figure 18-E15

Two blocks $A$ and $B$, each of mass $m$, are placed on the two sides of the stand. At $t = 0$, the separation between $A$ and the mirrors is $2R$ and also the separation between $B$ and the mirrors is $2R$. The block $B$ moves towards the mirror at a speed $v$. All collisions which take place are elastic. Taking the original position of the mirrors–stand system to be $x = 0$ and $X$-axis along $AB$, find the position of the images of $A$ and $B$ at $t =$

(a) $\frac{R}{v}$ (b) $\frac{3R}{v}$ (c) $\frac{5R}{v}$.

**79.** Consider the situation shown in figure (18-E16). The elevator is going up with an acceleration of 2·00 m s$^{-2}$ and the focal length of the mirror is 12·0 cm. All the surfaces are smooth and the pulley is light. The mass–pulley system is released from rest (with respect to the elevator) at $t = 0$ when the distance of $B$ from the mirror is 42·0 cm. Find the distance between the image of the block $B$ and the mirror at $t$ = 0·200 s. Take $g$ = 10 m s$^{-2}$.

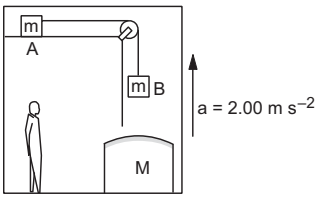


Figure 18-E16

□

## ANSWERS

### OBJECTIVE I

| | | | | | |
|---|---|---|---|---|---|
| 1. (a) | 2. (b) | 3. (c) | 4. (d) | 5. (a) | 6. (c) |
| 7. (d) | 8. (c) | 9. (b) | 10. (c) | 11. (d) | 12. (a) |
| 13. (c) | 14. (c) | 15. (a) | 16. (b) | 17. (b) | 18. (d) |

### OBJECTIVE II

| | | |
|---|---|---|
| 1. (a), (b) | 2. (b) | 3. (a), (c), (d) |
| 4. (c), (d) | 5. (c), (d) | 6. (a), (b) |
| 7. (b) | | |

### EXERCISES

1. 60 cm from the mirror on the side of the object
2. 1·43 m, 2·0 m
3. 10 cm or 30 cm from the mirror
4. 5 cm
5. 1·0 mm inside the ball bearing, 0·08 mm
6. $\frac{10}{3}$ cm from the mirror on the side opposite to the object, 1·33 cm, virtual and erect
7. 10 cm
8. 87·5 cm
9. 6·9 cm
10. 4·0 cm
11. $\frac{(R-h)}{\mu}$ above the water surface
12. $2f$, $4f$
13. (a) 50 cm (b) 10 cm from the diverging mirror farther from the converging mirror
14. 5·44 ns
15. 81·5 cm
16. 2·83 m shifted from the position directly below the piece of the wood.
17. 0·70 cm
18. 30·4 cm
19. 0·2 cm above $P$
20. $\frac{\sum_{i=1}^{k} t_i}{\sum_{i=1}^{k} (t_i/\mu_i)}$
21. 7·1 cm above the bottom
22. (a) $H\left(\mu+\frac{1}{2}\right)$ above itself, $H\left(\mu+\frac{3}{2}\right)$ below itself
    (b) $H\left(1+\frac{1}{2\mu}\right)$ below itself and $H\left(1+\frac{3}{2\mu}\right)$ below itself
23. 2·25 cm, 1·78

24. 26·7 cm
25. 0·62 cm
26. $\sin^{-1}\frac{75}{86}$
27. $\cos^{-1}(2/3)$
28. 90°
30. 0 to $\cos^{-1}(8/9)$
31. 45°
32. (b) $\sin^{-1}(1/\mu)$
33. (a) 2·8 m (b) 22·6 cm
34. 60°, 60°
35. 2°
36. $\mu \le \sqrt{2}$
37. 100 cm from the surface on the side of $S$
38. 266·0 cm away from the separating surface
39. (a) They are reflected
    (b) If the sphere is completed, the image forms at the point diametrically opposite to $A$
    (c) At the mirror image of $A$ in $BC$
40. (a) 2 cm left to the centre
    (b) 2·65 cm left to the centre
41. 9·1 cm from the farther surface on the other side of the lens
42. (a) 2,
    (b) not possible, it will focus close to the centre if the refractive index is large.
43. At infinity
44. No shift is observed
45. 1 cm
46. At the reflecting surface of the sphere
47. (a) 15 cm from the lens on the axis
    (b) 1·14 cm towards the lens
48. 18·75 cm, 37·5 cm
49. (a) 100 cm (b) 300 cm
50. ± 24 cm, ± 120 cm
51. (a) $\frac{\mu_3 R}{2\mu_2 - \mu_1 - \mu_3}$ (b) $\frac{\mu_1 R}{2\mu_2 - \mu_1 - \mu_3}$
52. (a) 490 cm on the side of the object, virtual
    (b) 510 cm on the other side, real
53. 17·2 cm
54. 2·3 cm
56. 8·89 cm, 26·7 cm
57. 16 cm
58. 3 cm
59. 0·93 mm
60. 15 cm
61. 60 cm from the lens further away from the mirror
62. 30 cm from the lens further away from the mirror
63. 1·67 cm from the lens
64. One at 15 cm and the other at 24 cm from the lens away from the mirror
65. 30 cm from the lens towards the mirror
66. At the object itself, of the same size
67. 30·33 cm from the lens
68. 1·0 cm if the light is incident from the side of concave lens and 2·5 mm if it is incident from the side of the convex lens
69. 60 cm from the diverging lens or 210 cm from the converging lens
70. (a) 10 cm from the second lens further away,
    (b) erect and real, (c) 10 mm
71. 10 cm for convex lens and 60 cm for concave lens
72. (b) 5 cm from the first lens towards the second lens
    (c) 20 cm
73. $\frac{\mu R^2 gt}{\left[(\mu-1)\left(h-\frac{1}{2}gt^2\right)-R\right]^2}$
74. $\frac{R^2 V}{(2x-R)^2}$
75. (a) $-\frac{R^2 V}{[2(d-Vt)-R]^2}$
    (b) $V\left[1+\frac{R^2}{[2(Vt-d)-R]^2}\right]$
76. $2(1+m/M)V$
77. 1·2 cm
78. (a) $x=-\frac{2R}{3}, R$ (b) $x=-2R, 0$ (c) $x=-3R, -\frac{4R}{3}$
79. 8·57 cm

□

## CHAPTER 19

# OPTICAL INSTRUMENTS

### 19.1 THE EYE

Optical instruments are used primarily to assist the eye in viewing an object. Let us first discuss in brief the construction of a human eye and the mechanism by which we see, the most common but most important experiment we do from the day we open our eyes.

Figure (19.1) shows schematically the basic components of an eye. The eye has a nearly spherical shape of diameter about an inch. The front portion is more sharply curved and is covered by a transparent protective membrane called the *cornea*. It is this portion which is visible from outside. Behind the cornea, we have a space filled with a liquid called the *aqueous humor* and behind that a *crystalline lens*.

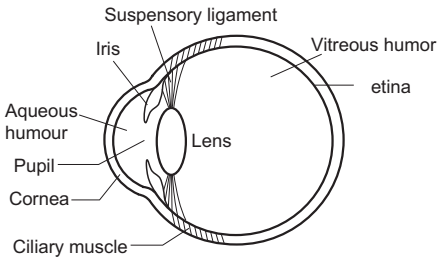



Figure 19.1

Between the aqueous humor and the lens, we have a muscular diaphragm called *iris*, which has a small hole in it called *pupil*. Iris is the coloured part that we see in an eye. The pupil appears black because any light falling on it goes into the eye and there is almost no chance of light coming back to the outside. The amount of light entering the eye, may be controlled by varying the aperture of the pupil with the help of the iris. In low-light condition, the iris expands the pupil to allow more light to go in. In good light conditions, it contracts the pupil.

The lens is hard in the middle and gradually becomes soft towards the outer edge. The curvature of the lens may be altered by the *ciliary muscles* to which it is attached. The light entering the eye forms an image on the *retina* which covers the inside of the rear part of the eyeball. The retina contains about 125 million receptors called *rods* and *cones* which receive the light signal and about one million optic-nerve fibres which transmit the information to the brain. The space between the lens and the retina is filled with another liquid called the *vitreous humor*.

The aqueous humor and the vitreous humor have almost same refractive index 1·336. The refractive index of the material of the lens is different in different portions but on the average it is about 1·396. When light enters the eye from air, most of the bending occurs at the cornea itself because there is a sharp change in the refractive index. Some additional bending is done by the lens which is surrounded by a fluid of somewhat lower refractive index. In normal conditions, the light should be focussed on the retina.

The cornea–lens–fluid system is equivalent to a single converging lens whose focal length may be adjusted by the ciliary muscles. Now onwards, we shall use the word eye-lens to mean this equivalent lens.

When the eye is focussed on a distant object, the ciliary muscles are relaxed so that the focal length of the eye-lens has its maximum value which is equal to its distance from the retina. The parallel rays coming into the eye are then focussed on the retina and we see the object clearly.

When the eye is focussed on a closer object, the ciliary muscles are strained and the focal length of the eye-lens decreases. The ciliary muscles adjust the focal length in such a way that the image is again formed on the retina and we see the object clearly. This process of adjusting focal length is called *accommodation*. However, the muscles cannot be strained beyond a limit and hence, if the object is brought too close to the eye, the focal length cannot be adjusted to form the image on the retina. Thus, there is a minimum distance for the clear vision of an object.

The nearest point for which the image can be focussed on the retina, is called the *near point* of the eye. The distance of the near point from the eye is called the *least distance for clear vision.* This varies from person to person and with age. At a young age (say below 10 years), the muscles are strong and flexible and can bear more strain. The near point may be as close as 7–8 cm at this age. In old age, the muscles cannot sustain a large strain and the near point shifts to large values, say, 1 to 2 m or even more. We shall discuss about these defects of vision and use of glasses in a later section. The average value of the least distance for clear vision for a normal eye is generally taken to be 25 cm.

## 19.2 APPARENT SIZE

The size of an object as sensed by us is related to the size of the image formed on the retina. A larger image on the retina activates larger number of rods and cones attached to it and the object looks larger. As is clear from figure (19.2), if an object is taken away from the eye, the size of the image on the retina decreases and hence, the same object looks smaller. It is also clear from figure (19.2) that the size of the image on the retina is roughly proportional to the angle subtended by the object on the eye. This angle is known as the *visual angle* and optical instruments are used to increase this angle artificially in order to improve the clarity.

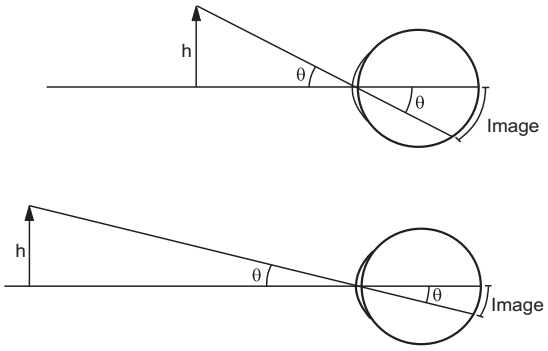



Figure 19.2

***Example 19.1***

*Two boys, one* 52 inches *tall and the other* 55 inches *tall, are standing at distances* 4·0 m *and* 5·0 m *respectively from an eye. Which boy will appear taller*?

***Solution*** **:** The angle subtended by the first boy on the eye is

$$\alpha_1 = \frac{52 \text{ inch}}{4{\cdot}0 \text{ m}} = 13 \text{ inch/m}$$

and the angle subtended by the second boy is

$$\alpha_2 = \frac{55 \text{ inch}}{5{\cdot}0 \text{ m}} = 11 \text{ inch/m}.$$

As $\alpha_1 > \alpha_2$, the first boy will look taller to the eye.

## 19.3 SIMPLE MICROSCOPE

When we view an object with naked eyes, the object must be placed somewhere between infinity and the near point. The maximum angle is subtended on the eye when the object is placed at the near point. This angle is (figure 19.3a)

$$\theta_o = \frac{h}{D}, \qquad \ldots \text{ (i)}$$

where $h$ is the size of the object and $D$ is the least distance for clear vision.

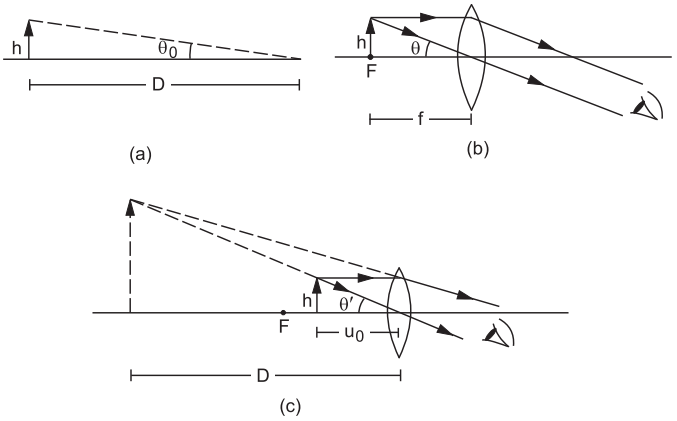



Figure 19.3

This angle can be further increased if a converging lens of short focal length is placed just in front of the eye. When a converging lens is used for this purpose, it is called a *simple microscope* or a *magnifier*.

Suppose, the lens has a focal length $f$ which is less than $D$ and let us move the object to the first focal point $F$. The eye receives rays which seem to come from infinity (figure 19.3b). The actual size of the image is infinite but the angle subtended on the lens (and hence on the eye) is

$$\theta = \frac{h}{f}. \qquad \ldots \text{ (ii)}$$

As $f < D$, equations (i) and (ii) show that $\theta > \theta_o$. Hence, the eye perceives a larger image than it could have had without the microscope. As the image is situated at infinity, the ciliary muscles are least strained to focus the final image on the retina. This situation is known as *normal adjustment.* We define *magnifying power* of a microscope as $\theta/\theta_o$ where $\theta$ is the angle subtended by the image on the eye when the microscope is used and $\theta_o$ is the angle subtended on the naked eye when the object is placed at the near point. This is also known as the *angular magnification.* Thus, *the magnifying power is the factor by which the*

*image on the retina can be enlarged by using the microscope.*

In normal adjustment, the magnifying power of a simple microscope is, by (i) and (ii),

$$m = \frac{\theta}{\theta_o} = \frac{h/f}{h/D}$$

or, $$m = \frac{D}{f} \cdot \quad \ldots \ (19.1)$$

If $f < D$, the magnifying power is greater than 1.

The magnifying power can be further increased by moving the object still closer to the lens. Suppose, we move the object to a distance $u_o$ from the lens such that the virtual erect image is formed at the near point (figure 19.3c). Though the eye is strained, it can still see the image clearly. The distance $u_o$ can be calculated using the lens formula,

$$\frac{1}{u} = \frac{1}{v} - \frac{1}{f} \cdot$$

Here $v = -D$ and $u = -u_o$, so that

$$\frac{1}{-u_o} = -\frac{1}{D} - \frac{1}{f}$$

or, $$\frac{D}{u_o} = 1 + \frac{D}{f} \cdot \quad \ldots \ (iii)$$

The angle subtended by the image on the lens (and hence on the eye) is

$$\theta' = \frac{h}{u_o} \cdot$$

The angular magnification or magnifying power in this case is

$$m = \frac{\theta'}{\theta_o} = \frac{h/u_o}{h/D}$$

$$= \frac{D}{u_o}$$

$$= 1 + \frac{D}{f} \cdot \quad \ldots \ (19.2)$$

Equations (19.1) and (19.2) show that the magnification can be made large by choosing the focal length $f$ small. However, due to several other aberrations the image becomes too defective at large magnification with a simple microscope. Roughly speaking, a magnification up to 4 is trouble-free.

The magnifying power is written with a unit X. Thus, if a magnifier produces an angular magnification of 10, it is called a 10 X magnifier.

## 19.4 COMPOUND MICROSCOPE

Figure (19.4) shows a simplified version of a compound microscope and the ray diagram for image formation. It consists of two converging lenses arranged coaxially. The one facing the object is called the *objective* and the one close to the eye is called the *eyepiece* or *ocular*. The objective has a smaller aperture and smaller focal length than those of the eyepiece. The separation between the objective and the eyepiece can be varied by appropriate screws fixed on the panel of the microscope.

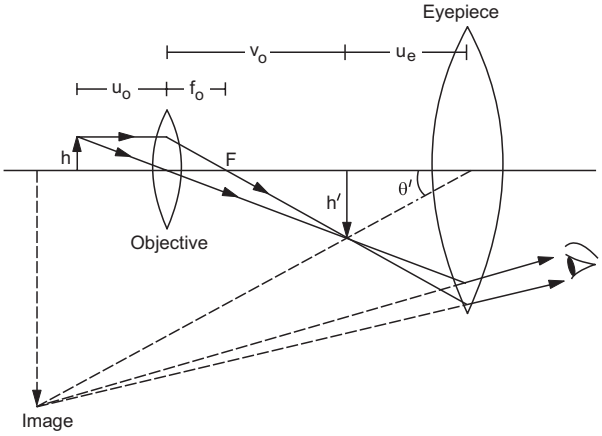



Figure 19.4

The object is placed at a distance $u_o$ from the objective which is slightly greater than its focal length $f_o$. A real and inverted image is formed at a distance $v_o$ on the other side of the objective. This image works as the object for the eyepiece. For normal adjustment, the position of the eyepiece is so adjusted that the image formed by the objective falls in the focal plane of the eyepiece. The final image is then formed at infinity. It is erect with respect to the first image and hence, inverted with respect to the object. The eye is least strained in this adjustment as it has to focus the parallel rays coming to it. The position of the eyepiece can also be adjusted in such a way that the final virtual image is formed at the near point. The angular magnification is increased in this case. The ray diagram in figure (19.4) refers to this case.

In effect, the eyepiece acts as a simple microscope used to view the first image. Thus, the magnification by a compound microscope is a two-step process. In the first step, the objective produces a magnified image of the given object. In the second step, the eyepiece produces an angular magnification. The overall angular magnification is the product of the two.

### Magnifying power

Refer to figure (19.4). If an object of height $h$ is seen by the naked eye, the largest image on the retina is formed when it is placed at the near point. The angle formed by the object on the eye in this situation is

$$\theta_o = \frac{h}{D} \cdot \quad \ldots \ (i)$$

When a compound microscope is used, the final image subtends an angle θ′ on the eyepiece (and hence on the eye) given by

$$\theta' = \frac{h'}{u_e}, \qquad \ldots \text{(ii)}$$

where $h'$ is the height of the first image and $u_e$ is its distance from the eyepiece.

The magnifying power of the compound microscope is, therefore,

$$m = \frac{\theta'}{\theta_o} = \frac{h'}{u_e} \times \frac{D}{h} = \left(\frac{h'}{h}\right)\left(\frac{D}{u_e}\right). \qquad \ldots \text{(iii)}$$

Also from figure (19.4),

$$\frac{h'}{h} = -\frac{v_o}{u_o} = \frac{v}{u}. \qquad \ldots \text{(iv)}$$

Now, $D/u_e$ is the magnifying power of the eyepiece treated as a simple microscope. Using (19.1) and (19.2), this is equal to $D/f_e$ in normal adjustment (image at infinity) and $1 + D/f_e$ for the adjustment when the image is formed at the least distance for clear vision, i.e., at $D$. Thus, the magnifying power of the compound microscope is, by (iii),

$$m = \frac{v}{u}\left(\frac{D}{f_e}\right) \qquad \ldots \text{(19.3)}$$

for normal adjustment and

$$m = \frac{v}{u}\left(1 + \frac{D}{f_e}\right) \qquad \ldots \text{(19.4)}$$

for the adjustment when the final image is formed at the least distance for clear vision.

Using lens equation for the objective,

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

or, $$1 - \frac{v}{u} = \frac{v}{f_o}$$

or, $$\frac{v}{u} = 1 - \frac{v}{f_o}.$$

In general, the focal length of the objective is very small so that $\frac{v}{f_o} >> 1$. Also, the first image is close to the eyepiece so that $v \approx l$, where $l$ is the tube-length (separation between the objective and the eyepiece).

Thus, $$\frac{v}{u} = 1 - \frac{v}{f_o} \approx -\frac{v}{f_o} \approx -\frac{l}{f_o}.$$

If these conditions are satisfied, the magnifying power of the compound microscope is, by (19.3) and (19.4),

$$m = -\frac{l}{f_o}\frac{D}{f_e}$$

for normal adjustment and

$$m = -\frac{l}{f_o}\left(1 + \frac{D}{f_e}\right)$$

for adjustment for the final image at the least distance for clear vision.

In an actual compound microscope each of the objective and the eyepiece consists of combination of several lenses instead of a single lens as assumed in the simplified version.

***Example 19.2***

*A compound microscope has an objective of focal length* 1 cm *and an eyepiece of focal length* 2·5 cm. *An object has to be placed at a distance of* 1·2 cm *away from the objective for normal adjustment. (a) Find the angular magnification. (b) Find the length of the microscope tube.*

***Solution*** :

(a) If the first image is formed at a distance $v$ from the objective, we have

$$\frac{1}{v} - \frac{1}{(-1{\cdot}2\text{ cm})} = \frac{1}{1\text{ cm}}$$

or, $v = 6$ cm.

The angular magnification in normal adjustment is

$$m = \frac{v}{u}\frac{D}{f_e} = -\frac{6\text{ cm}}{1{\cdot}2\text{ cm}} \cdot \frac{25\text{ cm}}{2{\cdot}5\text{ cm}} = -50.$$

(b) For normal adjustment, the first image must be in the focal plane of the eyepiece.

The length of the tube is, therefore,

$$L = v + f_e = 6\text{ cm} + 2{\cdot}5\text{ cm} = 8{\cdot}5\text{ cm}.$$

## 19.5 TELESCOPES

A microscope is used to view the objects placed close to it, say, within few centimeters. To look at distant objects such as a star, a planet or a distant tree, etc., we use another instrument called a *telescope*. We shall describe three types of telescopes which are in use.

### (A) Astronomical Telescope

Figure (19.5) shows the construction and working of a simplified version of an astronomical telescope.

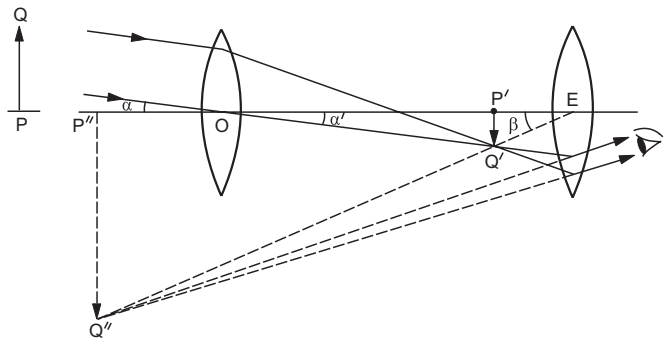



Figure 19.5

It consists of two converging lenses placed coaxially. The one facing the distant object is called the objective and has a large aperture and a large focal length. The other is called the eyepiece, as the eye is placed close to it. It has a smaller aperture and a smaller focal length. The lenses are fixed in tubes. The eyepiece tube can slide within the objective tube so that the separation between the objective and the eyepiece may be changed.

When the telescope is directed towards a distant object $PQ$, the objective forms a real image of the object in its focal plane. If the point $P$ is on the principal axis, the image point $P'$ is at the second focus of the objective. The rays coming from $Q$ are focussed at $Q'$. The eyepiece forms a magnified virtual image $P''Q''$ of $P'Q'$. This image is finally seen by the eye. In normal adjustment, the position is so adjusted that the final image is formed at infinity. In such a case, the first image $P'Q'$ is formed in the first focal plane of the eyepiece. The eye is least strained to focus this final image. The image can be brought closer by pushing the eyepiece closer to the first image. Maximum angular magnification is produced when the final image is formed at the near point.

**Magnifying Power**

Suppose, the objective and the eyepiece have focal lengths $f_o$ and $f_e$ respectively and the object is situated at a large distance $u_o$ from the objective. The object $PQ$ in figure (19.5) subtends an angle $\alpha$ on the objective. Since the object is far away, the angle it would subtend on the eye, if there were no telescope, is also essentially $\alpha$.

As $u_o$ is very large, the first image $P'Q'$ is formed in the focal plane of the objective.

From the figure,

$$| \alpha | = | \alpha' | \approx | \tan\alpha' | = \frac{P'Q'}{OP'} = \frac{P'Q'}{f_o} . \quad \ldots \text{(i)}$$

The final image $P''Q''$ subtends an angle $\beta$ on the eyepiece (and hence on the eye). We have from the triangle $P'Q'E$,

$$| \beta | \approx | \tan \beta | = \frac{P'Q'}{EP'}$$

or,
$$\left| \frac{\beta}{\alpha} \right| = \frac{f_o}{EP'} . \quad \ldots \text{(ii)}$$

If the telescope is set for normal adjustment so that the final image is formed at infinity, the first image $P'Q'$ must be in the focal plane of the eyepiece. Then $EP' = f_e$. Thus, equation (ii) becomes

$$\left| \frac{\beta}{\alpha} \right| = \frac{f_o}{f_e} . \quad \ldots \text{(iii)}$$

The angular magnification or the magnifying power of the telescope is defined as

$$m = \frac{\text{angle subtended by the final image on the eye}}{\text{angle subtended by the object on the unaided eye}} .$$

The angles $\beta$ and $\alpha$ are formed on the opposite sides of the axis. Hence, their signs are opposite and $\beta/\alpha$ is negative. Thus,

$$m = \frac{\beta}{\alpha} = - \left| \frac{\beta}{\alpha} \right| .$$

Using equation (iii),

$$m = - \frac{f_o}{f_e} . \quad \ldots \text{(19.5)}$$

If the telescope is adjusted so that the final image is formed at the near point of the eye, the angular magnification is further increased. Let us apply the lens equation to the eyepiece in this case.

Here $u = - EP'$

and $v = - EP'' = - D.$

The lens equation is

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

or,
$$\frac{1}{-D} - \frac{1}{-EP'} = \frac{1}{f_e}$$

or,
$$\frac{1}{EP'} = \frac{1}{f_e} + \frac{1}{D} = \frac{f_e + D}{f_e D} . \quad \ldots \text{(iv)}$$

By (ii),

$$\left| \frac{\beta}{\alpha} \right| = \frac{f_o(f_e + D)}{f_e D} .$$

The magnification is

$$m = \frac{\beta}{\alpha} = - \left| \frac{\beta}{\alpha} \right| = - \frac{f_o(f_e + D)}{f_e D} = - \frac{f_o}{f_e} \left( 1 + \frac{f_e}{D} \right) . \quad \ldots \text{(19.6)}$$

**Length of the Telescope**

From figure (19.5), we see that the length of the telescope is

$$L = OP' + P'E = f_o + P'E.$$

For normal adjustment, $P'E = f_e$ so that $L = f_o + f_e$. For adjustment for near point vision, we have, by (iv) above,

$$P'E = \frac{f_e D}{f_e + D}$$

so that the length is $L = f_o + \frac{f_e D}{f_e + D} .$

**(B) Terrestrial Telescope**

In an astronomical telescope, the final image is inverted with respect to the object. This creates some practical difficulty if the telescope is used to see earthly objects.

Imagine, how would you feel if you are viewing a cricket match from the spectator's gallery using an astronomical telescope. You would clearly see the turns and breaks of the ball, but the players would look like hanging from the field and not standing on the field.

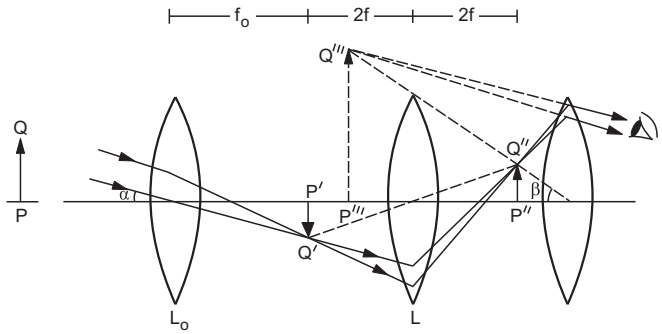



Figure 19.6

To remove this difficulty, a convex lens of focal length $f$ is included between the objective and the eyepiece in such a way that the focal plane of the objective is a distance $2f$ away from this lens (figure 19.6). The objective forms the image $P'Q'$ of a distant object in its focal plane. The lens $L$ forms an image $P''Q''$ which is inverted with respect to $P'Q'$. The eyepiece is adjusted in appropriate position to give the magnified view of $P''Q''$.

The role of the intermediate lens $L$ is only to invert the image. The magnification produced by it is, therefore, $-1$. The magnifying power of a terrestrial telescope is, therefore, obtained from (19.5) for normal adjustment and from (19.6) for near point vision by multiplying by $-1$ on the right-hand side. Thus, for normal adjustment,

$$m = \frac{f_o}{f_e} \qquad \ldots \quad (19.7)$$

and for final image at near point,

$$m = \frac{f_o}{f_e}\left(1 + \frac{f_e}{D}\right). \qquad \ldots \quad (19.8)$$

To have an inverted image of same size, the object should be placed at a distance of $2f$ from a convex lens of focal length $f$. Thus, $P'P'' = 4f$ in figure (19.6) so that the length of a terrestrial telescope is $f_o + 4f + u_e$. For normal adjustment, $u_e$ equals $f_e$ so that the length is

$$L = f_o + 4f + f_e.$$

If the final image is formed at the near point,

$$u_e = \frac{f_e D}{f_e + D}$$

as derived for astronomical telescope. Thus,

$$L = f_o + 4f + \frac{f_e D}{f_e + D}.$$

**(C) Galilean Telescope**

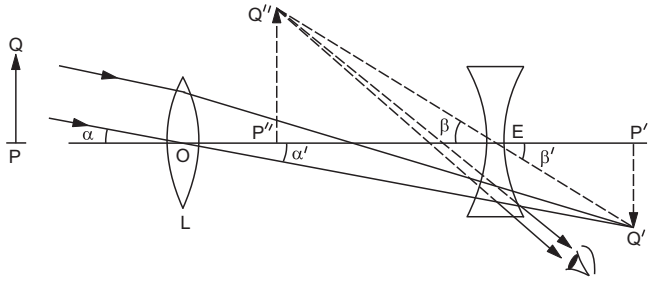



Figure 19.7

Figure (19.7) shows a simple model of Galilean telescope. A convergent lens is used as the objective and a divergent lens as the eyepiece. The objective $L$ would form a real inverted image $P'Q'$ of a distant object in its focal plane. The eyepiece intercepts the converging rays in between. $P'Q'$ then acts as a virtual object for the eyepiece. The position of the eyepiece is so adjusted that the final image is formed at the desired position. For normal adjustment, the final image is formed at infinity producing least strain on the eyes. If the final image is formed at the least distance of clear vision, the angular magnification is maximum.

**Magnifying Power**

Suppose the objective and the eyepiece have focal lengths $f_o$ and $f_e$ respectively and the object is situated at a large distance $u_o$ from the objective. The object $PQ$ subtends an angle $\alpha$ on the objective. Since the object is far away, the angle it would subtend on an unaided eye is also essentially $\alpha$.

As $u_o$ is very large, the first image $P'Q'$ is formed in the focal plane of the objective. Thus, from figure (19.7),

$$| \alpha | = | \alpha' | \approx | \tan \alpha' | = \frac{P'Q'}{OP'} = \frac{P'Q'}{f_o}. \qquad \ldots \quad (i)$$

The final image $P''Q''$ subtends an angle $\beta$ on the eyepiece. If the eye is placed close to the eyepiece, this is also the angle formed by the final image on the eye. From the figure,

$$| \beta | = | \beta' | \approx | \tan \beta' | = \frac{P'Q'}{EP'}$$

or, $$\left|\frac{\beta}{\alpha}\right| = \frac{f_o}{EP'}. \quad \ldots \text{(ii)}$$

As β and α are formed on the same side of the axis, β and α have same sign. Thus,

$$\frac{\beta}{\alpha} = \left|\frac{\beta}{\alpha}\right|.$$

The angular magnification is, therefore,

$$m = \frac{\beta}{\alpha} = \left|\frac{\beta}{\alpha}\right| = \frac{f_o}{P'E}. \quad \ldots \text{(iii)}$$

If the telescope is set for normal adjustment, the final image $P''Q''$ is formed at infinity. Then $P'E = -f_e$ and the angular magnification is

$$m = -\frac{f_o}{f_e}. \quad \ldots (19.9)$$

Note that the focal length $f_e$ is negative because the eyepiece is a diverging lens. Thus, $m$ is positive as expected for an erect image. If the final image is formed at the near point, the magnification is increased.

The lens formula is

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}.$$

For the eyepiece,

$$v = -P''E = -D,\ u = P'E$$

and $f = f_e$ ($f_e$ is itself negative).

Thus,

$$\frac{1}{-D} - \frac{1}{P'E} = \frac{1}{f_e}$$

or,
$$\frac{1}{P'E} = -\frac{1}{f_e} - \frac{1}{D}$$
$$= -\frac{1}{f_e}\left(1 + \frac{f_e}{D}\right). \quad \ldots \text{(iv)}$$

By (iii), the angular magnification is

$$m = -\frac{f_o}{f_e}\left(1 + \frac{f_e}{D}\right).$$

**Length of the Telescope**

The length of a Galilean telescope is

$$L = OE = OP' - P'E$$
$$= f_o - P'E.$$

For normal adjustment, $P'E = -f_e$ and hence the length of the tube is

$$L = f_o + f_e = f_o - |f_e|.$$

For the adjustment for near point vision, by (iv),

$$P'E = \frac{-f_e D}{D + f_e} \text{ and } L = f_o + \frac{f_e D}{D + f_e}$$
$$= f_o - \frac{|f_e| D}{D - |f_e|}.$$

## 19.6 RESOLVING POWER OF A MICROSCOPE AND A TELESCOPE

The resolving power of a microscope is defined as the reciprocal of the distance between two objects which can be just resolved when seen through the microscope. It depends on the wavelength λ of the light, the refractive index μ of the medium between the object and the objective of the microscope, and the angle θ subtended by a radius of the objective on one of the object. It is given by

$$R = \frac{1}{\Delta d} = \frac{2\mu \sin\theta}{\lambda}.$$

To increase the resolving power, the objective and the object are kept immersed in oil. This increases μ and hence $R$.

The resolving power of a telescope is defined as the reciprocal of the angular separation between two distant objects which are just resolved when viewed through a telescope. It is given by

$$R = \frac{1}{\Delta\theta} = \frac{a}{1{\cdot}22\,\lambda},$$

where $a$ is the diameter of the objective of the telescope. That is why, the telescopes with larger objective aperture (1 m or more) are used in astronomical studies.

## 19.7 DEFECTS OF VISION

As described earlier, the ciliary muscles control the curvature of the lens in the eye and hence can alter the effective focal length of the system. When the muscles are fully relaxed, the focal length is maximum. When the muscles are strained, the curvature of the lens increases and the focal length decreases. For a clear vision, the image must be formed on the retina. The image-distance is, therefore, fixed for clear vision and it equals the distance of the retina from the eye-lens. It is about 2·5 cm for a grown-up person. If we apply the lens formula to eye, the magnitudes of the object-distance, the image-distance and the effective focal length satisfy

$$\frac{1}{v_o} + \frac{1}{u_o} = \frac{1}{f}$$

or,
$$\frac{1}{u_o} = \frac{1}{f} - \frac{1}{v_o}. \quad \ldots \text{(i)}$$

Here $v_o$ is fixed, hence by changing $f$, the eye can be focussed on objects placed at different values of $u_o$.

We see from (i) that as $f$ increases, $u_o$ increases and as $f$ decreases, $u_o$ decreases. The maximum distance one can see is given by

$$\frac{1}{u_{\max}} = \frac{1}{f_{\max}} - \frac{1}{v_o}, \quad \ldots \text{(ii)}$$

where $f_{max}$ is the maximum focal length possible for the eye-lens.

The focal length is maximum when the ciliary muscles are fully relaxed. In a normal eye, this focal length equals the distance $v_o$ from the lens to the retina. Thus,

$v_o = f_{max}$ by (ii), $u_{max} = \infty$.

A person can theoretically have clear vision of objects situated at any large distance from the eye. For the closer objects, $u$ is smaller and hence $f$ should be smaller. The smallest distance at which a person can clearly see, is related to the minimum possible focal length $f$. The ciliary muscles are most strained in this position. By (ii), the closest distance for clear vision is given by

$$\frac{1}{u_{min}} = \frac{1}{f_{min}} - \frac{1}{v_o} . \quad \ldots \text{(iii)}$$

For an average grown-up person, $u_{min}$ should be around 25 cm or less. This is a convenient distance at which one can hold an object in his/her hand and see. Thus, a normal eye can clearly see objects placed in the range starting from about 25 cm from the eye to a large distance, say, of the order of several kilometers. The nearest point where an eye can clearly see is called the *near point* and the farthest point up to which an eye can clearly see is called the *far point.* For a normal eye, the distance of the near point should be around 25 cm or less and the far point should be at infinity. We now describe some common defects of vision.

**(A) Nearsightedness**

A person suffering from this defect cannot see distant objects clearly. This is because $f_{max}$ is less than the distance from the lens to the retina and the parallel rays coming from the distant object focus short of the retina. The ciliary muscles are fully relaxed in this case and any strain in it can only further decrease the focal length which is of no help to see distant objects.

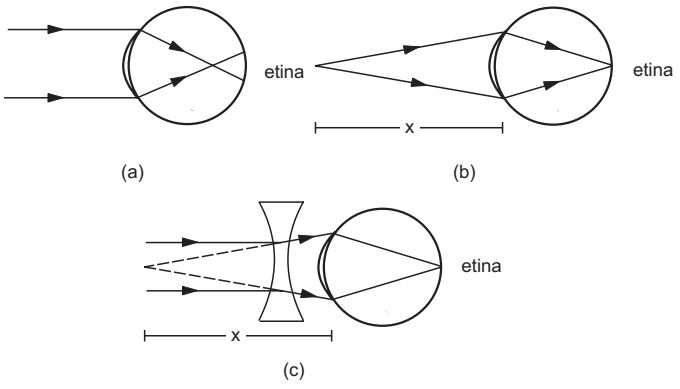



Figure 19.8

Nearsightedness is also called *myopia*. This may result because the lens is too thick or the diameter of the eyeball is larger than usual. The remedy of myopia is quite simple. The rays should be made a bit divergent before entering the eye so that they may focus a little later. Thus, a divergent lens should be given to a myopic person to enable him/her to see distant objects clearly.

**Power of the Lens Needed**

Suppose, a person can see an object at a maximum distance $x$. Thus, with fully relaxed muscles, rays coming from the distance $x$ converge on the retina. Figure (19.8) shows the situation. As is clear from the figure, if the eye is to see a distant object clearly, the diverging lens should form the virtual image of this distant object at a distance $x$. Thus, the required focal length of the diverging lens is $f = -x$ and the power is

$$P = \frac{1}{f} = \frac{1}{-x} .$$

***Example 19.3***

*A nearsighted man can clearly see objects up to a distance of* 1·5 m. *Calculate the power of the lens of the spectacles necessary for the remedy of this defect.*

***Solution*** **:** The lens should form a virtual image of a distant object at 1·5 m from the lens. Thus, it should be a divergent lens and its focal length should be – 1·5 m. Hence,

$$f = -1{\cdot}5 \text{ m}$$

or, $$P = \frac{1}{f} = -\frac{1}{1{\cdot}5} \text{ m}^{-1} = -0{\cdot}67 \text{ D}.$$

**(B) Farsightedness**

A person suffering from farsightedness cannot clearly see objects close to the eye. The least distance for clear vision is appreciably larger than 25 cm and the person has to keep the object inconveniently away from the eye. Thus, reading a newspaper or viewing a small thing held in the hands is difficult for such a person.

Farsightedness is also known as *hyperopia*. Generally, it occurs when the eye-lens is too thin at the centre and/or the eyeball is shorter than normal. The ciliary muscles even in their most strained position are not able to reduce the focal length to appropriate value. The defect can also arise if the ciliary muscles become weak and are not able to strain enough to reduce the focal length to appropriate value. When farsightedness develops due to this reason, it is known as *presbyopia*.

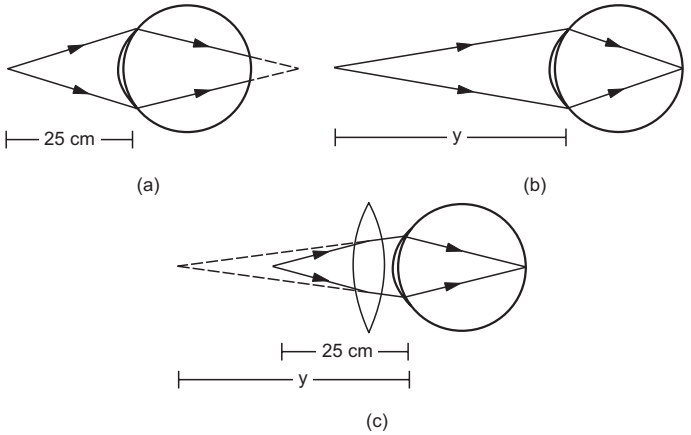



Figure 19.9

Figure (19.9) shows the situation and the remedy for farsightedness. The rays starting from the normal near point 25 cm would focus behind the retina. They should be made a bit less divergent before sending them to the eye so that they may focus on the retina. This can be achieved by putting a converging lens in front of the eye.

Suppose, the eye can clearly see an object at a minimum distance $y$. If the eye is to see clearly an object at 25 cm, the converging lens should form an image of this object at a distance $y$ (figure 19.9c).

Here $u = -25$ cm and $v = -y$.

Using the lens formula

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f},$$

we get

$$\frac{1}{-y} - \frac{1}{-25\text{ cm}} = \frac{1}{f}$$

or, $$P = \frac{1}{f} = \frac{1}{25\text{ cm}} - \frac{1}{y}.$$

**(C) Astigmatism**

Another kind of defect arises in the eye when the eye-lens develops different curvatures along different planes. Such a person cannot see all the directions equally well. A particular direction in the plane perpendicular to the line of sight is most visible. The direction perpendicular to this is least visible. Here is a 'do it yourself' test for astigmatism. Figure (19.10) shows four lines passing through a point. The lines are assumed to be drawn with equal intensity (you can draw such lines on a paper with equal intensity and do the test). If you can see all the lines equally distinct and intense, you are not astigmatic. If a particular line say (2)-(2) appears to be most intense and the perpendicular line (4)-(4) appears least intense, you are most likely astigmatic. If it is so, rotate the book through a right angle so that (2)-(2) takes the place of (4)-(4) and vice versa. If you are really astigmatic, you will find that now (4)-(4) appears most intense and (2)-(2) appears least intense.

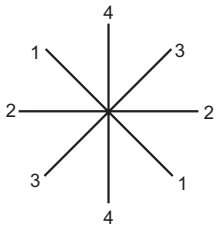



Figure 19.10

The remedy to astigmatism is also painless. Glasses with different curvatures in different planes are used to compensate for the deshaping of the eye-lens. Opticians call them cylindrical glasses.

A person may develop any of the above defects or a combination of more than one. Quite common in old age is the combination of nearsightedness and farsightedness. Such a person may need a converging glass for reading purpose and a diverging glass for seeing at a distance. Such persons either keep two sets of spectacles or a spectacle with upper portion divergent and lower portion convergent (bifocal).

### *Worked Out Examples*

**1.** *An object is seen through a simple microscope of focal length* 12 cm. *Find the angular magnification produced if the image is formed at the near point of the eye which is* 25 cm *away from it.*

***Solution*** **:** The angular magnification produced by a simple microscope when the image is formed at the near point of the eye is given by

$$m = 1 + \frac{D}{f}.$$

Here $f = 12$ cm, $D = 25$ cm. Hence,

$$m = 1 + \frac{25}{12} = 3{\cdot}08.$$

**2.** *A* 10 D *lens is used as a magnifier. Where should the object be placed to obtain maximum angular magnification for a normal eye (near point* = 25 cm*)*?

***Solution*** **:** Maximum angular magnification is achieved when the final image is formed at the near point. Thus, $v = -25$ cm. The focal length is $f = \frac{1}{10}\text{ m} = 10$ cm.

We have,

$$\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$$

or, $$-\frac{1}{25\text{ cm}}-\frac{1}{u}=\frac{1}{10\text{ cm}}$$

or, $$\frac{1}{u}=-\frac{1}{25\text{ cm}}-\frac{1}{10\text{ cm}}$$

or, $$u=-\frac{50}{7}\text{ cm}=-7{\cdot}1\text{ cm}.$$

**3.** *A small object is placed at a distance of* 3·6 cm *from a magnifier of focal length* 4·0 cm. *(a) Find the position of the image. (b) Find the linear magnification. (c) Find the angular magnification.*

***Solution*** **:**

(a) Using $\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$,

$$\frac{1}{v}=\frac{1}{u}+\frac{1}{f}=\frac{1}{-3{\cdot}6\text{ cm}}+\frac{1}{4{\cdot}0\text{ cm}}.$$

or, $v=-36$ cm.

(b) Linear magnification $=\frac{v}{u}$

$$=\frac{-36\text{ cm}}{-3{\cdot}6\text{ cm}}=10.$$

(c) If the object is placed at a distance $u_o$ from the lens, the angle subtended by the object on the lens is $\beta=\frac{h}{u_o}$ where $h$ is the height of the object. The maximum angle subtended on the unaided eye is $\alpha=\frac{h}{D}$.

Thus, the angular magnification is

$$m=\frac{\beta}{\alpha}=\frac{D}{u_o}=\frac{25\text{ cm}}{3{\cdot}6\text{ cm}}=7{\cdot}0.$$

**4.** *A compound microscope consists of an objective of focal length* 1·0 cm *and an eyepiece of focal length* 5·0 cm *separated by* 12·2 cm. *(a) At what distance from the objective should an object be placed to focus it properly so that the final image is formed at the least distance of clear vision (*25 cm*)? (b) Calculate the angular magnification in this case.*

***Solution*** **:**

(a) For the eyepiece, $v_e=-25$ cm and $f_e=+5$ cm.

Using $\frac{1}{v_e}-\frac{1}{u_e}=\frac{1}{f_e}$,

$$\frac{1}{u_e}=\frac{1}{v_e}-\frac{1}{f_e}$$

$$=-\frac{1}{25\text{ cm}}-\frac{1}{5\text{ cm}}$$

or, $$u_e=-\frac{25}{6}\text{ cm}=-4{\cdot}17\text{ cm}\approx-4{\cdot}2\text{ cm}.$$

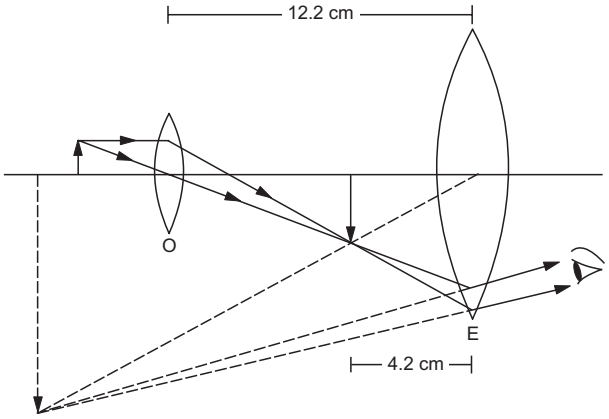



Figure 19-W1

As the objective is 12·2 cm away from the eyepiece, the image formed by the objective is 12·2 cm – 4·2 cm = 8·0 cm away from it. For the objective,

$$v=+8{\cdot}0\text{ cm},\quad f_o=+1{\cdot}0\text{ cm}.$$

Using

$$\frac{1}{v}-\frac{1}{u}=\frac{1}{f_o},$$

$$\frac{1}{u}=\frac{1}{v}-\frac{1}{f_o}$$

$$=\frac{1}{8{\cdot}0\text{ cm}}-\frac{1}{1{\cdot}0\text{ cm}}$$

or, $$u=-\frac{8{\cdot}0}{7{\cdot}0}\text{ cm}=-1{\cdot}1\text{ cm}.$$

(b) The angular magnification is

$$m=\frac{v}{u}\left(1+\frac{D}{f_e}\right)$$

$$=\frac{+8{\cdot}0\text{ cm}}{-1{\cdot}1\text{ cm}}\left(1+\frac{25\text{ cm}}{5\text{ cm}}\right)\approx-44.$$

**5**. *The separation L between the objective (*$f=0{\cdot}5$ cm*) and the eyepiece (*$f=5$ cm*) of a compound microscope is* 7 cm. *Where should a small object be placed so that the eye is least strained to see the image? Find the angular magnification produced by the microscope.*

***Solution*** **:** The eye is least strained if the final image is formed at infinity. In such a case, the image formed by the objective should fall at the focus of the eyepiece. As $f_e=5$ cm and $L=7$ cm, this first image should be formed at 7 cm – 5 cm = 2 cm from the objective. Thus, $v=+2$ cm. Also, $f_o=0{\cdot}5$ cm. For the objective, using

$$\frac{1}{v}-\frac{1}{u}=\frac{1}{f_o},$$

$$\frac{1}{u}=\frac{1}{v}-\frac{1}{f_o}$$

$$=\frac{1}{2\text{ cm}}-\frac{1}{0{\cdot}5\text{ cm}}$$

or, $u = -\frac{2}{3}$ cm.

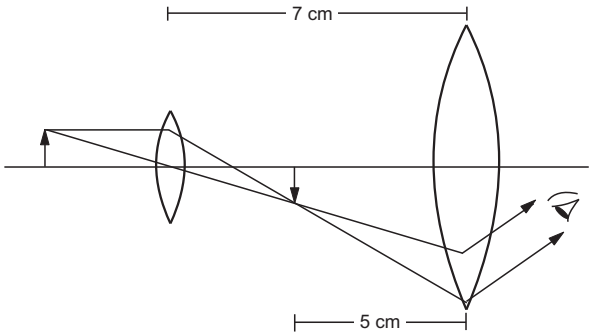



Figure 19-W2

The angular magnification in this case is

$$m = \frac{v}{u}\frac{D}{f_o}$$

$$= \frac{2\text{ cm}}{-(2/3)\text{ cm}}\frac{25\text{ cm}}{5\text{ cm}} = -15.$$

**6.** *An astronomical telescope has an objective of focal length* 200 cm *and an eyepiece of focal length* 4·0 cm. *The telescope is focussed to see an object* 10 km *from the objective. The final image is formed at infinity. Find the length of the tube and the angular magnification produced by the telescope.*

***Solution*** **:** As the object distance 10 km is much larger than the focal length 200 cm, the first image is formed almost at the focus of the objective. It is thus 200 cm from the objective. This image acts as the object for the eyepiece. To get the final image at infinity, this first image should be at the first focus of the eyepiece. The length of the tube is, therefore, 200 cm + 4 cm = 204 cm. The angular magnification in this case

$$m = -\frac{f_o}{f_e} = -\frac{200}{4} = -50.$$

**7.** *A Galilean telescope is constructed by an objective of focal length* 50 cm *and an eyepiece of focal length* 5·0 cm. *(a) Find the tube length and magnifying power when it is used to see an object at a large distance in normal adjustment. (b) If the telescope is to focus an object* 2·0 m *away from the objective, what should be the tube length and angular magnification, the image again forming at infinity*?

***Solution*** **:**

$$f_o = 50\text{ cm}, \quad f_e = -5\text{ cm}.$$

(a) $L = f_o - |f_e| = (50 - 5)\text{ cm} = 45\text{ cm}$

and $$m = -\frac{f_o}{f_e} = \frac{50}{5} = 10.$$

(b) Using the equation $\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$ for the objective,

$$\frac{1}{v} = \frac{1}{f_o} + \frac{1}{u}$$

$$= \frac{1}{50\text{ cm}} + \frac{1}{-200\text{ cm}}$$

or, $v = 66{\cdot}67$ cm.

The tube length $L = v - |f_e| = (66{\cdot}67 - 5)$ cm

or, $L = 61{\cdot}67$ cm.

To calculate the angular magnification, we assume that the object remains at large distance from the eye. In this case, the angular magnification

$$m = \frac{v}{f_e} = \frac{66{\cdot}67}{5} = 13{\cdot}33.$$

$v$ is the distance of the first image from the objective which is substituted for $f_o$.

**8.** *The image of the moon is focussed by a converging lens of focal length* 50 cm *on a plane screen. The image is seen by an unaided eye from a distance of* 25 cm. *Find the angular magnification achieved due to the converging lens.*

***Solution*** **:**

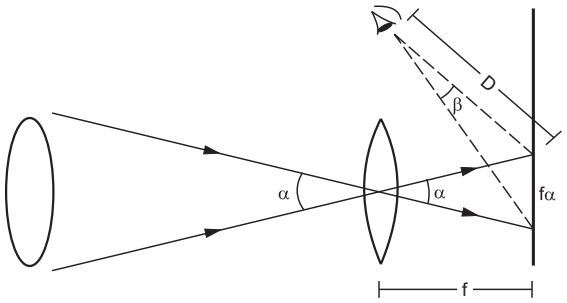



Figure 19-W3

Suppose the moon subtends an angle α on the lens. This will also be the angle subtended by the moon on the eye if the moon is directly viewed. The image is formed in the focal plane. The linear size of the image $\approx f\alpha = (50\text{ cm})\alpha$.

If this image is seen from a distance of 25 cm, the angle formed by the image on the eye

$$|\beta| \approx \frac{(50\text{ cm})|\alpha|}{25\text{ cm}} = 2|\alpha|.$$

The angular magnification is

$$\frac{\beta}{\alpha} = -\left|\frac{\beta}{\alpha}\right| = -2.$$

**9.** *The near and far points of a person are at* 40 cm *and* 250 cm *respectively. Find the power of the lens he/she should use while reading at* 25 cm. *With this lens on the eye, what maximum distance is clearly visible*?

***Solution*** : If an object is placed at 25 cm from the correcting lens, it should produce the virtual image at 40 cm. Thus, $u = -25$ cm, $v = -40$ cm.

$$\frac{1}{f} = \frac{1}{v} - \frac{1}{u}$$

$$= -\frac{1}{40 \text{ cm}} + \frac{1}{25 \text{ cm}}$$

or, $$f = \frac{200}{3} \text{ cm} = +\frac{2}{3} \text{ m}$$

or, $$P = \frac{1}{f} = +1{\cdot}5 \text{ D}.$$

The unaided eye can see a maximum distance of 250 cm. Suppose the maximum distance for clear vision is $d$ when the lens is used. Then the object at a distance $d$ is imaged by the lens at 250 cm. We have,

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f},$$

or, $$-\frac{1}{250 \text{ cm}} - \frac{1}{d} = \frac{3}{200 \text{ cm}}$$

or, $$d = -53 \text{ cm}.$$

Thus, the person will be able to see up to a maximum distance of 53 cm.

**10.** *A young boy can adjust the power of his eye-lens between* 50 D *and* 60 D. *His far point is infinity. (a) What is the distance of his retina from the eye-lens*? *(b) What is his near point*?

***Solution*** :

(a) When the eye is fully relaxed, its focal length is largest and the power of the eye-lens is minimum. This power is 50 D according to the given data. The focal length is $\frac{1}{50}$ m = 2 cm. As the far point is at infinity, the parallel rays coming from infinity are focussed on the retina in the fully relaxed condition. Hence, the distance of the retina from the lens equals the focal length which is 2 cm.

(b) When the eye is focussed at the near point, the power is maximum which is 60 D. The focal length in this case is $f = \frac{1}{60}$ m $= \frac{5}{3}$ cm. The image is formed on the retina and thus $v$ = 2 cm. We have,

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

or, $$\frac{1}{u} = \frac{1}{v} - \frac{1}{f} = \frac{1}{2 \text{ cm}} - \frac{3}{5 \text{ cm}}$$

or, $$u = -10 \text{ cm}.$$

The near point is at 10 cm.

□

## QUESTIONS FOR SHORT ANSWER

**1.** Can virtual image be formed on the retina in a seeing process?

**2.** Can the image formed by a simple microscope be projected on a screen without using any additional lens or mirror?

**3.** The angular magnification of a system is less than one. Does it mean that the image formed is inverted?

**4.** A simple microscope using a single lens often shows coloured image of a white source. Why?

**5.** A magnifying glass is a converging lens placed close to the eye. A farsighted person uses spectacles having converging lenses. Compare the functions of a converging lens used as a magnifying glass and as spectacles.

**6.** A person is viewing an extended object. If a converging lens is placed in front of his eyes, will he feel that the size has increased?

**7.** The magnifying power of a converging lens used as a simple microscope is $\left(1 + \frac{D}{f}\right)$. A compound microscope is a combination of two such converging lenses. Why don't we have magnifying power $\left(1 + \frac{D}{f_o}\right)\left(1 + \frac{D}{f_e}\right)$? In other words, why can the objective not be treated as a simple microscope but the eyepiece can?

**8.** By mistake, an eye surgeon puts a concave lens in place of the lens in the eye after a cataract operation. Will the patient be able to see clearly any object placed at any distance?

**9.** The magnifying power of a simple microscope is given by $1 + \frac{D}{f}$, where $D$ is the least distance for clear vision. For farsighted persons, $D$ is greater than the usual. Does it mean that the magnifying power of a simple microscope is greater for a farsighted person as compared to a normal person? Does it mean that a farsighted person can see an insect more clearly under a microscope than a normal person?

**10.** Why are the magnification properties of microscopes and telescopes defined in terms of the ratio of angles and not in terms of the ratio of sizes of objects and images?

**11.** An object is placed at a distance of 30 cm from a converging lens of focal length 15 cm. A normal eye (near point 25 cm, far point infinity) is placed close to the lens on the other side. (a) Can the eye see the object clearly? (b) What should be the minimum separation between the lens and the eye so that the eye can clearly see the

object? (c) Can a diverging lens, placed in contact with the converging lens, help in seeing the object clearly when the eye is close to the lens?

**12.** A compound microscope forms an inverted image of an object. In which of the following cases it is likely to create difficulties? (a) Looking at small germs. (b) Looking at circular spots. (c) Looking at a vertical tube containing some water.

## OBJECTIVE I

**1.** The size of an object as perceived by an eye depends primarily on
(a) actual size of the object
(b) distance of the object from the eye
(c) aperture of the pupil
(d) size of the image formed on the retina.

**2.** The muscles of a normal eye are least strained when the eye is focussed on an object
(a) far away from the eye
(b) very close to the eye
(c) at about 25 cm from the eye
(d) at about 1 m from the eye.

**3.** A normal eye is not able to see objects closer than 25 cm because
(a) the focal length of the eye is 25 cm
(b) the distance of the retina from the eye-lens is 25 cm
(c) the eye is not able to decrease the distance between the eye-lens and the retina beyond a limit
(d) the eye is not able to decrease the focal length beyond a limit.

**4.** When objects at different distances are seen by the eye, which of the following remain constant?
(a) The focal length of the eye-lens.
(b) The object-distance from the eye-lens.
(c) The radii of curvature of the eye-lens.
(d) The image-distance from the eye-lens.

**5.** A person $A$ can clearly see objects between 25 cm and 200 cm. Which of the following may represent the range of clear vision for a person $B$ having muscles stronger than $A$, but all other parameters of eye identical to that of $A$?
(a) 25 cm to 200 cm (b) 18 cm to 200 cm
(c) 25 cm to 300 cm (d) 18 cm to 300 cm

**6.** The focal length of a normal eye-lens is about
(a) 1 mm (b) 2 cm (c) 25 cm (d) 1 m.

**7.** The distance of the eye-lens from the retina is $x$. For a normal eye, the maximum focal length of the eye-lens
(a) $= x$ (b) $< x$ (c) $> x$ (d) $= 2x$.

**8.** A man wearing glasses of focal length +1 m cannot clearly see beyond 1 m
(a) if he is farsighted (b) if he is nearsighted
(c) if his vision is normal (d) in each of these cases

**9.** An object is placed at a distance $u$ from a simple microscope of focal length $f$. The angular magnification obtained depends
(a) on $f$ but not on $u$ (b) on $u$ but not on $f$
(c) on $f$ as well as $u$ (d) neither on $f$ nor on $u$.

**10.** To increase the angular magnification of a simple microscope, one should increase
(a) the focal length of the lens
(b) the power of the lens
(c) the aperture of the lens
(d) the object size.

**11.** A man is looking at a small object placed at his near point. Without altering the position of his eye or the object, he puts a simple microscope of magnifying power 5 X before his eyes. The angular magnification achieved is
(a) 5 (b) 2·5 (c) 1 (d) 0·2.

## OBJECTIVE II

**1.** When we see an object, the image formed on the retina is
(a) real (b) virtual (c) erect (d) inverted.

**2.** In which of the following the final image is erect?
(a) Simple microscope (b) Compound microscope
(c) Astronomical telescope (d) Galilean telescope

**3.** The maximum focal length of the eye-lens of a person is greater than its distance from the retina. The eye is
(a) always strained in looking at an object
(b) strained for objects at large distances only
(c) strained for objects at short distances only
(d) unstrained for all distances.

**4.** Mark the correct options.
(a) If the far point goes ahead, the power of the divergent lens should be reduced.
(b) If the near point goes ahead, the power of the convergent lens should be reduced.
(c) If the far point is 1 m away from the eye, divergent lens should be used.
(d) If the near point is 1 m away from the eye, divergent lens should be used.

**5.** The focal length of the objective of a compound microscope is $f_o$ and its distance from the eyepiece is $L$. The object is placed at a distance $u$ from the objective. For proper working of the instrument,
(a) $L < u$ (b) $L > u$ (c) $f_o < L < 2f_o$ (d) $L > 2f_o$.

## EXERCISES

**1.** A person looks at different trees in an open space with the following details. Arrange the trees in decreasing order of their apparent sizes.

| Tree | Height(m) | Distance from the eye(m) |
|---|---|---|
| *A* | 2·0 | 50 |
| *B* | 2·5 | 80 |
| *C* | 1·8 | 70 |
| *D* | 2·8 | 100 |

**2.** An object is to be seen through a simple microscope of focal length 12 cm. Where should the object be placed so as to produce maximum angular magnification? The least distance for clear vision is 25 cm.

**3.** A simple microscope has a magnifying power of 3·0 when the image is formed at the near point (25 cm) of a normal eye. (a) What is its focal length? (b) What will be its magnifying power if the image is formed at infinity?

**4.** A child has near point at 10 cm. What is the maximum angular magnification the child can have with a convex lens of focal length 10 cm?

**5.** A simple microscope is rated 5 X for a normal relaxed eye. What will be its magnifying power for a relaxed farsighted eye whose near point is 40 cm?

**6.** Find the maximum magnifying power of a compound microscope having a 25 diopter lens as the objective, a 5 diopter lens as the eyepiece and the separation 30 cm between the two lenses. The least distance for clear vision is 25 cm.

**7.** The separation between the objective and the eyepiece of a compound microscope can be adjusted between 9·8 cm to 11·8 cm. If the focal lengths of the objective and the eyepiece are 1·0 cm and 6 cm respectively, find the range of the magnifying power if the image is always needed at 24 cm from the eye.

**8.** An eye can distinguish between two points of an object if they are separated by more than 0·22 mm when the object is placed at 25 cm from the eye. The object is now seen by a compound microscope having a 20 D objective and 10 D eyepiece separated by a distance of 20 cm. The final image is formed at 25 cm from the eye. What is the minimum separation between two points of the object which can now be distinguished?

**9.** A compound microscope has a magnifying power of 100 when the image is formed at infinity. The objective has a focal length of 0·5 cm and the tube length is 6·5 cm. Find the focal length of the eyepiece.

**10.** A compound microscope consists of an objective of focal length 1 cm and an eyepiece of focal length 5 cm. An object is placed at a distance of 0·5 cm from the objective. What should be the separation between the lenses so that the microscope projects an inverted real image of the object on a screen 30 cm behind the eyepiece?

**11.** An optical instrument used for angular magnification has a 25 D objective and a 20 D eyepiece. The tube length is 25 cm when the eye is least strained. (a) Whether it is a microscope or a telescope? (b) What is the angular magnification produced?

**12.** An astronomical telescope is to be designed to have a magnifying power of 50 in normal adjustment. If the length of the tube is 102 cm, find the powers of the objective and the eyepiece.

**13.** The eyepiece of an astronomical telescope has a focal length of 10 cm. The telescope is focussed for normal vision of distant objects when the tube length is 1·0 m. Find the focal length of the objective and the magnifying power of the telescope.

**14.** A Galilean telescope is 27 cm long when focussed to form an image at infinity. If the objective has a focal length of 30 cm, what is the focal length of the eyepiece?

**15.** A farsighted person cannot see objects placed closer to 50 cm. Find the power of the lens needed to see the objects at 20 cm.

**16.** A nearsighted person cannot clearly see beyond 200 cm. Find the power of the lens needed to see objects at large distances.

**17.** A person wears glasses of power – 2·5 D. Is the person farsighted or nearsighted? What is the far point of the person without the glasses?

**18.** A professor reads a greeting card received on his 50th birthday with + 2·5 D glasses keeping the card 25 cm away. Ten years later, he reads his farewell letter with the same glasses but he has to keep the letter 50 cm away. What power of lens should he now use?

**19.** A normal eye has retina 2 cm behind the eye-lens. What is the power of the eye-lens when the eye is (a) fully relaxed, (b) most strained?

**20.** The near point and the far point of a child are at 10 cm and 100 cm. If the retina is 2·0 cm behind the eye-lens, what is the range of the power of the eye-lens?

**21.** A nearsighted person cannot see beyond 25 cm. Assuming that the separation of the glass from the eye is 1 cm, find the power of lens needed to see distant objects.

**22.** A person has near point at 100 cm. What power of lens is needed to read at 20 cm if he/she uses (a) contact lens, (b) spectacles having glasses 2·0 cm separated from the eyes?

**23.** A lady uses + 1·5 D glasses to have normal vision from 25 cm onwards. She uses a 20 D lens as a simple microscope to see an object. Find the maximum magnifying power if she uses the microscope (a) together with her glass (b) without the glass. Do the answers suggest that an object can be more clearly seen through a microscope without using the correcting glasses?

**24.** A lady cannot see objects closer than 40 cm from the left eye and closer than 100 cm from the right eye. While on a mountaineering trip, she is lost from her team. She tries to make an astronomical telescope from her reading glasses to look for her teammates. (a) Which glass should she use as the eyepiece? (b) What magnification can she get with relaxed eye?

## ANSWERS

### OBJECTIVE I

1. (d) 2. (a) 3. (d) 4. (d) 5. (b) 6. (b)
7. (a) 8. (d) 9. (c) 10. (b) 11. (c)

### OBJECTIVE II

1. (a), (d) 2. (a), (d) 3. (a)
4. (a), (c) 5. (b), (d)

### EXERCISES

1. *A, B, D, C*
2. 8·1 cm from the lens
3. (a) 12·5 cm (b) 2·0
4. 2
5. 8 X
6. 8·4
7. 20 to 30
8. 0·04 mm
9. 2 cm
10. 5 cm
11. (a) microscope (b) 20
12. 1 D, 50 D
13. 90 cm, 9
14. 3 cm
15. 3 D
16. – 0·5 D
17. nearsighted, 40 cm
18. + 4·5 D
19. (a) 50 D (b) 54 D
20. +60 D to +51 D
21. – 4·2 D
22. (a) +4 D (b) +4·53 D
23. (a) 6 (b) 9
24. (a) right lens (b) 2

□

## CHAPTER 20

# DISPERSION AND SPECTRA

### 20.1 DISPERSION

As mentioned earlier, the refractive index of a material depends slightly on the wavelength of light. The relation between the two may be approximately described by the equation

$$\mu = \mu_0 + \frac{A}{\lambda^2},$$

where $A$ is a small positive constant known as Cauchy's constant. The refractive index decreases as the wavelength increases. For visible light, it is maximum for the violet end and minimum for the red end. Figure (20.1) shows the variation of refractive index with wavelength for some transparent materials.

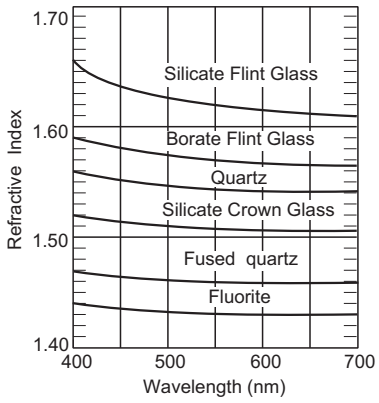



Figure 20.1

Because of the difference in refractive indices, light of different colours bend through different angles on refraction. If white light passes through a glass prism (figure 20.2), the violet rays deviate the most and the red rays deviate the least. Thus, white light is separated into its various component colours. This phenomenon of separation of different constituent colours of light while passing through a transparent medium is known as *dispersion of light.*

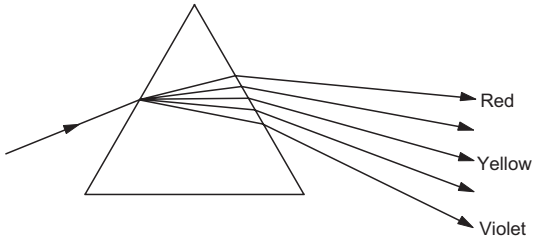



Figure 20.2

### 20.2 DISPERSIVE POWER

Consider a prism of a transparent material. When a beam of white light is passed through the prism, light of different wavelengths are deviated by different amounts. The overall deviation of the light beam is conventionally measured by the deviation of yellow light, as the wavelength of yellow light is roughly the average wavelength of the visible region. In figure (20.3), this deviation is shown by the symbol $\delta_y$. It is clear that if $\delta_r$ and $\delta_v$ are the deviations for red and violet components, the angular divergence of the transmitted beam is $\delta_v - \delta_r$. This divergence is called *angular dispersion*.

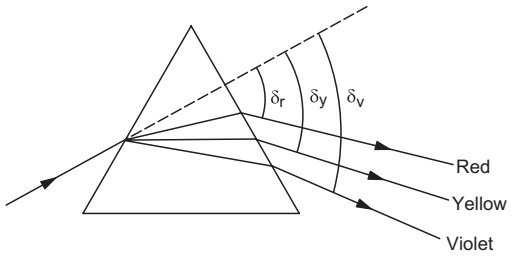



Figure 20.3

The mean deviation depends on the average refractive index $\mu$ and the angular dispersion depends on the difference $\mu_v - \mu_r$. It may be seen from figure (20.1) that if the average value of $\mu$ is small (fluorite), $\mu_v - \mu_r$ is also small and if the average value of $\mu$ is large (silicate flint glass), $\mu_v - \mu_r$ is also large. Thus, larger the mean deviation, larger will be the angular dispersion.

The *dispersive power* of a material is defined as the ratio of angular dispersion to the average deviation when a light beam is transmitted through a thin prism

placed in a position so that the mean ray (ray having the mean wavelength) passes symmetrically through it.

When a light ray passes symmetrically through a prism of refracting angle $A$, it suffers minimum deviation $\delta$ given by

$$\mu = \frac{\sin\frac{A+\delta}{2}}{\sin\frac{A}{2}}.$$

If the refracting angle $A$ is small, the deviation $\delta$ is also small. Then,

$$\mu \approx \frac{\frac{A+\delta}{2}}{\frac{A}{2}} = 1 + \frac{\delta}{A}$$

or, $$\delta = (\mu - 1)\, A.$$

This equation is also valid if the light ray does not pass symmetrically through the prism, but the angle $A$ and the angle of incidence $i$ are small.

Suppose, a beam of white light goes through such a prism. The deviation of violet, yellow and the red light are

$$\delta_v = (\mu_v - 1)\, A$$

$$\delta_y = (\mu_y - 1)\, A$$

and $$\delta_r = (\mu_r - 1)\, A.$$

The angular dispersion is $\delta_v - \delta_r = (\mu_v - \mu_r)\, A$. The average deviation is $\delta_y = (\mu_y - 1)A$. Thus, the dispersive power of the medium is

$$\omega = \frac{\mu_v - \mu_r}{\mu_y - 1}. \qquad \ldots (20.1)$$

This equation itself may be taken as the definition of dispersive power.

Refractive index is a continuous function of wavelength. Usually three wavelengths are selected, one from violet, one from yellow and one from red region and dispersive power is defined as (20.1) for these wavelengths.

***Example 20.1***

*Find the dispersive power of flint glass. The refractive indices of flint glass for red, yellow and violet light are* 1·613, 1·620 *and* 1·632 *respectively.*

***Solution*** : The dispersive power is $\omega = \dfrac{\mu_v - \mu_r}{\mu_y - 1}$

$$= \frac{1{\cdot}632 - 1{\cdot}613}{1{\cdot}620 - 1} = 0{\cdot}0306.$$

***Example 20.2***

*The focal lengths of a thin lens for red and violet light are* 90·0 cm *and* 86·4 cm *respectively. Find the dispersive power of the material of the lens. Make appropriate assumptions.*

***Solution*** : We have

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

or, $$\mu - 1 = \frac{1}{f} \cdot \frac{1}{\frac{1}{R_1} - \frac{1}{R_2}} = \frac{K}{f}.$$

Thus, $$\mu_v - 1 = \frac{K}{f_v}$$

and, $$\mu_r - 1 = \frac{K}{f_r}$$

so that $$\mu_v - \mu_r = K\left(\frac{1}{f_v} - \frac{1}{f_r}\right)$$

$$= K\left[\frac{1}{86{\cdot}4\text{ cm}} - \frac{1}{90\text{ cm}}\right] = K \times 4{\cdot}6 \times 10^{-4}\text{ cm}^{-1}.$$

Also, we can assume that

$$\mu_y - 1 = \frac{\mu_v + \mu_r}{2} - 1 = \frac{\mu_v - 1}{2} + \frac{\mu_r - 1}{2}$$

$$= \frac{K}{2}\left(\frac{1}{f_v} + \frac{1}{f_r}\right)$$

$$= \frac{K}{2}\left[\frac{1}{86{\cdot}4\text{ cm}} + \frac{1}{90\text{ cm}}\right] = K \times 1{\cdot}1 \times 10^{-2}\text{ cm}^{-1}.$$

Thus, the dispersive power of the material of the lens is

$$\omega = \frac{\mu_v - \mu_r}{\mu_y - 1} = \frac{4{\cdot}6 \times 10^{-4}}{1{\cdot}1 \times 10^{-2}} = 0{\cdot}042.$$

## 20.3 DISPERSION WITHOUT AVERAGE DEVIATION AND AVERAGE DEVIATION WITHOUT DISPERSION

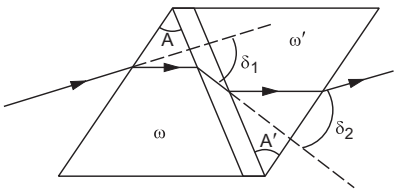



Figure 20.4

Figure (20.4) shows two thin prisms placed in contact in such a way that the two refracting angles are reversed with respect to each other. Suppose, the refracting angles of the two prisms are $A$ and $A'$ and their dispersive powers are $\omega$ and $\omega'$ respectively.

Consider a ray of light for which the refractive indices of the materials of the two prisms are $\mu$ and $\mu'$. Assuming that the ray passes through the prisms in symmetrical situation, the deviations produced by the two prisms are

$$\delta_1 = (\mu - 1)\, A$$

and $$\delta_2 = (\mu' - 1)\, A'.$$

As the two deviations are opposite to each other, the net deviation is

$$\delta = \delta_1 - \delta_2$$
$$= (\mu - 1)A - (\mu' - 1)A'. \quad \ldots \text{(i)}$$

If white light passes through the combination, the net deviation of the violet ray is

$$\delta_v = (\mu_v - 1)A - (\mu'_v - 1)A'$$

and that of the red ray is

$$\delta_r = (\mu_r - 1)A - (\mu'_r - 1)A'.$$

The angular dispersion produced by the combination is

$$\delta_v - \delta_r = (\mu_v - \mu_r)A - (\mu'_v - \mu'_r)A'. \quad \ldots \text{(ii)}$$

The dispersive powers are given by

$$\omega = \frac{\mu_v - \mu_r}{\mu_y - 1}$$

and $$\omega' = \frac{\mu'_v - \mu'_r}{\mu'_y - 1}.$$

Thus, by (ii), the net angular dispersion is

$$\delta_v - \delta_r = (\mu_y - 1)\omega A - (\mu'_y - 1)\omega' A'. \quad \ldots \text{(iii)}$$

The net deviation of the yellow ray, i.e., the average deviation, is, by (i),

$$\delta_y = (\mu_y - 1)A - (\mu'_y - 1)A'. \quad \ldots \text{(iv)}$$

**Dispersion without Average Deviation**

If the combination is not to produce a net average deviation in the beam, $\delta_y$ should be 0. By (iv), the required condition is

$$(\mu_y - 1)A = (\mu'_y - 1)A'. \quad \ldots (20.2)$$

Using this in (iii), the net angular dispersion produced is

$$\delta_v - \delta_r = (\mu_y - 1)A\,(\omega - \omega'). \quad \ldots (20.3)$$

By choosing $\omega$ and $\omega'$ different and the refracting angles to satisfy (20.2), one can get dispersion without average deviation.

**Average Deviation without Dispersion**

If the combination is not to produce a net dispersion, $\delta_v - \delta_r = 0$. By (iii),

$$(\mu_y - 1)\omega A = (\mu'_y - 1)\omega' A'. \quad \ldots (20.4)$$

By (ii), this condition may also be written as

$$(\mu_v - \mu_r)A = (\mu'_v - \mu'_r)A'. \quad \ldots (20.5)$$

The net average deviation produced is, by (i),

$$\delta = (\mu_y - 1)A - (\mu'_y - 1)A'$$
$$= (\mu_y - 1)A\left[1 - \frac{\mu'_y - 1}{\mu_y - 1}\frac{A'}{A}\right].$$

By (20.4),

$$\frac{(\mu'_y - 1)A'}{(\mu_y - 1)A} = \frac{\omega}{\omega'}$$

so that the net average deviation produced by the combination is

$$\delta = (\mu_y - 1)\, A\left(1 - \frac{\omega}{\omega'}\right). \quad \ldots (20.6)$$

## 20.4 SPECTRUM

When light coming from a source is dispersed by a prism or by any other dispersing element, light of different wavelengths are deviated through different angles and get separated. Such a dispersed light may be received on a screen, on a photographic plate or it may be viewed directly by the eye. A collection of dispersed light giving its wavelength composition is called a *spectrum*. As a very simple demonstration, let white light fall on a prism and collect the transmitted light on a white wall or a white paper. A spectrum consisting of different colours from red to violet is obtained.

**Pure and Impure Spectrum**

The spectrum of visible light shows different colours. In an ideal situation, light of one wavelength should occupy one particular spatial position in the spectrum. In such a case, no two wavelengths overlap in the dispersed beam. Each colour then gives its sharp impression. Such a spectrum is called a *pure spectrum*. To get a pure spectrum,

(a) the light beam incident on the dispersing element (prism, grating, etc.) should be parallel, and

(b) the dispersed light should be focussed in such a way that all the rays of a particular wavelength are collected at one place.

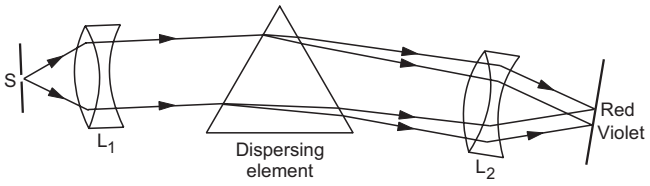



Figure 20.5

These conditions may be achieved to a good approximation using the arrangement shown in figure (20.5). A narrow slit $S$ allows a thin pencil of light to be analyzed. The slit is placed in the focal plane of an achromatic lens combination $L_1$. The light is dispersed by the dispersing element such as a prism or a grating. The emergent rays for a particular wavelength are all parallel. Another achromatic lens combination $L_2$ is used to focus the emergent rays in its focal plane. Rays

of one wavelength being parallel to each other are finally focussed at one place.

If the slit is wide, different points of the slit produce separate spectra which overlap each other. The colour impression gets diffused due to the overlap. Such a spectrum is called an *impure spectrum.*

## 20.5 KINDS OF SPECTRA

### A. Emission Spectra

Light is emitted by an object when it is suitably excited by heating or by passing an electric discharge. When a light beam emitted by such a source is dispersed to get the spectrum, it is called an *emission spectrum*. An emission spectrum carries information about the source material. An emission spectrum can be of three types :

#### (a) Continuous Spectrum

Quite often, a source emits light which has continuously varying wavelengths in it. An electric bulb, a candle or a red hot iron piece emits light of this type. When such a light is dispersed, a bright spectrum continuously distributed on a dark background is obtained. The colours gradually change and there are no sharp boundaries in between. Such a spectrum is known as a *continuous emission spectrum*.

#### (b) Line Spectrum

All objects are made of atoms and molecules. The atoms and molecules can have certain fixed energies. An atom or a molecule having the lowest possible energy is said to be in its ground state, otherwise, in an excited state. An atom or molecule, in an excited state, can emit light to lower its energy. Light emitted in such a process has certain fixed wavelengths. The light emitted by one kind of atoms generally have widely separated wavelength components (figure 20.6a). When such a light is dispersed, we get certain sharp bright lines on a dark background. Such a spectrum is called *line emission spectrum*. It carries information about the atoms of the source. For example, when electric discharge is passed through sodium vapour, the vapour emits light of the wavelengths 589·0 nm and 589·6 nm. When dispersed by a high resolution grating, one obtains two bright yellow lines on a dark background.

#### (c) Band Spectrum

The molecular energy levels are generally grouped into several bunches, each bunch widely separated from the other, and the levels in a bunch being close to each other. Thus, the wavelengths emitted by such molecules are also grouped, each group being well-separated from the other. The wavelengths in a group are close to one another and appear as continuous. The spectrum looks like separate bands of varying colours. Such a spectrum is called a *band emission spectrum*. Figure (20.6b) shows schematically the production and appearance of band spectra.

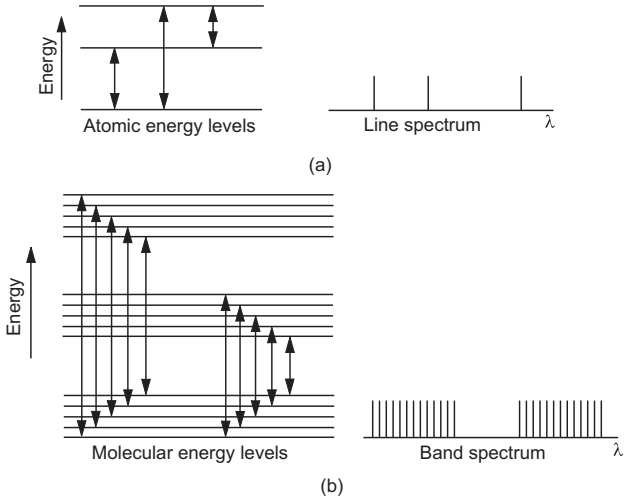



Figure 20.6

### B. Absorption Spectrum

When white light having all wavelengths is passed through an absorbing material, the material may absorb certain wavelengths selectively. When the transmitted light is dispersed, we get dark lines or bands at the positions of the missing wavelengths superposed on an otherwise bright continuous coloured background (figure 20.7).

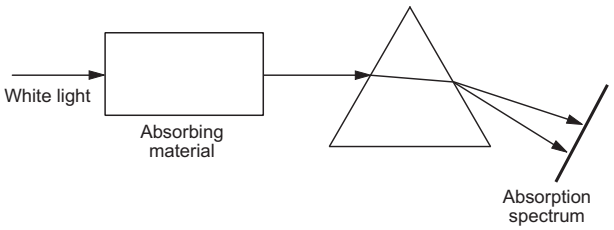



Figure 20.7

The missing wavelengths provide information about the absorbing material. Such a spectrum is called an *absorption spectrum.*

Absorption spectrum may be of two types depending on the absorbing material and the conditions, such as temperature, of the experiment.

**(a) Line Absorption Spectrum**

Light may be absorbed by atoms to take them from lower energy states to higher energy states. In this case, the missing wavelengths are widely separated and we get sharp dark lines on a continuous bright background. Such a spectrum is called a *line absorption spectrum*. When light coming from the sun is dispersed, it shows certain sharply defined dark lines. This shows that certain wavelengths are absent. These missing lines are called *Fraunhofer lines*.

**(b) Band Absorption Spectrum**

If light is absorbed by molecules of the absorbing material, exciting them from lower energy to higher energy states, the missing wavelengths are grouped into bunches. Thus, when the transmitted light is dispersed, we get separate dark bands on a continuous bright background. Such a spectrum is called a *band absorption spectrum*. Light passing through hydrogen gas at moderate temperature or through certain solutions of organic and inorganic compounds shows such a spectrum.

## 20.6 ULTRAVIOLET AND INFRARED SPECTRUM

When an object is suitably excited by heating or in some other way, it emits light. The light, that causes visual sensation to the eye, has a wavelength range from about 380 nm (violet) to about 780 nm (red). The light emitted by an excited object may have wavelengths beyond this visible region. We generally use the word light to mean visible light. That beyond the visible region is called by the general name *radiation*. The radiation with wavelength less than the lower end of the visible region (that is less than about 380 nm) is called *ultraviolet radiation* and the radiation with wavelength greater than the upper end of the visible region is called the *infrared radiation*. The range of ultraviolet radiation is roughly from 15 nm to 380 nm and that of infrared radiation is roughly 780 nm to 40000 nm. Beyond ultraviolet, we have *X*-rays and gamma-rays and above infrared, we have radiowaves.

Ordinary glass highly absorbs infrared and ultraviolet radiation. A prism made of quartz may be used for studying the spectrum in ultraviolet region. The dispersed radiation may be collected on a photographic plate. To study infrared spectrum, one can use a prism made of rocksalt. Infrared radiation considerably heats the object on which it falls. One way of detecting infrared radiation is from its heating effect. An instrument known as thermopile, which is sensitive to heat, is used to measure the dispersed infrared spectrum.

## 20.7 SPECTROMETER

Spectrometer is an instrument which is used to produce and study pure spectrum in visible region. It consists of basically three parts.

**(a) Collimator**

It consists of a long cylindrical tube fitted with an achromatic converging lens at one end. Another tube of slightly smaller diameter can slide into the first tube by a rack-and-pinion arrangement and has a linear slit at the outer end. The width of the slit may be adjusted by a screw. The distance between the slit and the lens may be changed by sliding the second tube into the first. The incident light is passed through the collimator to make it parallel before falling on the dispersing element.

**(b) Prism Table**

This is a horizontal platform which can be rotated about its axis and whose height may be adjusted. The dispersing element (prism, grating, etc.) is placed on the prism table. When the prism table rotates, a horizontal circular scale (graduated in degrees) rotates with it.

**(c) Telescope**

This is an astronomical telescope. The objective lens is fitted at one end of a long cylindrical tube. Another cylindrical tube can slide into it and contains the eyepiece. The dispersed light is passed through the telescope before falling to the eye which is placed just behind the eyepiece. A vernier scale is attached to the telescope which rotates on the horizontal circular scale when the telescope is rotated.

Levelling screws are provided under the main base, the collimator tube, the telescope tube and the prism table.

**Adjustment and Working**

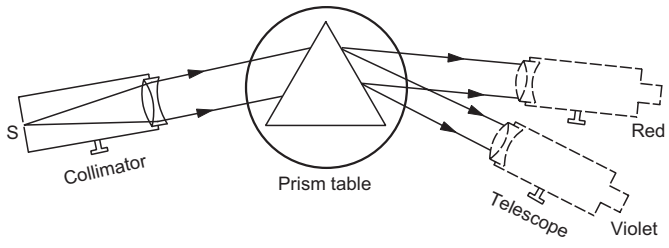



Figure 20.8

The collimator, the prism table and the telescope are fitted in one compact unit (figure 20.8). The prism table and the telescope can be independently rotated about the vertical axis of the prism table. The angle

of rotation can be accurately measured by the vernier scale and the horizontal circular scale.

The axis of the collimator tube, of the telescope tube and the surface of the prism table are made horizontal with the help of levelling screws. The light source to be examined is placed behind the slit of the collimator. The distance between the slit and the collimating lens is so adjusted that the slit lies in the first focal plane of the lens and the rays coming from the collimator become parallel. This parallel beam is incident upon the dispersing element (prism, grating, etc.) placed on the prism table. The dispersed beam is received by the telescope which is focussed for parallel rays, that is, for normal adjustment. The telescope tube is rotated and light rays of different wavelengths are received at different angular positions of the telescope.

**Application of Spectrometer**

The spectrometer can be used in a wide variety of applications. We mention here a few simple ones. In all these applications, the spectrometer is adjusted as described above.

**(a) Measuring the angle of a Prism**

The spectrometer is levelled and adjusted for parallel rays. The prism is placed on the prism table with its refracting edge facing the collimator. The slit is illuminated by a sodium vapour lamp. The parallel beam coming from the collimator is divided into two parts falling on the two surfaces of the prism (figure 20.9).

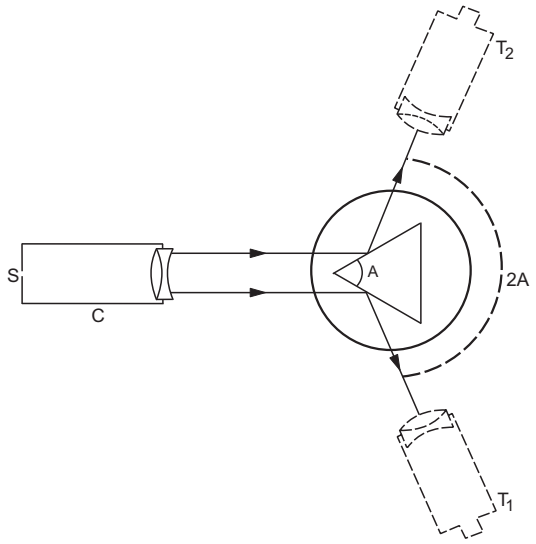



Figure 20.9

The telescope is rotated to a position $T_1$ where it receives the beam reflected by one surface. The image of the slit should coincide with the vertical crosswire of the telescope. The angular position is read on the base with the help of the vernier scale and the scale fixed in the base. The telescope is now rotated to the position $T_2$ where it receives the beam reflected by the other surface. The image of the slit should again coincide with the vertical crosswire. The angular position is again read on the base. The difference of these two readings gives the angle rotated by the telescope. This angle is equal to $2A$ where $A$ is the angle of prism. Sodium vapour lamp gives a nearly monochromatic light which makes the image identification easier.

**(b) Measuring the Angle of Minimum Deviation for a Prism for a Given Wavelength**

The spectrometer is adjusted as described before. The source emitting the light of the given wavelength is placed behind the slit.

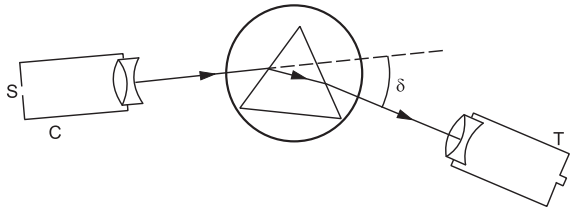



Figure 20.10

The telescope is rotated and placed at a position where the angle between the telescope axis and the collimator axis is large. The prism is placed on the prism table and the table is rotated to such a position that the refracted beam is received by the telescope (figure 20.10). The image of the slit is made to coincide with the crosswire. The angle between the axes of the collimator and the telescope is the angle of deviation δ. Now, the telescope is rotated slightly towards the collimator axis to decrease δ. The prism table is rotated to bring the image of the slit back at the crosswire. The process is repeated till a position comes where if the telescope is further rotated towards the collimator axis, it is not possible to bring the image at the crosswire for any position of the prism table. This position where the image can be last brought to the crosswire is the position of minimum deviation. The angle between the axes of the collimator and the telescope in this position is the angle of minimum deviation.

To measure this angle, the reading of vernier scale attached with the telescope is noted down. The prism is removed and the telescope is brought in line with the collimator so that the image of the slit forms at the crosswire. The reading of the vernier scale is again

noted. The difference between the two readings gives the angle of minimum deviation.

**(c) Variation of Refractive Index with Wavelength**

To study the variation of refractive index $\mu$ with the wavelength $\lambda$, a source is chosen which emits light of sharply defined discrete wavelengths. Neon discharge tube is one such source.

The spectrometer is adjusted as before and a prism is placed on the prism table. A particular colour is chosen and the angle of minimum deviation for that colour is obtained by the method described above. This is done by always focusing the image of the slit formed by that colour. The refractive index of the material of the prism is calculated by the formula

$$\mu = \frac{\sin \frac{A + \delta_m}{2}}{\sin \frac{A}{2}}.$$

The experiment is repeated for each colour and corresponding values of $\mu$ are obtained. Knowing the values of wavelengths, $\mu - \lambda$ variation is studied.

## 20.8 RAINBOW

When sunlight falls on small water droplets suspended in air during or after a rain, it suffers refraction, internal reflection and dispersion. If an observer has the sun at the back and the water droplets in front, he or she may see two rainbows, one inside the other. The inner one is called the *primary rainbow* and the outer one is called the *secondary rainbow*. Figure (20.11) shows the path of a typical ray forming the primary bow. It suffers a refraction followed by a internal reflection and then again a refraction. Dispersion takes place at both the refractions. It turns out that rays of a given colour are strongly returned by the droplet in a direction that corresponds to maximum deviation in its path. For light of red colour this maximum deviation is 137·8° so that the angle $\theta$ in figure (20.11a) is 180° – 137·8° = 42·2°. For violet this angle is 40·6° and for other colours it is in between 40·6° and 42·2°. Now consider an observer at $P$ (figure 20.11b). Suppose sunrays are incident in a direction parallel to $PX$. Consider a cone with $PX$ as its axis and semivertical angle 42·2°. All the droplets on the surface of this cone will return the light to $P$ at an angle of 42·2°. This light will be predominantly red. Thus, the red rays coming to the observer will appear to come from a circle which subtends an angle of 42·2° on the eye. Similarly, the violet rays coming to the observer will appear to come from a circle which subtends an angle of 40·6° on the eye. The other colours form their respective circles of intermediate radii. From the ground level, only an arc of the rainbow is usually visible. A complete circular rainbow may be seen from an elevated position such as from an aeroplane.

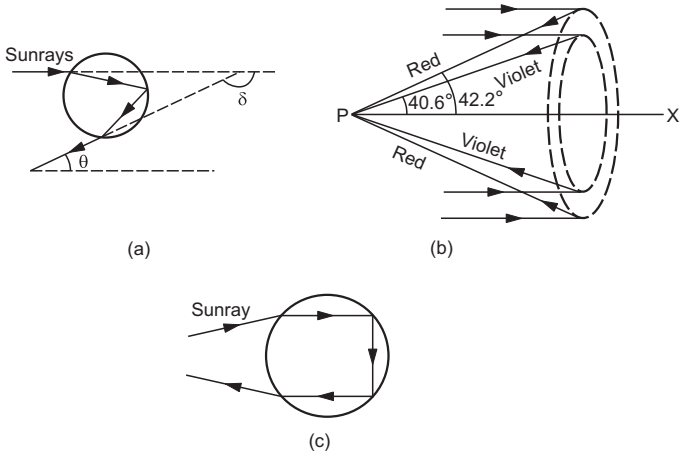



Figure 20.11

The secondary rainbow is formed by rays which suffer two internal reflections before coming out of the water drop (figure 20.11c). In this, the order of colours is reverse. Red appears on the inner circle in the rainbow and violet on the outer. The angle $\theta$ is 50·5° for red and 54° for violet.

### *Worked Out Examples*

**1.** *The refractive indices of flint glass for red and violet light are* 1·613 *and* 1·632 *respectively. Find the angular dispersion produced by a thin prism of flint glass having refracting angle* 5°.

***Solution*** **:** Deviation of the red light is $\delta_r = (\mu_r - 1)A$ and deviation of the violet light is $\delta_v = (\mu_v - 1)A$.

The dispersion $= \delta_v - \delta_r = (\mu_v - \mu_r)A$

$= (1{\cdot}632 - 1{\cdot}613) \times 5°$

$= 0{\cdot}095°$.

**2.** *A crown glass prism of angle* 5° *is to be combined with a flint glass prism in such a way that the mean ray passes undeviated. Find (a) the angle of the flint glass prism needed and (b) the angular dispersion produced by the combination when white light goes through it.*

*Refractive indices for red, yellow and violet light are* 1·514, 1·517 *and* 1·523 *respectively for crown glass and* 1·613, 1·620 *and* 1·632 *for flint glass.*

***Solution*** : The deviation produced by the crown prism is

$$\delta = (\mu - 1)A$$

and by the flint prism is

$$\delta' = (\mu' - 1)A'.$$

The prisms are placed with their angles inverted with respect to each other. The deviations are also in opposite directions. Thus, the net deviation is

$$D = \delta - \delta' = (\mu - 1)A - (\mu' - 1)A'. \quad \ldots \text{ (i)}$$

(a) If the net deviation for the mean ray is zero,

$$(\mu - 1)A = (\mu' - 1)A'.$$

or,
$$A' = \frac{(\mu - 1)}{(\mu' - 1)}A = \frac{1{\cdot}517 - 1}{1{\cdot}620 - 1} \times 5^\circ$$

$$= 4{\cdot}2^\circ.$$

(b) The angular dispersion produced by the crown prism is

$$\delta_v - \delta_r = (\mu_v - \mu_r)A$$

and that by the flint prism is

$$\delta'_v - \delta'_r = (\mu'_v - \mu'_r)A'.$$

The net angular dispersion is,

$$\delta = (\mu_v - \mu_r)A - (\mu'_v - \mu'_r)A'$$

$$= (1{\cdot}523 - 1{\cdot}514) \times 5^\circ - (1{\cdot}632 - 1{\cdot}613) \times 4{\cdot}2^\circ$$

$$= -\,0{\cdot}0348^\circ.$$

The angular dispersion has magnitude 0·0348°.

**3.** *The dispersive powers of crown and flint glasses are* 0·03 *and* 0·05 *respectively. The refractive indices for yellow light for these glasses are* 1·517 *and* 1·621 *respectively. It is desired to form an achromatic combination of prisms of crown and flint glasses which can produce a deviation of* 1° *in the yellow ray. Find the refracting angles of the two prisms needed.*

***Solution*** : Suppose, the angle of the crown prism needed is $A$ and that of the flint prism is $A'$. We have

$$\omega = \frac{\mu_v - \mu_r}{\mu - 1}$$

or,
$$\mu_v - \mu_r = (\mu - 1)\omega.$$

The angular dispersion produced by the crown prism is

$$(\mu_v - \mu_r)A = (\mu - 1)\omega A.$$

Similarly, the angular dispersion produced by the flint prism is

$$(\mu' - 1)\omega' A'.$$

For achromatic combination, the net dispersion should be zero. Thus,

$$(\mu - 1)\omega A = (\mu' - 1)\omega' A'$$

or,
$$\frac{A'}{A} = \frac{(\mu - 1)\omega}{(\mu' - 1)\omega'} = \frac{0{\cdot}517 \times 0{\cdot}03}{0{\cdot}621 \times 0{\cdot}05} = 0{\cdot}50. \quad \ldots \text{ (i)}$$

The deviation in the yellow ray produced by the crown prism is $\delta = (\mu - 1)A$ and by the flint prism is $\delta' = (\mu' - 1)A'$. The net deviation produced by the combination is

$$\delta - \delta' = (\mu - 1)A - (\mu' - 1)A'$$

or,
$$1^\circ = 0{\cdot}517\,A - 0{\cdot}621\,A'. \quad \ldots \text{ (ii)}$$

Solving (i) and (ii), $A = 4{\cdot}8^\circ$ and $A' = 2{\cdot}4^\circ$. Thus, the crown prism should have its refracting angle 4·8° and that of the flint prism should be 2·4°.

□

## QUESTIONS FOR SHORT ANSWER

**1.** The equation $\omega = \frac{\mu_v - \mu_r}{\mu - 1}$ was derived for a prism having small refracting angle. Is it also valid for a prism of large refracting angle ? Is it also valid for a glass slab or a glass sphere ?

**2.** Can the dispersive power $\omega = \frac{\mu_v - \mu_r}{\mu - 1}$ be negative ? What is the sign of ω if a hollow prism is immersed into water ?

**3.** If three identical prisms are combined, is it possible to pass a beam that emerges undeviated ? Undispersed ?

**4.** "Monochromatic light should be used to produce pure spectrum". Comment on this statement.

**5.** Does focal length of a lens depend on the colour of the light used ? Does focal length of a mirror depend on the colour ?

**6.** Suggest a method to produce a rainbow in your house.

## OBJECTIVE I

**1.** The angular dispersion produced by a prism
(a) increases if the average refractive index increases
(b) increases if the average refractive index decreases
(c) remains constant whether the average refractive index increases or decreases
(d) has no relation with average refractive index.

**2.** If a glass prism is dipped in water, its dispersive power
(a) increases (b) decreases (c) does not change
(d) may increase or decrease depending on whether the angle of the prism is less than or greater than 60°.

**3.** A prism can produce a minimum deviation δ in a light beam. If three such prisms are combined, the minimum deviation that can be produced in this beam is
(a) 0 (b) δ (c) 2δ (d) 3δ.

**4.** Consider the following two statements :
(A) Line spectra contain information about atoms.
(B) Band spectra contain information about molecules.
(a) Both *A* and *B* are wrong.
(b) *A* is correct but *B* is wrong.
(c) *B* is correct but *A* is wrong.
(d) Both *A* and *B* are correct.

**5.** The focal length of a converging lens are $f_v$ and $f_r$ for violet and red light respectively.
(a) $f_v > f_r$. (b) $f_v = f_r$. (c) $f_v < f_r$.
(d) Any of the three is possible depending on the value of the average refractive index μ.

## OBJECTIVE II

**1.** A narrow beam of white light goes through a slab having parallel faces.
(a) The light never splits in different colours.
(b) The emergent beam is white.
(c) The light inside the slab is split into different colours.
(d) The light inside the slab is white.

**2.** By properly combining two prisms made of different materials, it is possible to
(a) have dispersion without average deviation
(b) have deviation without dispersion
(c) have both dispersion and average deviation
(d) have neither dispersion nor average deviation.

**3.** In producing a pure spectrum, the incident light is passed through a narrow slit placed in the focal plane of an achromatic lens because a narrow slit
(a) produces less diffraction
(b) increases intensity
(c) allows only one colour at a time
(d) allows a more parallel beam when it passes through the lens.

**4.** Which of the following quantities related to a lens depend on the wavelength or wavelengths of the incident light ?
(a) Power (b) Focal length
(c) Chromatic aberration (d) Radii of curvature

5. Which of the following quantities increase when wavelength is increased ? Consider only the magnitudes.
(a) The power of a converging lens.
(b) The focal length of a converging lens.
(c) The power of a diverging lens.
(d) The focal length of a diverging lens.

## EXERCISES

**1.** A flint glass prism and a crown glass prism are to be combined in such a way that the deviation of the mean ray is zero. The refractive index of flint and crown glasses for the mean ray are 1·620 and 1·518 respectively. If the refracting angle of the flint prism is 6·0°, what would be the refracting angle of the crown prism ?

**2.** A certain material has refractive indices 1·56, 1·60 and 1·68 for red, yellow and violet light respectively. (a) Calculate the dispersive power. (b) Find the angular dispersion produced by a thin prism of angle 6° made of this material.

**3.** The focal lengths of a convex lens for red, yellow and violet rays are 100 cm, 98 cm and 96 cm respectively. Find the dispersive power of the material of the lens.

**4.** The refractive index of a material changes by 0·014 as the colour of the light changes from red to violet. A rectangular slab of height 2·00 cm made of this material is placed on a newspaper. When viewed normally in yellow light, the letters appear 1·32 cm below the top surface of the slab. Calculate the dispersive power of the material.

**5.** A thin prism is made of a material having refractive indices 1·61 and 1·65 for red and violet light. The dispersive power of the material is 0·07. It is found that a beam of yellow light passing through the prism suffers a minimum deviation of 4·0° in favourable conditions. Calculate the angle of the prism.

**6.** The minimum deviations suffered by red, yellow and violet beams passing through an equilateral transparent prism are 38·4°, 38·7° and 39·2° respectively. Calculate the dispersive power of the medium.

**7.** Two prisms of identical geometrical shape are combined with their refracting angles oppositely directed. The materials of the prisms have refractive indices 1·52 and 1·62 for violet light. A violet ray is deviated by 1·0° when passes symmetrically through this combination. What is the angle of the prisms ?

**8.** Three thin prisms are combined as shown in figure (20-E1). The refractive indices of the crown glass for red, yellow and violet rays are $\mu_r$ , $\mu_y$ and $\mu_v$ respectively and

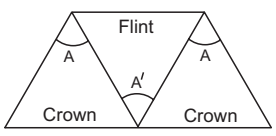



Figure 20-E1

those for the flint glass are $\mu'_r$, $\mu'_y$ and $\mu'_v$ respectively. Find the ratio $A'/A$ for which (a) there is no net angular dispersion, and (b) there is no net deviation in the yellow ray.

**9.** A thin prism of crown glass ($\mu_r = 1{\cdot}515$, $\mu_v = 1{\cdot}525$) and a thin prism of flint glass ($\mu_r = 1{\cdot}612$, $\mu_v = 1{\cdot}632$) are placed in contact with each other. Their refracting angles are 5·0° each and are similarly directed. Calculate the angular dispersion produced by the combination.

**10.** A thin prism of angle 6·0°, $\omega = 0{\cdot}07$ and $\mu_y = 1{\cdot}50$ is combined with another thin prism having $\omega = 0{\cdot}08$ and $\mu_y = 1{\cdot}60$. The combination produces no deviation in the mean ray. (a) Find the angle of the second prism. (b) Find the net angular dispersion produced by the combination when a beam of white light passes through it. (c) If the prisms are similarly directed, what will be the deviation in the mean ray ? (d) Find the angular dispersion in the situation described in (c).

**11.** The refractive index of a material $M1$ changes by 0·014 and that of another material $M2$ changes by 0·024 as the colour of the light is changed from red to violet. Two thin prisms, one made of $M1(A = 5{\cdot}3°)$ and the other made of $M2(A = 3.7°)$ are combined with their refracting angles oppositely directed. (a) Find the angular dispersion produced by the combination. (b) The prisms are now combined with their refracting angles similarly directed. Find the angular dispersion produced by the combination.

□

## ANSWERS

### OBJECTIVE I

1. (a) 2. (b) 3. (b) 4. (d) 5. (c)

### OBJECTIVE II

1. (b), (c) 2. (a), (b), (c) 3. (d) 4. (a), (b), (c)
5. (b), (d)

### EXERCISES

1. 7·2°
2. (a) 0·2 (b) 0·72°
3. 0·041
4. 0·026
5. 7°
6. 0·0206
7. 10°
8. (a) $\dfrac{2(\mu_v - \mu_r)}{\mu'_v - \mu'_r}$ (b) $\dfrac{2(\mu_y - 1)}{\mu'_y - 1}$
9. 0·15°
10. (a) 5° (b) 0·03° (c) 6° (d) 0·45°
11. (a) 0·0146° (b) 0·163°

□

## CHAPTER 21

# SPEED OF LIGHT

### 21.1 HISTORICAL INTRODUCTION

The speed of light in vacuum is a fundamental constant in physics. The most interesting fact about this speed is that if an object moves with this speed in one frame, it has the same speed in any other frame. This led to a major revision of our concept of space and time and is the key fact on which the special theory of relativity is based.

In 1983, the speed of light was defined to be exactly 299, 792, 458 m $s^{-1}$. In fact, the length of an object is now defined to be 299, 792, 458 m $s^{-1}$ multiplied by the time taken by the light to cross it. Thus, when one sends light from one place to another place and measures the time taken by the light to do so, one is not measuring the speed of light, rather one is measuring the distance between the two places.

Prior to 1983, the length was defined independently and one had a separate metre. The speed of light could then be measured as the length divided by the time taken by the light to cross it.

Perhaps, the great Indian talents in the Vedic age had the knowledge of the speed of light. G. V. Raghavrao in his book quotes a verse from Rigveda (I, 50-4) *Yojananam Sahastra Dwe Dwe Shate Dwe Cha Yojane Aken Nimishardhena Krammana Namostute.* In this verse, the author pays respects to the one (the reference is to the sun light) who moves 2202 *yojans* in half *nimish.* Yojan is a quite common unit in India, it means 4 *kose*, each kose measuring 8000 British yards and each yard measuring 0·9144 m. The definition of the time unit nimish can be found in Shrimadbhagwat (III, 11-3 to 10) where it is mentioned that 15 nimishas make 1 *kashta*, 15 kashtas make one *laghu*, 30 laghus make 1 *muhurta* and 30 muhurtas make 1 *diva–ratri*. A diva–ratri is, of course, a day–night which is 24 hours in modern language. When you convert 2202 yojans per half nimish into SI units, it turns out to be $3{\cdot}0 \times 10^8$ m $s^{-1}$ up to two significant digits, a value quite accurate as we know it today.

In the modern era, perhaps the first attempt to measure the speed of light was made by Galileo. The design of the experiment was as follows. Two experimenters $A$ and $B$, each having a lantern and a shutter, stand on two small hills. The shutter can cover or uncover the lantern. Initially, both the lanterns are covered. One of the persons $A$ uncovers the lantern. The second person $B$ uncovers his lantern when he sees the light from the lantern of $A$. The first person $A$ covers his lantern when he sees the light from the lantern of $B$. The time elapsed between the uncovering and covering of the first lantern is measured. During this time, the light travels from the first person to the second person and then back. Knowing the distance and time, the speed of light may be calculated.

The proposed method failed because the speed of light is so large that a human being cannot respond with the required accuracy of timing. If the distance between the hills is as large as 15 km, the time taken by light in going back and forth is only one ten thousandth part of a second. The first recorded speed of light in modern era came through the astronomical observations by the Danish astronomer Olaf Roemer in 1676. The value obtained was about $2{\cdot}1 \times 10^8$ m $s^{-1}$, somewhat smaller than the actual. In 1728, English astronomer Bradley measured the speed of light from his observations. The value was quite close to the correct one.

The first measurement of the speed of light from purely terrestrial experiments was reported by the French physicist Fizeau in 1849. The method was improved by another French physicist Foucault. Yet another method was proposed by American physicist Michelson. We now describe these three methods.

### 21.2 FIZEAU METHOD

Figure (21.1a) shows a schematic diagram of the arrangement used in this method. Light from a source $S$ passes through a convex lens $L_1$. The transmitted beam is intercepted by a semi-transparent inclined

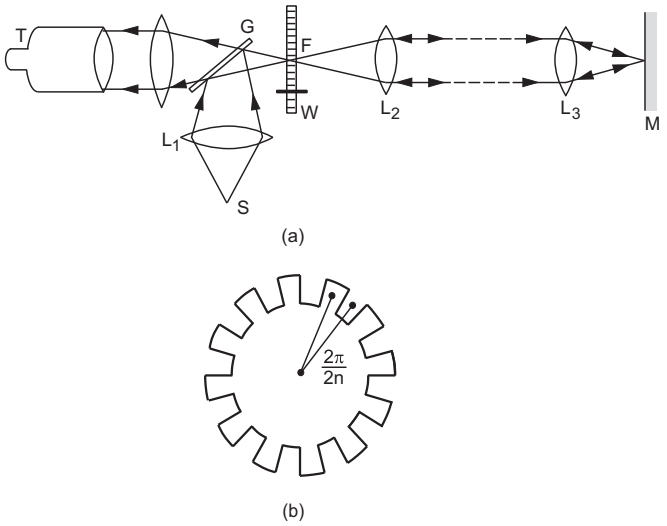



Figure 21.1

glass plate $G$. A part of the light is reflected and is converged near the rim of a toothed wheel $W$ which can be set into rapid rotation. The light passing through the space between two consecutive teeth is made parallel by a convex lens $L_2$. This parallel beam travels for several kilometers (in the original Fizeau experiment it was 8·6 km) and is then converged by a convex lens $L_3$. A plane mirror $M$ is placed in the focal plane of the lens $L_3$. The reflected light is again made parallel by the lens $L_3$ and it converges at the rim of the wheel. If it finds a gap, it falls on the glass plate $G$. The beam is partially transmitted and an observer receives these rays to see the image of $S$ through a telescope.

When the wheel is rotated, it allows light to pass through in separate bursts. Light is passed when a gap comes at $F$ and is stopped when a tooth comes there. The speed of rotation of the wheel is gradually increased while the observer keeps looking for the image. Initially, the image flickers but at a particular angular speed the image cannot be seen at all. This happens when the angular speed is such that by the time light passes through a gap, goes to the mirror $M$ and comes back, the next tooth comes at $F$. Any light passing through the wheel does not return to the observer and the image cannot be seen. The angular speed of the wheel is carefully measured in this state.

Suppose, $D$ = distance from the wheel $W$ to the mirror $M$,

$\omega$ = angular speed of rotation of the wheel when the image is completely unseen for the first time,

$n$ = number of teeth in the wheel.

The angle rotated by the wheel when a tooth comes in the place of its adjacent gap is $\theta = \frac{2\pi}{2n}$ (figure 21.1b). The time taken by the wheel in doing so is $\theta/\omega = \frac{\pi}{n\omega}$. In this time interval, the light travels a distance $2D$. The speed of light is, therefore,

$$c = \frac{2D}{\pi/n\omega} = \frac{2Dn\omega}{\pi}.$$

If the number of revolutions of the wheel per unit time is $\nu$, we have $\omega = 2\pi\nu$ and the speed of light is

$$c = 4Dn\nu. \qquad \ldots (21.1)$$

One can use a concave mirror in place of the plane mirror. If the radius of curvature of this mirror be equal to its distance from the convex lens $L_3$ (i.e., equal to the focal length of $L_3$), a slight error in orientation of lens $L_3$ does not seriously affect the accuracy of the experiment.

There are two serious difficulties in this method. Since the light has to travel a large distance, the intensity decreases considerably and the final image becomes very dim. Secondly, the experiment cannot be done inside a laboratory. It needs an open space of several kilometers. These difficulties are removed in Foucault method.

### 21.3 FOUCAULT METHOD

The basic principle of Foucault's method can be understood with the help of figure (21.2). Light from a source $S$ is partly transmitted by a glass plate $G$ and is incident on a convex lens $L$. The distance of the lens from $S$ is so adjusted that the beam transmitted through the lens is convergent. This beam is intercepted by a plane mirror $M_1$ which can be rotated about an axis perpendicular to the plane of the figure. The plane mirror reflects the light which converges on a concave mirror $M_2$. The distance between the two mirrors is equal to the radius of curvature of the concave mirror. The concave mirror reflects the light beam back to the plane mirror. The central ray is always incident on the concave mirror perpendicularly so that it retraces the path. If the plane mirror does not rotate, the rays retrace the path up to the glass plate $G$. A part of the beam is reflected by the glass plate and forms an image $I$ of the source. Now, suppose the plane mirror $M_1$ rotates by an angle $\Delta\theta$ by the time light goes from $M_1$ to $M_2$ and comes back to it. The light reflected by $M_1$ then makes an angle $2\Delta\theta$ with the direction of the rays reflected earlier. Because of this deviation, the returning rays (shown dotted in figure 21.2) form an image $I'$ of the source which is slightly shifted from the position $I$.

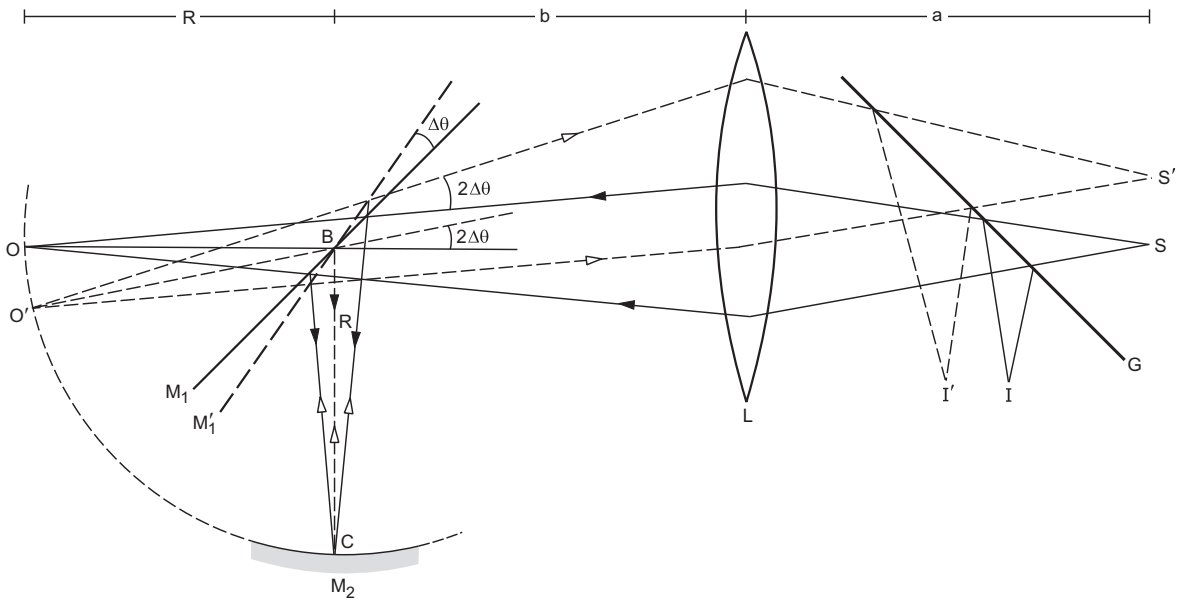



Figure 21.2

Suppose,

$R$ = radius of the concave mirror,

$\omega$ = the angular speed of the plane mirror,

$s$ = the shift $II'$,

$b$ = the distance from $M_1$ to $L$,

$a$ = the distance from $L$ to $S$.

When the mirror is in position $M_1$, the rays reflected by it to the lens seem to come from a point $O$ which is the image of the point $C$ in $M_1$. When it has rotated by an angle $\Delta\theta$, the rays reflected by it to the lens seem to come from a point $O'$ which is the image of $C$ in the new position $M'_1$ of the mirror. The distance $BO = BC = R$. It is clear from the figure that

$$OO' = R.(2\Delta\theta). \quad \ldots \text{(i)}$$

Now, the rays reflected by the position $M_1$ of the mirror retrace the path and would converge at the source $S$ itself. The glass plate partly reflects the beam to converge it at $I$. Thus, $I$ is the image of $S$ in the plate $G$ acting as a plane mirror. Similarly, the rays reflected by the position $M'_1$ of the mirror are converged by the lens at a point $S'$. The glass plate $G$ partly reflects the beam to converge it at $I'$ which is the image of $S'$ in $G$. It is clear that

$$SS' = II' = s. \quad \ldots \text{(ii)}$$

Thus, the lens $L$ forms an image of $O$ at $S$ and of $O'$ at $S'$. If we place an object of size $OO'$ at $O$, its image will have the size $SS'$ at $S$. Thus,

$$\frac{SS'}{OO'} = \text{magnification produced by } L$$

$$= \frac{\text{image-distance}}{\text{object-distance}}$$

$$= \frac{a}{R+b}.$$

Putting from (i) and (ii),

$$\frac{s}{2R\Delta\theta} = \frac{a}{R+b}. \quad \ldots \text{(iii)}$$

If the speed of light is $c$, it takes time $\Delta t = 2R/c$ to go from $M_1$ to $M_2$ and to come back. As the angular speed of $M_1$ is $\omega$, the angle rotated by it in time $\Delta t$ is

$$\Delta\theta = \omega\,\Delta t = \frac{2R\omega}{c}.$$

Putting in (iii),

$$\frac{s}{2R(2R\omega/c)} = \frac{a}{R+b}$$

or,
$$c = \frac{4R^2\omega\, a}{s(R+b)}. \quad \ldots \text{(21.2)}$$

All the quantities on the right side may be measured in the experiment and hence, the speed of light may be calculated. Foucault obtained a value $2{\cdot}98 \times 10^8$ m s$^{-1}$ from his measurement.

The space required in this experiment is quite small and hence, it may be performed inside a laboratory. Another advantage with this method is that one can put a tube of a transparent material between the two mirrors. The speed calculated by equation (21.2) then gives the speed of light in that material. It could be experimentally verified that light travels at a slower speed in a medium as compared to its speed in vacuum. This finding was contrary to the predictions of Newton's corpuscular theory.

### 21.4 MICHELSON METHOD

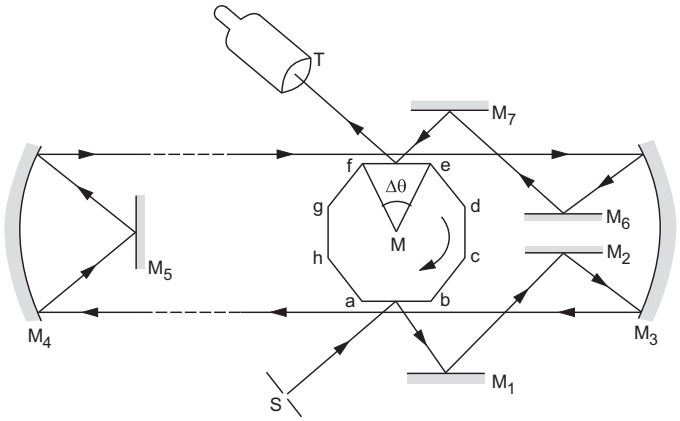



Figure 21.3

The scheme of Michelson method to measure the speed of light is shown in figure (21.3). Light from an intense source $S$ is incident upon one face of a polygon-shaped mirror $M$. The light reflected from this surface is sent to the lower portion of a concave mirror $M_3$ after reflections from two plane mirrors $M_1$ and $M_2$. The geometry is set so that the light reflected from the concave mirror becomes parallel. This parallel beam of light is allowed to travel through a long distance (several kilometers) and falls on the lower portion of another concave mirror $M_4$. The parallel beam is converged at the focus of $M_4$ where a plane mirror $M_5$ is placed. $M_5$ reflects the beam back to the concave mirror $M_4$, this time at the upper portion. As $M_5$ is at the focus, the beam reflected by $M_4$ becomes parallel and travels back to the concave mirror $M_3$. After proper reflections from $M_3$ and the plane mirrors, it is sent to the polygonal mirror. A telescope is adjusted to receive the rays reflected by the polygonal mirror and hence, to form an image of the source.

Suppose the polygonal mirror $M$ is stationary. Light from the source falls on the face $ab$ of the mirror $M$ and after reflections from all the mirrors, finally falls on the face $ef$ of the mirror $M$. The image of $S$ is seen in the telescope. If the polygonal mirror rotates, the face $ef$ also turns a little while light travels between the two reflections from the polygonal mirror. The light thus fails to enter into the telescope and the image is not seen. If the rotational speed of the mirror is gradually increased, a stage comes when the adjacent face $fg$ takes the place of $ef$ by the time light comes there. Then, the light is again sent into the telescope.

In the experiment, one looks through the telescope and gradually increases the angular speed of the polygonal mirror. The image flickers initially and becomes steady at a particular angular speed of the mirror. This angular speed is measured.

Suppose,

$N$ = the number of faces in the polygonal mirror,

$\omega$ = the angular speed of rotation of the mirror when the image becomes steady,

$D$ = the distance travelled by the light between the reflections from the polygonal mirror.

If the speed of light is $c$, the time taken by the light to travel the distance $D$ is $\Delta t = D/c$. The angle rotated by the mirror during this time is $\Delta\theta = 2\pi/N$.

The angular speed of the mirror is

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi/N}{D/c} = \frac{2\pi c}{DN}$$

or,
$$c = \frac{D\omega N}{2\pi}.$$

If $\nu$ be the frequency of rotation, $\omega/2\pi = \nu$ and

$$c = D\nu N. \qquad \ldots (21.3)$$

Michelson and his co-workers made a series of similar experiments. The first determination was made in 1879 with an octagonal rotating mirror. The latest in the series was underway at the time of the death of Michelson and was completed in 1935 by Pease and Pearson. This experiment used a rotating mirror with 32 faces.

□

### QUESTIONS FOR SHORT ANSWER

**1.** The speed of sound in air is 332 m $s^{-1}$. Is it advisable to define the length 1 m as the distance travelled by sound in 1/332 s ?

**2.** Consider Galileo's method of measuring the speed of light using two lanterns. To get an accuracy of about 10%, the time taken by the experimenter in closing or opening the shutter should be about one tenth of the time taken by the light in going from one experimenter to the other. Assume that it takes 1/100 second for an experimenter to close or open the shutter. How far should the two experimenters be to get a 10% accuracy ? What are the difficulties in having this separation ?

**3.** In Fizeau method of measuring the speed of light, the toothed wheel is placed in the focal plane of a converging

lens. How would the experiment be affected if the wheel is slightly away from the focal plane ?

**4.** In the original Fizeau method, the light travelled 8·6 km and then returned. What could be the difficulty if this distance is taken as 8·6 m ?

**5.** What is the advantage of using a polygonal mirror with larger number of faces in Michelson method of measuring the speed of light ?

## OBJECTIVE I

**1.** Light passes through a closed cylindrical tube containing a gas. If the gas is gradually pumped out, the speed of light inside the tube will
(a) increase (b) decrease (c) remain constant
(d) first increase and then decrease.

**2.** The speeds of red light and yellow light are exactly same
(a) in vacuum but not in air
(b) in air but not in vacuum
(c) in vacuum as well as in air
(d) neither in vacuum nor in air.

**3.** An illuminated object is placed on the principal axis of a converging lens so that a real image is formed on the other side of the lens. If the object is shifted a little,
(a) the image will be shifted simultaneously with the object
(b) the image will be shifted a little later than the object
(c) the image will be shifted a little earlier than the object
(d) the image will not shift.

## OBJECTIVE II

**1.** The speed of light is 299,792,458 m $s^{-1}$
(a) with respect to the earth
(b) with respect to the sun
(c) with respect to a train moving on the earth
(d) with respect to a spaceship going in outer space.

**2.** Which of the following methods can be used to measure the speed of light in laboratory ?
(a) Roemer method (b) Fizeau method
(c) Foucault method (d) Michelson method

**3.** Which of the following methods can be used to measure the speed of light in water ?
(a) Roemer method (b) Fizeau method
(c) Foucault method (d) Michelson method

## EXERCISES

**1.** In an experiment to measure the speed of light by Fizeau's apparatus, following data are used :
Distance between the mirrors = 12·0 km,
Number of teeth in the wheel = 180.
Find the minimum angular speed of the wheel for which the image is not seen.

**2.** In an experiment with Foucault's apparatus, the various distances used are as follows :
Distance between the rotating and the fixed mirror = 16 m
Distance between the lens and the rotating mirror = 6 m,
Distance between the source and the lens = 2 m.
When the mirror is rotated at a speed of 356 revolutions per second, the image shifts by 0·7 mm. Calculate the speed of light from these data.

**3.** In a Michelson experiment for measuring speed of light, the distance travelled by light between two reflections from the rotating mirror is 4·8 km. The rotating mirror has a shape of a regular octagon. At what minimum angular speed of the mirror (other than zero) the image is formed at the position where a nonrotating mirror forms it ?

□

## ANSWERS

OBJECTIVE I

1. (a) 2. (a) 3. (b)

OBJECTIVE II

1. (a), (b), (c), (d) 2. (c) 3. (c)

EXERCISES

1. $1{\cdot}25 \times 10^{4}$ deg $s^{-1}$
2. $2{\cdot}984 \times 10^{8}$ m $s^{-1}$
3. $7{\cdot}8 \times 10^{3}$ rev $s^{-1}$

□

## CHAPTER 22

# PHOTOMETRY

We see an object when light coming from the object enters our eyes and excites the sensation of vision. The brightness sensed by the eye depends on the amount of light energy entering into it and the wavelength distribution of this energy. In this chapter, we shall study the factors responsible for the sensation of brightness.

### 22.1 TOTAL RADIANT FLUX

The total energy of radiation emitted by a source per unit time is called its *total radiant flux.* This radiation contains components of various wavelengths extending even beyond the visible range. However, not all wavelengths have equal contribution in making up the total radiation. In calculating total radiant flux of a source, the total energy emitted per unit time in the whole range of wavelengths must be calculated.

The SI unit of total radiant flux of a source is *watt.*

### 22.2 LUMINOSITY OF RADIANT FLUX

The brightness produced by radiation depends on the wavelength of the radiation besides depending on the total radiant flux. For example, consider two 10 W sources of light, one emitting yellow light and the other red light. Though both emit equal energy per unit time, yellow will look brighter than the red. The *luminosity of radiant flux* measures the capacity to produce brightness sensation in eye. A relative comparison of luminosity of radiant flux of different wavelengths can be made by the curve in figure (22.1). The figure represents relative luminosity under normal light conditions for an average person. The scale on the vertical axis is chosen arbitrarily. We see that for normal light conditions, the luminosity is maximum for wavelength around 555 nm and falls off on both sides. Radiation is "visible" if its luminosity is not zero. As the luminosity falls off gradually, there are no sharp cut-offs of visible region.

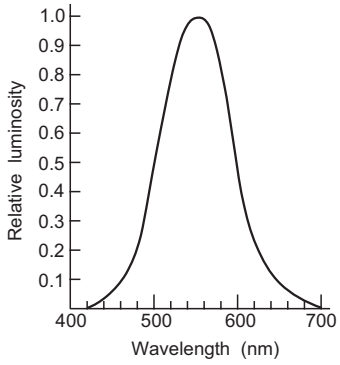



Figure 22.1

### 22.3 LUMINOUS FLUX : RELATIVE LUMINOSITY

In general, the radiation emitted by a source has components corresponding to a wide range of wavelengths. Different component wavelengths have different energies (in a given time) and different brightness producing capacities. The radiant flux is a quantity directly representing the total energy emitted per unit time. The *luminous flux* is a quantity directly representing the total brightness producing capacity of the source. Its unit is called *lumen.* The luminous flux of a source of 1/685 W emitting monochromatic light of wavelength 555 nm is called one lumen. In other words, a 1 W source emitting monochromatic light of wavelength 555 nm emits 685 lumen.

*Relative luminosity* of a wavelength refers to the fraction

$$\frac{\text{luminous flux of a source of given wavelength}}{\text{luminous flux of a 555 nm source of same power}}.$$

It is often represented as a percentage. Thus, figure (22.1) represents the relative luminosity as a function of wavelength.

It should be clear that the luminous flux depends on the radiant flux as well as on the wavelength distribution.

***Example 22.1***

*Find the luminous flux of a* 10 W *source of* 600 nm. *The relative luminosity at* 600 nm *is* 0·6.

***Solution*** : The luminous flux of a 1 W source of 555 nm = 685 lumen. Thus, the luminous flux of a 10 W source of 555 nm = 6850 lumen. The luminous flux of a 10 W source of 600 nm is, therefore, 0·6 × 6850 lumen = 4110 lumen.

For radiation having a range of wavelengths, the luminous flux gets contribution from each wavelength.

## 22.4 LUMINOUS EFFICIENCY

Total luminous flux per unit radiant flux is called *luminous efficiency.* Thus,

$$\text{Luminous efficiency} = \frac{\text{Total luminous flux}}{\text{Total radiant flux}}. \quad \ldots \quad (22.1)$$

The luminous efficiency of a monochromatic source of 555 nm is 685 lumen watt$^{-1}$ by definition. The luminous efficiency of a monochromatic source of any other wavelength is the relative luminosity of that wavelength multiplied by 685 lumen watt$^{-1}$.

An electric lamp glows when electric energy is given to it. However, not all the electric power given to it is converted into radiant flux. The term luminous efficiency is used in a slightly wider sense for such a light source. It is defined as the luminous flux divided by the power input to the source. Thus, it is the efficiency with which the power input to the source is used to produce brightness. We may call it *overall luminous efficiency.* Overall luminous efficiency

$$= \frac{\text{Luminous flux emitted}}{\text{Power input to the source}}. \quad \ldots \quad (22.2)$$

A good fraction of power given to a filament lamp is used to heat the filament to a certain temperature at which it glows. Also, a good fraction of the emitted radiation has a wavelength where the relative luminosity is small or zero. The overall luminous efficiency of a filament lamp is rarely more than 50 lumen watt$^{-1}$.

## 22.5 LUMINOUS INTENSITY OR ILLUMINATING POWER

In the chapter on Gauss's law, we shall describe in detail what is a solid angle. In brief, the solid angle measures the angular divergence of a cone and is defined as

$$\omega = \frac{A}{R^2},$$

where $A$ is the area intercepted by the cone on a sphere of radius $R$ centred at the apex of the cone (figure 22.2).

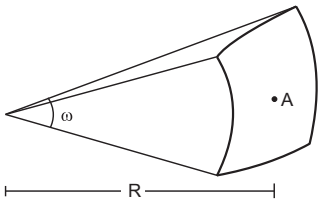



Figure 22.2

It is clear that the solid angle does not depend on the radius of the sphere. The SI unit of solid angle is called a *steradian* written in short as *sr*.

The *luminous intensity* of a source in a given direction is defined as

$$I = \frac{dF}{d\omega}, \quad \ldots \quad (22.3)$$

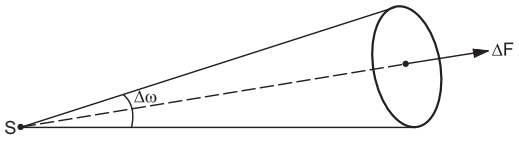



Figure 22.3

where $dF$ is the luminous flux of the radiation emitted by the source in a small cone of solid angle $d\omega$ constructed around the given direction. The luminous intensity is also called just *intensity* in short. An ideal point source emits radiation uniformly in all directions. If the total luminous flux of the source is $F$, its intensity in any direction is

$$I = \frac{F}{(4\pi \text{ sr})}$$

as the total solid angle at a point is $4\pi$ sr. For an extended source, the intensity is different in different directions.

The SI unit of luminous intensity is lumen/steradian. This is called a *candela* written in short as "cd". Luminous intensity is also called *illuminating power.*

Candela is one of the seven base units of SI. It is defined precisely as the luminous intensity of a blackbody of surface area $\frac{1}{60}$ cm$^2$ placed at the freezing temperature of platinum at a pressure of 101, 325 N m$^{-2}$ in the direction perpendicular to the surface.

## 22.6 ILLUMINANCE

When radiation strikes a surface, the surface gets illuminated. We define the *illuminance* of a small area as follows. If $dF$ be the luminous flux of the radiation

striking a surface area $dA$, the illuminance of the area is defined as

$$E = \frac{dF}{dA} \cdot \qquad \ldots \quad (22.4)$$

The illuminance is, therefore, the luminous flux incident per unit area.

It is the illuminance which is directly related to the brightness of an illuminated area. The SI unit of illuminance is lumen $m^{-2}$ and is called *lux*.

### 22.7 INVERSE SQUARE LAW

Consider a point source $S$ and a small area $\Delta A$ around the point $P$ at a distance $r$ from the source (figure 22.4). Suppose, the angle between $SP$ and the normal $PN$ to the area is $\theta$. Also suppose, the luminous intensity of the source in the direction $SP$ is $I$.

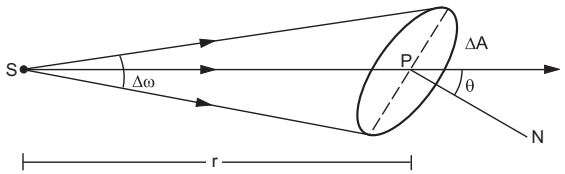



Figure 22.4

The solid angle subtended by the area $\Delta A$ at the source is

$$\Delta\omega = \frac{\Delta A \cos\theta}{r^2} \cdot$$

The luminous flux going through this solid angle is

$$\Delta F = I\Delta\omega$$
$$= I \frac{\Delta A \cos\theta}{r^2} \cdot$$

The illuminance at $\Delta A$ is

$$E = \frac{\Delta F}{\Delta A} \qquad \ldots \quad (22.5)$$

or,
$$E = \frac{I \cos\theta}{r^2} \cdot$$

We note that

(a) the illuminance of a small area is inversely proportional to the square of the distance of the area from the source and

(b) the illuminance of a small area is proportional to $\cos\theta$ where $\theta$ is the angle made by the normal to the area with the direction of incident radiation.

The first observation is known as the *inverse square law*.

### 22.8 LAMBERT'S COSINE LAW

An ideal point source emits radiation uniformly in all directions. In general, sources are extended and such a source has different luminous intensity in different directions. If the source is in the form of a small plane surface, the radiation is emitted only in the forward half that is in a solid angle $2\pi$ around the forward normal. Even in this half, the intensity is different in different directions. The intensity is maximum along the normal to the surface and decreases as we consider directions away from this normal. For many surfaces, if the luminous intensity along the normal is $I_0$, it is

$$I = I_0 \cos\theta \qquad \ldots \quad (22.6)$$

in a direction making an angle $\theta$ with the normal. Equation (22.6) is called *Lambert's cosine law*. The surfaces which radiate according to the Lambert's cosine law are called *perfectly diffused*.

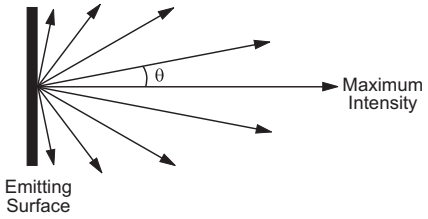



Figure 22.5

### 22.9 PHOTOMETERS

A photometer is used to compare the intensities of two point sources. The basic principle is as follows. Two screens are placed side by side. One screen is illuminated by the source $S_1$ only and the other screen by the source $S_2$ only. Light falls on the two screens at equal angles. The distances $d_1$ and $d_2$ of the sources from the screens are so adjusted that the two screens look equally bright. If $I_1$ and $I_2$ be the intensities of the sources, we must have for equal illuminance

$$\frac{I_1}{d_1^2} = \frac{I_2}{d_2^2}$$

or,
$$\frac{I_1}{I_2} = \frac{d_1^2}{d_2^2} \cdot \qquad \ldots \quad (22.7)$$

A simple design proposed by Bunsen is now described (figure 22.6). It consists of an optical bench fitted with three vertical stands. The stands can slide along a straight rail on the bench.

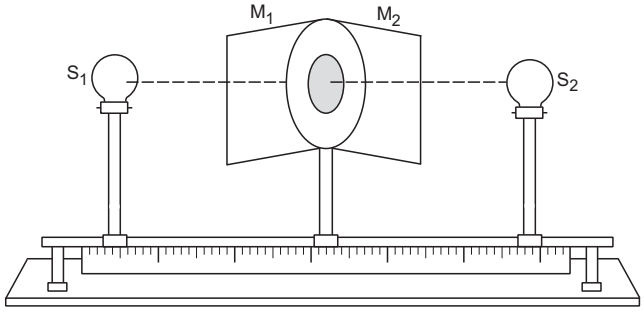



Figure 22.6

The distance between any two points on the rail may be read from a meter scale attached to the bench. The central stand contains a white paper with a grease spot. The other two stands carry the sources $S_1$ and $S_2$ to be compared. Two plane mirrors $M_1$ and $M_2$ are placed behind the central stand at proper inclination so that one side of the spot is imaged in one mirror and the other side of the spot is imaged in the other mirror. The two images can be seen simultaneously.

One of the sources is kept fixed at a distance from the spot and the position of the other is adjusted till the two spots seen in the mirrors appear equally bright. The distances $d_1$ and $d_2$ of the sources from the spot are measured in this condition. In this condition, the light falling on the spot from the two sources has equal intensity. If $I_1$ and $I_2$ be the intensity of the two sources, we have for equal illuminance,

$$\frac{I_1}{d_1^2} = \frac{I_2}{d_2^2} \quad \text{or,} \quad \frac{I_1}{I_2} = \frac{d_1^2}{d_2^2}.$$

### *Worked Out Examples*

**1.** *A source emits* 12·0 J *of light of wavelength* 620 nm *and* 8·0 J *of light of wavelength* 580 nm *per second. The relative luminosity at* 620 nm *is* 35% *and that at* 580 nm *is* 80%. *Find (a) the total radiant flux, (b) the total luminous flux and (c) the luminous efficiency.*

***Solution*** **:**

(a) The total radiant flux = Total energy radiated per unit time = 12 J/s + 8 J/s = 20 J/s = 20 W.

(b) The luminous flux corresponding to the 12 W of 620 nm radiation is

$$0{\cdot}35 \times (12\ \text{W}) \times 685\ \text{lumen W}^{-1} = 2877\ \text{lumen}.$$

Similarly, the luminous flux corresponding to the 8 W of 580 nm radiation is

$$0{\cdot}80 \times (8\ \text{W}) \times 685\ \text{lumen W}^{-1} = 4384\ \text{lumen}.$$

The luminous flux of the source is 2877 lumen + 4384 lumen

$$= 7261\ \text{lumen} \approx 7260\ \text{lumen}.$$

(c) The luminous efficiency $= \dfrac{\text{Total luminous flux}}{\text{Total radiant flux}}$

$$= \frac{7260\ \text{lumen}}{20\ \text{W}} = 363\ \text{lumen W}^{-1}.$$

**2.** *A circular area of radius* 1·0 cm *is placed at a distance of* 2·0 m *from a point source. The source emits light uniformly in all directions. The line joining the source to the centre of the area is normal to the area. It is found that* $2{\cdot}0 \times 10^{-3}$ lumen *of luminous flux is incident on the area. Calculate the total luminous flux emitted by the source and the luminous intensity of the source along the axis of the area.*

***Solution*** **:** The solid angle subtended by the area on the point source is

$$\Delta\omega = \frac{\pi(1{\cdot}0\ \text{cm})^2}{(2{\cdot}0\ \text{m})^2} = \frac{\pi}{4} \times 10^{-4}\ \text{sr}.$$

Thus, $2{\cdot}0 \times 10^{-3}$ lumen of flux is emitted in $\dfrac{\pi}{4} \times 10^{-4}$ sr.

The total solid angle at the source is $4\pi$. As the source radiates uniformly in all directions, the total luminous flux is

$$F = \frac{4\pi}{\frac{\pi}{4} \times 10^{-4}} \times 2{\cdot}0 \times 10^{-3}\ \text{lumen}$$

$$= 320\ \text{lumen}.$$

The luminous intensity $= \Delta F / \Delta\omega$

$$= \frac{2{\cdot}0 \times 10^{-3}\ \text{lumen}}{\frac{\pi}{4} \times 10^{-4}\ \text{sr}} = 25\ \text{cd}.$$

**3.** *The overall luminous efficiency of a* 100 W *electric lamp is* 25 lumen $\text{W}^{-1}$. *Assume that light is emitted by the lamp only in the forward half, and is uniformly distributed in all directions in this half. Calculate the luminous flux falling on a plane object of area* 1 $\text{cm}^2$ *placed at a distance of* 50 cm *from the lamp and perpendicular to the line joining the lamp and the object.*

***Solution*** **:** The power input to the bulb = 100 W.

The luminous flux emitted by the bulb

$$= (25\ \text{lumen W}^{-1}) \times 100\ \text{W}$$

$$= 2500\ \text{lumen}.$$

Since light is emitted only in the forward half and is distributed uniformly in this half, the luminous intensity is

$$I = \Delta F / \Delta\omega$$

$$= \frac{2500\ \text{lumen}}{2\pi\ \text{sr}}.$$

The solid angle subtended by the object on the lamp is

$$\Delta\omega = \frac{1\ \text{cm}^2}{(50\ \text{cm})^2} = \frac{1}{2500}\ \text{sr}.$$

The luminous flux emitted in this solid angle is

$$\Delta F = I\,\Delta\omega$$

$$= \left(\frac{2500}{2\pi}\ \frac{\text{lumen}}{\text{sr}}\right)\left(\frac{1}{2500}\ \text{sr}\right)$$

$$= \frac{1}{2\pi}\ \text{lumen} = 0{\cdot}16\ \text{lumen}.$$

**4.** *A point source emitting uniformly in all directions is placed above a table-top at a distance of* 0·50 m *from it. The luminous flux of the source is* 1570 lumen. *Find the illuminance at a small surface area of the table-top (a) directly below the source and (b) at a distance of* 0·80 m *from the source.*

***Solution*** **:** Consider the situation shown in figure (22-W1). Let $A$ be the point directly below the source $S$ and $B$ be the point at 0·80 m from the source.

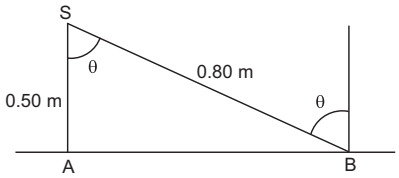



Figure 22-W1

The luminous flux of 1570 lumen is emitted uniformly in the solid angle $4\pi$. The luminous intensity of the source in any direction is

$$I = \frac{1570 \text{ lumen}}{4\pi \text{ sr}}$$

$$= 125 \text{ cd}.$$

The illuminance is

$$E = \frac{I \cos\theta}{r^2}.$$

At the point $A$, $r = 0{\cdot}50$ m and $\theta = 0$. Thus,

$$E_A = \frac{125 \text{ cd}}{0{\cdot}25 \text{ m}^2} = 500 \text{ lux}.$$

At the point $B$, $r = 0{\cdot}80$ m and $\cos\theta = \frac{SA}{SB} = \frac{0{\cdot}50}{0{\cdot}80} = \frac{5}{8}$.

Thus,
$$E_B = \frac{(125 \text{ cd}) \times \frac{5}{8}}{0{\cdot}64 \text{ m}^2}$$

$$= 122 \text{ lux}.$$

**5.** *The luminous intensity of a small plane source of light along the forward normal is* 160 candela. *Assuming the source to be perfectly diffused, find the luminous flux emitted into a cone of solid angle* 0·02 sr *around a line making an angle of* 60° *with the forward normal.*

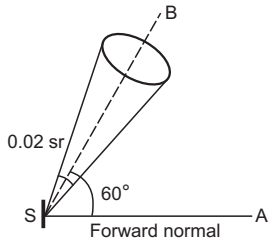



Figure 22-W2

***Solution*** **:** The situation is shown in figure (22-W2). By Lambert's cosine law, the intensity in the direction $SB$ is

$$I = I_0 \cos 60°,$$

where $I_0 = 160$ candela is the intensity along the forward normal.

Thus,
$$I = (160 \text{ candela}) \left(\frac{1}{2}\right)$$

$$= 80 \text{ candela}.$$

The luminous flux emitted in the cone shown in the figure is

$$\Delta F = I \, \Delta\omega$$

$$= (80 \text{ candela}) (0{\cdot}02 \text{ sr})$$

$$= 1{\cdot}6 \text{ lumen}.$$

□

## QUESTIONS FOR SHORT ANSWER

**1.** What is the luminous flux of a source emitting radio waves ?

**2.** The luminous flux of a 1 W sodium vapour lamp is more than that of a 10 kW source of ultraviolet radiation. Comment.

**3.** Light is incident normally on a small plane surface. If the surface is rotated by an angle of 30° about the incident light, does the illuminance of the surface increase, decrease or remain same ? Does your answer change if the light did not fall normally on the surface ?

**4.** A bulb is hanging over a table. At which portion of the table is the illuminance maximum ? If a plane mirror is placed above the bulb facing the table, will the illuminance on the table increase ?

**5.** The sun is less bright at morning and evening as compared to at noon although its distance from the observer is almost the same. Why ?

**6.** Why is the luminous efficiency small for a filament bulb as compared to a mercury vapour lamp ?

**7.** The yellow colour has a greater luminous efficiency as compared to the other colours. Can we increase the illuminating power of a white light source by putting a yellow plastic paper around this source ?

## OBJECTIVE I

**1.** The one parameter that determines the brightness of a light source sensed by an eye is
(a) energy of light entering the eye per second
(b) wavelength of the light
(c) total radiant flux entering the eye
(d) total luminous flux entering the eye.

**2.** Three light sources *A, B* and *C* emit equal amount of radiant energy per unit time. The wavelengths emitted by the three sources are 450 nm, 555 nm and 700 nm respectively. The brightness sensed by an eye for the sources are $X_A$, $X_B$ and $X_C$ respectively. Then,
(a) $X_A > X_B,\ X_C > X_B$ (b) $X_A > X_B,\ X_B > X_C$
(c) $X_B > X_A,\ X_B > X_C$ (d) $X_B > X_A,\ X_C > X_B$.

**3.** As the wavelength is increased from violet to red, the luminosity
(a) continuously increases
(b) continuously decreases
(c) increases then decreases
(d) decreases then increases.

**4.** An electric bulb is hanging over a table at a height of 1 m above it. The illuminance on the table directly below the bulb is 40 lux. The illuminance at a point on the table 1 m away from the first point will be about
(a) 10 lux (b) 14 lux (c) 20 lux (d) 28 lux.

**5.** Light from a point source falls on a screen. If the separation between the source and the screen is increased by 1%, the illuminance will decrease (nearly) by
(a) 0·5% (b) 1% (c) 2% (d) 4%.

**6.** A battery-operated torch is adjusted to send an almost parallel beam of light. It produces an illuminance of 40 lux when the light falls on a wall 2 m away. The illuminance produced when it falls on a wall 4 m away is close to
(a) 40 lux (b) 20 lux (c) 10 lux (d) 5 lux.

**7.** The intensity produced by a long cylindrical light source at a small distance *r* from the source is proportional to
(a) $\frac{1}{r^2}$ (b) $\frac{1}{r^3}$ (c) $\frac{1}{r}$ (d) none of these.

**8.** A photographic plate placed at a distance of 5 cm from a weak point source is exposed for 3 s. If the plate is kept at a distance of 10 cm from the source, the time needed for the same exposure is
(a) 3 s (b) 12 s (c) 24 s (d) 48 s.

**9.** A photographic plate is placed directly in front of a small diffused source in the shape of a circular disc. It takes 12 s to get a good exposure. If the source is rotated by 60° about one of its diameters, the time needed to get the same exposure will be
(a) 6 s (b) 12 s (c) 24 s (d) 48 s.

**10.** A point source of light moves in a straight line parallel to a plane table. Consider a small portion of the table directly below the line of movement of the source. The illuminance at this portion varies with its distance *r* from the source as
(a) $I \propto \frac{1}{r}$ (b) $I \propto \frac{1}{r^2}$ (c) $I \propto \frac{1}{r^3}$ (d) $I \propto \frac{1}{r^4}$.

**11.** Figure (22-Q1) shows a glowing mercury tube. The intensities at point *A, B* and *C* are related as
(a) $B > C > A$ (b) $A > C > B$
(c) $B = C > A$ (d) $B = C < A$.

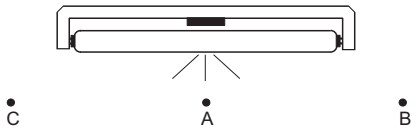



Figure 22-Q1

## OBJECTIVE II

**1.** The brightness-producing capacity of a source
(a) does not depend on its power
(b) does not depend on the wavelength emitted
(c) depends on its power
(d) depends on the wavelength emitted.

**2.** A room is illuminated by an extended source. The illuminance at a particular portion of a wall can be increased by
(a) moving the source (b) rotating the source
(c) bringing some mirrors in proper positions
(d) changing the colour of the source.

**3.** Mark the correct options.
(a) The luminous efficiency of a monochromatic source is always greater than that of a white light source of same power.
(b) The luminous efficiency of a monochromatic source of wavelength 555 nm is always greater than that of a white light source of same power.
(c) The illuminating power of a monochromatic source of wavelength 555 nm is always greater than that of a white light source of same power.
(d) The illuminating power of a monochromatic source is always greater than that of a white light source of same power.

**4.** Mark out the correct options.
(a) Luminous flux and radiant flux have same dimensions.
(b) Luminous flux and luminous intensity have same dimensions.
(c) Radiant flux and power have same dimensions.
(d) Relative luminosity is a dimensionless quantity.

## EXERCISES

**1.** A source emits 45 joules of energy in 15 s. What is the radiant flux of the source ?

**2.** A photographic plate records sufficiently intense lines when it is exposed for 12 s to a source of 10 W. How long should it be exposed to a 12 W source radiating the light of same colour to get equally intense lines ?

**3.** Using figure (22.1), find the relative luminosity of wavelength (a) 480 nm, (b) 520 nm (c) 580 nm and (d) 600 nm.

**4.** The relative luminosity of wavelength 600 nm is 0·6. Find the radiant flux of 600 nm needed to produce the same brightness sensation as produced by 120 W of radiant flux at 555 nm.

**5.** The luminous flux of a monochromatic source of 1 W is 450 lumen. Find the relative luminosity at the wavelength emitted.

**6.** A source emits light of wavelengths 555 nm and 600 nm. The radiant flux of the 555 nm part is 40 W and of the 600 nm part is 30 W. The relative luminosity at 600 nm is 0·6. Find (a) the total radiant flux, (b) the total luminous flux, (c) the luminous efficiency.

**7.** A light source emits monochromatic light of wavelength 555 nm. The source consumes 100 W of electric power and emits 35 W of radiant flux. Calculate the overall luminous efficiency.

**8.** A source emits 31·4 W of radiant flux distributed uniformly in all directions. The luminous efficiency is 60 lumen watt$^{-1}$. What is the luminous intensity of the source ?

**9.** A point source emitting 628 lumen of luminous flux uniformly in all directions is placed at the origin. Calculate the illuminance on a small area placed at (1·0 m, 0, 0) in such a way that the normal to the area makes an angle of 37° with the $X$-axis.

**10.** The illuminance of a small area changes from 900 lumen m$^{-2}$ to 400 lumen m$^{-2}$ when it is shifted along its normal by 10 cm. Assuming that it is illuminated by a point source placed on the normal, find the distance between the source and the area in the original position.

**11.** A point source emitting light uniformly in all directions is placed 60 cm above a table-top. The illuminance at a point on the table-top, directly below the source, is 15 lux. Find the illuminance at a point on the table-top 80 cm away from the first point.

**12.** Light from a point source falls on a small area placed perpendicular to the incident light. If the area is rotated about the incident light by an angle of 60°, by what fraction will the illuminance change ?

**13.** A student is studying a book placed near the edge of a circular table of radius $R$. A point source of light is suspended directly above the centre of the table. What should be the height of the source above the table so as to produce maximum illuminance at the position of the book ?

**14.** Figure (22-E1) shows a small diffused plane source $S$ placed over a horizontal table-top at a distance of 2·4 m with its plane parallel to the table-top. The illuminance at the point $A$ directly below the source is 25 lux. Find the illuminance at a point $B$ of the table at a distance of 1·8 m from $A$.

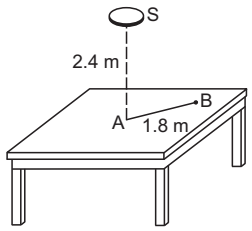



Figure 22-E1

**15.** An electric lamp and a candle produce equal illuminance at a photometer screen when they are placed at 80 cm and 20 cm from the screen respectively. The lamp is now covered with a thin paper which transmits 49% of the luminous flux. By what distance should the lamp be moved to balance the intensities at the screen again ?

**16.** Two light sources of intensities 8 cd and 12 cd are placed on the same side of a photometer screen at a distance of 40 cm from it. Where should a 80 cd source be placed to balance the illuminance ?

□

## ANSWERS

### OBJECTIVE I

1. (d) 2. (c) 3. (c) 4. (b) 5. (c) 6. (a)
7. (c) 8. (b) 9. (c) 10. (c) 11. (d)

### OBJECTIVE II

1. (c), (d) 2. (a), (b), (c), (d) 3. (b), (c)
4. (b), (c), (d)

### EXERCISES

1. 3 W
2. 10 s
3. (a) 0·14 (b) 0·68 (c) 0·92 (d) 0·66
4. 200 W
5. 66%
6. (a) 70 W (b) 39730 lumen (c) 568 lumen W$^{-1}$

7. 240 lumen $W^{-1}$
8. 150 cd
9. 40 lux
10. 20 cm
11. 3·24 lux
12. it will not change
13. $R/\sqrt{2}$
14. 10·2 lux
15. 24 cm
16. 80 cm

□

# APPENDIX A

## Units and Dimensions of Physical Quantities

| Quantity | Common Symbol | SI Unit | Dimension |
|---|---|---|---|
| Displacement | $s$ | METRE (m) | L |
| Mass | $m, M$ | KILOGRAM (kg) | M |
| Time | $t$ | SECOND (s) | T |
| Area | $A$ | $m^2$ | $L^2$ |
| Volume | $V$ | $m^3$ | $L^3$ |
| Density | $\rho$ | $kg\ m^{-3}$ | $M\ L^{-3}$ |
| Velocity | $v, u$ | $m\ s^{-1}$ | $L\ T^{-1}$ |
| Acceleration | $a$ | $m\ s^{-2}$ | $L\ T^{-2}$ |
| Force | $F$ | newton (N) | $ML\ T^{-2}$ |
| Work | $W$ | joule (J)(= N–m) | $ML^2\ T^{-2}$ |
| Energy | $E, U, K$ | joule (J) | $ML^2\ T^{-2}$ |
| Power | $P$ | watt (W)(= $J\ s^{-1}$) | $ML^2\ T^{-3}$ |
| Momentum | $p$ | $kg–m\ s^{-1}$ | $ML\ T^{-1}$ |
| Gravitational constant | $G$ | $N–m^2\ kg^{-2}$ | $L^3\ M^{-1}T^{-2}$ |
| Angle | $\theta, \varphi$ | radian | |
| Angular velocity | $\omega$ | $rad\ s^{-1}$ | $T^{-1}$ |
| Angular acceleration | $\alpha$ | $rad\ s^{-2}$ | $T^{-2}$ |
| Angular momentum | $L$ | $kg–m^2\ s^{-1}$ | $ML^2\ T^{-1}$ |
| Moment of inertia | $I$ | $kg–m^2$ | $ML^2$ |
| Torque | $\tau$ | N–m | $ML^2\ T^{-2}$ |
| Angular frequency | $\omega$ | $rad\ s^{-1}$ | $T^{-1}$ |
| Frequency | $\nu$ | hertz (Hz) | $T^{-1}$ |
| Period | $T$ | s | T |
| Young's modulus | $Y$ | $N\ m^{-2}$ | $M\ L^{-1}T^{-2}$ |
| Bulk modulus | $B$ | $N\ m^{-2}$ | $M\ L^{-1}T^{-2}$ |
| Shear modulus | $\eta$ | $N\ m^{-2}$ | $M\ LT^{-2}$ |
| Surface tension | $S$ | $N\ m^{-1}$ | $M\ T^{-2}$ |
| Coefficient of viscosity | $\eta$ | $N–s\ m^{-2}$ | $M\ L^{-1}T^{-1}$ |
| Pressure | $P, p$ | $N\ m^{-2}$, Pa | $M\ L^{-1}T^{-2}$ |
| Wavelength | $\lambda$ | m | L |
| Intensity of wave | $I$ | $W\ m^{-2}$ | $M\ T^{-3}$ |
| Temperature | $T$ | KELVIN (K) | $\Theta$ |
| Specific heat capacity | $c$ | $J\ kg^{-1}–K^{-1}$ | $L^2\ T^{-2}\ K^{-1}$ |
| Stefan's constant | $\sigma$ | $W\ m^{-2}–K^{-4}$ | $M\ T^{-3}K^{-4}$ |
| Heat | $Q$ | J | $ML^2\ T^{-2}$ |
| Thermal conductivity | $K$ | $W\ m^{-1}–K^{-1}$ | $ML\ T^{-3}K^{-1}$ |
| Current | $I$ | AMPERE(A) | I |
| Charge | $q, Q$ | coulomb (C) | IT |
| Current density | $j$ | $A\ m^{-2}$ | $IL^{-2}$ |
| Electrical conductivity | $\sigma$ | $1/\Omega–m$ (= mho/m) | $I^2T^3\ M^{-1}L^{-3}$ |
| Dielectric constant | $k$ | | |
| Electric dipole moment | $p$ | C–m | LIT |
| Electric field | $E$ | $V\ m^{-1}$(= $N\ C^{-1}$) | $ML\ I^{-1}T^{-3}$ |
| Potential (voltage) | $V$ | volt (V)(= $J\ C^{-1}$) | $ML^2\ I^{-1}T^{-3}$ |
| Electric flux | $\Phi$ | V–m | $ML^3\ I^{-1}T^{-3}$ |
| Capacitance | $C$ | farad (F) | $I^2T^4\ M^{-1}L^{-2}$ |
| Electromotive force | $E$ | volt (V) | $ML^2\ I^{-1}T^{-3}$ |
| Resistance | $R$ | ohm ($\Omega$) | $ML^2\ I^{-2}T^{-3}$ |
| Permittivity of space | $\varepsilon_0$ | $C^2\ N^{-1}–m^{-2}$ (= $F\ m^{-1}$) | $I^2T^4\ M^{-1}L^{-3}$ |
| Permeability of space | $\mu_0$ | $N\ A^{-2}$ | $ML\ I^{-2}T^{-2}$ |
| Magnetic field | $B$ | tesla (T)(= $Wb\ m^{-2}$) | $M\ I^{-1}T^{-2}$ |
| Magnetic flux | $\Phi_B$ | weber (Wb) | $ML^2\ I^{-1}T^{-2}$ |
| Magnetic dipole moment | $\mu$ | $N–m\ T^{-1}$ | $IL^2$ |
| Inductance | $L$ | henry (H) | $ML^2\ I^{-2}T^{-2}$ |

# APPENDIX B

## Universal Constants (as revised in 1986)

| Quantity | Symbol | Value | Unit | Uncertainty in the last two digits |
|---|---|---|---|---|
| Constant of gravitation | $G$ | $6{\cdot}67259 \times 10^{-11}$ | $N–m^2 kg^{-2}$ | 85 |
| Speed of light in vacuum | $c$ | $2{\cdot}99792458 \times 10^{8}$ | $m\ s^{-1}$ | exact |
| Avogadro constant | $N_A$ | $6{\cdot}0221367 \times 10^{23}$ | $mol^{-1}$ | 36 |
| Gas constant | $R$ | 8·314510 | $J\ K^{-1}–mol^{-1}$ | 70 |
| Boltzmann constant | $k$ | $1{\cdot}380658 \times 10^{-23}$<br>$8{\cdot}617385 \times 10^{-5}$ | $J\ K^{-1}$<br>$eV\ K^{-1}$ | 12<br>73 |
| Stefan–Boltzmann constant | $\sigma$ | $5{\cdot}67051 \times 10^{-8}$ | $W\ m^{-2}–K^{-4}$ | 19 |
| Wien's displacement law constant | $b$ | $2{\cdot}897756 \times 10^{-3}$ | m–K | 24 |
| Charge of proton | $e$ | $1{\cdot}60217733 \times 10^{-19}$ | C | 49 |
| Mass of electron | $m_e$ | $9{\cdot}1093897 \times 10^{-31}$<br>$5{\cdot}48579903 \times 10^{-4}$ | kg<br>u | 54<br>13 |
| Mass of proton | $m_p$ | $1{\cdot}6726231 \times 10^{-27}$<br>1·007276470 | kg<br>u | 10<br>12 |
| Mass of neutron | $m_n$ | $1{\cdot}6749286 \times 10^{-27}$<br>1·008664904 | kg<br>u | 10<br>14 |
| Permeability of vacuum | $\mu_0$ | $4\pi \times 10^{-7}$<br>$= 12{\cdot}566370614\ldots \times 10^{-7}$ | $N\ A^{-2}$ | exact |
| Permittivity of vacuum | $\epsilon_0$ | $\frac{1}{\mu_0 c^2}$<br>$= 8{\cdot}854187817\ldots \times 10^{-12}$ | $C^2\ N^{-1}–m^{-2} = F\ m^{-1}$ | exact |
| Faraday constant | $F$ | 96485·3029 | $C\ mol^{-1}$ | 29 |
| Planck constant | $h$ | $6{\cdot}6260755 \times 10^{-34}$<br>$4{\cdot}1356692 \times 10^{-15}$ | J–s<br>eV–s | 40<br>12 |
| Rydberg constant | $R$ | $1{\cdot}0973731534 \times 10^{7}$ | $m^{-1}$ | 13 |
| Ground state energy of hydrogen atom | | 13·605698 | eV | 40 |
| Bohr radius | $a_0$ | $5{\cdot}29177249 \times 10^{-11}$ | m | 24 |

## Astronomical Constant

| Quantity | Value | Unit |
|---|---|---|
| Mass of the sun | $1.99 \times 10^{30}$ | kg |
| Radius of the sun | $6.95 \times 10^{8}$ | m |
| Mass of the earth | $5.98 \times 10^{24}$ | kg |
| Mean radius of the earth | $6.37 \times 10^{6}$ | m |
| Mass of the moon | $7.36 \times 10^{22}$ | kg |
| Radius of the moon | $1.74 \times 10^{6}$ | m |
| Mean earth–sun distance | $1.50 \times 10^{11}$ | m |
| Mean earth–moon distance | $3.84 \times 10^{8}$ | m |
| Escape speed from the earth | 11.2 | $km\ s^{-1}$ |
| Escape speed from the moon | 2.38 | $km\ s^{-1}$ |

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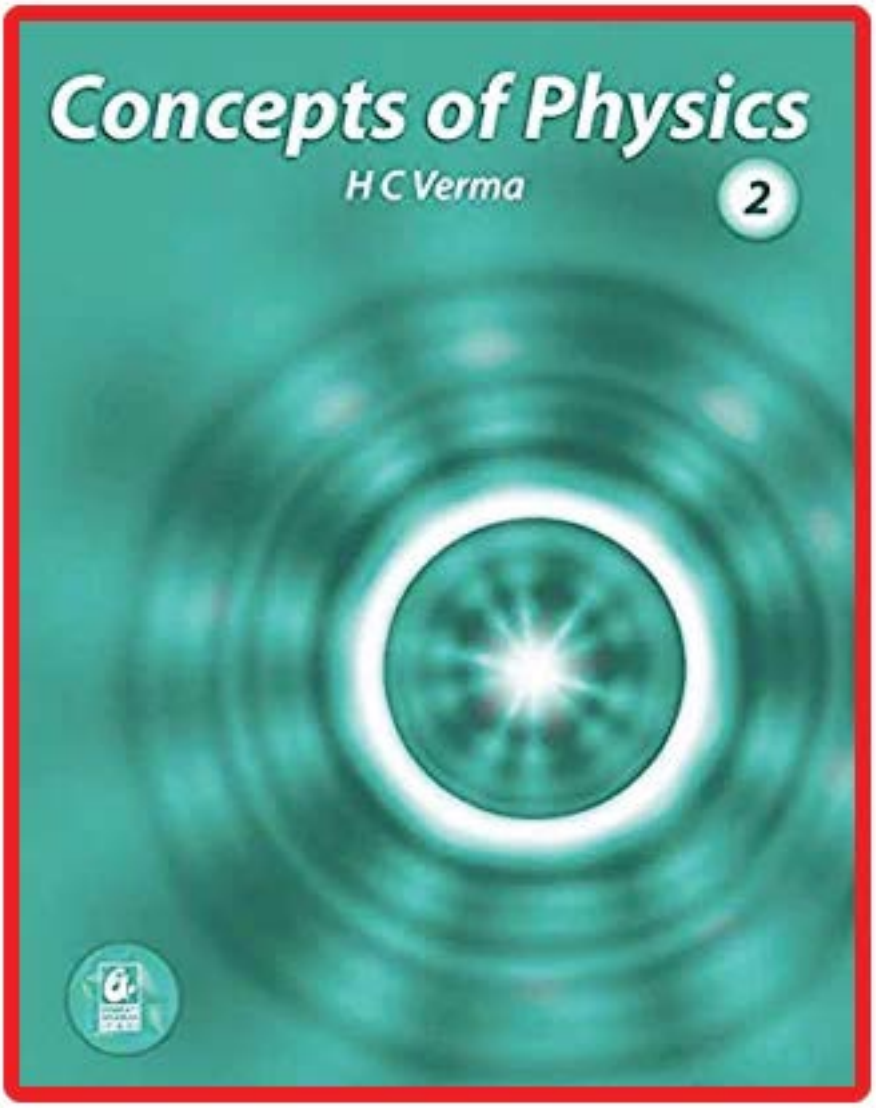
Concepts of Physics
H C Verma
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# CONCEPTS OF PHYSICS

[VOLUME 2]

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Bharati Bhawan
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*Dedicated to*
*Indian Philosophy & Way of Life*
*of which*
*my parents were*
*an integral part*

# FOREWORD

A few years ago I had an occasion to go through the book *Calculus* by L V Terasov. It unravels intricacies of the subject through a dialogue between Teacher and Student. I thoroughly enjoyed reading it. For me this seemed to be one of the few books which teach a difficult subject through inquisition, and using programmed concept for learning. After that book, Dr Harish Chandra Verma's book on physics, *CONCEPTS OF PHYSICS* is another such attempt, even though it is not directly in the dialogue form. I have thoroughly appreciated it. It is clear that Dr Verma has spent considerable time in formulating the structure of the book, besides its contents. I think he has been successful in this attempt. Dr Verma's book has been divided into two parts because of the size of the total manuscript. There have been several books on this subject, each one having its own flavour. However, the present book is a totally different attempt to teach physics, and I am sure it will be extremely useful to the undergraduate students. The exposition of each concept is extremely lucid. In carefully formatted chapters, besides problems and short questions, a number of objective questions have also been included. This book can certainly be extremely useful not only as a textbook, but also for preparation of various competitive examinations.

Those who have followed Dr Verma's scientific work always enjoyed the outstanding contributions he has made in various research areas. He was an outstanding student of Physics Department of IIT Kanpur during his academic career. An extremely methodical, sincere person as a student, he has devoted himself to the task of educating young minds and inculcating scientific temper amongst them. The present venture in the form of these two volumes is another attempt in that direction. I am sure that young minds who would like to *learn physics in an appropriate manner* will find these volumes extremely useful.

I must heartily congratulate Dr Harish Chandra Verma for the magnificent job he has done.

**Y R Waghmare**
Professor of Physics
IIT Kanpur.

# PREFACE

**Why a new book ?**

Excellent books exist on physics at an introductory college level so why a new one ? Why so many books exist at the same level, in the first place, and why each of them is highly appreciated ? It is because each of these books has the privilege of having an author or authors who have *experienced* physics and have their own method of communicating with the students. During my years as a physics teacher, I have developed a somewhat different methodology of presenting physics to the students. *Concepts of Physics* is a translation of this methodology into a textbook.

**Prerequisites**

The book presents a calculus-based physics course which makes free use of algebra, trigonometry and co-ordinate geometry. The level of the latter three topics is quite simple and high school mathematics is sufficient. Calculus is generally done at the introductory college level and I have assumed that the student is enrolled in a concurrent first calculus course. The relevant portions of calculus have been discussed in Chapter 2 so that the student may start using it from the beginning.

Almost no knowledge of physics is a prerequisite. I have attempted to start each topic from the zero level. A receptive mind is all that is needed to use this book.

**Basic philosophy of the book**

The motto underlying the book is *physics is enjoyable.*

Being a description of the nature around us, physics is our best friend from the day of our existence. I have extensively used this aspect of physics to introduce the physical principles starting with common day occurrences and examples. The subject then appears to be friendly and enjoyable. I have taken care that numerical values of different quantities used in problems correspond to real situations to further strengthen this approach.

**Teaching and training**

The basic aim of physics teaching has been to let the student know and understand the principles and equations of physics and their applications in real life.

However, to be able to use these principles and equations correctly in a given physical situation, one needs further training. A large number of *questions and solved and unsolved problems* are given for this purpose. Each question or problem has a specific purpose. It may be there to bring out a subtle point which might have passed unnoticed while doing the text portion. It may be a further elaboration of a concept developed in the text. It may be there to make the student react when several concepts introduced in different chapters combine and show up as a physical situation and so on. Such tools have been used to develop a culture: *analyse the situation, make a strategy to invoke correct principles and work it out.*

**Conventions**

I have tried to use symbols, names, etc., which are popular nowadays. SI units have been consistently used throughout the book. SI prefixes such as *micro, milli, mega,* etc., are used whenever they make the presentation more readable. Thus, 20 μF is preferred over $20 \times 10^{-6}$ F. Co-ordinate sign convention is used in geometrical optics. Special emphasis has been given to dimensions of physical quantities. Numerical values of physical quantities have been mentioned with the units even in equations to maintain dimensional consistency.

I have tried my best to keep errors out of this book. I shall be grateful to the readers who point out any errors and/or make other constructive suggestions.

**H C Verma**

# ACKNOWLEDGEMENTS

The work on this book started in 1984. Since then, a large number of teachers, students and physics lovers have made valuable suggestions which I have incorporated in this work. It is not possible for me to acknowledge all of them individually. I take this opportunity to express my gratitude to them. However, to Dr S B Mathur, who took great pains in going through the entire manuscript and made valuable comments, I am specially indebted. I am also beholden to my colleagues Dr A Yadav, Dr Deb Mukherjee, Mr M M R Akhtar, Dr Arjun Prasad, Dr S K Sinha and others who gave me valuable advice and were good enough to find time for fruitful discussions. To Dr T K Dutta of B E College, Sibpur I am grateful for having taken time to go through portions of the book and making valuable comments.

I thank my student Mr Shailendra Kumar who helped me in checking the answers. I am grateful to Dr B C Rai, Mr Sunil Khijwania & Mr Tejaswi Khijwania for helping me in the preparation of rough sketches for the book.

Finally, I thank the members of my family for their support and encouragement.

**H C Verma**

# TO THE STUDENTS

Here is a brief discussion on the organisation of the book which will help you in using the book most effectively. The book contains 47 chapters divided in two volumes. Though I strongly believe in the underlying unity of physics, a broad division may be made in the book as follows:

Chapters 1–14: Mechanics

15–17: Waves including wave optics

18–22: Optics

23–28: Heat and thermodynamics

29–40: Electric and magnetic phenomena

41–47: Modern physics

Each chapter contains a description of the physical principles related to that chapter. It is well supported by mathematical derivations of equations, descriptions of laboratory experiments, historical background, etc. There are "in-text" solved examples. These examples explain the equation just derived or the concept just discussed. These will help you in fixing the ideas firmly in your mind. Your teachers may use these in-text examples in the classroom to encourage students to participate in discussions.

After the theory section, there is a section on *Worked Out Examples*. These numerical examples correspond to various thinking levels and often use several concepts introduced in that chapter or even in previous chapters. You should read the statement of a problem and try to solve it yourself. In case of difficulty, look at the solution given in the book. Even if you solve the problem successfully, you should look into the solution to compare it with your method of solution. You might have thought of a better method, but knowing more than one method is always beneficial.

Then comes the part which tests your understanding as well as develops it further. *Questions for Short Answer* generally touch very minute points of your understanding. It is not necessary that you answer these questions in a single sitting. They have great potential to initiate very fruitful dicussions. So, freely discuss these questions with your friends and see if they agree with your answer. Answers to these questions are not given for the simple reason that the answers could have cut down the span of such discussions and that would have sharply reduced the utility of these questions.

There are two sections on multiple-choice questions, namely OBJECTIVE I and OBJECTIVE II. There are four options following each of these questions. Only one option is correct for OBJECTIVE I questions. Any number of options, zero to four, may be correct for OBJECTIVE II questions. Answers to all these questions are provided.

Finally, a set of numerical problems are given for your practice. Answers to these problems are also provided. The problems are generally arranged according to the sequence of the concepts developed in the chapter but they are not grouped under section-headings. I don't want to bias your ideas beforehand by telling you that this problem belongs to that section and hence use that particular equation. You should yourself look into the problem and decide which equations or which methods should be used to solve it. Many of the problems use several concepts developed in different sections of the chapter. Many of them even use the concepts from the previous chapters. Hence, you have to plan out the strategy after understanding the problem.

Remember, no problem is difficult. Once you understand the theory, each problem will become easy. So, don't jump to exercise problems before you have gone through the theory, the worked-out problems and the objectives. Once you feel confident in theory, do the exercise problems. The exercise problems are so arranged that they gradually require more thinking.

I hope you will enjoy *Concepts of Physics*.

**H C Verma**

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## CHAPTER 23

# HEAT AND TEMPERATURE

### 23.1 HOT AND COLD BODIES

When we rub our hands for some time, they become warm. When a block slides on a rough surface, it becomes warm. Press against a rapidly spinning wheel. The wheel slows down and becomes warm. While going on a bicycle, touch the road with your shoe. The bicycle slows down and the shoe becomes warm. When two vehicles collide with each other during an accident, they become very hot. When an aeroplane crashes, it becomes so hot that it catches fire.

In each of these examples, mechanical energy is lost and the bodies in question become hot. Where does the mechanical energy vanish? It goes into the internal energy of the bodies. We conclude that the cold bodies absorb energy to become hot. In other words, a hot body has more internal energy than an otherwise identical cold body.

When a hot body is kept in contact with a cold body, the cold body warms up and the hot body cools down. The internal energy of the hot body decreases and the internal energy of the cold body increases. Thus, energy is transferred from the hot body to the cold body when they are placed in contact. Notice that no mechanical work is done during this transfer of energy (neglect any change in volume of the body). This is because there are no displacements involved. This is different from the case when we lift a ball vertically and the energy of the ball–earth system increases or when a compressed spring relaxes and a block attached to its end speeds up. In the case of lifting the ball, we do some work on the ball and the energy is increased by that amount. In the case of spring–block example, the spring does some work and the kinetic energy of the block increases.

The transfer of energy from a hot body to a cold body is a nonmechanical process. The energy that is transferred from one body to the other, without any mechanical work involved, is called *heat.*

### 23.2 ZEROTH LAW OF THERMODYNAMICS

Two bodies are said to be in *thermal equilibrium* if no transfer of heat takes place when they are placed in contact. We can now state the Zeroth law of thermodynamics as follows:

*If two bodies A and B are in thermal equilibrium and A and C are also in thermal equilibrium then B and C are also in thermal equilibrium.*

It is a matter of observation and experience that is described in the Zeroth law. It should not be taken as obvious. For example, if two persons $A$ and $B$ know each other and $A$ and $C$ know each other, it is not necessary that $B$ and $C$ know each other.

The Zeroth law allows us to introduce the concept of temperature to measure the hotness or coldness of a body. All bodies in thermal equilibrium are assigned equal temperature. A hotter body is assigned higher temperature than a colder body. Thus, the temperatures of two bodies decide the direction of heat-flow when the two bodies are put in contact. Heat flows from the body at higher temperature to the body at lower temperature.

### 23.3 DEFINING SCALE OF TEMPERATURE: MERCURY AND RESISTANCE THERMOMETERS

We are now in a position to say whether two given bodies are at the same temperature or not. If they are not at the same temperature, we also know which is at higher temperature and which is at lower temperature. Our next task is to define a scale of temperature so that we can give numerical value to the temperature of a body. To do this, we can choose a substance and look for a measurable property of the substance which monotonically changes with temperature. The temperature can then be defined as a chosen function of this property. As an example, take a mass of mercury in a glass bulb terminating in a long capillary. The length of the mercury column in the capillary changes with temperature. Each length

corresponds to a particular temperature of the mercury. How can we assign a numerical value corresponding to an observed length of the mercury column ?

Figure 23.1

The earlier method was to choose two fixed points of temperature which can be easily reproduced in laboratory. The temperature of melting ice at 1 atm (called ice point) and the temperature of boiling water at 1 atm (called steam point) are often chosen as the fixed points. We arbitrarily assign a temperature $t_1$ to the ice point and $t_2$ to the steam point. Suppose the length of the mercury column is $l_1$ when the bulb is kept in melting ice and it is $l_2$ when the bulb is kept in boiling water. Thus, a length $l_1$ of mercury column means that the temperature of the bulb is $t_1$ and a length $l_2$ of the column means that the temperature is $t_2$. The temperature corresponding to any length $l$ may be defined by assuming a linear relation between $l$ and $t$,

$$t = al + b. \qquad \ldots (23.1)$$

A change of one degree in temperature will mean a change of $\frac{l_2 - l_1}{t_2 - t_1}$ in the length of mercury column. Thus, we can graduate the length of the capillary directly in degrees.

The centigrade system assumes ice point at 0°C and the steam point at 100°C. If the length of the mercury column between its values for 0°C and 100°C is divided equally in 100 parts, each part will correspond to a change of 1°C.

Let $l_0$ and $l_{100}$ denote the lengths of the mercury column at 0°C and 100°C respectively. From equation (23.1),

$$0 = al_0 + b$$

$$\text{and } 100 = al_{100} + b$$

$$\text{giving } b = -al_0$$

$$\text{and } a = \frac{100}{l_{100} - l_0}$$

Putting in equation (23.1), the temperature corresponding to a length $l$ is given by

$$t = \frac{l - l_0}{l_{100} - l_0} \times 100 \text{ degree.} \qquad \ldots (23.2)$$

To measure the temperature of a body, the bulb containing mercury is kept in contact with the body and sufficient time is allowed so that the mercury comes to thermal equilibrium with the body. The temperature of the body is then the same as that of the mercury. The length of the column then gives the temperature according to equation (23.2).

Another popular system known as Fahrenheit system assumes 32°F for the ice point and 212°F for the steam point. A change of 1°F means $\frac{1}{180}$ of the interval between the steam point and the ice point. The average temperature of a normal human body is around 98°F. The conversion formula from centigrade to Fahrenheit scale is

$$F = 32 + \frac{9}{5} C.$$

The expansion of mercury is just one thermometric property that can be used to define a temperature scale and prepare thermometers. There may be many others. Electric resistance of a metal wire increases monotonically with temperature and may be used to define a temperature scale. If $R_0$ and $R_{100}$ denote the resistances of a metal wire at ice point and steam point respectively, we can define temperature $t$ corresponding to the resistance $R_t$ as

$$t = \frac{R_t - R_0}{R_{100} - R_0} \times 100 \text{ degree.} \qquad \ldots (23.3)$$

The temperatures of the ice point and the steam point are chosen to be 0°C and 100°C as in centigrade scale. A platinum wire is often used to construct a thermometer based on this scale. Such a thermometer is called platinum resistance thermometer and the temperature scale is called the *platinum scale.*

**Platinum Resistance Thermometer**

The platinum resistance thermometer works on the principle of *Wheatstone bridge* used to measure a resistance. In a Wheatstone bridge, four resistances $P$, $Q$, $R$ and $X$ are joined in a loop as shown in figure (23.2a). A galvanometer and a battery are also joined as shown. If

$$\frac{P}{Q} = \frac{R}{X},$$

there is no deflection in the galvanometer and the bridge is called balanced. If the condition is not fulfilled, there is a deflection.

Figure (23.2b) represents the arrangement for a platinum resistance thermometer. A thin platinum wire is coiled on a mica base and placed in a glass tube. Two connecting wires $YY$ going through the

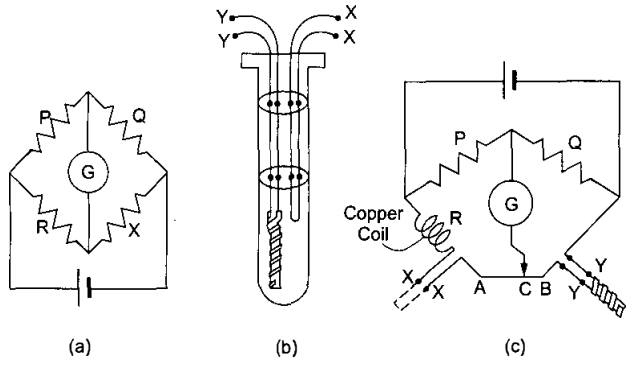



Figure 23.2

ebonite lid of the tube are connected to the platinum coil. A similar copper wire $XX$, called compensating wire, also goes into the tube as shown in the figure. A Wheatstone bridge is arranged as shown in figure (23.2c). Two equal resistances $P$ and $Q$ are connected in two arms of the bridge. A copper coil having resistance roughly equal to that of the platinum coil is connected in the third arm through the compensating wire $XX$. The platinum coil is inserted in the fourth arm of the bridge by connecting the points $Y,Y$ in that arm. The end $C$ of the wire connected to the galvanometer can slide on a uniform wire $AB$ of length $l$. The end $A$ of this wire is connected to the wire $XX$ and the end $B$ is connected to the wire $YY$. Thus, the copper coil, the compensating wire and the wire $AC$ are in the third arm and the platinum coil, the connecting wire $YY$ and the wire $CB$ are in the fourth arm of the bridge. The test tube containing the platinum coil is immersed in the bath of which we want to measure the temperature. The end $C$ is slid on $AB$ till the deflection in the galvanometer becomes zero. Let $AC = x$.

Suppose the resistance of the copper coil connected in the third arm is $R$, that of the compensating wire is $R_c$ and that of the wire $AB$ is $r$. The resistance of the connecting wire $YY$ is the same as $R_c$ and that of the platinum coil is $R_t$. The resistance of the wire $AC = \frac{r}{l}x$ and that of the wire $CB = \frac{r}{l}(l-x)$. The net resistance in the third arm is $R + R_c + \frac{r}{l}x$ and that in the fourth arm is $R_t + R_c + \frac{r}{l}(l-x)$. For no deflection in the galvanometer,

$$\frac{P}{Q} = \frac{R + R_c + \frac{r}{l}x}{R_t + R_c + \frac{r}{l}(l-x)}.$$

As $P = Q$, we get

$$R_t + \frac{r}{l}(l-x) = R + \frac{r}{l}x$$

or,
$$R_t = (R - r) + \frac{2r}{l}x.$$

If $x_0$, $x_{100}$ and $x_t$ are the values of $x$ at the ice point, steam point and the temperature $t$ respectively,

$$R_{100} = (R - r) + \frac{2r}{l}x_{100}$$

$$R_0 = (R - r) + \frac{2r}{l}x_0$$

and
$$R_t = (R - r) + \frac{2r}{l}x_t.$$

This gives

$$t = \frac{R_t - R_0}{R_{100} - R_0} \times 100 = \frac{x_t - x_0}{x_{100} - x_0} \times 100.$$

**Absolute Scale and Ideal Gas Scale**

Equation (23.3) tells us that the change in resistance $\Delta R$ when the temperature increases by one degree ($\Delta t = 1°$) is

$$\Delta R = \frac{R_{100} - R_0}{100}$$

and is the same for all temperatures. This means that the resistance increases uniformly as the temperature increases. Note that this conclusion follows from the particular definition (23.3) of temperature and is not the property of platinum or the other metal used to form the resistance. Similarly, it follows that the expansion of mercury is uniform on the mercury scale defined by (23.2). If we measure the change in resistance of a platinum wire against the temperature measured on mercury scale, we shall find that the resistance varies slowly at lower temperatures and slightly more rapidly at higher temperatures. Similarly, if the expansion of mercury is measured against the temperature defined on platinum scale (23.3), we shall find that mercury does not expand at uniform rate as the temperature varies.

Thus, the two scales of temperature do not agree with each other. They are forced to agree at ice point and steam point but for other physical states the readings of the two thermometers will be different. In general, the scale depends on the properties of the thermometric substance used to define the scale.

It is possible to define an absolute scale of temperature which does not depend on the thermometric substance and its properties. We now define the ideal gas temperature scale which happens to be identical to the absolute temperature scale.

## 23.4 CONSTANT VOLUME GAS THERMOMETER

A gas enclosed in a container has a definite volume and a definite pressure in any given state. If the volume is kept constant, the pressure of a gas

increases monotonically with increasing temperature. This property of a gas may be used to construct a thermometer. Figure (23.3) shows a schematic diagram of a constant volume gas thermometer.

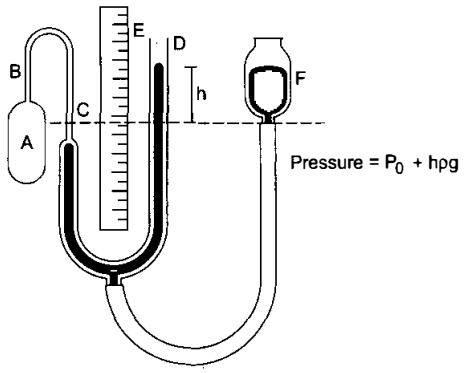



Figure 23.3

A mass of gas is enclosed in a bulb $A$ connected to a capillary $BC$. The capillary is connected to the manometer $CD$ which contains mercury. The other end of the manometer is open to atmosphere. A vertical metre scale $E$ is fixed in such a way that the height of the mercury column in the tube $D$ can easily be measured. The mercury in the manometer is connected to the mercury reservoir $F$ through a rubber tube.

The capillary $BC$ has a fixed mark at $C$. By raising or lowering the reservoir $F$, the mercury level in the left part of the manometer is maintained at $C$. This ensures that the volume of the gas enclosed in the bulb $A$ (and the capillary $BC$) remains constant. The pressure of the gas is equal to the atmospheric pressure plus the pressure due to the difference of the mercury columns in the manometer. Thus, $p = p_0 + h\rho g$, where $p_0$ = atmospheric pressure, $h$ = difference of mercury levels in the manometer tube, $\rho$ = density of mercury and $g$ = acceleration due to gravity.

If the temperature of the bulb is increased and its volume is kept constant by adjusting the height of the reservoir $F$, the pressure of the gas $p = p_0 + h\rho g$ increases. Thus, a temperature scale may be defined by choosing some suitable function of this pressure. Let us assume that the temperature is proportional to the pressure, i.e.,

$$T = cp\,, \qquad \ldots (23.4)$$

where $c$ is some constant.

In addition, the temperature of triple point of water is assigned a value 273·16 K. (Triple point is a state in which ice, water and water vapour can stay together in equilibrium.) The unit is called a kelvin and is denoted by the symbol K. To get the value of the constant $c$ in equation (23.4), we can put the bulb $A$ in a triple point cell and measure the pressure $p_{tr}$ of the gas. From equation (23.4),

$$273{\cdot}16\ \text{K} = cp_{tr}$$

or,
$$c = \frac{273{\cdot}16\ \text{K}}{p_{tr}}\,.$$

The temperature of the gas when the pressure is $p$ is obtained by putting this value of $c$ in equation (23.4). It is

$$T = \frac{p}{p_{tr}} \times 273{\cdot}16\ \text{K}. \qquad \ldots (23.5)$$

To use the thermometer, we must first determine the pressure of the gas $p_{tr}$ at the triple point. This is a fixed value for the thermometer and is used in any measurement. To measure the temperature of a bath of extended volume, the bulb $A$ is dipped in the bath. Sufficient time is allowed so that the gas in the bulb comes to thermal equilibrium with the bath. The reservoir $F$ is adjusted to bring the volume of the gas to its original value and the pressure $p$ of the gas is measured with the manometer. The temperature $T$ on the gas scale is then obtained from equation (23.5).

One can also define a centigrade scale with gas thermometers. Suppose the pressure of the gas is $p_0$ when the bulb $A$ is placed in melting ice (ice point) and it is $p_{100}$ when the bulb is placed in a steam bath (steam point). We assign 0°C to the temperature of the ice point and 100°C to the steam point. The temperature $t$ corresponding to a pressure $p$ of the gas is defined by

$$t = \frac{p - p_0}{p_{100} - p_0} \times 100°\text{C}. \qquad \ldots (23.6)$$

The constant volume gas thermometer allows several errors in the temperature measurement. The main sources of error are the following:

(a) The space in the capillary tube $BC$ generally remains out of the heat bath in which the bulb $A$ is placed. The gas in $BC$ is, therefore, not at the same temperature as the gas in $A$.

(b) The volume of the glass bulb changes slightly with temperature allowing the volume of the gas to change.

***Example 23.1***

*The pressure of air in the bulb of a constant volume gas thermometer is* 73 cm *of mercury at* 0°C, 100·3 cm *of mercury at* 100°C *and* 77·8 cm *of mercury at room temperature. Find the room temperature in centigrades.*

***Solution*** : We have $t = \dfrac{p - p_0}{p_{100} - p_0} \times 100°\text{C}$

$$= \frac{77{\cdot}8 - 73}{100{\cdot}3 - 73} \times 100°\text{C} = 17°\text{C}.$$

## 23.5 IDEAL GAS TEMPERATURE SCALE

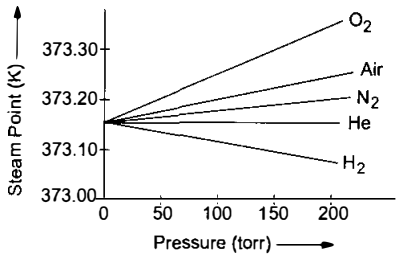



Figure 23.4

The temperature scale defined by equation (23.5) depends slightly on the gas used and its amount present in the thermometer. Figure (23.4) shows the temperature of a steam bath (steam point) as measured by placing different gases in different amounts. On the horizontal axis, we have plotted the pressure $p_{tr}$ of the gas in the thermometer at the triple point of water. This is almost proportional to the mass of the gas present. From the figure, we see that although different gas thermometers differ in their measurement of steam point but the difference decreases as the amount of the gas and hence $p_{tr}$ decreases. In the limit $p_{tr} \to 0$, all the different gas thermometers give the same value 373·15 K for steam point. So we define a temperature scale by the equation

$$T = \lim_{p_{tr} \to 0} \frac{p}{p_{tr}} \times 273{\cdot}16 \text{ K}. \qquad \ldots (23.7)$$

When we use small amount of gas in a gas thermometer, the scale (23.5) is almost identical to (23.7).

The temperature scale defined by (23.7) is called *ideal gas temperature scale* and is independent of the gas chosen. However, it may depend on the properties of gases in general. As mentioned above, it is possible to define an *absolute temperature scale* which does not depend on any property of any substance. Such a temperature scale has been defined and is in use. The unit of this temperature is called kelvin and is abbreviated as K. The ideal gas temperature scale (23.7) happens to be identical with the absolute scale and hence we have used K to denote the unit.

## 23.6 CELSIUS TEMPERATURE SCALE

The temperature of the ice point on the ideal gas scale is 273·15 K and of the steam point is $T = 373{\cdot}15$ K. The interval between the two is 100 K. The centigrade scale discussed earlier has 0°C for the ice point and 100°C for the steam point. However, dividing the temperature interval into 100 parts needs a thermometric substance like mercury in glass or resistance of platinum wire etc. Thus, the scales are different for different thermometers. The use of the term "centrigrade scale" is now replaced by "Celsius scale". Celsius scale is defined to have the ice point at 0°C and the steam point at 100°C. The size of a degree in Celsius scale is defined to be the same as the size of a degree in the ideal gas scale. The Celsius scale is shifted from the ideal gas scale by $-273{\cdot}15$. If $\theta$ denotes the Celsius temperature and $T$ denotes the kelvin temperature,

$$\theta = T - 273{\cdot}15 \text{ K}. \qquad \ldots (23.8)$$

The mercury centigrade scale described earlier is quite close to the Celsius scale.

## 23.7 IDEAL GAS EQUATION

We have seen that the pressure of all gases changes with temperature in a similar fashion for low pressures. Many of the properties of gases are common at low pressures (and high temperatures, that is, far above their condensation point). The pressure, volume and the temperature in kelvin of such a gas obey the equation

$$pV = nRT, \qquad \ldots (23.9)$$

where $n$ is the amount of the gas in number of moles and $R$ is a universal constant having value 8·314 J K$^{-1}$ mol$^{-1}$. The constant $R$ is called the gas constant. Equation (23.9) is known as *ideal gas equation*. A gas obeying this equation is called an *ideal gas*.

## 23.8 CALLENDAR'S COMPENSATED CONSTANT PRESSURE THERMOMETER

In constant volume gas thermometer, the gas in the capillary tube connecting the bulb and the manometer remains outside the heat bath. The temperature of this part is different from the main bulk of the gas and this introduces some error. Callender's compensated constant pressure thermometer avoids this problem by a special design.

Figure (23.5) shows a schematic diagram of the Callender's thermometer. An ideal gas is filled in a bulb $A$ connected to a manometer $M$ and another bulb $B$ through a capillary $cd$. The bulb $B$ is filled with mercury and is graduated in volume. Mercury may be

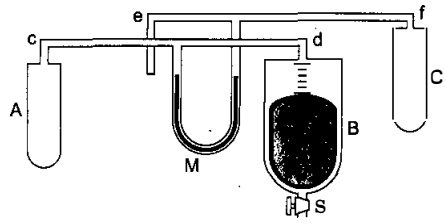



Figure 23.5

taken out from the bulb by opening a stopcock fitted in the lower part of $B$. The volume of the mercury taken out may be measured with the help of the graduations on $B$. The other arm of the manometer is connected to a capillary tube $ef$ and a bulb $C$. The capillary $ef$ and the capillary $cd$ have equal volumes. The bulb $A$ and the bulb $C$ also have equal volumes.

The same ideal gas (as filled in $A$) is filled in the bulb $C$. The amount of gas in the bulb $C$ (and the capillary $ef$) is kept the same as the amount in the bulb $A$ (and the capillary $cd$). When all the parts of the thermometer are at the same temperature and the bulb $B$ is completely filled with mercury, the levels of the liquid in the two arms of the manometer are equal.

To measure the temperature of a heat bath (at temperature larger than the ice point), the bulb $A$ is placed in the heat bath and the bulbs $B$ and $C$ are placed in melting-ice baths. The pressure in the bulb $A$ becomes more than the pressure in the bulb $C$ and the levels in the manometer tubes become different. Mercury is taken out from $B$ till the levels in the manometer tubes become equal. When mercury is taken out, the total volume of the gas in bulb $A$, capillary $cd$ and bulb $B$ increases. This decreases the pressure on this side. When sufficient amount of mercury is taken out, the pressure becomes equal to the pressure in the bulb $C$ and hence, the levels in the manometer tubes again become equal.

Suppose,

the volume of $A$ = volume of $C = V$,

volume of capillary $cd$ = volume of capillary $ef = v_0$,

volume of the mercury taken out $= v'$,

temperature of the heat bath $= T$,

temperature of the ice bath $= T_0$,

and the temperature of $cd$ and $ef = T'$.

Using $pV = nRT$, the number of moles in the bulb $A$, the capillary $cd$ and the bulb $B$

$$= \frac{pV}{RT} + \frac{pv_0}{RT'} + \frac{pv'}{RT_0}$$

and, the number of moles in the bulb $C$ and the capillary $ef$

$$= \frac{pV}{RT_0} + \frac{pv_0}{RT'}.$$

As equal amounts of the gas are filled on the two sides,

$$\frac{pV}{RT} + \frac{pv_0}{RT'} + \frac{pv'}{RT_0} = \frac{pV}{RT_0} + \frac{pv_0}{RT'}$$

or, $$\frac{V}{T} = \frac{V}{T_0} - \frac{v'}{T_0}$$

or, $$T = \frac{V}{V - v'} T_0. \quad \ldots (23.10)$$

All the quantities on the right side are known and hence, the temperature of the bath is obtained.

### 23.9 ADIABATIC AND DIATHERMIC WALLS

We have seen that heat flows from a high-temperature body to a low-temperature body when they are put in contact. There are certain materials which resist the flow of heat through them. When two bodies at different temperatures are separated by such a material, the heat-flow is very slow. We assume an idealised wall or separator which does not allow any heat-flow through it. The bodies on the two sides of such a wall may remain at different temperatures. Such a wall is called an *adiabatic wall*.

Opposite is the concept of a *diathermic wall* which allows heat transfer through it rapidly. If two bodies at different temperatures are separated by such a wall, their temperatures will become equal in a short time.

### 23.10 THERMAL EXPANSION

If the temperature of a body increases, in general, its size also increases. We used this expansion property to define a temperature scale. Now, we have an absolute scale of temperature independent of any property of any susbstance. We can study the thermal expansion of a body as a function of temperature. Consider a rod at temperature $T$ and suppose its length at this temperature is $L$. As the temperature is changed to $T + \Delta T$, the length is changed to $L + \Delta L$. We define *average coefficient of linear expansion* in the temperature range $\Delta T$ as

$$\bar{\alpha} = \frac{1}{L} \frac{\Delta L}{\Delta T}.$$

The *coefficient of linear expansion* at temperature $T$ is limit of average coefficient as $\Delta T \to 0$, i.e.,

$$\alpha = \lim_{\Delta T \to 0} \frac{1}{L} \frac{\Delta L}{\Delta T} = \frac{1}{L} \frac{dL}{dT}.$$

Suppose the length of a rod is $L_0$ at 0°C and $L_\theta$ at temperature $\theta$ measured in Celsius. If $\alpha$ is small and constant over the given temperature interval,

$$\alpha = \frac{L_\theta - L_0}{L_0 \theta}$$

or, $$L_\theta = L_0 (1 + \alpha\theta). \quad \ldots (23.11)$$

This equation is widely used in solving problems.

The coefficient of volume expansion $\gamma$ is defined in a similar way. If $V$ is the volume of a body at temperature $T$, the *coefficient of volume expansion* at temperature $T$ is

$$\gamma = \frac{1}{V}\frac{dV}{dT}.$$

It is also known as coefficient of cubical expansion.

If $V_0$ and $V_\theta$ denote the volumes at 0°C and $\theta$ (measured in Celsius) respectively and $\gamma$ is small and constant over the given temperature range, we have

$$V_\theta = V_0 (1 + \gamma\theta). \quad \ldots (23.12)$$

It is easy to show that $\gamma = 3\alpha$.

The change in volume of water as temperature increases is slightly complicated. Figure (23.6) shows the volume of 1 g of water as the temperature increases from 0°C to 10°C. The volume of water

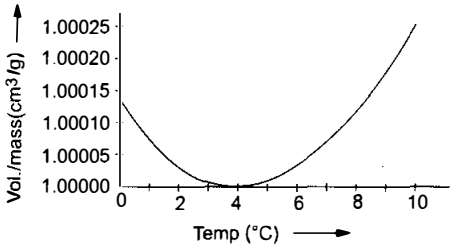



Figure 23.6

decreases from about 1·00013 $cm^3$ to 1·00000 $cm^3$ as the temperature increases from 0°C to 4°C. This means $\gamma$ is negative in this temperature range. The volume again increases as the temperature is increased further from 4°C. The density (mass/volume) of water is thus maximum at 4°C.

The anomalous expansion of water has a favourable effect for animals living in water. Since the density of water is maximum at 4°C, water at the bottom of lakes remain at 4°C in winter even if that at the surface freezes. This allows marine animals to remain alive and move near the bottom.

The study of expansion of a liquid presents another difficulty due to the expansion of the container. Consider a liquid kept in a flask with a graduated stem. As the temperature is increased, the volume of the flask expands faster than the liquid in the beginning and the level of liquid in the stem goes down as if the liquid has contracted. As the temperature of the liquid increases, the volume of the liquid increases and rises in the stem. The apparent increase in the volume is equal to the real increase in the volume of the liquid minus the increase in the volume of the container.

### *Worked Out Examples*

**1.** *The pressure of the gas in a constant volume gas thermometer at steam point* (373·15 K) *is* $1{\cdot}50 \times 10^4$ Pa. *What will be the pressure at the triple point of water ?*

***Solution*** : The temperature in kelvin is defined as

$$T = \frac{p}{p_{tr}} \times 273{\cdot}16 \text{ K}.$$

Thus,

$$373{\cdot}15 = \frac{1{\cdot}50 \times 10^4 \text{ Pa}}{p_{tr}} \times 273{\cdot}16$$

or,
$$p_{tr} = 1{\cdot}50 \times 10^4 \text{ Pa} \times \frac{273{\cdot}16}{373{\cdot}15}$$
$$= 1{\cdot}10 \times 10^4 \text{ Pa}.$$

**2.** *The pressure of air in the bulb of a constant volume gas thermometer at* 0°C *and* 100°C *are* 73·00 cm *and* 100 cm *of mercury respectively. Calculate the pressure at the room temperature* 20°C.

***Solution*** : The room temperature on the scale measured by the thermometer is

$$t = \frac{p_t - p_0}{p_{100} - p_0} \times 100°\text{C}.$$

Thus,

$$20°\text{C} = \frac{p_t - 73{\cdot}00 \text{ cm of Hg}}{100 \text{ cm of Hg} - 73{\cdot}00 \text{ cm of Hg}} \times 100°\text{C}$$

or, $p_t = 78{\cdot}4$ cm of mercury.

**3.** *The pressure of the gas in a constant volume gas thermometer is* 80 cm *of mercury in melting ice at* 1 atm. *When the bulb is placed in a liquid, the pressure becomes* 160 cm *of mercury. Find the temperature of the liquid.*

***Solution*** : For an ideal gas at constant volume,

$$\frac{T_1}{T_2} = \frac{p_1}{p_2}$$

or,
$$T_2 = \frac{p_2}{p_1} T_1.$$

The temperature of melting ice at 1 atm is 273·15 K. Thus, the temperature of the liquid is

$$T_2 = \frac{160}{80} \times 273{\cdot}15 \text{ K} = 546{\cdot}30 \text{ K}.$$

**4.** *In a constant volume gas thermometer, the pressure of the working gas is measured by the difference in the levels of mercury in the two arms of a U-tube connected to the gas at one end. When the bulb is placed at the room temperature* 27·0°C, *the mercury column in the arm open to atmosphere stands* 5·00 cm *above the level of mercury in the other arm. When the bulb is placed in a hot liquid, the difference of mercury levels becomes* 45·0 cm. *Calculate the temperature of the liquid. (Atmospheric pressure* = 75·0 cm of mercury.)

***Solution*** : The pressure of the gas = atmospheric pressure + the pressure due to the difference in mercury levels.

At 27°C, the pressure is 75 cm + 5 cm = 80 cm of mercury. At the liquid temperature, the pressure is 75 cm + 45 cm = 120 cm of mercury. Using $T_2 = \frac{P_2}{P_1} T_1$, the temperature of the liquid is

$$T = \frac{120}{80} \times (27{\cdot}0 + 273{\cdot}15)\ \text{K} = 450{\cdot}22\ \text{K}.$$

$$= 177{\cdot}07°\text{C} \approx 177°\text{C}.$$

**5.** *The resistances of a platinum resistance thermometer at the ice point, the steam point and the boiling point of sulphur are 2·50, 3·50 and 6·50 Ω respectively. Find the boiling point of sulphur on the platinum scale. The ice point and the steam point measure 0° and 100° respectively.*

***Solution*** : The temperature on the platinum scale is defined as

$$t = \frac{R_t - R_0}{R_{100} - R_0} \times 100°.$$

The boiling point of sulphur on this scale is

$$t = \frac{6{\cdot}50 - 2{\cdot}50}{3{\cdot}50 - 2{\cdot}50} \times 100° = 400°.$$

**6.** *A platinum resistance thermometer reads 0° and 100° at the ice point and the boiling point of water respectively. The resistance of a platinum wire varies with Celsius temperature θ as $R_t = R_0 (1 + \alpha\theta + \beta\theta^2)$, where $\alpha = 3{\cdot}8 \times 10^{-3}\ °\text{C}^{-1}$ and $\beta = -5{\cdot}6 \times 10^{-7}\ °\text{C}^{-2}$. What will be the reading of this thermometer if it is placed in a liquid bath maintained at 50°C ?*

***Solution*** : The resistances of the wire in the thermometer at 100°C and 50°C are

$$R_{100} = R_0 [1 + \alpha \times 100°\text{C} + \beta \times (100°\text{C})^2]$$

and, $$R_{50} = R_0 [1 + \alpha \times 50°\text{C} + \beta \times (50°\text{C})^2].$$

The temperature $t$ measured on the platinum thermometer is given by

$$t = \frac{R_{50} - R_0}{R_{100} - R_0} \times 100°$$

$$= \frac{\alpha \times 50°\text{C} + \beta \times (50°\text{C})^2}{\alpha \times 100°\text{C} + \beta \times (100°\text{C})^2} \times 100°$$

$$= 50{\cdot}4°.$$

**7.** *A platinum resistance thermometer is constructed which reads 0° at ice point and 100° at steam point. Let $t_p$ denote the temperature on this scale and let t denote the temperature on a mercury thermometer scale. The resistance of the platinum coil varies with t as $R_t = R_0 (1 + \alpha t + \beta t^2)$. Derive an expression for the resistance as a function of $t_p$.*

***Solution*** : Let $R_{t_p}$ denote the resistance of the coil at the platinum scale temperature $t_p$. Then

$$t_p = \frac{R_{t_p} - R_0}{R_{100} - R_0} \times 100$$

or, $$R_{t_p} = \frac{t_p}{100}\left(R_{100} - R_0\right) + R_0$$

$$= \frac{t_p}{100}\left[R_0\{1 + \alpha \times 100 + \beta \times (100)^2\} - R_0\right] + R_0$$

$$= \frac{t_p}{100}\left[\alpha \times 100 + \beta \times (100)^2\right] R_0 + R_0$$

$$= R_0\left[1 + \{\alpha \times 100 + \beta \times (100)^2\} \frac{t_p}{100}\right]$$

$$= R_0\left[1 + \alpha t_p + \beta \times (100)\, t_p\right].$$

Only numerical values of α and β are to be used.

**8.** *An iron rod of length 50 cm is joined at an end to an aluminium rod of length 100 cm. All measurements refer to 20°C. Find the length of the composite system at 100°C and its average coefficient of linear expansion. The coefficient of linear expansion of iron and aluminium are $12 \times 10^{-6}\ °\text{C}^{-1}$ and $24 \times 10^{-6}\ °\text{C}^{-1}$ respectively.*

***Solution*** : The length of the iron rod at 100°C is

$$l_1 = (50\ \text{cm})\ [1 + (12 \times 10^{-6}\ °\text{C}^{-1})\ (100°\text{C} - 20°\text{C})]$$

$$= 50{\cdot}048\ \text{cm}.$$

The length of the aluminium rod at 100°C is

$$l_2 = (100\ \text{cm})\ [1 + (24 \times 10^{-6}\ °\text{C}^{-1})\ (100°\text{C} - 20°\text{C})]$$

$$= 100{\cdot}192\ \text{cm}.$$

The length of the composite system at 100°C is

$$50{\cdot}048\ \text{cm} + 100{\cdot}192\ \text{cm} = 150{\cdot}24\ \text{cm}.$$

The length of the composite system at 20°C is 150 cm. So, the average coefficient of linear expansion of the composite rod is

$$\alpha = \frac{0{\cdot}24\ \text{cm}}{150\ \text{cm} \times (100°\text{C} - 20°\text{C})}$$

$$= 20 \times 10^{-6}\ °\text{C}^{-1}.$$

**9.** *An iron ring measuring 15·00 cm in diameter is to be shrunk on a pulley which is 15·05 cm in diameter. All measurements refer to the room temperature 20°C. To what minimum temperature should the ring be heated to make the job possible ? Calculate the strain developed in the ring when it comes to the room temperature. Coefficient of linear expansion of iron = $12 \times 10^{-6}\ °\text{C}^{-1}$.*

***Solution*** : The ring should be heated to increase its diameter from 15·00 cm to 15·05 cm.

Using $$l_2 = l_1 (1 + \alpha\, \Delta\theta),$$

$$= \frac{0{\cdot}05 \text{ cm}}{15{\cdot}00 \text{ cm} \times 12 \times 10^{-6} \text{ °C}^{-1}}$$

$$= 278\text{°C}$$

The temperature $= 20\text{°C} + 278\text{°C} = 298\text{°C}$.

The strain developed $= \frac{l_2 - l_1}{l_1} = 3{\cdot}33 \times 10^{-3}$.

**10.** *A pendulum clock consists of an iron rod connected to a small, heavy bob. If it is designed to keep correct time at* 20°C, *how fast or slow will it go in* 24 hours *at* 40°C ? *Coefficient of linear expansion of iron* $= 1.2 \times 10^{-5} \text{ °C}^{-1}$.

***Solution*** : The time period at temperature θ is

$$T = 2\pi\sqrt{l_\theta / g}$$

$$= 2\pi \sqrt{l_0(1 + \alpha\theta)/g}$$

$$= 2\pi \sqrt{l_0 / g}\ (1 + \alpha\theta)^{1/2}$$

$$\approx T_0(1 + \frac{1}{2}\alpha\theta).$$

Thus, $T_{20} = T_0[1 + \frac{1}{2}\alpha(20\text{°C})]$

and, $T_{40} = T_0[1 + \frac{1}{2}\alpha(40\text{°C})]$

or, $\frac{T_{40}}{T_{20}} = [1 + (20\text{°C})\alpha]\ [1 + (10\text{°C})\alpha]^{-1}$

$$\approx [1 + (20\text{°C})\alpha]\ [1 - (10\text{°C})\alpha]$$

$$\approx 1 + (10\text{°C})\ \alpha$$

or, $\frac{T_{40} - T_{20}}{T_{20}} = (10\text{°C})\ \alpha = 1{\cdot}2 \times 10^{-4}$. ... (i)

This is fractional loss of time. As the temperature increases, the time period also increases. Thus, the clock goes slow. The time lost in 24 hours is, by (i),

$$\Delta t = (24 \text{ hours})\ (1{\cdot}2 \times 10^{-4}) = 10{\cdot}4 \text{ s}.$$

**11.** *A pendulum clock having copper rod keeps correct time at* 20°C. *It gains* 15 seconds *per day if cooled to* 0°C. *Calculate the coefficient of linear expansion of copper.*

***Solution*** : The time period at temperature θ is

$$T = 2\pi \sqrt{l_\theta / g}$$

$$\approx T_0\ (1 + \frac{1}{2}\alpha\theta)$$

Thus, $T_{20} = T_0\ [1 + \alpha\ (10\text{°C})]$

or, $\frac{(T_{20} - T_0)}{T_0} = \alpha\ (10\text{°C})$. ... (i)

$T_{20}$ is the correct time period. The period at 0°C is smaller so that the clock runs fast. Equation (i) gives approximately the fractional gain in time. The time gained in 24 hours is

$$\Delta T = (24 \text{ hours})\ [(10\text{°C})\alpha]$$

or, $15 \text{ s} = (24 \text{ hours})\ [(10\text{°C})\alpha]$

or, $\alpha = \frac{15 \text{ s}}{(24 \text{ hours})\ (10\text{°C})}$

$$= 1{\cdot}7 \times 10^{-5} \text{ °C}^{-1}.$$

**12.** *A piece of metal weighs* 46 g *in air and* 30 g *in a liquid of density* $1{\cdot}24 \times 10^3 \text{ kg m}^{-3}$ *kept at* 27°C. *When the temperature of the liquid is raised to* 42°C, *the metal piece weighs* 30·5 g. *The density of the liquid at* 42°C *is* $1{\cdot}20 \times 10^3 \text{ kg m}^{-3}$. *Calculate the coefficient of linear expansion of the metal.*

***Solution*** : Let the volume of the metal piece be $V_0$ at 27°C and $V_\theta$ at 42°C. The density of the liquid at 27°C is $\rho_0 = 1{\cdot}24 \times 10^3 \text{ kg m}^{-3}$ and the density of the liquid at 42°C is $\rho_\theta = 1{\cdot}20 \times 10^3 \text{ kg m}^{-3}$.

The weight of the liquid displaced = apparent loss in the weight of the metal piece when dipped in the liquid. Thus,

$$V_0\ \rho_0 = 46 \text{ g} - 30 \text{ g} = 16 \text{ g}$$

and, $V_\theta\ \rho_\theta = 46 \text{ g} - 30{\cdot}5 \text{ g} = 15{\cdot}5 \text{ g}$ .

Thus,

$$\frac{V_\theta}{V_0} = \frac{\rho_0}{\rho_\theta} \times \frac{15{\cdot}5}{16}$$

or, $1 + 3\ \alpha\Delta\theta = \frac{1{\cdot}24 \times 10^3 \times 15{\cdot}5}{1{\cdot}20 \times 10^3 \times 16}$

or, $1 + 3\alpha(42\text{°C} - 27\text{°C}) = 1{\cdot}00104$

or, $\alpha = 2{\cdot}3 \times 10^{-5} \text{ °C}^{-1}$.

**13.** *A sphere of diameter* 7·0 cm *and mass* 266·5 g *floats in a bath of liquid. As the temperature is raised, the sphere begins to sink at a temperature of* 35°C. *If the density of the liquid is* $1{\cdot}527 \text{ g cm}^{-3}$ *at* 0°C, *find the coeffiecient of cubical expansion of the liquid. Neglect the expansion of the sphere.*

***Solution*** : It is given that the expansion of the sphere is negligible as compared to the expansion of the liquid. At 0°C, the density of the liquid is $\rho_0 = 1{\cdot}527 \text{ g cm}^{-3}$. At 35°C, the density of the liquid equals the density of the sphere. Thus,

$$\rho_{35} = \frac{266{\cdot}5 \text{ g}}{\frac{4}{3}\pi\ (3{\cdot}5 \text{ cm})^3}$$

$$= 1{\cdot}484 \text{ g cm}^{-3}$$

We have $\frac{\rho_\theta}{\rho_0} = \frac{V_0}{V_\theta} = \frac{1}{(1 + \gamma\theta)}$

or, $\rho_\theta = \frac{\rho_0}{1 + \gamma\theta}$

Thus, $\gamma = \frac{\rho_0 - \rho_{35}}{\rho_{35}\ (35\text{°C})}$

$$= \frac{(1{\cdot}527 - 1{\cdot}484)\ \text{g cm}^{-3}}{(1{\cdot}484\ \text{g cm}^{-3})\ (35°\text{C})}$$

$$= 8{\cdot}28 \times 10^{-4}\ °\text{C}^{-1}.$$

**14.** *An iron rod and a copper rod lie side by side. As the temperature is changed, the difference in the lengths of the rods remains constant at a value of* 10 cm. *Find the lengths at* 0°C. *Coefficients of linear expansion of iron and copper are* $1{\cdot}1 \times 10^{-5}\ °\text{C}^{-1}$ *and* $1{\cdot}7 \times 10^{-5}\ °\text{C}^{-1}$ *respectively.*

***Solution*** : Suppose the length of the iron rod at 0°C is $l_{i0}$ and the length of the copper rod at 0°C is $l_{c0}$. The lengths at temperature $\theta$ are

$$l_{i\theta} = l_{i0}\ (1 + \alpha_i \theta) \qquad \ldots \text{(i)}$$

and $$l_{c\theta} = l_{c0}\ (1 + \alpha_c \theta). \qquad \ldots \text{(ii)}$$

Subtracting,

$$l_{i\theta} - l_{c\theta} = (l_{i0} - l_{c0}) + (l_{i0}\ \alpha_i - l_{c0}\ \alpha_c)\ \theta. \qquad \ldots \text{(iii)}$$

Now,

$$l_{i\theta} - l_{c\theta} = l_{i0} - l_{c0}\ (= 10\ \text{cm}).$$

Thus, from (iii), $l_{i0}\ \alpha_i = l_{c0}\ \alpha_c$

or, $$\frac{l_{i0}}{l_{c0}} = \frac{\alpha_c}{\alpha_i}$$

or, $$\frac{l_{i0}}{l_{i0} - l_{c0}} = \frac{\alpha_c}{\alpha_c - \alpha_i}$$

$$= \frac{1{\cdot}7 \times 10^{-5}\ °\text{C}^{-1}}{0{\cdot}6 \times 10^{-5}\ °\text{C}^{-1}} = \frac{17}{6}$$

This shows that $l_{i0} - l_{c0}$ is positive. Its value is 10 cm as given in the question.

Hence, $$l_{i0} = \frac{17}{6} \times (l_{i0} - l_{c0})$$

$$= \frac{17}{6} \times 10\ \text{cm} = 28{\cdot}3\ \text{cm}$$

and $$l_{c0} = l_{i0} - 10\ \text{cm} = 18{\cdot}3\ \text{cm}.$$

**15.** *A uniform steel wire of cross-sectional area* $0{\cdot}20\ \text{mm}^2$ *is held fixed by clamping its two ends. Find the extra force exerted by each clamp on the wire if the wire is cooled from* 100°C *to* 0°C. *Young's modulus of steel* $= 2.0 \times 10^{11}\ \text{N m}^{-2}$. *Coefficient of linear expansion of steel* $= 1{\cdot}2 \times 10^{-5}\ °\text{C}^{-1}$.

***Solution*** : Let us assume that the tension is zero at 100°C so that $l_\theta$ is the natural length of the wire at 100°C. As the wire cools down, its natural length decreases to $l_0$. As the wire is fixed at the clamps, its length remains the same as the length at 100°C. Thus, the extension of the wire over its natural length at 0°C is

$$l_\theta - l_0 = l_0\ (1 + \alpha\theta) - l_0 = l_0 \alpha\theta.$$

The strain developed is $\dfrac{l_\theta - l_0}{l_\theta} \approx \dfrac{l_\theta - l_0}{l_0} = \alpha\theta.$

The stress developed = $Y \times$ strain = $Y\alpha\theta$.

The tension in the wire at 0°C is

$T$ = stress × area

$$= Y \alpha t \times 0{\cdot}20\ \text{mm}^2$$

$$= (2{\cdot}0 \times 10^{11}\ \text{N m}^{-2}) \times (1{\cdot}2 \times 10^{-5}\ °\text{C}^{-1})$$

$$\times 100°\text{C} \times (0{\cdot}20 \times 10^{-6}\ \text{m}^{-2})$$

$$= 48\ \text{N}.$$

This is equal to the extra force exerted by each clamp.

**16.** *A glass vessel of volume* $100\ \text{cm}^{-3}$ *is filled with mercury and is heated from* 25°C *to* 75°C. *What volume of mercury will overflow ? Coefficient of linear expansion of glass* $= 1.8 \times 10^{-6}\ °\text{C}^{-1}$ *and coefficient of volume expansion of mercury is* $1.8 \times 10^{-4}\ °\text{C}^{-1}$.

***Solution*** : The volume of mercury at 25°C is

$$V_0 = 100\ \text{cm}^{-3}.$$

The coefficient of volume expansion of mercury

$$\gamma_L = 1{\cdot}8 \times 10^{-4}\ °\text{C}^{-1}.$$

The coefficient of volume expansion of glass

$$\gamma_S = 3 \times 1{\cdot}8 \times 10^{-6}\ °\text{C}^{-1}$$

$$= 5{\cdot}4 \times 10^{-6}\ °\text{C}^{-1}.$$

Thus, the volume of mercury at 75°C is

$$V_{L\theta} = V_0(1 + \gamma_L\ \Delta\theta)$$

and the volume of the vessel at 75°C is

$$V_{S\theta} = V_0\ (1 + \gamma_S\ \Delta\theta).$$

The volume of mercury overflown

$$= V_{L\theta} - V_{S\theta} = V_0\ (\gamma_L - \gamma_S)\ \Delta\theta \qquad \ldots \text{(i)}$$

$$= (100\ \text{cm}^{-3})\ (1{\cdot}8 \times 10^{-4} - 5{\cdot}4 \times 10^{-6})/°\text{C} \times (50°\text{C})$$

$$= 0{\cdot}87\ \text{cm}^3.$$

Note that $\gamma_a = (\gamma_L - \gamma_S)$ acts as the effective coefficient of expansion of the liquid with respect to the solid. The expansion of mercury 'as seen from the glass' can be written as

$$V_\theta - V_0 = V_0\ \gamma_a \theta$$

or, $$V_\theta = V_0(1 + \gamma_a \theta).$$

The constant $\gamma_a$ is called the 'apparent coefficient of expansion' of the liquid with respect to the solid.

**17.** *A barometer reads* 75·0 cm *on a steel scale. The room temperature is* 30°C. *The scale is correctly graduated for* 0°C. *The coefficient of linear expansion of steel is* $\alpha = 1{\cdot}2 \times 10^{-5}\ °\text{C}^{-1}$ *and the coefficient of volume expansion of mercury is* $\gamma = 1{\cdot}8 \times 10^{-4}\ °\text{C}^{-1}$. *Find the correct atmospheric pressure.*

***Solution*** : The 75 cm length of steel at 0°C will become $l_\theta$ at 30°C where,

$$l_\theta = (75\ \text{cm})\ [1 + \alpha\ (30°\text{C})]. \qquad \ldots \text{(i)}$$

The length of mercury column at 30°C is $l_\theta$. Suppose the length of the mercury column, if it were at 0°C, is $l_0$. Then,

$$l_\theta = l_0 [1 + \frac{1}{3}\gamma (30°C)]. \quad \ldots \text{(ii)}$$

By (i) and (ii),

$$l_0[1 + \frac{1}{3}\gamma (30°C)] = 75 \text{ cm}[1 + \alpha(30°C)]$$

or, $$l_0 = 75 \text{ cm} \frac{[1 + \alpha(30°C)]}{[1 + \frac{1}{3}\gamma (30°C)]}$$

$$\approx 75 \text{ cm } [1 + (\alpha - \frac{\gamma}{3}) (30°C)]$$

$$= 74{\cdot}89 \text{ cm}.$$

□

## QUESTIONS FOR SHORT ANSWER

1. If two bodies are in thermal equilibrium in one frame, will they be in thermal equilibrium in all frames ?
2. Does the temperature of a body depend on the frame from which it is observed ?
3. It is heard sometimes that mercury is used in defining the temperature scale because it expands uniformly with the temperature. If the temperature scale is not yet defined, is it logical to say that a substance expands uniformly with the temperature ?
4. In defining the ideal gas temperature scale, it is assumed that the pressure of the gas at constant volume is proportional to the temperature $T$. How can we verify whether this is true or not ? Are we using the kinetic theory of gases ? Are we using the experimental result that the pressure is proportional to temperature ?
5. Can the bulb of a thermometer be made of an adiabatic wall ?
6. Why do marine animals live deep inside a lake when the surface of the lake freezes ?
7. The length of a brass rod is found to be smaller on a hot summer day than on a cold winter day as measured by the same aluminium scale. Do we conclude that brass shrinks on heating ?
8. If mercury and glass had equal coefficient of volume expansion, could we make a mercury thermometer in a glass tube ?
9. The density of water at 4°C is supposed to be 1000 kg $m^{-3}$. Is it same at the sea level and at a high altitude ?
10. A tightly closed metal lid of a glass bottle can be opened more easily if it is put in hot water for some time. Explain.
11. If an automobile engine is overheated, it is cooled by putting water on it. It is advised that the water should be put slowly with engine running. Explain the reason.
12. Is it possible for two bodies to be in thermal equilibrium if they are not in contact ?
13. A spherical shell is heated. The volume changes according to the equation $V_\theta = V_0 (1 + \gamma\theta)$. Does the volume refer to the volume enclosed by the shell or the volume of the material making up the shell ?

## OBJECTIVE I

1. A system $X$ is neither in thermal equilibrium with $Y$ nor with $Z$. The systems $Y$ and $Z$
   (a) must be in thermal equilibrium
   (b) cannot be in thermal equilibrium
   (c) may be in thermal equilibrium.
2. Which of the curves in figure (23-Q1) represents the relation between Celsius and Fahrenheit temperatures ?

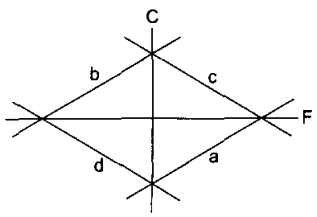



Figure 23-Q1

3. Which of the following pairs may give equal numerical values of the temperature of a body ?
   (a) Fahrenheit and kelvin (b) Celsius and kelvin
   (c) Kelvin and platinum
4. For a constant volume gas thermometer, one should fill the gas at
   (a) low temperature and low pressure
   (b) low temperature and high pressure
   (c) high temperature and low pressure
   (d) high temperature and high pressure.
5. Consider the following statements.
   (A) The coefficient of linear expansion has dimension $K^{-1}$.
   (B) The coefficient of volume expansion has dimension $K^{-1}$.
   (a) $A$ and $B$ are both correct.
   (b) $A$ is correct but $B$ is wrong.

(c) $B$ is correct but $A$ is wrong.
(d) $A$ and $B$ are both wrong.

6. A metal sheet with a circular hole is heated. The hole
(a) gets larger (b) gets smaller
(c) remains of the same size (d) gets deformed.

7. Two identical rectangular strips, one of copper and the other of steel, are rivetted together to form a bimetallic strip ($\alpha_{copper} > \alpha_{steel}$). On heating, this strip will
(a) remain straight
(b) bend with copper on convex side
(c) bend with steel on convex side
(d) get twisted.

8. If the temperature of a uniform rod is slightly increased by $\Delta t$, its moment of inertia $I$ about a perpendicular bisector increases by
(a) zero (b) $\alpha I \Delta t$ (c) $2\alpha I \Delta t$ (d) $3\alpha I \Delta t$.

9. If the temperature of a uniform rod is slightly increased by $\Delta t$, its moment of inertia $I$ about a line parallel to itself will increase by
(a) zero (b) $\alpha I \Delta t$ (c) $2\alpha I \Delta t$ (d) $3\alpha I \Delta t$.

10. The temperature of water at the surface of a deep lake is 2°C. The temperature expected at the bottom is
(a) 0°C (b) 2°C (c) 4°C (d) 6°C.

11. An aluminium sphere is dipped into water at 10°C. If the temperature is increased, the force of buoyancy
(a) will increase (b) will decrease
(c) will remain constant
(d) may increase or decrease depending on the radius of the sphere.

## OBJECTIVE II

1. A spinning wheel is brought in contact with an identical wheel spinning at identical speed. The wheels slow down under the action of friction. Which of the following energies of the first wheel decrease ?
(a) Kinetic (b) Total (c) Mechanical (d) Internal

2. A spinning wheel $A$ is brought in contact with another wheel $B$ initially at rest. Because of the friction at contact, the second wheel also starts spinning. Which of the following energies of the wheel $B$ increase ?
(a) Kinetic (b) Total (c) Mechanical (d) Internal

3. A body $A$ is placed on a railway platform and an identical body $B$ in a moving train. Which of the following energies of $B$ are greater than those of $A$ as seen from the ground ?
(a) Kinetic (b) Total (c) Mechanical (d) Internal

4. In which of the following pairs of temperature scales, the size of a degree is identical ?
(a) Mercury scale and ideal gas scale
(b) Celsius scale and mercury scale
(c) Celsius scale and ideal gas scale
(d) Ideal gas scale and absolute scale

5. A solid object is placed in water contained in an adiabatic container for some time. The temperature of water falls during the period and there is no appreciable change in the shape of the object. The temperature of the solid object
(a) must have increased (b) must have decreased
(c) may have increased
(d) may have remained constant.

6. As the temperature is increased, the time period of a pendulum
(a) increases proportionately with temperature
(b) increases (c) decreases
(d) remains constant.

## EXERCISES

1. The steam point and the ice point of a mercury thermometer are marked as 80° and 20°. What will be the temperature in centigrade mercury scale when this thermometer reads 32° ?

2. A constant volume thermometer registers a pressure of $1{\cdot}500 \times 10^4$ Pa at the triple point of water and a pressure of $2{\cdot}050 \times 10^4$ Pa at the normal boiling point. What is the temperature at the normal boiling point ?

3. A gas thermometer measures the temperature from the variation of pressure of a sample of gas. If the pressure measured at the melting point of lead is 2·20 times the pressure measured at the triple point of water, find the melting point of lead.

4. The pressure measured by a constant volume gas thermometer is 40 kPa at the triple point of water. What will be the pressure measured at the boiling point of water (100°C) ?

5. The pressure of the gas in a constant volume gas thermometer is 70 kPa at the ice point. Find the pressure at the steam point.

6. The pressures of the gas in a constant volume gas thermometer are 80 cm, 90 cm and 100 cm of mercury at the ice point, the steam point and in a heated wax bath respectively. Find the temperature of the wax bath.

7. In a Callender's compensated constant pressure air thermometer, the volume of the bulb is 1800 cc. When the bulb is kept immersed in a vessel, 200 cc of mercury has to be poured out. Calculate the temperature of the vessel.

8. A platinum resistance thermometer reads 0° when its resistance is 80 Ω and 100° when its resistance is 90 Ω. Find the temperature at the platinum scale at which the resistance is 86 Ω.

**9.** A resistance thermometer reads $R = 20{\cdot}0\ \Omega$, $27{\cdot}5\ \Omega$, and $50{\cdot}0\ \Omega$ at the ice point (0°C), the steam point (100°C) and the zinc point (420°C) respectively. Assuming that the resistance varies with temperature as $R_\theta = R_0(1 + \alpha\theta + \beta\theta^2)$, find the values of $R_0$, $\alpha$ and $\beta$. Here $\theta$ represents the temperature on Celsius scale.

**10.** A concrete slab has a length of 10 m on a winter night when the temperature is 0°C. Find the length of the slab on a summer day when the temperature is 35°C. The coefficient of linear expansion of concrete is $1{\cdot}0 \times 10^{-5}\ °C^{-1}$.

**11.** A metre scale made of steel is calibrated at 20°C to give correct reading. Find the distance between 50 cm mark and 51 cm mark if the scale is used at 10°C. Coefficient of linear expansion of steel is $1{\cdot}1 \times 10^{-5}\ °C^{-1}$.

**12.** A railway track (made of iron) is laid in winter when the average temperature is 18°C. The track consists of sections of 12·0 m placed one after the other. How much gap should be left between two such sections so that there is no compression during summer when the maximum temperature goes to 48°C ? Coefficient of linear expansion of iron $= 11 \times 10^{-6}\ °C^{-1}$.

**13.** A circular hole of diameter 2·00 cm is made in an aluminium plate at 0°C. What will be the diameter at 100°C ? $\alpha$ for aluminium $= 2{\cdot}3 \times 10^{-5}\ °C^{-1}$.

**14.** Two metre scales, one of steel and the other of aluminium, agree at 20°C. Calculate the ratio aluminium-centimetre/steel-centimetre at (a) 0°C, (b) 40°C and (c) 100°C. $\alpha$ for steel $= 1{\cdot}1 \times 10^{-5}\ °C^{-1}$ and for aluminium $= 2{\cdot}3 \times 10^{-5}\ °C^{-1}$.

**15.** A metre scale is made up of steel and measures correct length at 16°C. What will be the percentage error if this scale is used (a) on a summer day when the temperature is 46°C and (b) on a winter day when the temperature is 6°C ? Coefficient of linear expansion of steel $= 11 \times 10^{-6}\ °C^{-1}$.

**16.** A metre scale made of steel reads accurately at 20°C. In a sensitive experiment, distances accurate up to 0·055 mm in 1 m are required. Find the range of temperature in which the experiment can be performed with this metre scale. Coefficient of linear expansion of steel $= 11 \times 10^{-6}\ °C^{-1}$.

**17.** The density of water at 0°C is $0{\cdot}998\ g\ cm^{-3}$ and at 4°C is $1{\cdot}000\ g\ cm^{-3}$. Calculate the average coefficient of volume expansion of water in the temperature range 0 to 4°C.

**18.** Find the ratio of the lengths of an iron rod and an aluminium rod for which the difference in the lengths is independent of temperature. Coefficients of linear expansion of iron and aluminium are $12 \times 10^{-6}\ °C^{-1}$ and $23 \times 10^{-6}\ °C^{-1}$ respectively.

**19.** A pendulum clock gives correct time at 20°C at a place where $g = 9{\cdot}800\ m\ s^{-2}$. The pendulum consists of a light steel rod connected to a heavy ball. It is taken to a different place where $g = 9{\cdot}788\ m\ s^{-2}$. At what temperature will it give correct time ? Coefficient of linear expansion of steel $= 12 \times 10^{-6}\ °C^{-1}$.

**20.** An aluminium plate fixed in a horizontal position has a hole of diameter 2·000 cm. A steel sphere of diameter 2·005 cm rests on this hole. All the lengths refer to a temperature of 10°C. The temperature of the entire system is slowly increased. At what temperature will the ball fall down ? Coefficient of linear expansion of aluminium is $23 \times 10^{-6}\ °C^{-1}$ and that of steel is $11 \times 10^{-6}\ °C^{-1}$.

**21.** A glass window is to be fit in an aluminium frame. The temperature on the working day is 40°C and the glass window measures exactly 20 cm × 30 cm. What should be the size of the aluminium frame so that there is no stress on the glass in winter even if the temperature drops to 0°C ? Coefficients of linear expansion for glass and aluminium are $9{\cdot}0 \times 10^{-6}\ °C^{-1}$ and $24 \times 10^{-6}\ °C^{-1}$ respectively.

**22.** The volume of a glass vessel is 1000 cc at 20°C. What volume of mercury should be poured into it at this temperature so that the volume of the remaining space does not change with temperature ? Coefficients of cubical expansion of mercury and glass are $1{\cdot}8 \times 10^{-4}\ °C^{-1}$ and $9{\cdot}0 \times 10^{-6}\ °C^{-1}$ respectively.

**23.** An aluminium can of cylindrical shape contains $500\ cm^3$ of water. The area of the inner cross section of the can is $125\ cm^2$. All measurements refer to 10°C. Find the rise in the water level if the temperature increases to 80°C. The coefficient of linear expansion of aluminium $= 23 \times 10^{-6}\ °C^{-1}$ and the average coefficient of volume expansion of water $= 3.2 \times 10^{-4}\ °C^{-1}$ respectively.

**24.** A glass vessel measures exactly 10 cm × 10 cm × 10 cm at 0°C. It is filled completely with mercury at this temperature. When the temperature is raised to 10°C, $1{\cdot}6\ cm^3$ of mercury overflows. Calculate the coefficient of volume expansion of mercury. Coefficient of linear expansion of glass $= 6{\cdot}5 \times 10^{-6}\ °C^{-1}$.

**25.** The densities of wood and benzene at 0°C are $880\ kg\ m^3$ and $900\ kg\ m^{-3}$ respectively. The coefficients of volume expansion are $1{\cdot}2 \times 10^{-3}\ °C^{-1}$ for wood and $1{\cdot}5 \times 10^{-3}\ °C^{-1}$ for benzene. At what temperature will a piece of wood just sink in benzene ?

**26.** A steel rod of length 1 m rests on a smooth horizontal base. If it is heated from 0°C to 100°C, what is the longitudinal strain developed ?

**27.** A steel rod is clamped at its two ends and rests on a fixed horizontal base. The rod is unstrained at 20°C. Find the longitudinal strain developed in the rod if the temperature rises to 50°C. Coefficient of linear expansion of steel $= 1{\cdot}2 \times 10^{-5}\ °C^{-1}$.

**28.** A steel wire of cross-sectional area $0{\cdot}5\ mm^2$ is held between two fixed supports. If the wire is just taut at 20°C, determine the tension when the temperature falls to 0°C. Coefficient of linear expansion of steel is $1{\cdot}2 \times 10^{-5}\ °C^{-1}$ and its Young's modulus is $2{\cdot}0 \times 10^{11}\ N\ m^{-2}$.

**29.** A steel rod is rigidly clamped at its two ends. The rod is under zero tension at 20°C. If the temperature rises to 100°C, what force will the rod exert on one of the

clamps? Area of cross section of the rod = 2·00 mm$^2$. Coefficient of linear expansion of steel = $12.0 \times 10^{-6}$ °C$^{-1}$ and Young's modulus of steel = $2{\cdot}00 \times 10^{11}$ N m$^{-2}$.

**30.** Two steel rods and an aluminium rod of equal length $l_o$ and equal cross section are joined rigidly at their ends as shown in the figure below. All the rods are in a state of zero tension at 0°C. Find the length of the system when the temperature is raised to θ. Coefficient of linear expansion of aluminium and steel are $\alpha_a$ and $\alpha_s$ respectively. Young's modulus of aluminium is $Y_a$ and of steel is $Y_s$.

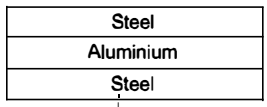



Figure 23-E1

**31.** A steel ball initially at a pressure of $1{\cdot}0 \times 10^5$ Pa is heated from 20°C to 120°C keeping its volume constant. Find the pressure inside the ball. Coefficient of linear expansion of steel = $12 \times 10^{-6}$ °C$^{-1}$ and bulk modulus of steel = $1{\cdot}6 \times 10^{11}$ N m$^{-2}$.

**32.** Show that moment of inertia of a solid body of any shape changes with temperature as $I = I_0 (1 + 2\alpha\theta)$, where $I_0$ is the moment of inertia at 0°C and α is the coefficient of linear expansion of the solid.

**33.** A torsional pendulum consists of a solid disc connected to a thin wire ($\alpha = 2{\cdot}4 \times 10^{-5}$ °C$^{-1}$) at its centre. Find the percentage change in the time period between peak winter (5°C) and peak summer (45°C).

**34.** A circular disc made of iron is rotated about its axis at a constant velocity ω. Calculate the percentage change in the linear speed of a particle of the rim as the disc is slowly heated from 20°C to 50°C keeping the angular velocity constant. Coefficient of linear expansion of iron = $1{\cdot}2 \times 10^{-5}$ °C$^{-1}$.

□

## ANSWERS

### OBJECTIVE I

1. (c) 2. (a) 3. (a) 4. (c) 5. (a) 6. (a)
7. (b) 8. (c) 9. (c) 10. (c) 11. (b)

### OBJECTIVE II

1. (a), (c) 2. all 3. (a), (b), (c)
4. (c), (d) 5. (a) 6. (b)

### EXERCISES

1. 20°C
2. 373·3 K
3. 601 K
4. 55 kPa
5. 96 kPa
6. 200°C
7. 307 K
8. 60°
9. 20·0 Ω, $3{\cdot}8 \times 10^{-3}$ °C$^{-1}$, $-5{\cdot}6 \times 10^{-7}$ °C$^{-2}$
10. 10·0035 m
11. 0·99989 cm
12. 0·4 cm
13. 2·0046 cm
14. (a) 0·99977 (b) 1·00025 (c) 1·00096
15. (a) –0·033% (b) 0·011%
16. 15°C to 25°C
17. $-5 \times 10^{-4}$ °C$^{-1}$
18. 23 : 12
19. – 82°C
20. 219°C
21. 20·012 cm × 30·018 cm
22. 50 cc
23. 0·089 cm
24. $1{\cdot}8 \times 10^{-4}$ °C$^{-1}$
25. 83°C
26. zero
27. $-3{\cdot}6 \times 10^{-4}$
28. 24 N
29. 384 N
30. $l_o\left[1 + \dfrac{\alpha_a Y_a + 2\alpha_S Y_S}{Y_a + 2Y_S}\theta\right]$
31. $5{\cdot}8 \times 10^8$ Pa
33. $9{\cdot}6 \times 10^{-2}$
34. $3{\cdot}6 \times 10^{-2}$

□

CHAPTER 24

# KINETIC THEORY OF GASES

## 24.1 INTRODUCTION

We have seen in the previous chapter that the pressure $p$, the volume $V$ and the temperature $T$ of any gas at low densities obey the equation

$$pV = nRT,$$

where $n$ is the number of moles in the gas and $R$ is the gas constant having value 8·314 $JK^{-1}mol^{-1}$. The temperature $T$ is defined on the absolute scale. We define the term *ideal gas* to mean a gas which always obeys this equation. The real gases available to us are good approximation of an ideal gas at low density but deviate from it when the density is increased.

Any sample of a gas is made of molecules. A molecule is the smallest unit having all the chemical properties of the sample. The observed behaviour of a gas results from the detailed behaviour of its large number of molecules. The kinetic theory of gases attempts to develop a model of the molecular behaviour which should result in the observed behaviour of an ideal gas.

## 24.2 ASSUMPTIONS OF KINETIC THEORY OF GASES

1. All gases are made of molecules moving randomly in all directions.

2. The size of a molecule is much smaller than the average separation between the molecules.

3. The molecules exert no force on each other or on the walls of the container except during collision.

4. All collisions between two molecules or between a molecule and a wall are perfectly elastic. Also, the time spent during a collision is negligibly small.

5. The molecules obey Newton's laws of motion.

6. When a gas is left for sufficient time, it comes to a steady state. The density and the distribution of molecules with different velocities are independent of position, direction and time. This assumption may be justified if the number of molecules is very large.

The last assumption needs some explanation. Suppose there are $2 \times 10^{19}$ molecules in a particular $1\ cm^3$. Our assumption means that there are $2 \times 10^{19}$ molecules in any other $1\ cm^3$ in the container and this number does not change as time passes. Similarly, if there are 400 molecules having velocities nearly parallel to the $x$-axis in a particular $1\ cm^3$, there are 400 molecules having velocities in this direction in any other $1\ cm^3$ and this number does not change with time. Also, there are 400 molecules in $1\ cm^3$ that are going in $y$-direction. The fact that the distribution of molecules does not change with time has an interesting consequence. Consider a molecule in a small volume $\Delta V$ having a velocity $\vec{v}$. A collision occurs and the velocity of this molecule changes. But the number of molecules in $\Delta V$ having velocity $\vec{v}$ should not depend on time. So there must be another collision which results in a nearby molecule taking up velocity $\vec{v}$. Effectively, we may neglect both the collisions and say that the molecule continues with the same velocity $\vec{v}$. This greatly simplifies calculations.

The assumptions of kinetic theory are close to the real situation at low densities. The molecular size is roughly 100 times smaller than the average separation between the molecules at 0·1 atm and room temperature. The real molecules do exert electric forces on each other but these forces may be neglected as the average separation between the molecules is large as compared to their size. The collisions between real molecules are indeed elastic if no permanent deformation is caused to a molecule. This is true when the temperature is not too high. The collisions with the walls are elastic if the temperature of the walls is the same as the temperature of the gas. If the gas is left in the container for sufficient time, this assumption will be valid. The fact that the motion of molecules may be described by Newton's laws may be taken as a pure chance for the time being. The last assumption is very nearly true if the number of molecules is very large. As there are about $6 \times 10^{23}$ molecules per mole, this condition is almost always true in a practical situation.

### 24.3 CALCULATION OF THE PRESSURE OF AN IDEAL GAS

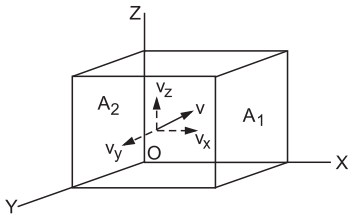



Figure 24.1

Consider an ideal gas enclosed in a cubical vessel of edge $L$. Take a corner of the vessel as the origin $O$ and the $x$-, $y$-, $z$- axes along the edges (figure 24.1). Let $A_1$ and $A_2$ be the parallel faces perpendicular to the $x$-axis. Consider a molecule moving with velocity $\vec{v}$. The components of the velocity along the axes are $v_x$, $v_y$ and $v_z$. When the molecule collides with the face $A_1$, the $x$-component of the velocity is reversed whereas the $y$- and the $z$-components remain unchanged. This follows from our assumption that the collisions of the molecules with the wall are perfectly elastic. The change in momentum of the molecule is

$$\Delta p = (-mv_x) - (mv_x) = -2mv_x.$$

As the momentum remains conserved in a collision, the change in momentum of the wall is

$$\Delta p' = 2mv_x. \qquad \ldots \text{(i)}$$

After rebound, this molecule travels towards $A_2$ with the $x$-component of velocity equal to $-v_x$. Any collision of the molecule with any other face (except for $A_2$) does not change the value of $v_x$. So, it travels between $A_1$ and $A_2$ with a constant $x$-component of velocity which is equal to $-v_x$. Note that we can neglect any collision with the other molecules in view of the last assumption discussed in the previous section.

The distance travelled parallel to the $x$-direction between $A_1$ and $A_2 = L$. Thus, the time taken by the molecule to go from $A_1$ to $A_2 = L/v_x$. The molecule rebounds from $A_2$, travels towards $A_1$ and collides with it after another time interval $L/v_x$. Thus, the time between two consecutive collisions of this molecule with $A_1$ is $\Delta t = 2L/v_x$. The number of collisions of this molecule with $A_1$ in unit time is

$$n = \frac{1}{\Delta t} = \frac{v_x}{2L}. \qquad \ldots \text{(ii)}$$

The momentum imparted per unit time to the wall by this molecule is, from (i) and (ii),

$$\Delta F = n\Delta p'$$

$$= \frac{v_x}{2L} \times 2mv_x = \frac{m}{L} v_x^2.$$

This is also the force exerted on the wall $A_1$ due to this molecule. The total force on the wall $A_1$ due to all the molecules is

$$F = \sum \frac{m}{L} v_x^2$$

$$= \frac{m}{L} \sum v_x^2. \qquad \ldots \text{(iii)}$$

As all directions are equivalent, we have

$$\sum v_x^2 = \sum v_y^2 = \sum v_z^2$$

$$= \frac{1}{3} \sum (v_x^2 + v_y^2 + v_z^2)$$

$$= \frac{1}{3} \sum v^2.$$

Thus, from (iii), $F = \frac{1}{3} \frac{m}{L} \sum v^2$.

If $N$ is the total number of molecules in the sample, we can write

$$F = \frac{1}{3} \frac{mN}{L} \frac{\Sigma v^2}{N}.$$

The pressure is force per unit area so that

$$p = \frac{F}{L^2}$$

$$= \frac{1}{3} \frac{mN}{L^3} \frac{\Sigma v^2}{N}$$

$$= \frac{1}{3} \frac{M}{L^3} \frac{\Sigma v^2}{N} = \frac{1}{3} \rho \frac{\Sigma v^2}{N},$$

where $M$ is the total mass of the gas taken and $\rho$ is its density. Also, $\Sigma v^2/N$ is the average of the speeds squared. It is written as $\overline{v^2}$ and is called *mean square speed*. Thus, the pressure is

$$p = \frac{1}{3} \rho \overline{v^2} \qquad \ldots \text{(24.1)}$$

or, $$pV = \frac{1}{3} M \overline{v^2} \qquad \ldots \text{(24.2)}$$

or, $$pV = \frac{1}{3} Nm \overline{v^2}. \qquad \ldots \text{(24.3)}$$

### 24.4 RMS SPEED

The square root of mean square speed is called *root-mean-square speed* or rms speed. It is denoted by the symbol $v_{rms}$. Thus,

$$v_{rms} = \sqrt{\Sigma v^2/N}$$

or, $$\overline{v^2} = (v_{rms})^2.$$

Equation (24.1) may be written as

$$p = \frac{1}{3} \rho v_{rms}^2$$

so that $$v_{rms} = \sqrt{\frac{3p}{\rho}} = \sqrt{\frac{3pV}{M}}.$$

***Example 24.1***

*Calculate the rms speed of nitrogen at STP (pressure = 1 atm and temperature = 0°C). The density of nitrogen in these conditions is* 1·25 kg m$^{-3}$.

***Solution :*** At STP, the pressure is $1{\cdot}0 \times 10^5$ N m$^{-2}$. The rms speed is

$$v_{rms} = \sqrt{\frac{3p}{\rho}} = \sqrt{\frac{3 \times 10^5 \text{ N m}^{-2}}{1{\cdot}25 \text{ kg m}^{-3}}} = 490 \text{ m s}^{-1}.$$

**Translational Kinetic Energy of a Gas**

The total translational kinetic energy of all the molecules of the gas is

$$K = \sum \frac{1}{2} m v^2 = \frac{1}{2} mN \frac{\Sigma v^2}{N} = \frac{1}{2} M v_{rms}^2. \quad \ldots (24.4)$$

The average kinetic energy of a molecule is

$$K/N = \frac{1}{2} \frac{M}{N} v_{rms}^2 = \frac{1}{2} m v_{rms}^2.$$

From equation (24.2),

$$pV = \frac{2}{3} \cdot \frac{1}{2} M v_{rms}^2$$

or, $$pV = \frac{2}{3} K$$

or, $$K = \frac{3}{2} pV.$$

## 24.5 KINETIC INTERPRETATION OF TEMPERATURE

We know that a hotter body has larger internal energy than an otherwise similar colder body. Thus, higher temperature means higher internal energy and lower temperature means lower internal energy. According to the kinetic theory of gases, the internal energy of an ideal gas is the same as the total translational kinetic energy of its molecules which is, from equation (24.4),

$$K = \frac{1}{2} M v_{rms}^2.$$

Thus, for a given sample of a gas, higher temperature means higher value of $v_{rms}$ and lower temperature means lower value of $v_{rms}$. We can write,

$$T = f(v_{rms})$$

for a given sample of a gas.

Let $p$ and $v$ be the pressure of the gas and the rms speed of the molecules at temperature $T$ respectively. Let $p_{tr}$ and $v_{tr}$ be the values of these quantities at temperature 273·16 K, keeping the volume $V$ the same as that at $T$.

From equation (24.2),

$$pV = \frac{1}{3} M v^2$$

and $$p_{tr} V = \frac{1}{3} M v_{tr}^2.$$

Thus, $$\frac{p}{p_{tr}} = \frac{v^2}{v_{tr}^2}. \quad \ldots (i)$$

From the definition of absolute temperature scale,

$$\frac{p}{p_{tr}} = \frac{T}{273{\cdot}16 \text{ K}}. \quad \ldots (ii)$$

From (i) and (ii),

$$T = \left( \frac{273{\cdot}16 \text{ K}}{v_{tr}^2} \right) v^2. \quad \ldots (24.5)$$

Now, $v_{tr}$ is the rms speed of the molecules at 273·16 K and hence is a constant for a given gas. Equation (24.5) shows that *the absolute temperature of a given gas is proportional to the square of the rms speed of its molecules.* As the total translational kinetic energy of the molecules is $K = \frac{1}{2} M v_{rms}^2$, we see that $T \propto K$ for a given sample of a gas.

Thus, *the absolute temperature of a given sample of a gas is proportional to the total translational kinetic energy of its molecules.*

Now consider a mixture of two gases $A$ and $B$. Let $m_1$ be the mass of a molecule of the first gas and $m_2$ be that of the second. As the molecules collide with each other, they exchange energy. On an average, the molecules with higher kinetic energy lose energy to those with lower kinetic energy. In thermal equilibrium, the average kinetic energy of all molecules are equal. If $v_1$ and $v_2$ be the rms speeds of the molecules of $A$ and $B$ respectively,

$$\frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_2 v_2^2. \quad \ldots (24.6)$$

We find that for different kinds of gases, it is not the rms speed but average kinetic energy of individual molecules that has a fixed value at a given temperature. The heavier molecules move with smaller rms speed and the lighter molecules move with larger rms speed.

***Example 24.2***

*If the rms speed of nitrogen molecules is* 490 m s$^{-1}$ *at* 273 K, *find the rms speed of hydrogen molecules at the same temperature.*

***Solution :*** The molecular weight of nitrogen is 28 g mol$^{-1}$ and that of hydrogen is 2 g mol$^{-1}$. Let $m_1$, $m_2$ be the

masses and $v_1, v_2$ be the rms speeds of a nitrogen molecule and a hydrogen molecule respectively. Then $m_1 = 14m_2$. Using equation (24.6),

$$\frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_2 v_2^2$$

or, $v_2 = v_1\sqrt{m_1/m_2} = 490 \text{ m s}^{-1} \times \sqrt{14} \approx 1830 \text{ m s}^{-1}$.

## 24.6 DEDUCTIONS FROM KINETIC THEORY

### Boyle's Law

*At a given temperature, the pressure of a given mass of a gas is inversely proportional to its volume.* This is known as Boyle's law.

From equation (24.3), we have

$$pV = \frac{1}{3} mN v_{rms}^2. \quad \ldots \text{(i)}$$

As for a given gas $v_{rms}^2 \propto T$, the value of $v_{rms}^2$ is constant at a given temperature. Also, for a given mass of the gas, $m$ and $N$ are constants. Thus, from (i),

$$pV = \text{constant}$$

or, $$p \propto \frac{1}{V}$$

which is Boyle's law.

### Charles's Law

*At a given pressure, the volume of a given mass of a gas is proportional to its absolute temperature.* This is known as Charles's law.

From (i), if $p$ is constant,

$$V \propto v_{rms}^2.$$

As $v_{rms}^2 \propto T$, we get $V \propto T$

which is Charles's law.

### Charles's Law of Pressure

*At a given volume, the pressure of a given mass of a gas is proportional to its absolute temperature.* This is known as Charles's law for pressure.

In fact, this is the definition of the absolute temperature $T$. If one starts from the fact that $v_{rms}^2 \propto T$ and uses the fact that $V$ is constant, one gets from (i),

$$p \propto v_{rms}^2$$

or, $$p \propto T.$$

### Avogadro's Law

*At the same temperature and pressure, equal volumes of all gases contain equal number of molecules.* This is known as Avogadro's law.

Consider equal volumes of two gases kept at the same pressure and temperature. Let,

$m_1$ = mass of a molecule of the first gas
$m_2$ = mass of a molecule of the second gas
$N_1$ = number of molecules of the first gas
$N_2$ = number of molecules of the second gas
$p$ = common pressure of the two gases
$V$ = common volume of the two gases.

From equation (24.3),

$$pV = \frac{1}{3} N_1 m_1 v_1^2$$

and $$pV = \frac{1}{3} N_2 m_2 v_2^2,$$

where $v_1$ and $v_2$ are rms speeds of the molecules of the first and the second gas respectively. Thus,

$$N_1 m_1 v_1^2 = N_2 m_2 v_2^2. \quad \ldots \text{(i)}$$

As the temperatures of the gases are the same, the average kinetic energy of the molecules is same for the two gases (equation 24.6), i.e.,

$$\frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_2 v_2^2. \quad \ldots \text{(ii)}$$

From (i) and (ii),

$$N_1 = N_2$$

which proves Avogadro's law.

### Graham's Law of Diffusion

*When two gases at the same pressure and temperature are allowed to diffuse into each other, the rate of diffusion of each gas is inversely proportional to the square root of the density of the gas.* This is known as Graham's law of diffusion.

It is reasonable to assume that the rate of diffusion is proportional to the rms speed of the molecules of the gas. Then if $r_1$ and $r_2$ be the rates of diffusion of the two gases,

$$\frac{r_1}{r_2} = \frac{v_{1,rms}}{v_{2,rms}}. \quad \ldots \text{(i)}$$

From equation (24.1),

$$v_{rms} = \sqrt{\frac{3p}{\rho}}.$$

If the pressure of the two gases are the same,

$$\frac{v_{1,rms}}{v_{2,rms}} = \sqrt{\frac{\rho_2}{\rho_1}}$$

so that from (i)

$$\frac{r_1}{r_2} = \sqrt{\frac{\rho_2}{\rho_1}}$$

which is Graham's law.

### Dalton's Law of Partial Pressure

Dalton's law of partial pressure says that *the pressure exerted by a mixture of several gases equals*

*the sum of the pressures exerted by each gas occupying the same volume as that of the mixture.*

In kinetic theory, we assume that the pressure exerted by a gas on the walls of a container is due to the collisions of the molecules with the walls. The total force on the wall is the sum of the forces exerted by the individual molecules. Suppose there are $N_1$ molecules of gas 1, $N_2$ molecules of gas 2, etc., in the mixture.

Thus, the force on a wall of surface area $A$ is

$$F = \text{ force by } N_1 \text{ molecules of gas 1} + \text{force by } N_2 \text{ molecules of gas 2} + \ldots$$
$$= F_1 + F_2 + \ldots$$

Thus, the pressure is

$$p = \frac{F_1}{A} + \frac{F_2}{A} + \ldots$$

If the first gas alone is kept in the container, its $N_1$ molecules will exert a force $F_1$ on the wall. If the pressure in this case is $p_1$,

$$p_1 = F_1 / A.$$

Similar is the case for other gases.

Thus,

$$p = p_1 + p_2 + p_3 + \ldots$$

## 24.7 IDEAL GAS EQUATION

Consider a sample of an ideal gas at pressure $P$, volume $V$ and temperature $T$. Let $m$ be the mass of each molecule and $v$ be the rms speed of the molecules. Also, let $v_{tr}$ be the rms speed of the gas at the triple point 273·16 K. From equation (24.3),

$$pV = \frac{1}{3} N m v^2 \quad \ldots \text{ (i)}$$

and from equation (24.5)

$$T = \left(\frac{273{\cdot}16 \text{ K}}{v_{tr}^2}\right) v^2$$

or,
$$v^2 = \left(\frac{v_{tr}^2}{273{\cdot}16 \text{ K}}\right) T.$$

Putting this expression for $v^2$ in (i),

$$pV = N\left(\frac{1}{3}\frac{m v_{tr}^2}{273{\cdot}16 \text{ K}}\right) T. \quad \ldots \text{ (ii)}$$

Now $\frac{1}{2} m v_{tr}^2$ is the average kinetic energy of a molecule at the triple point 273·16 K. As the average kinetic energy of a molecule is the same for all gases at a fixed temperature (equation 24.6), $\frac{1}{2} m v_{tr}^2$ is a universal constant. Accordingly, the quantity in bracket in equation (ii) above is also a universal constant. Writing this constant as $k$, equation (ii) becomes,

$$pV = NkT. \quad \ldots (24.7)$$

The universal constant $k$ is known as the *Boltzmann constant* and its value is

$$k = 1{\cdot}38 \times 10^{-23} \text{ J K}^{-1}$$

up to three significant digits. If the gas contains $n$ moles, the number of molecules is

$$N = n N_A$$

where $N_A = 6{\cdot}02 \times 10^{23} \text{ mol}^{-1}$ is the Avogadro constant.

Using this, equation (24.7) becomes

$$pV = nN_A kT$$

or,
$$pV = nRT \quad \ldots (24.8)$$

where $R = N_A k$ is another universal constant known as the *universal gas constant.* Its value is

$$R = 8{\cdot}314 \text{ J K}^{-1} \text{ mol}^{-1}.$$

Equation (24.8) is known as the *equation of state* of an ideal gas.

***Example 24.3***

*Calculate the number of molecules in each cubic metre of a gas at* 1 atm *and* 27 °C.

***Solution*** : We have $pV = NkT$

or,
$$N = \frac{pV}{kT}$$
$$= \frac{(1{\cdot}0 \times 10^5 \text{ N m}^{-2})\,(1 \text{ m}^3)}{(1{\cdot}38 \times 10^{-23} \text{ J K}^{-1})\,(300 \text{ K})}$$
$$\approx 2{\cdot}4 \times 10^{25}.$$

### Rms Speed in Terms of Temperature

We are now in a position to write the rms speed of the molecules in terms of the absolute temperature. From equation (24.3),

$$pV = \frac{1}{3} N m v_{rms}^2$$

and from equation (24.7),

$$pV = NkT.$$

From these two,

$$\frac{1}{3} m v_{rms}^2 = kT$$

or,
$$v_{rms} = \sqrt{\frac{3kT}{m}}. \quad \ldots (24.9)$$

This may also be written as,

$$v_{rms} = \sqrt{\frac{3kN_A T}{mN_A}}$$

$$= \sqrt{\frac{3RT}{M_o}} \qquad \ldots \quad (24.10)$$

where $M_o = mN_A$ is the molecular weight.

**Average Kinetic Energy of a Molecule**

Average kinetic energy of a molecule is

$$\frac{1}{2} m v_{rms}^2 = \frac{1}{2} m \cdot \frac{3kT}{m}$$

$$= \frac{3}{2} kT. \qquad \ldots \quad (24.11)$$

The total kinetic energy of all the molecules is

$$U = N\left(\frac{3}{2} kT\right) = \frac{3}{2} nRT. \qquad \ldots \quad (24.12)$$

The *average speed* $\bar{v} = \Sigma v / N$ is somewhat smaller than the rms speed. It can be shown that

$$\bar{v} = \sqrt{\frac{8kT}{\pi m}} = \sqrt{\frac{8RT}{\pi M_o}}.$$

***Example 24.4***

*Find the rms speed of oxygen molecules in a gas at* 300 K.

***Solution*** : $v_{rms} = \sqrt{\frac{3RT}{M_o}}$

$$= \sqrt{\frac{3 \times (8{\cdot}3 \text{ J K}^{-1} \text{mol}^{-1}) \times (300 \text{ K})}{32 \text{ g mol}^{-1}}}$$

$$= \sqrt{\frac{3 \times 8{\cdot}3 \times 300}{0{\cdot}032}} \text{ m s}^{-1} = 483 \text{ m s}^{-1}.$$

It should be clearly understood that the motion of molecules discussed here is truly random motion. In other words, the centre of mass of the gas is assumed to be at rest and any rotation about the centre of mass is assumed to be absent. Any systematic motion of a gas sample has no effect on temperature. For example, if we place a gas jar in a moving train, the increase in translational kinetic energy does not increase the temperature of the gas.

## 24.8 MAXWELL'S SPEED DISTRIBUTION LAW

The rms speed of an oxygen molecule in a sample at 300 K is about 480 m $s^{-1}$. This does not mean that the speed of each molecule is 480 m $s^{-1}$. Many of the molecules have speed less than 480 m $s^{-1}$ and many have speed more than 480 m $s^{-1}$. Maxwell derived an equation giving the distribution of molecules in different speeds. If $dN$ represents the number of molecules with speeds between $v$ and $v + dv$ then

$$dN = 4\pi N \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-mv^2/2kT} dv \qquad \ldots \quad (24.13)$$

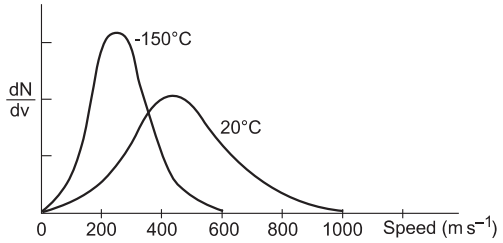



Figure 24.2

Figure (24.2) shows plots of $dN/dv$ against $v$ at two different temperatures. We see that there are some molecules which have speeds many times greater than the mean speed. This fact helps in making nuclear fusion reactions in a laboratory. The speed $v_p$ at which $dN/dv$ is maximum is called the *most probable speed*. Its value is given by

$$v_p = \sqrt{\frac{2kT}{m}}. \qquad \ldots \quad (24.14)$$

## 24.9 THERMODYNAMIC STATE

A given sample of a substance has a number of parameters which can be physically measured. When these parameters are uniquely specified, we say that the *thermodynamic state* of the system is specified. However, not all of these parameters are independent of each other. For example, we can measure pressure, volume, temperature, internal energy and many other parameters of an ideal gas. But if pressure and volume are specified, the rest of the parameters may be calculated. Thus, a thermodynamic state of a given sample of an ideal gas is completely described if its pressure and its volume are given. When a process is performed on a system, it passes from one thermodynamic state to another.

**Equation of State**

The pressure, volume and temperature of a given sample of a substance are related to each other. An equation describing this relation is called the *equation of state for that substance*. For an ideal gas it is

$$pV = nRT$$

where the symbols have their usual meanings. For a real gas the equation of state is different. The size of a molecule is not negligible in comparison to the average separation between them. Also, the molecular attraction is not negligible. Taking these two facts into account, van der Waals derived the following equation of state for a real gas:

$$\left(p + \frac{a}{V^2}\right)(V - b) = nRT \qquad \ldots \quad (24.15)$$

where $a$ and $b$ are small positive constants. The constant $a$ is related to the average force of attraction between the molecules and $b$ is related to the total volume of the molecules.

## 24.10 BROWNIAN MOTION

We have assumed in kinetic theory of gases that the molecules of a gas are in constant random motion, colliding with each other and with the walls of the container. This is also valid for a liquid. Robert Brown, a botanist, accidentally came across an evidence of this type of molecular motion in 1827. He was observing small pollen grains suspended in water, under a powerful microscope. He observed that although the water appeared to be at complete rest, the grains were moving randomly in the water, occasionally changing their directions of motion. A typical path of a grain looks as shown in figure (24.3). Such a phenomenon is called *Brownian motion*. The molecules strike the particles of the pollen grains and kick them to move in a direction. Another collision with some other molecules changes the direction of the grain.

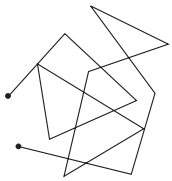


Figure 24.3

The molecules are too small to be directly seen under a normal microscope, but the grains can be seen. A piece of wood floating in water can be seen with naked eyes but its mass is so large that it does not respond quickly to the molecular collisions. Hence, to observe Brownian motion one should have light suspended particles. Brownian motion increases if we increase the temperature. Comparing between different liquids, one with smaller viscosity and smaller density will show more intense Brownian motion.

Einstein developed a theoretical model for Brownian motion in 1905 and deduced the average size of the molecules from it.

## 24.11 VAPOUR

The kinetic theory of gases described above is strictly valid only for ideal gases. These concepts are also useful in qualitatively understanding several behaviours of nonideal gases, liquids and even solids. We have assumed in kinetic theory that the molecules do not exert a force on each other except at the time of collision. The molecules, in fact, exert an attractive force on each other. For a gas at a low pressure and a high temperature, the average separation between the molecules is quite large. The molecular attraction is quite small and hence the associated attraction energy is small as compared to the average kinetic energy. If the pressure is increased or the temperature is decreased, the attraction energy gradually becomes more important and finally the gas liquefies. Thus, in general, a gas can be liquefied either by increasing the pressure (by compressing it) or by decreasing the temperature. However, if the temperature is sufficiently high so that the kinetic energy of the molecules is large, no amount of pressure can liquefy the gas. The temperature above which this behaviour occurs is called the *critical temperature* of the substance. A gas below its critical temperature is called *vapour*.

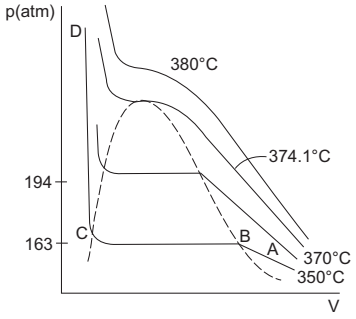



Figure 24.4

Figure (24.4) shows the gas–liquid transition of water in a $p$–$V$ diagram. Each solid curve represents the variation in pressure with the volume of a sample of water while the temperature is kept fixed. It, therefore, represents an isothermal process and is called an *isotherm*. The dotted curve represents the region where the transition actually takes place. Consider the isotherm *ABCD*. Suppose the water is in its gas form at 350°C and its pressure and volume are represented by the point *A*. We say that the state of the system is represented by *A*. As the pressure is increased keeping the temperature fixed, the state changes to *B* where it enters the region bounded by the dotted curve. Liquefaction starts at 350°C at 163 atm of pressure and the volume rapidly decreases. When the entire water is converted to liquid phase, the state is represented by the point *C*. Now, a large increase in pressure is needed for even a small compression so that the part *CD* is very steep.

At 370°C, liquefaction starts at a higher pressure that is around 194 atm. If the temperature is higher than 374·1°C (say 380°C), the isotherm does not enter the region bounded by the dotted curve. This means water cannot be liquefied at a temperature greater

than 374·1°C, no matter how large the pressure is. Thus, 374·1°C is the critical temperature of water. Water in its gas form at a temperature lower than 374·1°C is called *water vapour* and that above 374·1°C is *water gas*.

So, a vapour is a gas which can be liquefied by increasing the pressure without changing the temperature. Vapour obeys Dalton's law of partial pressure. Thus, the pressure exerted by the vapour present in air is the same as what it would be if the vapour were alone in the space with the same density and at the same temperature.

### 24.12 EVAPORATION

Let us take equal amounts of ether in three identical vessels. One is closed by a lid, one by a big jar and one is left open to air.

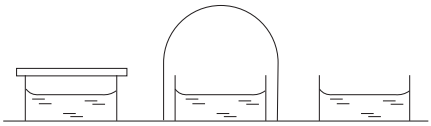


Figure 24.5

If we examine the vessels after a few hours, we shall find that the quantity of ether in the first vessel is almost the same, that in the second is reduced to some extent and that in the third has almost vanished. This can be explained with the help of kinetic theory. The molecules of ether move with random speeds and in random directions. A molecule collides frequently with other molecules and the container walls to change its direction and speed. Occasionally, a molecule starting in upward direction near the surface of the liquid may escape collisions and move out of the liquid. This process is called *evaporation*. Thus, *evaporation is a process in which molecules escape slowly from the surface of a liquid.*

We can now understand why the first vessel contains almost the same amount of ether as time passes. Surface molecules still move out of the liquid, but they cannot move very far away because of the lid. These molecules collide with the lid, with the air molecules, with the surface of the water and among themselves. Some of these molecules may be directed back into the liquid. The number going back depends on the density of the escaped molecules which keeps on increasing as more and more evaporation takes place. An equilibrium is reached when the number of molecules escaping from the liquid per second equals the number returning to the liquid. The volume of the liquid then becomes constant.

The same process takes place in the vessel covered by the jar. But now there is much more space for the escaped molecules to move around and many more molecules must escape before the equilibrium is reached.

In the vessel open to air, there is no restriction on the escaped molecules. They can go far away and never return. Occasionally, some molecules may return after colliding with an air molecule but the number escaped is always greater than the number returned. Thus, the liquid will keep on evaporating till the entire liquid is evaporated. If air blows over the liquid surface in the open vessel, the number of returning molecules is further reduced. This is because any molecule escaping from the surface is blown away from the vicinity of the liquid. This increases the rate of net evaporation. This is why clothes dry faster when a wind is blowing.

When a molecule comes out of the liquid surface, it has to oppose the attraction of the surface molecules it is leaving behind. This needs extra energy. Thus, only those molecules can escape which have kinetic energy sufficiently larger than the average. The average kinetic energy of the remaining liquid decreases and hence, its temperature goes down. This effect is used in cooling water in pitchers having porous walls.

### 24.13 SATURATED AND UNSATURATED VAPOUR: VAPOUR PRESSURE

Let us consider the vessel of figure (24.5) closed by the jar. After sufficient time an equilibrium is reached when the volume of the liquid becomes constant. The rate of transformation from liquid to vapour equals the rate of transformation from vapour to liquid. If we inject some vapour from outside into the space above the liquid, the rate of the returning molecules will increase while the rate of evaporation will still be the same. The net result is that the extra amount of vapour will convert into liquid. Thus, the space cannot contain more than a certain maximum amount of vapour. If the amount of vapour is less than this, the return rate is less than the rate of evaporation and the amount of vapour will increase to its maximum value.

When a space actually contains the maximum possible amount of vapour, the vapour is called *saturated*. If the amount is less than the maximum possible, the vapour is called *unsaturated*.

This maximum amount depends on the temperature. If we increase the temperature of the vessel, the liquid molecules will have higher average speed and the chances of escaping increases. Thus, the rate of evaporation increases and equilibrium is reached after more vapour has gone in the space provided.

The pressure exerted by a saturated vapour is called *saturation vapour pressure*. We shall denote it by the symbol SVP. The saturation vapour pressure of a substance is constant at a given temperature. It increases when the temperature is increased. Figure (24.6) shows the saturation vapour pressure as a function of temperature for methyl alcohol and water.

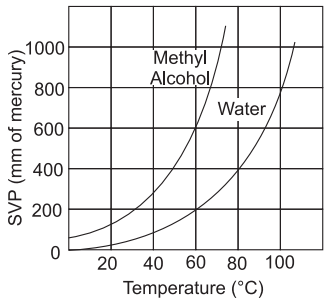



Figure 24.6

Table (24.1) gives the saturation vapour pressure of water at different temperatures.

**Table 24.1 :** ***Saturation vapour pressure of water***

| *T* °C | Vapour pressure torr (mmHg) | *T* °C | Vapour pressure torr (mmHg) |
|---|---|---|---|
| 0 | 4·58 | 100 | 760 |
| 5 | 6·51 | 120 | 1490 |
| 10 | 8·94 | 140 | 2710 |
| 15 | 12·67 | 160 | 4630 |
| 20 | 17·5 | 180 | 7510 |
| 40 | 55·1 | 200 | 11650 |
| 60 | 149 | 220 | 17390 |
| 80 | 355 | | |

The concept of saturated vapour is valid even if there is no liquid below. Consider a closed vessel in which we gradually inject vapour. When the amount of vapour is small, on an average, the molecules are far apart. The average attraction energy is much smaller than the average kinetic energy. As the amount is increased, the average separation decreases and the attraction energy becomes more and more important. At a certain state, the attraction becomes sufficient to draw several molecules close enough to form liquid. Thus, the vapour starts condensing. This is the case of saturation. Any vapour further injected will condense and the pressure inside the container will remain equal to the saturation vapour pressure.

In atmosphere, air and vapour are mixed with each other. If a given volume contains maximum amount of vapour possible, the air is called saturated with vapour. Otherwise, it is called unsaturated.

## 24.14 BOILING

The energy of a certain amount of substance is more in its vapour state than in its liquid state. This is because energy has to be supplied to separate the molecules against the attractive forces operating in the liquid phase. If we heat the liquid, the average kinetic energy of the entire liquid increases and at a certain stage the energy becomes sufficient to break the molecular attraction. The molecules anywhere in the liquid can form vapour bubbles. These bubbles float to the surface of the liquid and finally come out of the liquid. This phenomenon is called *boiling* and the temperature at which boiling occurs is called *boiling point*. Thus, in evaporation, only the molecules near the surface which have kinetic energy greater than the average escape from the liquid, whereas, in boiling, the molecules all over the liquid gain enough energy to become vapour.

The boiling point of a liquid depends on the external pressure over its surface. In fact, boiling occurs at a temperature where the SVP equals the external pressure. Thus, from figure (24.6), the boiling point of water at 1 atm (760 mm of mercury) is 100°C but at 0·5 atm it is 82°C.

***Example 24.5***

*At what external pressure will water boil at* 140°C ? *Use table (24.1) for vapour pressure data and express the answer in* atm.

***Solution*** **:** The saturation vapour pressure of water at 140°C is 2710 mm of Hg. Thus, water will boil at 140°C at this pressure. Now 760 mm of Hg = 1 atm.

Thus, 2710 mm of Hg $= \frac{2710}{760}$ atm = 3·56 atm.

The pressure inside a pressure cooker is of this order when it whistles. So, the temperature inside is of the order of 140°C which helps in cooking the food much faster.

## 24.15 DEW POINT

Table (24.1) gives the saturation vapour pressure of water as a function of temperature. Suppose air at temperature 20°C contains some vapour which exerts a pressure of 8·94 mm of mercury. The air is unsaturated because a vapour pressure of 17·5 mm of mercury is needed to saturate the air at 20°C. If we decrease the temperature from 20°C to 10°C, the air will become saturated with vapour because at 10°C the saturated vapour pressure is 8·94 mm of mercury.

The temperature at which the saturation vapour pressure is equal to the present vapour pressure is called the *dew point*.

If the temperature is decreased below the dew point, some of the vapour condenses.

## 24.16 HUMIDITY

The amount of water vapour present in a unit volume of air is called the *absolute humidity* of air. It is generally mentioned in terms of g m$^{-3}$. The ratio of the amount of water vapour present in a given volume to the amount of water vapour required to saturate the volume at the same temperature is called the *relative humidity*. Thus, relative humidity is defined as

$$RH = \frac{\text{Amount of water vapour present in a given volume of air at a given temperature}}{\text{Amount of water vapour required to saturate the same volume of air at the same temperature}}. \quad \ldots (24.16)$$

Relative humidity is generally expressed as a percentage. Thus, if the above ratio is 0·6, the relative humidity is 60%. If the air is already saturated, the relative humidity is 100%.

As the pressure exerted by the vapour is directly proportional to the amount of vapour present in a given volume, the relative humidity may also be defined as

$$RH = \frac{\text{Vapour pressure of air}}{\text{SVP at the same temperature}}. \quad \ldots (24.17)$$

The vapour pressure of air at the actual temperature is equal to the saturation vapour pressure at the dew point. Thus, the relative humidity may be redefined as

$$RH = \frac{\text{SVP at the dew point}}{\text{SVP at the air-temperature}}. \quad \ldots (24.18)$$

***Example 24.6***

*The vapour pressure of air at* 20°C *is found to be* 12 mm of Hg *on a particular day. Find the relative humidity. Use the data of table (24.1).*

***Solution :*** The saturation vapour pressure of water at 20°C is 17·5 mm of Hg. Thus, the relative humidity is

$$\frac{\text{vapour pressure of air}}{\text{SVP at the same temperature}} = \frac{12 \text{ mm of Hg}}{17{\cdot}5 \text{ mm of Hg}} = 0{\cdot}69,$$

that is, 69%.

## 24.17 DETERMINATION OF RELATIVE HUMIDITY

A simple method to measure the relative humidity is to find the dew point and then use equation (24.18). We describe below the Regnault's hygrometer to find the dew point.

The apparatus consists of two test tubes *A* and *B* fitted with a hollow stand *C*. The test tube *A* can communicate with the hollow space in *C* but the tube *B* cannot. Both the tubes have silvered outer surfaces in the lower part. A hollow tube *D* goes into the test tube *A*. The other end of the tube *D* is open to the atmosphere. Sensitive thermometers $T_1$ and $T_2$ are inserted in the test tubes. The hollow space of the stand is connected to a vessel *E* through a rubber tube. The vessel has an outlet.

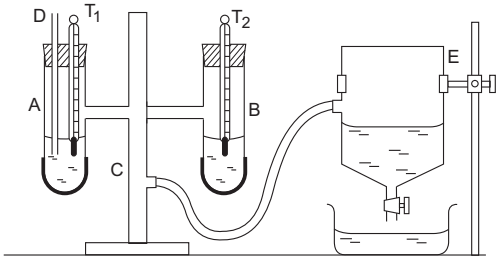



Figure 24.7

Some ether (about half the test tube) is taken in the test tube *A* and the vessel *E* is filled with water. The outlet below the vessel *E* is opened so that the water slowly comes out of the vessel. As a partial vacuum is created over the water surface, air is sucked through the tube *D*, the hollow stand and the rubber tube. Thus, air passes through the ether and evaporates it in the process. The temperature of the test tube *A* gradually decreases and becomes equal to the dew point at a certain time. The vapour in the air near the silvered surface of the test tube *A* starts condensing. Tiny water droplets in the form of *dew* appear on the silvered surface. This surface becomes hazy while the silvered surface of the test tube *B* remains shining. Both the surfaces are continuously observed from a distance (a telescope may be used for the purpose). As soon as the difference in shine is observed, the temperature of the test tube *A* is noted with the thermometer $T_1$. The reading of $T_2$ gives the air-temperature.

The outlet to the vessel *D* is closed. The evaporation stops and the temperature of *A* which had gone below the dew point, starts rising. As it just crosses the dew point, the surface again starts shining. The temperature of $T_1$ at which the two silvered surfaces start looking similar, is noted. The average of the two readings of $T_1$ is taken as the dew point.

If $f$ and $F$ be the saturation vapour pressures at the dew point and at the air-temperature respectively, the relative humidity is

$$\frac{f}{F} \times 100\%.$$

***Example 24.7***

*In an experiment with Regnault's hygrometer, dew appears at* 10°C *when the atmospheric temperature is* 40°C. *Using table (24.1), find the relative humidity.*

***Solution*** **:** The dew point is 10°C. The saturation vapour pressure at this temperature is 8·94 mm of Hg from table (24.1). Also, the saturation vapour pressure of air at 40°C is 55·1 mm of Hg. The relative humidity expressed in percentage

$$= \frac{\text{vapour pressure at the dew point}}{\text{SVP at the air-temperature}} \times 100\%$$

$$= \frac{8{\cdot}94}{55{\cdot}1} \times 100\% = 16{\cdot}2\%.$$

## 24.18 PHASE DIAGRAMS : TRIPLE POINT

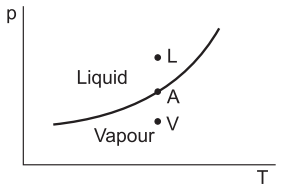



Figure 24.8

When a liquid and its vapour remain together in equilibrium, the vapour is saturated. The saturation vapour pressure depends on temperature. Figure (24.8) shows a curve representing the saturation vapour pressure as a function of temperature. If the vapour pressure and the temperature of a liquid–vapour system are represented by a point on the curve, such as $A$, the vapour is saturated. The liquid and the vapour can coexist in this case. Now suppose, the vapour pressure is increased by injecting more vapour into the space so that the situation is represented by the point $L$ in figure (24.8). As the vapour is already saturated, the extra vapour will condense into liquid and the system will return to the point $A$. Similarly, suppose the vapour pressure is decreased by taking out some vapour from the space so that the situation is represented by the point $V$ in figure (24.8). The vapour will become unsaturated and some liquid will evaporate to take the system back to the point $A$.

Thus, the liquid phase and the vapour phase can coexist only along the curve shown. At points above this curve a pure liquid can exist in equilibrium and at points below this curve a pure vapour can exist in equilibrium.

At the boiling point, the saturation vapour pressure equals the external pressure. The curve in figure (24.8), therefore, also represents the boiling point as a function of external pressure. The horizontal axis then represents the boiling point and the vertical axis represents the external pressure.

Similar curves also exist for solid–liquid transition and for solid–vapour transition. Figure (24.9) shows qualitatively these curves for water and carbon dioxide. The curve $PB$ represents solid–liquid transition and $PC$ represents solid–vapour transition. Solid and liquid phases may coexist along the curve $PB$ and solid and vapour phases can coexist along $PC$. These curves also represent, respectively, the melting point as a function of pressure and sublimation point as a function of pressure. Thus, the $p$–$T$ space is divided in three regions labelled solid, liquid and vapour.

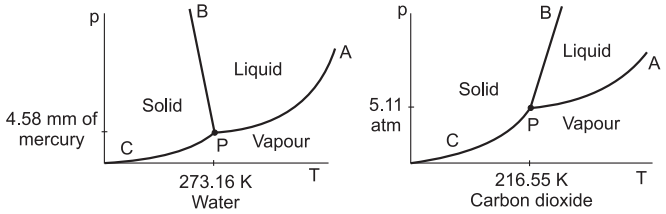



Figure 24.9

The three curves meet at one point labelled $P$. At the pressure and temperature corresponding to this point, all the three phases may remain together in equilibrium. This point is known as the *triple point*. For water, the triple point occurs at the pressure 4·58 mm of mercury and temperature 273·16 K.

For carbon dioxide, the temperature at the triple point is 216·55 K and the corresponding pressure is 5·11 atm. Thus, at atmospheric pressure it can remain either in solid phase or in vapour phase (figure 24.9). When solid $CO_2$, open to atmosphere, is heated, it becomes vapour directly without passing through the liquid phase. It is, therefore, called *dry ice*.

## 24.19 DEW AND FOG

In winter nights, the atmospheric temperature goes down. The surfaces of windowpanes, flowers, grass, etc., become still colder due to radiation. The air near them becomes saturated and condensation begins. The droplets condensed on such surfaces are known as *dew*.

If the temperature falls further, the whole atmosphere in that region may become saturated. Small droplets then condense on the dust particles present in the air. These droplets keep floating in the air and form a thick mist which restricts visibility. This thick mist is called *fog*.

## *Worked Out Examples*

**1.** *A vessel of volume* $8{\cdot}0 \times 10^{-3}$ m$^3$ *contains an ideal gas at* 300 K *and* 200 kPa. *The gas is allowed to leak till the pressure falls to* 125 kPa . *Calculate the amount of the gas (in moles) leaked assuming that the temperature remains constant.*

***Solution*** : As the gas leaks out, the volume and the temperature of the remaining gas do not change. The number of moles of the gas in the vessel is given by $n = \frac{pV}{RT}$. The number of moles in the vessel before the leakage is

$$n_1 = \frac{p_1 V}{RT}$$

and that after the leakage is

$$n_2 = \frac{p_2 V}{RT}.$$

Thus, the amount leaked is

$$n_1 - n_2 = \frac{(p_1 - p_2)\, V}{RT}$$

$$= \frac{(200 - 125) \times 10^3 \text{ N m}^{-2} \times 8{\cdot}0 \times 10^{-3} \text{ m}^3}{(8{\cdot}3 \text{ J K}^{-1} \text{ mol}^{-1}) \times (300 \text{ K})}$$

$$= 0{\cdot}24 \text{ mol}^{-1}.$$

**2.** *A vessel of volume* 2000 cm$^3$ *contains* 0·1 mol *of oxygen and* 0·2 mol *of carbon dioxide. If the temperature of the mixture is* 300 K, *find its pressure.*

***Solution*** : We have $p = \frac{nRT}{V}$ .

The pressure due to oxygen is

$$p_1 = \frac{(0{\cdot}1 \text{ mol})\,(8{\cdot}3 \text{ J K}^{-1} \text{ mol}^{-1})\,(300 \text{ K})}{(2000 \times 10^{-6} \text{ m}^{-3})} = 1{\cdot}25 \times 10^5 \text{ Pa}.$$

Similarly, the pressure due to carbon dioxide is

$$p_2 = 2{\cdot}50 \times 10^5 \text{ Pa}.$$

The total pressure in the vessel is

$$p = p_1 + p_2$$

$$= (1{\cdot}25 + 2{\cdot}50) \times 10^5 \text{ Pa} = 3{\cdot}75 \times 10^5 \text{ Pa}.$$

**3.** *A mixture of hydrogen and oxygen has volume* 2000 cm$^3$, *temperature* 300 K, *pressure* 100 kPa *and mass* 0·76 g. *Calculate the masses of hydrogen and oxygen in the mixture.*

***Solution*** : Suppose there are $n_1$ moles of hydrogen and $n_2$ moles of oxygen in the mixture. The pressure of the mixture will be

$$p = \frac{n_1 RT}{V} + \frac{n_2 RT}{V} = (n_1 + n_2)\frac{RT}{V}$$

$$\text{or,} \quad 100 \times 10^3 \text{ Pa} = (n_1 + n_2)\,\frac{(8{\cdot}3 \text{ J K}^{-1} \text{ mol}^{-1})\,(300 \text{ K})}{2000 \times 10^{-6} \text{ m}^{-3}}$$

$$\text{or,} \quad n_1 + n_2 = 0{\cdot}08 \text{ mol.} \quad \ldots \text{ (i)}$$

The mass of the mixture is

$$n_1 \times 2 \text{ g mol}^{-1} + n_2 \times 32 \text{ g mol}^{-1} = 0{\cdot}76 \text{ g}$$

$$\text{or,} \quad n_1 + 16 n_2 = 0{\cdot}38 \text{ mol.} \quad \ldots \text{ (ii)}$$

from (i) and (ii),

$$n_1 = 0{\cdot}06 \text{ mol} \text{ and } n_2 = 0{\cdot}02 \text{ mol}.$$

The mass of hydrogen = $0{\cdot}06 \times 2$ g = 0·12 g and the mass of oxygen = $0{\cdot}02 \times 32$ g = 0·64 g .

**4.** *A mercury manometer (figure 24-W1) consists of two unequal arms of equal cross section* 1 cm$^2$ *and lengths* 100 cm *and* 50 cm. *The two open ends are sealed with air in the tube at a pressure of* 80 cm of mercury. *Some amount of mercury is now introduced in the manometer through the stopcock connected to it. If mercury rises in the shorter tube to a length* 10 cm *in steady state, find the length of the mercury column risen in the longer tube.*

***Solution*** : Let $p_1$ and $p_2$ be the pressures in centimetre of mercury in the two arms after introducing mercury in the tube. Suppose the mercury column rises in the second arm to $l_0$ cm.

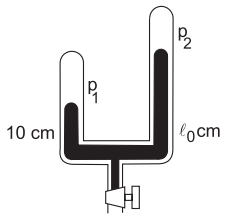



Figure 24-W1

Using $pV$ = constant for the shorter arm,

(80 cm) (50 cm) = $p_1$(50 cm − 10 cm)

$$\text{or,} \quad p_1 = 100 \text{ cm.} \quad \ldots \text{ (i)}$$

Using $pV$ = constant for the longer arm,

$$(80 \text{ cm})\,(100 \text{ cm}) = p_2(100 - l_0) \text{ cm.} \quad \ldots \text{ (ii)}$$

From the figure,

$$p_1 = p_2 + (l_0 - 10) \text{ cm.}$$

Thus by (i),

$$100 \text{ cm} = p_2 + (l_0 - 10) \text{ cm}$$

$$\text{or,} \quad p_2 = 110 \text{ cm} - l_0 \text{ cm.}$$

Putting in (ii),

$$(110 - l_0)\,(100 - l_0) = 8000$$

$$\text{or,} \quad l_0^2 - 210 l_0 + 3000 = 0$$

$$\text{or,} \quad l_0 = 15{\cdot}5.$$

The required length is 15·5 cm.

**5.** *An ideal gas has pressure* $p_0$ , *volume* $V_0$ *and temperature* $T_0$. *It is taken through an isochoric process till its*

*pressure is doubled. It is now isothermally expanded to get the original pressure. Finally, the gas is isobarically compressed to its original volume $V_0$. (a) Show the process on a p–V diagram. (b) What is the temperature in the isothermal part of the process? (c) What is the volume at the end of the isothermal part of the process?*

***Solution*** : (a) The process is shown in a $p$–$V$ diagram in figure (24-W2). The process starts from $A$ and goes through $ABCA$.

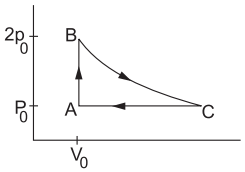



Figure 24-W2

(b) Applying $pV = nRT$ at $A$ and $B$,

$$p_0V_0 = nRT_0$$

and $$(2p_0)V_0 = nRT_B .$$

Thus, $$T_B = 2\,T_0.$$

This is the temperature in the isothermal part $BC$.

(c) As the process $BC$ is isothermal, $T_C = T_B = 2T_0$. Applying $pV = nRT$ at $A$ and $C$,

$$p_0V_0 = nRT_0$$

and $$p_0V_C = nR(2T_0)$$

or, $$V_C = 2V_0 .$$

**6.** *A cyclic process ABCA shown in the V–T diagram (figure 24-W3) is performed with a constant mass of an ideal gas. Show the same process on a p–V diagram. In the figure, CA is parallel to the V-axis and BC is parallel to the T-axis.*

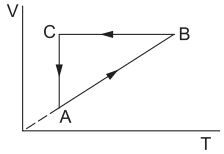



Figure 24-W3

***Solution*** : The $p$–$V$ diagram is shown in figure (24-W4). During the part $AB$ of figure (24-W3), $V$ is proportional to $T$. Thus, $\frac{V}{T}$ is constant. Using $\frac{pV}{T} = nR$, we see that the pressure $p$ is constant in this part. This is represented by the part $A'B'$ in the $p$–$V$ diagram. During the part $BC$, volume is constant. Thus, $\frac{p}{T}$ is constant. As the temperature decreases, pressure also decreases. This is represented by the part $B'C'$ in the $p$–$V$ diagram. During the part $CA$, the temperature remains constant so that $pV =$ constant. Thus, $p$ is inversely proportional to $V$. This is represented by the part $C'A'$ in the $p$–$V$ diagram.

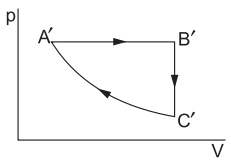



Figure 24-W4

**7.** *Two closed vessels of equal volume contain air at* 105 kPa, 300 K *and are connected through a narrow tube. If one of the vessels is now maintained at* 300 K *and the other at* 400 K, *what will be the pressure in the vessels?*

***Solution*** : Let the initial pressure, volume and temperature in each vessel be $p_0$(= 105 kPa), $V_0$ and $T_0$(= 300 K). Let the number of moles in each vessel be $n$. When the first vessel is maintained at temperature $T_0$ and the other is maintained at $T' = 400$ K, the pressures change. Let the common pressure become $p'$ and the number of moles in the two vessels become $n_1$ and $n_2$. We have

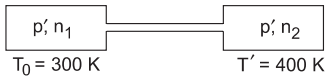



Figure 24-W5

$$p_0V_0 = nRT_0 \quad \ldots \text{(i)}$$

$$p'V_0 = n_1RT_0 \quad \ldots \text{(ii)}$$

$$p'V_0 = n_2RT' \quad \ldots \text{(iii)}$$

and $$n_1 + n_2 = 2n. \quad \ldots \text{(iv)}$$

Putting $n$, $n_1$ and $n_2$ from (i), (ii) and (iii) in (iv),

$$\frac{p'V_0}{RT_0} + \frac{p'V_0}{RT'} = 2\frac{p_0V_0}{RT_0}$$

or, $$p'\left(\frac{T' + T_0}{T_0T'}\right) = \frac{2p_0}{T_0}$$

or, $$p' = \frac{2p_0T'}{T' + T_0}$$

$$= \frac{2 \times 105\ \text{kPa} \times 400\ \text{K}}{400\ \text{K} + 300\ \text{K}} = 120\ \text{kPa} .$$

**8.** *A vessel contains* 14 g *of hydrogen and* 96 g *of oxygen at STP. (a) Find the volume of the vessel. (b) Chemical reaction is induced by passing electric spark in the vessel till one of the gases is consumed. The temperature is brought back to its starting value* 273 K. *Find the pressure in the vessel.*

***Solution*** : (a) The number of moles of hydrogen = 14 g/2 g = 7 and the number of moles of oxygen = 96 g/32 g = 3. The total number of moles in the vessel = 7 + 3 = 10. The pressure is 1 atm = $1{\cdot}0 \times 10^5$ N m$^{-2}$ and the temperature = 273 K.

Now $$pV = nRT \quad \ldots \text{(i)}$$

or, $V = \dfrac{nRT}{p}$

$$= \frac{(10 \text{ mol}) \times (8{\cdot}3 \text{ J K}^{-1} \text{ mol}^{-1}) \times (273 \text{ K})}{1{\cdot}0 \times 10^5 \text{ N m}^{-2}}$$

$$= 0{\cdot}23 \text{ m}^3.$$

(b) When electric spark is passed, hydrogen reacts with oxygen to form water ($H_2O$). Each gram of hydrogen reacts with eight grams of oxygen. Thus, 96 g of oxygen will be totally consumed together with 12 g of hydrogen. The gas left in the vessel will be 2 g of hydrogen which is $n' = 1$ mole .

Neglecting the volume of the water formed,

$$p'V = n'RT. \quad \ldots \text{ (ii)}$$

From (i) and (ii),

$$\frac{p'}{p} = \frac{n'}{n} = \frac{1}{10}$$

or, $p' = p \times 0{\cdot}10$

$= 0{\cdot}10$ atm.

**9.** *A barometer reads* 75 cm of mercury. *When* $2{\cdot}0 \text{ cm}^3$ *of air at atmospheric pressure is introduced into the space above the mercury level, the volume of this space becomes* $50 \text{ cm}^3$. *Find the length by which the mercury column descends.*

***Solution*** : Let the pressure of the air in the barometer be $p$. We have,

$$p \times 50 \text{ cm}^3 = (75 \text{ cm of mercury}) \times (2{\cdot}0 \text{ cm}^3)$$

or, $p = 3{\cdot}0$ cm of mercury.

The atmospheric pressure is equal to the pressure due to the mercury column plus the pressure due to the air inside. Thus, the mercury column descends by 3·0 cm.

**10.** *A barometer tube is* 1 m *long and* $2 \text{ cm}^2$ *in cross section. Mercury stands to a height of* 75 cm *in the tube. When a small amount of oxygen is introduced in the space above the mercury level, the level falls by* 5 cm. *Calculate the mass of the oxygen introduced. Room temperature* = 27°C, $g = 10 \text{ m s}^{-2}$ *and density of mercury* $= 13600 \text{ kg m}^{-3}$.

***Solution*** : The pressure of oxygen in the space above the mercury level = 5 cm of mercury

$$= 0{\cdot}05 \text{ m} \times 13600 \text{ kg m}^{-3} \times 10 \text{ m s}^{-2}$$

$$= 6800 \text{ N m}^{-2}.$$

The volume of oxygen $= (2 \text{ cm}^2) \times (25 \text{ cm} + 5 \text{ cm})$

$$= 60 \text{ cm}^3 = 6 \times 10^{-5} \text{ m}^3.$$

The temperature $= (273 + 27) \text{ K} = 300 \text{ K}.$

The amount of oxygen is

$$n = \frac{pV}{RT}$$

$$= \frac{(6800 \text{ N m}^{-2}) \times 6 \times 10^{-5} \text{ m}^3}{(8{\cdot}3 \text{ J K}^{-1} \text{ mol}^{-1}) \times (300 \text{ K})}$$

$$= 16{\cdot}4 \times 10^{-5} \text{ mol}.$$

The mass of oxygen is

$$(16{\cdot}4 \times 10^{-5} \text{ mol}) \times (32 \text{ g mol}^{-1})$$

$$= 5{\cdot}24 \times 10^{-3} \text{ g}.$$

**11.** *Figure (24-W6) shows a vertical cylindrical vessel separated in two parts by a frictionless piston free to move along the length of the vessel. The length of the cylinder is* 90 cm *and the piston divides the cylinder in the ratio of* 5 : 4. *Each of the two parts of the vessel contains* 0.1 *mole of an ideal gas. The temperature of the gas is* 300 K *in each part. Calculate the mass of the piston.*

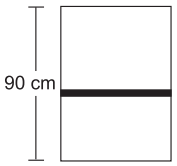



Figure 24-W6

***Solution*** : Let $l_1$ and $l_2$ be the lengths of the upper part and the lower part of the cylinder respectively. Clearly, $l_1 = 50$ cm and $l_2 = 40$ cm. Let the pressures in the upper and lower parts be $p_1$ and $p_2$ respectively. Let the area of cross section of the cylinder be $A$. The temperature in both parts is $T = 300$ K.

Consider the equilibrium of the piston. The forces acting on the piston are

(a) its weight $mg$

(b) $p_1 A$ downward, by the upper part of the gas and (c) $p_2 A$ upward, by the lower part of the gas.

Thus, $p_2 A = p_1 A + mg \quad \ldots \text{ (i)}$

Using $pV = nRT$ for the upper and the lower parts

$$p_1 l_1 A = nRT \quad \ldots \text{ (ii)}$$

and $p_2 l_2 A = nRT. \quad \ldots \text{ (iii)}$

Putting $p_1 A$ and $p_2 A$ from (ii) and (iii) into (i),

$$\frac{nRT}{l_2} = \frac{nRT}{l_1} + mg.$$

Thus, $m = \dfrac{nRT}{g}\left[\dfrac{1}{l_2} - \dfrac{1}{l_1}\right]$

$$= \frac{(0{\cdot}1 \text{ mol}) (8{\cdot}3 \text{ J K}^{-1} \text{ mol}^{-1}) (300 \text{ K})}{9{\cdot}8 \text{ m s}^{-2}}\left[\frac{1}{0{\cdot}4 \text{ m}} - \frac{1}{0{\cdot}5 \text{ m}}\right]$$

$= 12{\cdot}7$ kg.

**12.** *Figure (24-W7) shows a cylindrical tube of volume* $V_0$ *divided in two parts by a frictionless separator. The walls of the tube are adiabatic but the separator is conducting. Ideal gases are filled in the two parts. When the separator is kept in the middle, the pressures are* $p_1$ and $p_2$ *in the*

*left part and the right part respectively. The separator is slowly slid and is released at a position where it can stay in equilibrium. Find the volumes of the two parts.*

Figure 24-W7

***Solution*** : As the separator is conducting, the temperatures in the two parts will be the same. Suppose the common temperature is $T$ when the separator is in the middle. Let $n_1$ and $n_2$ be the number of moles of the gas in the left part and the right part respectively. Using ideal gas equation,

$$p_1\frac{V_0}{2} = n_1RT$$

and $$p_2\frac{V_0}{2} = n_2RT.$$

Thus, $$\frac{n_1}{n_2} = \frac{p_1}{p_2}. \quad \ldots \text{(i)}$$

The separator will stay in equilibrium at a position where the pressures on the two sides are equal. Suppose the volume of the left part is $V_1$ and of the right part is $V_2$ in this situation. Let the common pressure be $p'$. Also, let the common temperature in this situation be $T'$. Using ideal gas equation,

$$p'V_1 = n_1RT'$$

and $$p'V_2 = n_2RT'$$

or, $$\frac{V_1}{V_2} = \frac{n_1}{n_2} = \frac{p_1}{p_2}. \quad \text{[using (i)]}$$

Also, $$V_1 + V_2 = V_0.$$

Thus, $$V_1 = \frac{p_1V_0}{p_1+p_2} \text{ and } V_2 = \frac{p_2V_0}{p_1+p_2}.$$

**13.** *A thin tube of uniform cross section is sealed at both ends. It lies horizontally, the middle* 5 cm *containing mercury and the parts on its two sides containing air at the same pressure p. When the tube is held at an angle of* 60° *with the vertical, the length of the air column above and below the mercury pellet are* 46 cm *and* 44·5 cm *respectively. Calculate the pressure p in* centimetres of mercury. *The temperature of the system is kept at* 30°C.

***Solution*** **:** When the tube is kept inclined to the vertical, the length of the upper part is $l_1 = 46$ cm and that of the lower part is $l_2 = 44{\cdot}5$ cm. When the tube lies horizontally, the length on each side is

$$l_0 = \frac{l_1+l_2}{2} = \frac{46\text{ cm} + 44{\cdot}5\text{ cm}}{2} = 45{\cdot}25\text{ cm}.$$

Let $p_1$ and $p_2$ be the pressures in the upper and the lower parts when the tube is kept inclined. As the temperature is constant throughout, we can apply Boyle's law. For the upper part,

$$p_1l_1A = pl_0A$$

or, $$p_1 = \frac{pl_0}{l_1}. \quad \ldots \text{(i)}$$

Similarly, for the lower part,

$$p_2 = \frac{pl_0}{l_2}. \quad \ldots \text{(ii)}$$

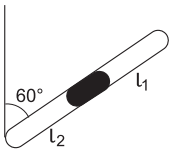



Figure 24-W8

Now consider the equilibrium of the mercury pellet when the tube is kept in inclined position. Let $m$ be the mass of the mercury. The forces along the length of the tube are

(a) $p_1A$ down the tube

(b) $p_2A$ up the tube

and (c) $mg\ \cos 60°$ down the tube.

Thus, $$p_2 = p_1 + \frac{mg}{A}\cos 60°.$$

Putting from (i) and (ii),

$$\frac{pl_0}{l_2} = \frac{pl_0}{l_1} + \frac{mg}{2A}$$

or, $$pl_0\left(\frac{1}{l_2} - \frac{1}{l_1}\right) = \frac{mg}{2A}$$

or, $$p = \frac{mg}{2Al_0\left(\frac{1}{l_2} - \frac{1}{l_1}\right)}.$$

If the pressure $p$ is equal to a height $h$ of mercury,

$$p = h\rho g.$$

Also, $$m = (5\text{ cm})A\rho$$

so that $$h\rho g = \frac{(5\text{ cm})A\rho g}{2Al_0\left(\frac{1}{l_2} - \frac{1}{l_1}\right)}$$

or, $$h = \frac{(5\text{ cm})}{2(45{\cdot}25\text{ cm})\left(\frac{1}{44{\cdot}5\text{ cm}} - \frac{1}{46\text{ cm}}\right)}$$

$$= 75{\cdot}39\text{ cm}.$$

The pressure $p$ is equal to 75·39 cm of mercury.

**14.** *An ideal monatomic gas is confined in a cylinder by a spring-loaded piston of cross section* $8{\cdot}0 \times 10^{-3}\text{ m}^2$. *Initially the gas is at* 300 K *and occupies a volume of* $2{\cdot}4 \times 10^{-3}\text{ m}^3$ *and the spring is in its relaxed state (figure 24-W9). The gas is heated by a small heater until the piston moves out slowly by* 0·1 m. *Calculate the final temperature of the gas. The force constant of the spring is* $8000\text{ N m}^{-1}$, *and the atmospheric pressure is* $1{\cdot}0 \times 10^5\text{ N m}^{-2}$. *The cylinder and the piston are*

*thermally insulated. The piston and the spring are massless and there is no friction between the piston and the cylinder. Neglect any heat-loss through the lead wires of the heater. The heat capacity of the heater coil is negligible.*

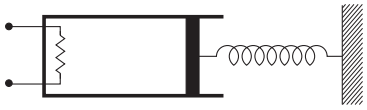


Figure 24-W9

***Solution*** **:** Initially the spring is in its relaxed state. So, the pressure of the gas equals the atmospheric pressure.

Initial pressure $= p_1 = 1{\cdot}0 \times 10^5 \text{ N m}^{-2}$.

Final pressure $= p_2 = p_1 + \dfrac{kx}{A}$

$$= 1{\cdot}0 \times 10^5 \text{ N m}^{-2} + \frac{(8000 \text{ N m}^{-1})\,(0{\cdot}1 \text{ m})}{8{\cdot}0 \times 10^{-3} \text{ m}^2}$$

$$= 2{\cdot}0 \times 10^5 \text{ N m}^{-2}.$$

Final volume $= V_2 = V_1 + Ax$

$= 2{\cdot}4 \times 10^{-3} \text{ m}^3 + 8{\cdot}0 \times 10^{-3} \text{ m}^2 \times 0{\cdot}1 \text{ m} = 3{\cdot}2 \times 10^{-3} \text{ m}^3$.

Using $\dfrac{p_1V_1}{T_1} = \dfrac{p_2V_2}{T_2}$,

$$T_2 = \frac{p_2V_2}{p_1V_1}\,T_1$$

$$= \frac{(2{\cdot}0 \times 10^5 \text{ N m}^{-2})\,(3{\cdot}2 \times 10^{-3} \text{ m}^3)}{(1{\cdot}0 \times 10^5 \text{ N m}^{-2})\,(2{\cdot}4 \times 10^{-3} \text{ m}^3)} \times 300 \text{ K}$$

$$= 800 \text{ K}.$$

**15.** *Assume that the temperature remains essentially constant in the upper part of the atmosphere. Obtain an expression for the variation in pressure in the upper atmosphere with height. The mean molecular weight of air is M.*

***Solution*** **:** Suppose the pressure at height $h$ is $p$ and that at $h + dh$ is $p + dp$. Then

$$dp = -\rho g\, dh. \qquad \ldots \text{(i)}$$

Now considering any small volume $\Delta V$ of air of mass $\Delta m$,

$$p\Delta V = nRT = \frac{\Delta m}{M}\,RT$$

or, $$p = \frac{\Delta m}{\Delta V}\,\frac{RT}{M} = \frac{\rho RT}{M}$$

or, $$\rho = \frac{M}{RT}p.$$

Putting in (i),

$$dp = -\frac{M}{RT}pg\, dh$$

or, $$\int_{p_0}^{p} \frac{dp}{p} = \int_0^h -\frac{M}{RT}g\, dh$$

or, $$\ln \frac{p}{p_0} = -\frac{Mgh}{RT}$$

where $p_0$ is the pressure at $h = 0$.

Thus, $$p = p_0\, e^{-\frac{Mgh}{RT}}.$$

**16.** *A horizontal tube of length l closed at both ends contains an ideal gas of molecular weight M. The tube is rotated at a constant angular velocity ω about a vertical axis passing through an end. Assuming the temperature to be uniform and constant, show that*

$$p_2 = p_1\, e^{\frac{M\omega^2 l^2}{2RT}},$$

*where $p_2$ and $p_1$ denote the pressures at the free end and the fixed end respectively.*

***Solution*** **:** Consider an element of the gas between the cross sections at distances $x$ and $x + dx$ from the fixed end (figure 24-W10). If $p$ be the pressure at $x$ and $p + dp$ at $x + dx$, the force acting on the element towards the centre is $Adp$, where $A$ is the cross sectional area. As this element is going in a circle of radius $x$,

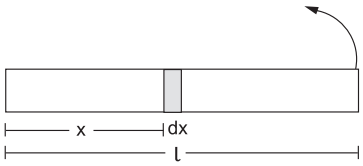



Figure 24-W10

$$Adp = (dm)\omega^2 x \qquad \ldots \text{(i)}$$

where $dm$ = mass of the element. Using $pV = nRT$ on this element,

$$pAdx = \frac{dm}{M}\,RT$$

or, $$dm = \frac{MpA}{RT}\,dx.$$

Putting in (i),

$$Adp = \frac{MpA}{RT}\,\omega^2 x\, dx$$

or, $$\int_{p_1}^{p_2} \frac{dp}{p} = \int_0^l \frac{M\omega^2}{RT}\,x\, dx$$

or, $$\ln \frac{p_2}{p_1} = \frac{M\omega^2 l^2}{2RT}$$

or, $$p_2 = p_1\, e^{\frac{M\omega^2 l^2}{2RT}}.$$

**17.** *A barometer tube contains a mixture of air and saturated water vapour in the space above the mercury column. It reads* 70 cm *when the actual atmospheric pressure is* 76 cm of mercury. *The saturation vapour pressure at room temperature is* 1·0 cm of mercury. *The tube is now lowered in the reservoir till the space above the mercury column is reduced to half its original volume. Find the*

*reading of the barometer. Assume that the temperature remains constant.*

***Solution*** **:** The pressure due to the air + vapour is 76 cm – 70 cm = 6 cm of mercury. The vapour is saturated and the pressure due to it is 1 cm of mercury. The pressure due to the air is, therefore, 5 cm of mercury.

As the tube is lowered and the volume above the mercury is decreased, some of the vapour will condense. The remaining vapour will again exert a pressure of 1 cm of mercury. The pressure due to air is doubled as the volume is halved. Thus, $p_{air} = 2 \times 5$ cm = 10 cm of mercury. The pressure due to the air + vapour = 10 cm + 1 cm = 11 cm of mercury. The barometer reading is 76 cm – 11 cm = 65 cm.

**18.** *Find the mass of water vapour per cubic metre of air at temperature* 300 K *and relative humidity* 50%. *The saturation vapour pressure at* 300 K *is* 3·6 kPa *and the gas constant* $R = 8{\cdot}3$ J K$^{-1}$ mol$^{-1}$.

***Solution*** : At 300 K, the saturation vapour pressure = 3·6 kPa. Considering 1 m$^3$ of volume,

$$pV = nRT = \frac{m}{M}RT$$

where $m$ = mass of vapour and $M$ = molecular weight of water.

Thus, $$m = \frac{MpV}{RT}$$

$$= \frac{(18 \text{ g mol}^{-1})\,(3{\cdot}6 \times 10^3 \text{ Pa})\,(1 \text{ m}^3)}{(8{\cdot}3 \text{ J K}^{-1} \text{ mol}^{-1})\,(300 \text{ K})} \approx 26 \text{ g}.$$

As the relative humidity is 50%, the amount of vapour present in 1 m$^3$ is 26 g × 0·50 = 13 g.

**19.** *The temperature and the relative humidity of air are* 20°C and 80% *on a certain day. Find the fraction of the mass of water vapour that will condense if the temperature falls to* 5°C. *Saturation vapour pressures at* 20°C *and* 5°C *are* 17·5 mm *and* 6·5 mm of mercury *respectively.*

***Solution*** : The relative humidity is

$$\frac{\text{vapour pressure of the air}}{\text{SVP at the same temperature}}.$$

Thus, the vapour pressure at 20°C

$$= 0{\cdot}8 \times 17{\cdot}5 \text{ mm of Hg}$$
$$= 14 \text{ mm of Hg}.$$

Consider a volume $V$ of air. If the vapour pressure is $p$ and the temperature is $T$, the mass $m$ of the vapour present is given by

$$pV = \frac{m}{M}RT$$

or, $$m = \frac{MV}{R}\frac{p}{T}. \quad \ldots \text{(i)}$$

The mass present at 20°C is

$$m_1 = \frac{MV}{R}\frac{14 \text{ mm of Hg}}{293 \text{ K}}.$$

When the air is cooled to 5°C, some vapour condenses and the air gets saturated with the remaining vapour. The vapour pressure at 5°C is, therefore, 6·5 mm of mercury. The mass of vapour present at 5°C is, therefore,

$$m_2 = \frac{MV}{R}\frac{6{\cdot}5 \text{ mm of Hg}}{278 \text{ K}}.$$

The fraction condensed

$$= \frac{m_1 - m_2}{m_1} = 1 - \frac{m_2}{m_1}$$

$$= 1 - \frac{6{\cdot}5}{278} \times \frac{293}{14} = 0{\cdot}51.$$

**20.** *A vessel containing water is put in a dry sealed room of volume* 76 m$^3$ *at a temperature of* 15°C. *The saturation vapour pressure of water at* 15°C is 15 mm of mercury. *How much water will evaporate before the water is in equilibrium with the vapour* ?

***Solution*** : Water will be in equilibrium with its vapour when the vapour gets saturated. In this case, the pressure of vapour = saturation vapour pressure = 15 mm of mercury

$$= (15 \times 10^{-3} \text{ m})\,(13600 \text{ kg m}^{-3})\,(9{\cdot}8 \text{ m s}^{-2})$$
$$= 2000 \text{ N m}^{-2}.$$

Using gas law, $pV = \frac{m}{M}RT$

$$m = \frac{MpV}{RT}$$

$$= \frac{(18 \text{ g mol}^{-1})\,(2000 \text{ N m}^{-2})\,(76 \text{ m}^3)}{(8{\cdot}3 \text{ J K}^{-1} \text{ mol}^{-1})\,(288 \text{ K})}$$

$$= 1145 \text{ g} = 1{\cdot}14 \text{ kg}.$$

Thus, 1·14 kg of water will evaporate.

**21.** *A jar contains a gas and a few drops of water at absolute temperature* $T_1$. *The pressure in the jar is* 830 mm of mercury. *The temperature of the jar is reduced by* 1%. *The saturation vapour pressures of water at the two temperatures are* 30 mm of mercury *and* 25 mm of mercury. *Calculate the new pressure in the jar.*

***Solution*** : At temperature $T_1$, the total pressure is 830 mm of mercury. Out of this, 30 mm of mercury is due to the vapour and 800 mm of mercury is due to the gas. As the temperature decreases, the pressure due to the gas decreases according to the gas law. Here the volume is constant, so,

$$\frac{p_2}{T_2} = \frac{p_1}{T_1}$$

or, $$p_2 = \frac{T_2}{T_1}p_1.$$

As $T_2$ is 1% less than $T_1$

$$T_2 = 0{\cdot}99\ T_1$$

and hence,

$$p_2 = 0{\cdot}99\ p_1$$

$= 0{\cdot}99 \times 800$ mm of mercury $= 792$ mm of mercury.

The vapour is still saturated and hence, its pressure is 25 mm of mercury. The total pressure at the reduced temperature is

$p = (792 + 25)$ mm of mercury
$= 817$ mm of mercury.

**22.** *Calculate the mass of* 1 litre *of moist air at* 27°C *when the barometer reads* 753·6 mm of mercury *and the dew point is* 16·1°C. *Saturation vapour pressure of water at* 16·1°C = 13·6 mm of mercury, *density of air at STP* $= 0{\cdot}001293$ g $(cc)^{-1}$, *density of saturated water vapour at STP* $= 0{\cdot}000808$ g $(cc)^{-1}$.

***Solution*** : We have $pV = \frac{m}{M} RT$

or, $$\rho = \frac{m}{V} = \frac{Mp}{RT}. \quad \ldots \text{(i)}$$

The dew point is 16·1°C and the saturation vapour pressure is 13·6 mm of mercury at the dew point. This means that the present vapour pressure is 13·6 mm of mercury.

At this pressure and temperature, the density of vapour will be

$$\rho = \frac{Mp}{RT}$$

$$= \frac{(18 \text{ g mol}^{-1})\ (13{\cdot}6 \times 10^{-3} \text{ m})\ (13600 \text{ kg m}^{-3})\ (9{\cdot}8 \text{ m s}^{-2})}{(8{\cdot}3 \text{ J K}^{-1} \text{ mol}^{-1})\ (300 \text{ K})}$$

$= 13{\cdot}1$ g $m^{-3}$.

Thus, 1 litre of moist air at 27°C contains 0·0131 g of vapour.

The pressure of dry air at 27°C is 753·6 mm – 13·6 mm = 740 mm of mercury. The density of air at STP is 0·001293 g $(cc)^{-1}$. The density at 27°C is given by equation (i),

$$\frac{\rho_1}{\rho_2} = \frac{p_1 / T_1}{p_2 / T_2}$$

or, $$\rho_2 = \frac{p_2 T_1}{T_2 p_1} \rho_1$$

$$= \frac{740 \times 273}{300 \times 760} \times 0{\cdot}001293 \text{ g } (cc)^{-1}.$$

$$= {\cdot}001457 \text{ g } (cc)^{-1}.$$

Thus, 1 litre of moist air contains 1·145 g of dry air. The mass of 1 litre of moist air is 1·1457 g + 0·0131 g ≈ 1·159 g.

□

## QUESTIONS FOR SHORT ANSWER

**1.** When we place a gas cylinder on a van and the van moves, does the kinetic energy of the molecules increase ? Does the temperature increase ?

**2.** While gas from a cooking gas cylinder is used, the pressure does not fall appreciably till the last few minutes. Why ?

**3.** Do you expect the gas in a cooking gas cylinder to obey the ideal gas equation ?

**4.** Can we define the temperature of (a) vacuum, (b) a single molecule ?

**5.** Comment on the following statement: the temperature of all the molecules in a sample of a gas is the same.

**6.** Consider a gas of neutrons. Do you expect it to behave much better as an ideal gas as compared to hydrogen gas at the same pressure and temperature ?

**7.** A gas is kept in a rigid cubical container. If a load of 10 kg is put on the top of the container, does the pressure increase ?

**8.** If it were possible for a gas in a container to reach the temperature 0 K, its pressure would be zero. Would the molecules not collide with the walls ? Would they not transfer momentum to the walls ?

**9.** It is said that the assumptions of kinetic theory are good for gases having low densities. Suppose a container is so evacuated that only one molecule is left in it. Which of the assumptions of kinetic theory will not be valid for such a situation ? Can we assign a temperature to this gas ?

**10.** A gas is kept in an enclosure. The pressure of the gas is reduced by pumping out some gas. Will the temperature of the gas decrease by Charles's law ?

**11.** Explain why cooking is faster in a pressure cooker.

**12.** If the molecules were not allowed to collide among themselves, would you expect more evaporation or less evaporation ?

**13.** Is it possible to boil water at room temperature, say 30°C ? If we touch a flask containing water boiling at this temperature, will it be hot ?

**14.** When you come out of a river after a dip, you feel cold. Explain.

## OBJECTIVE I

**1.** Which of the following parameters is the same for molecules of all gases at a given temperature ?
(a) Mass (b) Speed
(c) Momentum (d) Kinetic energy.

**2.** A gas behaves more closely as an ideal gas at
(a) low pressure and low temperature
(b) low pressure and high temperature
(c) high pressure and low temperature
(d) high pressure and high temperature.

**3.** The pressure of an ideal gas is written as $p = \frac{2E}{3V}$. Here $E$ refers to
(a) translational kinetic energy
(b) rotational kinetic energy
(c) vibrational kinetic energy
(d) total kinetic energy.

**4.** The energy of a given sample of an ideal gas depends only on its
(a) volume (b) pressure (c) density (d) temperature.

**5.** Which of the following gases has maximum rms speed at a given temperature ?
(a) hydrogen (b) nitrogen
(c) oxygen (d) carbon dioxide.

**6.** Figure 24-Q1 shows graphs of pressure vs density for an ideal gas at two temperatures $T_1$ and $T_2$.
(a) $T_1 > T_2$ (b) $T_1 = T_2$
(c) $T_1 < T_2$ (d) Any of the three is possible.

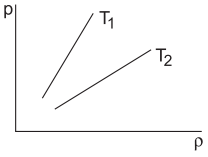



Figure 24-Q1

**7.** The mean square speed of the molecules of a gas at absolute temperature $T$ is proportional to
(a) $\frac{1}{T}$ (b) $\sqrt{T}$ (c) $T$ (d) $T^2$.

**8.** Suppose a container is evacuated to leave just one molecule of a gas in it. Let $v_a$ and $v_{rms}$ represent the average speed and the rms speed of the gas.
(a) $v_a > v_{rms}$ (b) $v_a < v_{rms}$
(c) $v_a = v_{rms}$ (d) $v_{rms}$ is undefined.

**9.** The rms speed of oxygen at room temperature is about 500 m/s. The rms speed of hydrogen at the same temperature is about
(a) 125 m $s^{-1}$ (b) 2000 m $s^{-1}$ (c) 8000 m $s^{-1}$ (d) 31 m $s^{-1}$.

**10.** The pressure of a gas kept in an isothermal container is 200 kPa. If half the gas is removed from it, the pressure will be
(a) 100 kPa (b) 200 kPa (c) 400 kPa (d) 800 kPa.

**11.** The rms speed of oxygen molecules in a gas is $v$. If the temperature is doubled and the oxygen molecules dissociate into oxygen atoms, the rms speed will become
(a) $v$ (b) $v\sqrt{2}$ (c) $2v$ (d) $4v$.

**12.** The quantity $\frac{pV}{kT}$ represents
(a) mass of the gas
(b) kinetic energy of the gas
(c) number of moles of the gas
(d) number of molecules in the gas.

**13.** The process on an ideal gas, shown in figure (24-Q2), is
(a) isothermal (b) isobaric (c) isochoric (d) none of these.

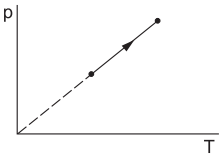



Figure 24-Q2

**14.** There is some liquid in a closed bottle. The amount of liquid is continuously decreasing. The vapour in the remaining part
(a) must be saturated (b) must be unsaturated
(c) may be saturated (d) there will be no vapour.

**15.** There is some liquid in a closed bottle. The amount of liquid remains constant as time passes. The vapour in the remaining part
(a) must be saturated (b) must be unsaturated
(c) may be unsaturated (d) there will be no vapour.

**16.** Vapour is injected at a uniform rate in a closed vessel which was initially evacuated. The pressure in the vessel
(a) increases continuously
(b) decreases continuously
(c) first increases and then decreases
(d) first increases and then becomes constant.

**17.** A vessel $A$ has volume $V$ and a vessel $B$ has volume $2V$. Both contain some water which has a constant volume. The pressure in the space above water is $p_a$ for vessel $A$ and $p_b$ for vessel $B$.
(a) $p_a = p_b$ (b) $p_a = 2p_b$
(c) $p_b = 2p_a$ (d) $p_b = 4p_a$

## OBJECTIVE II

**1.** Consider a collision between an oxygen molecule and a hydrogen molecule in a mixture of oxygen and hydrogen kept at room temperature. Which of the following are possible ?
(a) The kinetic energies of both the molecules increase.
(b) The kinetic energies of both the molecules decrease.

(c) kinetic energy of the oxygen molecule increases and that of the hydrogen molecule decreases.
(d) The kinetic energy of the hydrogen molecule increases and that of the oxygen molecule decreases.

**2.** Consider a mixture of oxygen and hydrogen kept at room temperature. As compared to a hydrogen molecule an oxygen molecule hits the wall
(a) with greater average speed
(b) with smaller average speed
(c) with greater average kinetic energy
(d) with smaller average kinetic energy.

**3.** Which of the following quantities is zero on an average for the molecules of an ideal gas in equilibrium ?
(a) Kinetic energy (b) Momentum
(c) Density (d) Speed.

**4.** Keeping the number of moles, volume and temperature the same, which of the following are the same for all ideal gases ?
(a) Rms speed of a molecule (b) Density
(c) Pressure
(d) Average magnitude of momentum.

**5.** The average momentum of a molecule in a sample of an ideal gas depends on
(a) temperature (b) number of moles
(c) volume (d) none of these.

**6.** Which of the following quantities is the same for all ideal gases at the same temperature ?
(a) The kinetic energy of 1 mole
(b) The kinetic energy of 1 g
(c) The number of molecules in 1 mole
(d) The number of molecules in 1 g

**7.** Consider the quantity $\frac{MkT}{pV}$ of an ideal gas where $M$ is the mass of the gas. It depends on the
(a) temperature of the gas (b) volume of the gas
(c) pressure of the gas (d) nature of the gas.

## EXERCISES

Use $R = 8{\cdot}3$ J K$^{-1}$ mol$^{-1}$ wherever required.

**1.** Calculate the volume of 1 mole of an ideal gas at STP.

**2.** Find the number of molecules of an ideal gas in a volume of $1{\cdot}000$ cm$^3$ at STP.

**3.** Find the number of molecules in 1 cm$^3$ of an ideal gas at 0°C and at a pressure of $10^{-5}$ mm of mercury.

**4.** Calculate the mass of 1 cm$^3$ of oxygen kept at STP.

**5.** Equal masses of air are sealed in two vessels, one of volume $V_0$ and the other of volume $2V_0$. If the first vessel is maintained at a temperature 300 K and the other at 600 K, find the ratio of the pressures in the two vessels.

**6.** An electric bulb of volume 250 cc was sealed during manufacturing at a pressure of $10^{-3}$ mm of mercury at 27°C. Compute the number of air molecules contained in the bulb. Avogadro constant = $6 \times 10^{23}$ mol$^{-1}$, density of mercury = 13600 kg m$^{-3}$ and $g = 10$ m s$^{-2}$.

**7.** A gas cylinder has walls that can bear a maximum pressure of $1{\cdot}0 \times 10^6$ Pa. It contains a gas at $8{\cdot}0 \times 10^5$ Pa and 300 K. The cylinder is steadily heated. Neglecting any change in the volume, calculate the temperature at which the cylinder will break.

**8.** 2 g of hydrogen is sealed in a vessel of volume $0{\cdot}02$ m$^3$ and is maintained at 300 K. Calculate the pressure in the vessel.

**9.** The density of an ideal gas is $1{\cdot}25 \times 10^{-3}$ g cm$^{-3}$ at STP. Calculate the molecular weight of the gas.

**10.** The temperature and pressure at Simla are 15·0°C and 72·0 cm of mercury and at Kalka these are 35·0°C and 76·0 cm of mercury. Find the ratio of air density at Kalka to the air density at Simla.

**11.** Figure (24-E1) shows a cylindrical tube with adiabatic walls and fitted with a diathermic separator. The separator can be slid in the tube by an external mechanism. An ideal gas is injected into the two sides at equal pressures and equal temperatures. The separator remains in equilibrium at the middle. It is now slid to a position where it divides the tube in the ratio of 1:3. Find the ratio of the pressures in the two parts of the vessel.

Figure 24-E1

**12.** Find the rms speed of hydrogen molecules in a sample of hydrogen gas at 300 K. Find the temperature at which the rms speed is double the speed calculated in the previous part.

**13.** A sample of $0{\cdot}177$ g of an ideal gas occupies 1000 cm$^3$ at STP. Calculate the rms speed of the gas molecules.

**14.** The average translational kinetic energy of air molecules is 0·040 eV (1 eV = $1{\cdot}6 \times 10^{-19}$ J). Calculate the temperature of the air. Boltzmann constant $k = 1{\cdot}38 \times 10^{-23}$ J K$^{-1}$.

**15.** Consider a sample of oxygen at 300 K. Find the average time taken by a molecule to travel a distance equal to the diameter of the earth.

**16.** Find the average magnitude of linear momentum of a helium molecule in a sample of helium gas at 0°C. Mass of a helium molecule = $6{\cdot}64 \times 10^{-27}$ kg and Boltzmann constant = $1{\cdot}38 \times 10^{-23}$ J K$^{-1}$.

**17.** The mean speed of the molecules of a hydrogen sample equals the mean speed of the molecules of a helium sample. Calculate the ratio of the temperature of the hydrogen sample to the temperature of the helium sample.

**18.** At what temperature the mean speed of the molecules of hydrogen gas equals the escape speed from the earth ?

**19.** Find the ratio of the mean speed of hydrogen molecules to the mean speed of nitrogen molecules in a sample containing a mixture of the two gases.

**20.** Figure (24-E2) shows a vessel partitioned by a fixed diathermic separator. Different ideal gases are filled in the two parts. The rms speed of the molecules in the left part equals the mean speed of the molecules in the right part. Calculate the ratio of the mass of a molecule in the left part to the mass of a molecule in the right part.

Figure 24-E2

**21.** Estimate the number of collisions per second suffered by a molecule in a sample of hydrogen at STP. The mean free path (average distance covered by a molecule between successive collisions) = $1{\cdot}38 \times 10^{-5}$ cm.

**22.** Hydrogen gas is contained in a closed vessel at 1 atm (100 kPa) and 300 K. (a) Calculate the mean speed of the molecules. (b) Suppose the molecules strike the wall with this speed making an average angle of 45° with it. How many molecules strike each square metre of the wall per second ?

**23.** Air is pumped into an automobile tyre's tube up to a pressure of 200 kPa in the morning when the air temperature is 20°C. During the day the temperature rises to 40°C and the tube expands by 2%. Calculate the pressure of the air in the tube at this temperature.

**24.** Oxygen is filled in a closed metal jar of volume $1{\cdot}0 \times 10^{-3}$ m$^3$ at a pressure of $1{\cdot}5 \times 10^5$ Pa and temperature 400 K. The jar has a small leak in it. The atmospheric pressure is $1{\cdot}0 \times 10^5$ Pa and the atmospheric temperature is 300 K. Find the mass of the gas that leaks out by the time the pressure and the temperature inside the jar equalise with the surrounding.

**25.** An air bubble of radius 2·0 mm is formed at the bottom of a 3·3 m deep river. Calculate the radius of the bubble as it comes to the surface. Atmospheric pressure = $1{\cdot}0 \times 10^5$ Pa and density of water = 1000 kg m$^{-3}$.

**26.** Air is pumped into the tubes of a cycle rickshaw at a pressure of 2 atm. The volume of each tube at this pressure is 0·002 m$^3$. One of the tubes gets punctured and the volume of the tube reduces to 0·0005 m$^3$. How many moles of air have leaked out ? Assume that the temperature remains constant at 300 K and that the air behaves as an ideal gas.

**27.** 0·040 g of He is kept in a closed container initially at 100·0°C. The container is now heated. Neglecting the expansion of the container, calculate the temperature at which the internal energy is increased by 12 J.

**28.** During an experiment, an ideal gas is found to obey an additional law $pV^2$ = constant. The gas is initially at a temperature $T$ and volume $V$. Find the temperature when it expands to a volume $2V$.

**29.** A vessel contains 1·60 g of oxygen and 2·80 g of nitrogen. The temperature is maintained at 300 K and the volume of the vessel is 0·166 m$^3$. Find the pressure of the mixture.

**30.** A vertical cylinder of height 100 cm contains air at a constant temperature. The top is closed by a frictionless light piston. The atmospheric pressure is equal to 75 cm of mercury. Mercury is slowly poured over the piston. Find the maximum height of the mercury column that can be put on the piston.

**31.** Figure (24-E3) shows two vessels $A$ and $B$ with rigid walls containing ideal gases. The pressure, temperature and the volume are $p_A$, $T_A$, $V$ in the vessel $A$ and $p_B$, $T_B$, $V$ in the vessel $B$. The vessels are now connected through a small tube. Show that the pressure $p$ and the temperature $T$ satisfy

$$\frac{p}{T} = \frac{1}{2}\left(\frac{p_A}{T_A} + \frac{p_B}{T_B}\right)$$

when equilibrium is achieved.

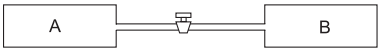



Figure 24-E3

**32.** A container of volume 50 cc contains air (mean molecular weight = 28·8 g) and is open to atmosphere where the pressure is 100 kPa. The container is kept in a bath containing melting ice (0°C). (a) Find the mass of the air in the container when thermal equilibrium is reached. (b) The container is now placed in another bath containing boiling water (100°C). Find the mass of air in the container. (c) The container is now closed and placed in the melting-ice bath. Find the pressure of the air when thermal equilibrium is reached.

**33.** A uniform tube closed at one end, contains a pellet of mercury 10 cm long. When the tube is kept vertically with the closed-end upward, the length of the air column trapped is 20 cm. Find the length of the air column trapped when the tube is inverted so that the closed-end goes down. Atmospheric pressure = 75 cm of mercury.

**34.** A glass tube, sealed at both ends, is 100 cm long. It lies horizontally with the middle 10 cm containing mercury. The two ends of the tube contain air at 27°C and at a pressure 76 cm of mercury. The air column on one side is maintained at 0°C and the other side is maintained at 127°C. Calculate the length of the air column on the cooler side. Neglect the changes in the volume of mercury and of the glass.

**35.** An ideal gas is trapped between a mercury column and the closed-end of a narrow vertical tube of uniform base containing the column. The upper end of the tube is open to the atmosphere. The atmospheric pressure equals 76 cm of mercury. The lengths of the mercury column and the trapped air column are 20 cm and 43 cm respectively. What will be the length of the air column when the tube is tilted slowly in a vertical plane through an angle of 60° ? Assume the temperature to remain constant.

**36.** Figure (24-E4) shows a cylindrical tube of length 30 cm which is partitioned by a tight-fitting separator. The separator is very weakly conducting and can freely slide along the tube. Ideal gases are filled in the two parts of the vessel. In the beginning, the temperatures in the parts $A$ and $B$ are 400 K and 100 K respectively. The separator slides to a momentary equilibrium position shown in the figure. Find the final equilibrium position of the separator, reached after a long time.

Figure 24-E4

**37.** A vessel of volume $V_0$ contains an ideal gas at pressure $p_0$ and temperature $T$. Gas is continuously pumped out of this vessel at a constant volume-rate $dV/dt = r$ keeping the temperature constant. The pressure of the gas being taken out equals the pressure inside the vessel. Find (a) the pressure of the gas as a function of time, (b) the time taken before half the original gas is pumped out.

**38.** One mole of an ideal gas undergoes a process

$$p = \frac{p_0}{1 + (V/V_0)^2}$$

where $p_0$ and $V_0$ are constants. Find the temperature of the gas when $V = V_0$.

**39.** Show that the internal energy of the air (treated as an ideal gas) contained in a room remains constant as the temperature changes between day and night. Assume that the atmospheric pressure around remains constant and the air in the room maintains this pressure by communicating with the surrounding through the windows, doors, etc.

**40.** Figure (24-E5) shows a cylindrical tube of radius 5 cm and length 20 cm. It is closed by a tight-fitting cork. The friction coefficient between the cork and the tube is 0·20. The tube contains an ideal gas at a pressure of 1 atm and a temperature of 300 K. The tube is slowly heated and it is found that the cork pops out when the temperature reaches 600 K. Let $dN$ denote the magnitude of the normal contact force exerted by a small length $dl$ of the cork along the periphery (see the figure). Assuming that the temperature of the gas is uniform at any instant, calculate $\frac{dN}{dl}$.

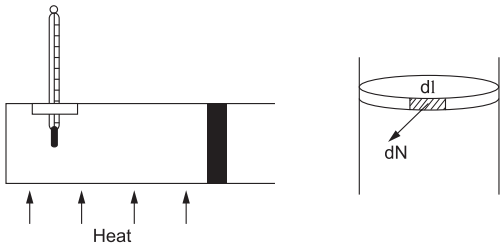



Figure 24-E5

**41.** Figure (24-E6) shows a cylindrical tube of cross-sectional area $A$ fitted with two frictionless pistons. The pistons are connected to each other by a metallic wire. Initially, the temperature of the gas is $T_0$ and its pressure is $p_0$ which equals the atmospheric pressure. (a) What is the tension in the wire ? (b) What will be the tension if the temperature is increased to $2T_0$ ?

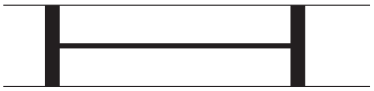


Figure 24-E6

**42.** Figure (24-E7) shows a large closed cylindrical tank containing water. Initially the air trapped above the water surface has a height $h_0$ and pressure $2p_0$ where $p_0$ is the atmospheric pressure. There is a hole in the wall of the tank at a depth $h_1$ below the top from which water comes out. A long vertical tube is connected as shown. (a) Find the height $h_2$ of the water in the long tube above the top initially. (b) Find the speed with which water comes out of the hole. (c) Find the height of the water in the long tube above the top when the water stops coming out of the hole.

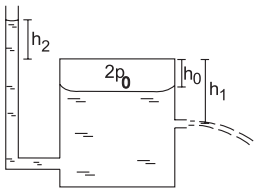



Figure 24-E7

**43.** An ideal gas is kept in a long cylindrical vessel fitted with a frictionless piston of cross-sectional area 10 cm$^2$ and weight 1 kg (figure 24-E8). The vessel itself is kept in a big chamber containing air at atmospheric pressure 100 kPa. The length of the gas column is 20 cm. If the chamber is now completely evacuated by an exhaust pump, what will be the length of the gas column ? Assume the temperature to remain constant throughout the process.

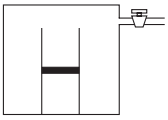


Figure 24-E8

**44.** An ideal gas is kept in a long cylindrical vessel fitted with a frictionless piston of cross-sectional area 10 cm$^2$ and weight 1 kg. The length of the gas column in the vessel is 20 cm. The atmospheric pressure is 100 kPa. The vessel is now taken into a spaceship revolving round the earth as a satellite. The air pressure in the spaceship is maintained at 100 kPa. Find the length of the gas column in the cylinder.

**45.** Two glass bulbs of equal volume are connected by a narrow tube and are filled with a gas at 0°C at a pressure of 76 cm of mercury. One of the bulbs is then placed in melting ice and the other is placed in a water bath maintained at 62°C. What is the new value of the

pressure inside the bulbs ? The volume of the connecting tube is negligible.

**46.** The weather report reads, "Temperature 20°C : Relative humidity 100%". What is the dew point ?

**47.** The condition of air in a closed room is described as follows. Temperature = 25°C, relative humidity = 60%, pressure = 104 kPa. If all the water vapour is removed from the room without changing the temperature, what will be the new pressure ? The saturation vapour pressure at 25°C = 3·2 kPa.

**48.** The temperature and the dew point in an open room are 20°C and 10°C. If the room temperature drops to 15°C, what will be the new dew point ?

**49.** Pure water vapour is trapped in a vessel of volume 10 $cm^3$. The relative humidity is 40%. The vapour is compressed slowly and isothermally. Find the volume of the vapour at which it will start condensing.

**50.** A barometer tube is 80 cm long (above the mercury reservoir). It reads 76 cm on a particular day. A small amount of water is introduced in the tube and the reading drops to 75·4 cm. Find the relative humidity in the space above the mercury column if the saturation vapour pressure at the room temperature is 1·0 cm.

**51.** Using figure (24.6) of the text, find the boiling point of methyl alcohol at 1 atm (760 mm of mercury) and at 0·5 atm.

**52.** The human body has an average temperature of 98 °F. Assume that the vapour pressure of the blood in the veins behaves like that of pure water. Find the minimum atmospheric pressure which is necessary to prevent the blood from boiling. Use figure (24.6) of the text for the vapour pressures.

**53.** A glass contains some water at room temperature 20°C. Refrigerated water is added to it slowly. When the temperature of the glass reaches 10°C, small droplets condense on the outer surface. Calculate the relative humidity in the room. The boiling point of water at a pressure of 17·5 mm of mercury is 20°C and at 8·9 mm of mercury it is 10°C.

**54.** 50 $m^3$ of saturated vapour is cooled down from 30°C to 20°C. Find the mass of the water condensed. The absolute humidity of saturated water vapour is 30 g $m^{-3}$ at 30°C and 16 g $m^{-3}$ at 20°C.

**55.** A barometer correctly reads the atmospheric pressure as 76 cm of mercury. Water droplets are slowly introduced into the barometer tube by a dropper. The height of the mercury column first decreases and then becomes constant. If the saturation vapour pressure at the atmospheric temperature is 0·80 cm of mercury, find the height of the mercury column when it reaches its minimum value.

**56.** 50 cc of oxygen is collected in an inverted gas jar over water. The atmospheric pressure is 99·4 kPa and the room temperature is 27°C. The water level in the jar is same as the level outside. The saturation vapour pressure at 27°C is 3·4 kPa. Calculate the number of moles of oxygen collected in the jar.

**57.** A faulty barometer contains certain amount of air and saturated water vapour. It reads 74·0 cm when the atmospheric pressure is 76·0 cm of mercury and reads 72·10 cm when the atmospheric pressure is 74·0 cm of mercury. Saturation vapour pressure at the air temperature = 1·0 cm of mercury. Find the length of the barometer tube above the mercury level in the reservoir.

**58.** On a winter day, the outside temperature is 0°C and relative humidity 40%. The air from outside comes into a room and is heated to 20°C. What is the relative humidity in the room ? The saturation vapour pressure at 0°C is 4·6 mm of mercury and at 20°C it is 18 mm of mercury.

**59.** The temperature and humidity of air are 27°C and 50% on a particular day. Calculate the amount of vapour that should be added to 1 cubic metre of air to saturate it. The saturation vapour pressure at 27°C = 3600 Pa.

**60.** The temperature and relative humidity in a room are 300 K and 20% respectively. The volume of the room is 50 $m^3$. The saturation vapour pressure at 300 K is 3·3 kPa. Calculate the mass of the water vapour present in the room.

**61.** The temperature and the relative humidity are 300 K and 20% in a room of volume 50 $m^3$. The floor is washed with water, 500 g of water sticking on the floor. Assuming no communication with the surrounding, find the relative humidity when the floor dries. The changes in temperature and pressure may be neglected. Saturation vapour pressure at 300 K = 3·3 kPa.

**62.** A bucket full of water is placed in a room at 15°C with initial relative humidity 40%. The volume of the room is 50 $m^3$. (a) How much water will evaporate ? (b) If the room temperature is increased by 5°C, how much more water will evaporate ? The saturation vapour pressure of water at 15°C and 20°C are 1·6 kPa and 2·4 kPa respectively.

□

## ANSWERS

### OBJECTIVE I

| | | | | | |
|---|---|---|---|---|---|
| 1. (d) | 2. (b) | 3. (a) | 4. (d) | 5. (a) | 6. (a) |
| 7. (c) | 8. (c) | 9. (b) | 10. (a) | 11. (c) | 12. (d) |
| 13. (c) | 14. (b) | 15. (a) | 16. (d) | 17. (a) | |

### OBJECTICE II

| | | |
|---|---|---|
| 1. (c), (d) | 2. (b) | 3. (b) |
| 4. (c) | 5. (d) | 6. (a), (c) |
| 7. (d) | | |

## EXERCISES

1. $2{\cdot}24 \times 10^{-2}\ \text{m}^3$
2. $2{\cdot}685 \times 10^{19}$
3. $3{\cdot}53 \times 10^{11}$
4. 1·43 mg
5. 1 : 1
6. $8{\cdot}0 \times 10^{15}$
7. 375 K
8. $1{\cdot}24 \times 10^{5}$ Pa
9. $28{\cdot}3\ \text{g mol}^{-1}$
10. 0·987
11. 3 : 1
12. $1930\ \text{m s}^{-1}$, 1200 K
13. $1300\ \text{m s}^{-1}$
14. 310 K
15. 8·0 hour
16. $8{\cdot}0 \times 10^{-24}\ \text{kg m s}^{-1}$
17. 1 : 2
18. 11800 K
19. 3·74
20. 1·18
21. $1{\cdot}23 \times 10^{10}$
22. (a) $1780\ \text{m s}^{-1}$ (b) $1{\cdot}2 \times 10^{28}$
23. 209 kPa
24. 0·16 g
25. 2·2 mm
26. 0·14
27. 196°C
28. $T/2$
29. $2250\ \text{N m}^{-2}$
30. 25 cm
32. (a) 0·058 g (b) 0·0468 g (c) 73·0 kPa
33. 15 cm
34. 36·5 cm
35. 48 cm
36. 10 cm from the left end
37. (a) $p = p_0 e^{-\gamma t/V_0}$ (b) $\dfrac{V_0 \ln 2}{\gamma}$
38. $\dfrac{p_0 V_0}{2R}\ \text{mol}^{-1}$
40. $1{\cdot}25 \times 10^{4}\ \text{N m}^{-1}$
41. (a) zero (b) $p_0 A$
42. (a) $\dfrac{p_o}{\rho g} - h_0$ (b) $\left[\dfrac{2}{\rho}\left(p_0 + \rho g\,(h_1 - h_0)\right)\right]^{1/2}$ (c) $-h_1$
43. 2·2 m
44. 22 cm
45. 84 cm of mercury
46. 20°C
47. 102 kPa
48. 10°C
49. $4{\cdot}0\ \text{cm}^3$
50. 60%
51. 65°C, 48°C
52. 50 mm of mercury
53. 51%
54. 700 g
55. 75·2 cm
56. $1{\cdot}93 \times 10^{-3}$
57. 91·1 cm
58. 9·5%
59. 13 g
60. 238 g
61. 62%
62. (a) 361 g (b) 296 g

□

## CHAPTER 25

# CALORIMETRY

### 25.1 HEAT AS A FORM OF ENERGY

When two bodies at different temperatures are placed in contact, the hotter body cools down and the colder body warms up. Energy is thus transferred from a body at higher temperature to a body at lower temperature when they are brought in contact.

The energy being transferred between two bodies or between adjacent parts of a body as a result of temperature difference is called *heat*.

Thus, heat is a form of energy. It is energy in transit whenever temperature differences exist. Once it is transferred it becomes the internal energy of the receiving body. It should be clearly understood that the word "heat" is meaningful only as long as the energy is being transferred. Thus, expressions like "heat in a body" or "heat of a body" are meaningless.

### 25.2 UNITS OF HEAT

As heat is just energy in transit, its unit in SI is joule. However, another unit of heat "calorie" is in wide use. This unit was formulated much before it was recognised that heat is a form of energy. The old day definition of calorie is as follows:

*The amount of heat needed to increase the temperature of* 1 g *of water from* 14·5°C to 15·5°C *at a pressure of* 1 atm *is called* 1 calorie.

The amount of heat needed to raise the temperature of 1 g of water by 1°C depends slightly on the actual temperature of water and the pressure. That is why, the range 14·5°C to 15·5°C and the pressure 1 atm was specified in the definition. We shall ignore this small variation and use one calorie to mean the amount of heat needed to increase the temperature of 1 g of water by 1°C at any region of temperature and pressure.

The calorie is now defined in terms of joule as 1 cal = 4·186 joule. We also use the unit kilocalorie which is equal to 1000 calorie as the name indicates.

***Example 25.1***

*What is the kinetic energy of a* 10 kg *mass moving at a speed of* 36 km $h^{-1}$ *in calorie* ?

***Solution*** :

The kinetic energy is

$$\frac{1}{2}mv^2 = \frac{1}{2}\times 10\text{ kg}\times\left(\frac{36\times 10^3\text{ m}}{3600\text{ s}}\right)^2$$

$$= 500\text{ J} = \frac{500}{4{\cdot}186}\text{ cal} \approx 120\text{ cal}.$$

### 25.3 PRINCIPLE OF CALORIMETRY

A simple calorimeter is a vessel generally made of copper with a stirrer of the same material. The vessel is kept in a wooden box to isolate it thermally from the surrounding. A thermometer is used to measure the temperature of the contents of the calorimeter.

Objects at different temperatures are made to come in contact with each other in the calorimeter. As a result, heat is exchanged between the objects as well as with the calorimeter. Neglecting any heat exchange with the surrounding, the principle of calorimetry states that *the total heat given by the hot objects equals the total heat received by the cold objects.*

### 25.4 SPECIFIC HEAT CAPACITY AND MOLAR HEAT CAPACITY

When we supply heat to a body, its temperature increases. The amount of heat absorbed depends on the mass of the body, the change in temperature, the material of the body as well as the surrounding conditions, such as pressure. We write the equation

$$Q = ms\Delta\theta \qquad \ldots\ (25.1)$$

where $\Delta\theta$ is the change in temperature, $m$ is the mass of the body, $Q$ is the heat supplied, and $s$ is a constant for the given material under the given surrounding conditions. The constant $s$ is called *specific heat capacity* of the substance. When a solid or a liquid is

kept open in the atmosphere and heated, the pressure remains constant. Table (25.1) gives the specific heat capacities of some of the solids and liquids under constant pressure condition. As can be seen from equation (25.1), the SI unit for specific heat capacity is J $kg^{-1}$ $K^{-1}$ which is the same as J $kg^{-1}$ $°C^{-1}$. The specific heat capacity may also be expressed in cal $g^{-1}$ $K^{-1}$ (same as cal $g^{-1}$ $°C^{-1}$). Specific heat capacity is also called *specific heat* in short.

***Table 25.1 : Specific heat capacities of some materials***

| Material | cal $g^{-1}$ $°C^{-1}$ | J $kg^{-1}$ $K^{-1}$ | Material | Cal $g^{-1}$ $°C^{-1}$ | J $kg^{-1}$ $K^{-1}$ |
|---|---|---|---|---|---|
| Water | 1·00 | 4186 | Glass | 0·1–0·2 | 419–837 |
| Ethanol | 0·55 | 2302 | Iron | 0·112 | 470 |
| Paraffin | 0·51 | 2135 | Copper | 0·093 | 389 |
| Ice | 0·50 | 2093 | Mercury | 0·033 | 138 |
| Steam | 0·46 | 1926 | Lead | 0·031 | 130 |
| Aluminium | 0·215 | 900 | | | |

The amount of substance in the given body may also be measured in terms of the number of moles. Equation (25.1) may be rewritten as

$$Q = nC\Delta\theta$$

where $n$ is the number of moles in the sample. The constant $C$ is called *molar heat capacity*.

***Example 25.2***

*A copper block of mass* 60 g *is heated till its temperature is increased by* 20°C. *Find the heat supplied to the block. Specific heat capacity of copper* = 0·09 cal $g^{-1}$ $°C^{-1}$ .

***Solution :***

The heat supplied is $Q = ms\Delta\theta$

$= (60\ g)\ (0{\cdot}09\ cal\ g^{-1}\ °C^{-1})\ (20°C) = 108\ cal.$

The quantity $ms$ is called the *heat capacity of the body*. Its unit is J $K^{-1}$. The mass of water having the same heat capacity as a given body is called the *water equivalent* of the body.

## 25.5 DETERMINATION OF SPECIFIC HEAT CAPACITY IN LABORATORY

Figure (25.1) shows Regnault's apparatus to determine the specific heat capacity of a solid heavier than water, and insoluble in it. A wooden partition $P$ separates a steam chamber $O$ and a calorimeter $C$. The steam chamber $O$ is a double-walled cylindrical vessel. Steam can be passed in the space between the two walls through an inlet $A$ and it can escape through an outlet $B$. The upper part of the vessel is closed by a cork. The given solid may be suspended in the vessel by a thread passing through the cork. A thermometer $T_1$ is also inserted into the vessel to record the temperature of the solid. The steam chamber is kept on a wooden platform with a removable wooden disc $D$ closing the bottom hole of the chamber.

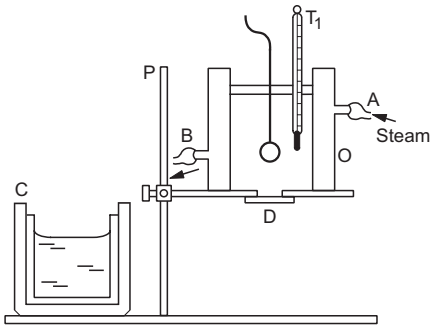



Figure 25.1

To start with, the experimental solid (in the form of a ball or a block) is weighed and then suspended in the steam chamber. Steam is prepared by boiling water in a separate boiler and is passed through the steam chamber. A calorimeter with a stirrer is weighed and sufficient amount of water is kept in it so that the solid may be completely immersed in it. The calorimeter is again weighed with water to get the mass of the water. The initial temperature of the water is noted.

When the temperature of the solid becomes constant (say for 15 minutes), the partition $P$ is removed, the calorimeter is taken below the steam chamber, the wooden disc $D$ is removed and the thread is cut to drop the solid in the calorimeter. The calorimeter is taken to its original place and is stirred. The maximum temperature of the mixture is noted.

***Calculation***:

Let the mass of the solid $= m_1$

mass of the calorimeter and the stirrer $= m_2$

mass of the water $= m_3$

specific heat capacity of the solid $= s_1$

specific heat capacity of the material of the calorimeter (and stirrer) $= s_2$

specific heat capacity of water $= s_3$

initial temperature of the solid $= \theta_1$

initial temperature of the calorimeter, stirrer and water $= \theta_2$

final temperature of the mixture $= \theta$.

We have,

heat lost by the solid $= m_1 s_1(\theta_1 - \theta)$

heat gained by the calorimeter (and the stirrer) $= m_2 s_2(\theta - \theta_2)$

and heat gained by the water = $m_3 s_3(\theta - \theta_2)$.

Assuming no loss of heat to the surrounding, the heat lost by the solid goes into the calorimeter, stirrer and water. Thus,

$$m_1 s_1(\theta_1 - \theta) = m_2 s_2(\theta - \theta_2) + m_3 s_3(\theta - \theta_2) \quad \ldots \text{ (i)}$$

or, $$s_1 = \frac{(m_2 s_2 + m_3 s_3)(\theta - \theta_2)}{m_1(\theta_1 - \theta)}.$$

Knowing the specific heat capacity of water ($s_3 = 4186$ J kg$^{-1}$ K$^{-1}$) and that of the material of the calorimeter and the stirrer ($s_2 = 389$ J kg$^{-1}$ K$^{-1}$ if the material be copper), one can calculate $s_1$.

Specific heat capacity of a liquid can also be measured with the Regnault's apparatus. Here a solid of known specific heat capacity $s_1$ is used and the experimental liquid is taken in the calorimeter in place of water. The solid should be denser than the liquid. Using the same procedure and with the same symbols we get an equation identical to equation (i) above, that is,

$$m_1 s_1(\theta_1 - \theta) = m_2 s_2(\theta - \theta_2) + m_3 s_3(\theta - \theta_2)$$

in which $s_3$ is the specific heat capacity of the liquid. We get,

$$s_3 = \frac{m_1 s_1(\theta_1 - \theta)}{m_3(\theta - \theta_2)} - \frac{m_2 s_2}{m_3}.$$

## 25.6 SPECIFIC LATENT HEAT OF FUSION AND VAPORIZATION

Apart from raising the temperature, heat supplied to a body may cause a phase change such as solid to liquid or liquid to vapour.

During this process of melting or vaporization, the temperature remains constant. The amount of heat needed to melt a solid of mass $m$ may be written as

$$Q = mL \quad \ldots \text{ (25.2)}$$

where $L$ is a constant for the given material (and surrounding conditions). This constant $L$ is called *specific latent heat of fusion*. The term *latent heat of fusion* is also used to mean the same thing. Equation (25.2) is also valid when a liquid changes its phase to vapour. The constant $L$ in this case is called the *specific latent heat of vaporization* or simply *latent heat of vaporization*. When a vapour condenses or a liquid solidifies, heat is released to the surrounding.

In solids, the forces between the molecules are large and the molecules are almost fixed in their positions inside the solid. In a liquid, the forces between the molecules are weaker and the molecules may move freely inside the volume of the liquid. However, they are not able to come out of the surface. In vapours or gases, the intermolecular forces are almost negligible and the molecules may move freely anywhere in the container. When a solid melts, its molecules move apart against the strong molecular attraction. This needs energy which must be supplied from outside. Thus, the internal energy of a given body is larger in liquid phase than in solid phase. Similarly, the internal energy of a given body in vapour phase is larger than that in liquid phase.

***Example 25.3***

*A piece of ice of mass* 100 g *and at temperature* 0°C *is put in* 200 g *of water at* 25°C. *How much ice will melt as the temperature of the water reaches* 0°C ? *The specific heat capacity of water* = 4200 J kg$^{-1}$ K$^{-1}$ *and the specific latent heat of fusion of ice* = $3{\cdot}4 \times 10^5$ J kg$^{-1}$.

***Solution*** :

The heat released as the water cools down from 25°C to 0°C is

$$Q = ms\Delta\theta = (0{\cdot}2 \text{ kg})\,(4200 \text{ J kg}^{-1} \text{ K}^{-1})\,(25 \text{ K}) = 21000 \text{ J}.$$

The amount of ice melted by this much heat is given by

$$m = \frac{Q}{L} = \frac{21000 \text{ J}}{3{\cdot}4 \times 10^5 \text{ J kg}^{-1}} = 62 \text{ g}.$$

## 25.7 MEASUREMENT OF SPECIFIC LATENT HEAT OF FUSION OF ICE

An empty calorimeter (together with a stirrer) is weighed. About two third of the calorimeter is filled with water and is weighed again. Thus, one gets the mass of the water. The initial temperature of the water is measured with the help of a thermometer. A piece of ice is taken and as it starts melting it is dried with a blotting paper and put into the calorimeter. The water is stirred keeping the ice always fully immersed in it. The minimum temperature reached is recorded. This represents the temperature when all the ice has melted. The calorimeter with its contents is weighed again. Thus, one can get the mass of the ice that has melted in the calorimeter.

***Calculation*:**

Let the mass of the calorimeter (with stirrer) = $m_1$
mass of water = $m_2$
mass of ice = $m_3$
initial temperature of the calorimeter and the water (in Celsius) = $\theta_1$
final temperature of the calorimeter and the water (in Celsius) = $\theta_2$
temperature of the melting ice = 0°C
specific latent heat of fusion of ice = $L$

specific heat capacity of the material of the calorimeter (and stirrer) $= s_1$

specific heat capacity of water $= s_2$.

We have,

heat lost by the calorimeter (and the stirrer) $= m_1 s_1(\theta_1 - \theta_2)$

heat lost by the water kept initially in the calorimeter $= m_2 s_2(\theta_1 - \theta_2)$

heat gained by the ice during its fusion to water $= m_3 L$

heat gained by this water in coming from 0°C to $\theta_2$ $= m_3 s_2 \theta_2$.

Assuming no loss of heat to the surrounding,

$$m_1 s_1(\theta_1 - \theta_2) + m_2 s_2(\theta_1 - \theta_2) = m_3 L + m_3 s_2 \theta_2$$

or, $$L = \frac{(m_1 s_1 + m_2 s_2)(\theta_1 - \theta_2)}{m_3} - s_2 \theta_2.$$

Knowing the specific heat capacity of water and that of the material of the calorimeter and the stirrer, one can calculate the specific latent heat of fusion of ice $L$.

## 25.8 MEASUREMENT OF SPECIFIC LATENT HEAT OF VAPORIZATION OF WATER

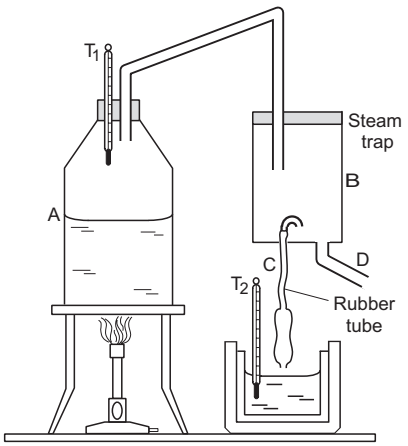



Figure 25.2

Figure (25.2) shows the arrangement used to measure the specific latent heat of vaporization of water. Steam is prepared by boiling water in a boiler $A$. The cork of the boiler has two holes. A thermometer $T_1$ is inserted into one to measure the temperature of the steam and the other contains a bent glass tube to carry the steam to a steam trap $B$. A tube $C$ with one end bent and the other end terminated in a jet is fitted in the steam trap. Another tube $D$ is fitted in the trap which is used to drain out the extra steam and water condensed at the bottom.

To start the experiment, an empty calorimeter (together with the stirrer) is weighed. About half of it is filled with water and is weighed again. Thus, one gets the mass of the water. The initial temperature of the water and the calorimeter is measured by a thermometer $T_2$ . Water is kept in the boiler $A$ and is heated. As it boils, the steam passes to the steam trap and then comes out through the tubes $C$ and $D$. After some steam has gone out (say for 5 minutes), the temperature of the steam is noted. The calorimeter with the water is kept below the tube $C$ so that steam goes into the calorimeter. The water in the calorimeter is continuously stirred and the calorimeter is removed after the temperature in it increases by about 5°C. The final temperature of the water in the calorimeter is noted. The calorimeter together with the water (including the water condensed) is weighed. From this, one gets the mass of the steam that condensed in the calorimeter.

Let the mass of the calorimeter (with the stirrer) $= m_1$

mass of the water $= m_2$

mass of the steam condensed $= m_3$

temperature of the steam $= \theta_1$

initial temperature of the water in the calorimeter $= \theta_2$

final temperature of the water in the calorimeter $= \theta_3$

specific latent heat of vaporization of water $= L$

specific heat capacity of the material of the calorimeter (and the stirrer) $= s_1$

specific heat capacity of water $= s_2$.

We have,

heat gained by the calorimeter (and the stirrer) $= m_1 s_1(\theta_3 - \theta_2)$

heat gained by the water kept initially in the calorimeter $= m_2 s_2(\theta_3 - \theta_2)$

heat lost by the steam in condensing $= m_3 L$

heat lost by the condensed water in cooling from temperature $\theta_1$ to $\theta_3 = m_3 s_2(\theta_1 - \theta_3)$.

Assuming no loss of heat to the surrounding,

$$m_1 s_1(\theta_3 - \theta_2) + m_2 s_2(\theta_3 - \theta_2) = m_3 L + m_3 s_2(\theta_1 - \theta_3)$$

or, $$L = \frac{(m_1 s_1 + m_2 s_2)(\theta_3 - \theta_2)}{m_3} - s_2(\theta_1 - \theta_3).$$

Knowing the specific heat capacity of water and that of the material of the calorimeter, one can

calculate the specific latent heat of vaporization of water $L$.

***Example 25.4***

*A calorimeter of water equivalent* 15 g *contains* 165 g *of water at* 25°C. *Steam at* 100°C *is passed through the water for some time. The temperature is increased to* 30°C *and the mass of the calorimeter and its contents is increased by* 1·5 g. *Calculate the specific latent heat of vaporization of water. Specific heat capacity of water is* 1 cal $g^{-1}$ $°C^{-1}$.

***Solution :*** Let $L$ be the specific latent heat of vaporization of water. The mass of the steam condensed is 1·5 g. Heat lost in condensation of steam is

$$Q_1 = (1{\cdot}5 \text{ g})\, L.$$

The condensed water cools from 100°C to 30°C. Heat lost in this process is

$$Q_2 = (1{\cdot}5 \text{ g})\,(1 \text{ cal g}^{-1} °C^{-1})\,(70°C) = 105 \text{ cal}.$$

Heat supplied to the calorimeter and to the cold water during the rise in temperature from 25°C to 30°C is

$$Q_3 = (15 \text{ g} + 165 \text{ g})\,(1 \text{ cal g}^{-1} °C^{-1})\,(5°C) = 900 \text{ cal}.$$

If no heat is lost to the surrounding,

$$(1{\cdot}5 \text{ g})\, L + 105 \text{ cal} = 900 \text{ cal}$$

or, $$L = 530 \text{ cal g}^{-1}.$$

## 25.9 MECHANICAL EQUIVALENT OF HEAT

In early days heat was not recognised as a form of energy. Heat was supposed to be something needed to raise the temperature of a body or to change its phase. Calorie was defined as the unit of heat. A number of experiments were performed to show that the temperature may also be increased by doing mechanical work on the system. These experiments established that heat is equivalent to mechanical energy and measured how much mechanical energy is equivalent to a calorie. If mechanical work $W$ produces the same temperature change as heat $H$, we write,

$$W = JH \qquad \ldots \quad (25.3)$$

where $J$ is called *mechanical equivalent of heat*. It is clear that if $W$ and $H$ are both measured in the same unit then $J = 1$. If $W$ is measured in joule (work done by a force of 1 N in displacing an object by 1 m in its direction) and $H$ in calorie (heat required to raise the temperature of 1 g of water by 1°C) then $J$ is expressed in joule per calorie. The value of $J$ gives how many joules of mechanical work is needed to raise the temperature of 1 g of water by 1°C. We describe below a laboratory method to measure the mechanical equivalent of heat.

**Searle's Cone Method**

Figure (25.3) shows the apparatus. A conical vessel $B$ just fits in another conical vessel $A$ of the same material. The outer vessel $A$ is connected to a spindle $C$ which may be rotated at high speed by an electric motor or by hand. The number of rotations made in a given time can be recorded.

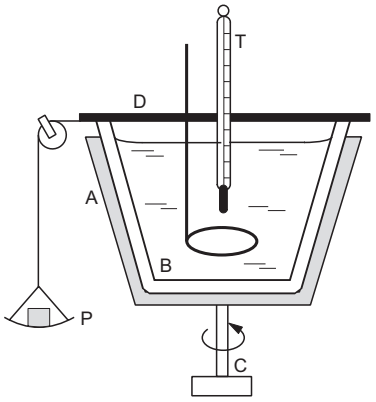



Figure 25.3

The inner vessel $B$ is fitted with a grooved wooden disc $D$. A cord is wound around the groove and is connected to a hanging pan $P$ after passing through a fixed pulley. Weights can be put on this pan.

The wooden disc $D$ contains two holes through which a thermometer and a stirrer can pass into the inner vessel. If the outer vessel is rotated, it tries to drag the inner vessel with it due to the friction between the surfaces of the cones. The friction produces a net torque $\Gamma$ about the central axis. The hanging weights also produce a torque about the central axis which is equal to $Mgr$ where $M$ is the total mass of the pan and the weights on it. If the direction of rotation is properly chosen, these torques may be opposite to each other. Also, the value of $M$ may be adjusted for a given speed of the outer vessel so that $Mgr = \Gamma$. In such a case the inner vessel does not move.

To start the experiment, a measured mass of water is taken in the inner vessel and the thermometer and the stirrer are placed in their positions. The masses of the vessels $A$ and $B$ are also known. The outer vessel $A$ is rotated by rotating the spindle either by a motor or by hand. The direction of rotation is chosen to make sure that the frictional torque and the torque due to the weight $Mg$ oppose each other. The value of $M$ is so adjusted that the inner vessel does not move. The temperature of the water is noted at an initial instant after the adjustments are made. The water is continuously stirred with the help of the stirrer and the temperature is noted at the final instant when it is increased roughly by 5°C. The number of revolutions made by the spindle during this period is noted.

Suppose,

the mass of the water taken $= m_1$

mass of the two vessels taken together $= m_2$

mass of the pan and the weights on it $= M$

initial temperature of water $= \theta_1$

final temperature of water $= \theta_2$

number of revolutions made by the outer vessel $= n$

radius of the disc $D$ $= r$

specific heat capacity of water $= s_1$

specific heat capacity of the material of the vessels $= s_2$.

The torque due to friction = The torque due to the weights $= Mgr$.

Work done by this torque as the outer vessel slides on the inner one $= \Gamma\theta$

$$= Mgr.2\pi n.$$

The heat needed to raise the temperature of water

$$= m_1 s_1(\theta_2 - \theta_1).$$

Heat needed to raise the temperature of the vessels

$$= m_2 s_2(\theta_2 - \theta_1).$$

The total amount of heat needed to raise the temperature is $(m_1 s_1 + m_2 s_2)(\theta_2 - \theta_1)$. Thus, the mechanical work $2\pi n\, Mgr$ produces the same effect as the heat $(m_1 s_1 + m_2 s_2)(\theta_2 - \theta_1)$.

Thus, $2\pi n\, Mgr = J\,(m_1 s_1 + m_2 s_2)(\theta_2 - \theta_1)$

or, $$J = \frac{2\pi n\, Mgr}{(m_1 s_1 + m_2 s_2)(\theta_2 - \theta_1)}.$$

Putting $s_1$, $s_2$ in cal gm$^{-1}$ °C$^{-1}$ (for water $s = 1$ cal g$^{-1}$ °C$^{-1}$), one gets $J$ in joule per calorie. Experiments give a value $J = 4{\cdot}186$ J cal$^{-1}$.

Although heat is the energy in transit due to temperature difference, the word "heat" is also used for the mechanical work that raises the temperature of a body or that which causes a phase change. Thus, if a block slides on a rough surface, its kinetic energy may be used to increase the temperature of the block and the surface. We say that "heat is developed" when the block slides on the surface. Such a use of the word "heat" is made only due to tradition, though it is not strictly correct. It is better to say that thermal energy is produced.

### *Worked Out Examples*

**1.** *Calculate the amount of heat required to convert* 1·00 kg *of ice at* −10°C *into steam at* 100°C *at normal pressure. Specific heat capacity of ice* = 2100 J kg$^{-1}$ K$^{-1}$, *latent heat of fusion of ice* = $3{\cdot}36 \times 10^5$ J kg$^{-1}$, *specific heat capacity of water* = 4200 J kg$^{-1}$ K$^{-1}$ *and latent heat of vaporization of water* = $2{\cdot}25 \times 10^6$ J kg$^{-1}$.

***Solution*** : Heat required to take the ice from −10°C to 0°C

$$= (1\text{ kg})\,(2100\text{ J kg}^{-1}\text{ K}^{-1})\,(10\text{ K}) = 21000\text{ J}.$$

Heat required to melt the ice at 0°C to water

$$= (1\text{ kg})\,(3{\cdot}36 \times 10^5\text{ J kg}^{-1}) = 336000\text{ J}.$$

Heat required to take 1 kg of water from 0°C to 100°C

$$= (1\text{ kg})\,(4200\text{ J kg}^{-1}\text{ K}^{-1})\,(100\text{ K}) = 420000\text{ J}.$$

Heat required to convert 1 kg of water at 100°C into steam

$$= (1\text{ kg})\,(2{\cdot}25 \times 10^6\text{ J kg}^{-1}) = 2{\cdot}25 \times 10^6\text{ J}.$$

Total heat required = $3{\cdot}03 \times 10^6$ J.

**2.** *A* 5 g *piece of ice at* −20°C *is put into* 10 g *of water at* 30°C. *Assuming that heat is exchanged only between the ice and the water, find the final temperature of the mixture. Specific heat capacity of ice* = 2100 J kg$^{-1}$ °C$^{-1}$, *specific heat capacity of water* = 4200 J kg$^{-1}$ °C$^{-1}$ *and latent heat of fusion of ice* = $3{\cdot}36 \times 10^5$ J kg$^{-1}$.

***Solution*** : The heat given by the water when it cools down from 30°C to 0°C is

$$(0{\cdot}01\text{ kg})\,(4200\text{ J kg}^{-1}\text{ °C}^{-1})\,(30\text{°C}) = 1260\text{ J}.$$

The heat required to bring the ice to 0°C is

$$(0{\cdot}005\text{ kg})\,(2100\text{ J kg}^{-1}\text{ °C}^{-1})\,(20\text{°C}) = 210\text{ J}.$$

The heat required to melt 5 g of ice is

$$(0{\cdot}005\text{ kg})\,(3{\cdot}36 \times 10^5\text{ J kg}^{-1}) = 1680\text{ J}.$$

We see that whole of the ice cannot be melted as the required amount of heat is not provided by the water. Also, the heat is enough to bring the ice to 0°C. Thus the final temperature of the mixture is 0°C with some of the ice melted.

**3.** *An aluminium container of mass* 100 g *contains* 200 g *of ice at* −20°C. *Heat is added to the system at a rate of* 100 cal s$^{-1}$. *What is the temperature of the system after* 4 *minutes* ? *Draw a rough sketch showing the variation in the temperature of the system as a function of time. Specific heat capacity of ice* = 0.5 cal g$^{-1}$ °C$^{-1}$, *specific heat capacity of aluminium* = 0·2 cal g$^{-1}$ °C$^{-1}$, *specific heat capacity of water* = 1 cal g$^{-1}$ °C$^{-1}$ *and latent heat of fusion of ice* = 80 cal g$^{-1}$.

***Solution*** **:** Total heat supplied to the system in 4 minutes is $Q = 100 \text{ cal s}^{-1} \times 240 \text{ s} = 2{\cdot}4 \times 10^4 \text{ cal}$.

The heat required to take the system from $-20°C$ to $0°C$

$$= (100 \text{ g}) \times (0{\cdot}2 \text{ cal g}^{-1} \text{ °C}^{-1}) \times (20°C) +$$
$$(200 \text{ g}) \times (0{\cdot}5 \text{ cal g}^{-1} \text{ °C}^{-1}) \times (20°C)$$
$$= 400 \text{ cal} + 2000 \text{ cal} = 2400 \text{ cal}.$$

The time taken in this process $= \dfrac{2400}{100} \text{ s} = 24 \text{ s}$.

The heat required to melt the ice at $0°C$

$$= (200 \text{ g}) (80 \text{ cal g}^{-1}) = 16000 \text{ cal}.$$

The time taken in this process $= \dfrac{16000}{100} \text{ s} = 160 \text{ s}$.

If the final temperature is $\theta$, the heat required to take the system to the final temperature is

$$= (100 \text{ g}) (0{\cdot}2 \text{ cal g}^{-1} \text{ °C}^{-1}) \theta + (200 \text{ g}) (1 \text{ cal g}^{-1} \text{ °C}^{-1}) \theta.$$

Thus,

$$2{\cdot}4 \times 10^4 \text{ cal} = 2400 \text{ cal} + 16000 \text{ cal} + (220 \text{ cal °C}^{-1}) \theta$$

or, $$\theta = \frac{5600 \text{ cal}}{220 \text{ cal °C}^{-1}} = 25{\cdot}5°C.$$

The variation in the temperature as a function of time is sketched in figure (25-W1).

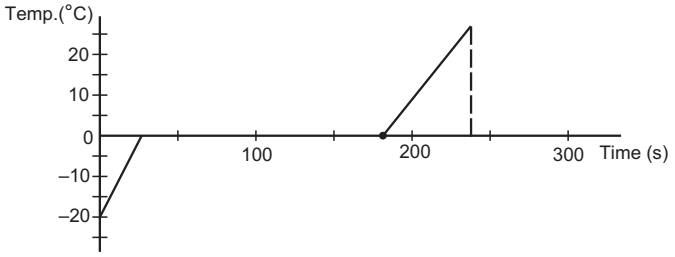



Figure 25-W1

*4. A thermally isolated vessel contains* 100 g *of water at* 0°C. *When air above the water is pumped out, some of the water freezes and some evaporates at* 0°C *itself. Calculate the mass of the ice formed if no water is left in the vessel. Latent heat of vaporization of water at* $0°C = 2{\cdot}10 \times 10^6 \text{ J kg}^{-1}$ *and latent heat of fusion of ice* $= 3{\cdot}36 \times 10^5 \text{ J kg}^{-1}$.

***Solution*** **:** Total mass of the water $= M = 100$ g.

Latent heat of vaporization of water at $0°C$

$$= L_1 = 21{\cdot}0 \times 10^5 \text{ J kg}^{-1}.$$

Latent heat of fusion of ice $= L_2 = 3{\cdot}36 \times 10^5 \text{ J kg}^{-1}$.

Suppose, the mass of the ice formed $= m$.

Then the mass of water evaporated $= M - m$.

Heat taken by the water to evaporate $= (M - m) L_1$

and heat given by the water in freezing $= mL_2$.

Thus, $mL_2 = (M - m)L_1$

or, $$m = \frac{ML_1}{L_1 + L_2}$$

$$= \frac{(100 \text{ g}) (21{\cdot}0 \times 10^5 \text{ J kg}^{-1})}{(21{\cdot}0 + 3{\cdot}36) \times 10^5 \text{ J kg}^{-1}} = 86 \text{ g}.$$

*5. A lead bullet penetrates into a solid object and melts. Assuming that* 50% *of its kinetic energy was used to heat it, calculate the initial speed of the bullet. The initial temperature of the bullet is* 27°C *and its melting point is* 327°C. *Latent heat of fusion of lead* $= 2{\cdot}5 \times 10^4 \text{ J kg}^{-1}$ *and specific heat capacity of lead* $= 125 \text{ J kg}^{1} \text{ K}^{1}$.

***Solution*** **:** Let the mass of the bullet $= m$.

Heat required to take the bullet from $27°C$ to $327°C$

$$= m \times (125 \text{ J kg}^{-1} \text{ K}^{-1}) (300 \text{ K})$$

$$= m \times (3{\cdot}75 \times 10^4 \text{ J kg}^{-1}).$$

Heat required to melt the bullet

$$= m \times (2{\cdot}5 \times 10^4 \text{ J kg}^{-1}).$$

If the initial speed be $v$, the kinetic energy is $\frac{1}{2} mv^2$ and hence the heat developed is $\frac{1}{2}\left(\frac{1}{2} mv^2\right) = \frac{1}{4} mv^2$. Thus,

$$\frac{1}{4} mv^2 = m(3{\cdot}75 + 2{\cdot}5) \times 10^4 \text{ J kg}^{-1}$$

or, $$v = 500 \text{ m s}^{-1}.$$

**6.** *A lead ball at* 30°C *is dropped from a height of* 6·2 km. *The ball is heated due to the air resistance and it completely melts just before reaching the ground. The molten substance falls slowly on the ground. Calculate the latent heat of fusion of lead. Specific heat capacity of lead* $= 126 \text{ J kg}^{-1} \text{ °C}^{-1}$ *and melting point of lead* $= 330°C$. *Assume that any mechanical energy lost is used to heat the ball. Use* $g = 10 \text{ m s}^{-2}$.

***Solution*** **:** The initial gravitational potential energy of the ball

$$= mgh$$
$$= m \times (10 \text{ m s}^{-2}) \times (6{\cdot}2 \times 10^3 \text{ m})$$
$$= m \times (6{\cdot}2 \times 10^4 \text{ m}^2 \text{ s}^{-2}) = m \times (6{\cdot}2 \times 10^4 \text{ J kg}^{-1}).$$

All this energy is used to heat the ball as it reaches the ground with a small velocity. Energy required to take the ball from $30°C$ to $330°C$ is

$$m \times (126 \text{ J kg}^{-1} \text{ °C}^{-1}) \times (300°C)$$
$$= m \times 37800 \text{ J kg}^{-1}$$

and energy required to melt the ball at $330°C$

$$= mL$$

where $L =$ latent heat of fusion of lead.

Thus,

$$m \times (6{\cdot}2 \times 10^4 \text{ J kg}^{-1}) = m \times 37800 \text{ J kg}^{-1} + mL$$

or, $$L = 2{\cdot}4 \times 10^4 \text{ J kg}^{-1}.$$

## QUESTIONS FOR SHORT ANSWER

**1.** Is heat a conserved quantity ?

**2.** The calorie is defined as 1 cal = 4·186 joule. Why not as 1 cal = 4 J to make the conversions easy ?

**3.** A calorimeter is kept in a wooden box to insulate it thermally from the surroundings. Why is it necessary ?

**4.** In a calorimeter, the heat given by the hot object is assumed to be equal to the heat taken by the cold object. Does it mean that heat of the two objects taken together remains constant ?

**5.** In Regnault's apparatus for measuring specific heat capacity of a solid, there is an inlet and an outlet in the steam chamber. The inlet is near the top and the outlet is near the bottom. Why is it better than the opposite choice where the inlet is near the bottom and the outlet is near the top ?

**6.** When a solid melts or a liquid boils, the temperature does not increase even when heat is supplied. Where does the energy go ?

**7.** What is the specific heat capacity of (a) melting ice (b) boiling water ?

**8.** A person's skin is more severely burnt when put in contact with 1 g of steam at 100°C than when put in contact with 1 g of water at 100°C. Explain.

**9.** The atmospheric temperature in the cities on sea-coast change very little. Explain.

**10.** Should a thermometer bulb have large heat capacity or small heat capacity ?

## OBJECTIVE I

**1.** The specific heat capacity of a body depends on
(a) the heat given (b) the temperature raised
(c) the mass of the body (d) the material of the body.

**2.** Water equivalent of a body is measured in
(a) kg (b) calorie (c) kelvin (d) $m^3$.

**3.** When a hot liquid is mixed with a cold liquid, the temperature of the mixture
(a) first decreases then becomes constant
(b) first increases then becomes constant
(c) continuously increases
(d) is undefined for some time and then becomes nearly constant.

**4.** Which of the following pairs represent units of the same physical quantity ?
(a) Kelvin and joule (b) Kelvin and calorie
(c) Newton and calorie (d) Joule and calorie

**5.** Which of the following pairs of physical quantities may be represented in the same unit ?
(a) Heat and temperature (b) Temperature and mole
(c) Heat and work (d) Specific heat and heat

**6.** Two bodies at different temperatures are mixed in a calorimeter. Which of the following quantities remains conserved ?
(a) Sum of the temperatures of the two bodies
(b) Total heat of the two bodies
(c) Total internal energy of the two bodies
(d) Internal energy of each body

**7.** The mechanical equivalent of heat
(a) has the same dimension as heat
(b) has the same dimension as work
(c) has the same dimension as energy
(d) is dimensionless.

## OBJECTIVE II

**1.** The heat capacity of a body depends on
(a) the heat given (b) the temperature raised
(c) the mass of the body (d) the material of the body.

**2.** The ratio of specific heat capacity to molar heat capacity of a body
(a) is a universal constant
(b) depends on the mass of the body
(c) depends on the molecular weight of the body
(d) is dimensionless.

**3.** If heat is supplied to a solid, its temperature
(a) must increase (b) may increase
(c) may remain constant (d) may decrease.

**4.** The temperature of a solid object is observed to be constant during a period. In this period
(a) heat may have been supplied to the body
(b) heat may have been extracted from the body
(c) no heat is supplied to the body
(d) no heat is extracted from the body.

**5.** The temperature of an object is observed to rise in a period. In this period
(a) heat is certainly supplied to it
(b) heat is certainly not supplied to it
(c) heat may have been supplied to it
(d) work may have been done on it.

**6.** Heat and work are equivalent. This means,
(a) when we supply heat to a body we do work on it
(b) when we do work on a body we supply heat to it
(c) the temperature of a body can be increased by doing work on it
(d) a body kept at rest may be set into motion along a line by supplying heat to it.

## EXERCISES

**1.** An aluminium vessel of mass 0·5 kg contains 0·2 kg of water at 20°C. A block of iron of mass 0·2 kg at 100°C is gently put into the water. Find the equilibrium temperature of the mixture. Specific heat capacities of aluminium, iron and water are 910 J kg$^{-1}$ K$^{-1}$, 470 J kg$^{-1}$ K$^{-1}$ and 4200 J kg$^{-1}$ K$^{-1}$ respectively.

**2.** A piece of iron of mass 100 g is kept inside a furnace for a long time and then put in a calorimeter of water equivalent 10 g containing 240 g of water at 20°C. The mixture attains an equilibrium temperature of 60°C. Find the temperature of the furnace. Specific heat capacity of iron = 470 J kg$^{-1}$ °C$^{-1}$.

**3.** The temperatures of equal masses of three different liquids *A*, *B* and *C* are 12°C, 19°C and 28°C respectively. The temperature when *A* and *B* are mixed is 16°C, and when *B* and *C* are mixed, it is 23°C. What will be the temperature when *A* and *C* are mixed ?

**4.** Four 2 cm × 2 cm × 2 cm cubes of ice are taken out from a refrigerator and are put in 200 ml of a drink at 10°C. (a) Find the temperature of the drink when thermal equilibrium is attained in it. (b) If the ice cubes do not melt completely, find the amount melted. Assume that no heat is lost to the outside of the drink and that the container has negligible heat capacity. Density of ice = 900 kg m$^{-3}$, density of the drink = 1000 kg m$^{-3}$, specific heat capacity of the drink = 4200 J kg$^{-1}$ K$^{-1}$, latent heat of fusion of ice = 3·4 × 10$^{5}$ J kg$^{-1}$.

**5.** Indian style of cooling drinking water is to keep it in a pitcher having porous walls. Water comes to the outer surface very slowly and evaporates. Most of the energy needed for evaporation is taken from the water itself and the water is cooled down. Assume that a pitcher contains 10 kg of water and 0·2 g of water comes out per second. Assuming no backward heat transfer from the atmosphere to the water, calculate the time in which the temperature decreases by 5°C. Specific heat capacity of water = 4200 J kg$^{-1}$ °C$^{-1}$ and latent heat of vaporization of water = 2·27 × 10$^{6}$ J kg$^{-1}$.

**6.** A cube of iron (density = 8000 kg m$^{-3}$, specific heat capacity = 470 J kg$^{-1}$ K$^{-1}$) is heated to a high temperature and is placed on a large block of ice at 0°C. The cube melts the ice below it, displaces the water and sinks. In the final equilibrium position, its upper surface just goes inside the ice. Calculate the initial temperature of the cube. Neglect any loss of heat outside the ice and the cube. The density of ice = 900 kg m$^{-3}$ and the latent heat of fusion of ice = 3·36 × 10$^{5}$ J kg$^{-1}$.

**7.** 1 kg of ice at 0°C is mixed with 1 kg of steam at 100°C. What will be the composition of the system when thermal equilibrium is reached ? Latent heat of fusion of ice = 3·36 × 10$^{5}$ J kg$^{-1}$ and latent heat of vaporization of water = 2·26 × 10$^{6}$ J kg$^{-1}$.

**8.** Calculate the time required to heat 20 kg of water from 10°C to 35°C using an immersion heater rated 1000 W. Assume that 80% of the power input is used to heat the water. Specific heat capacity of water = 4200 J kg$^{-1}$ K$^{-1}$.

**9.** On a winter day the temperature of the tap water is 20°C whereas the room temperature is 5°C. Water is stored in a tank of capacity 0·5 m$^{3}$ for household use. If it were possible to use the heat liberated by the water to lift a 10 kg mass vertically, how high can it be lifted as the water comes to the room temperature ? Take $g$ = 10 m s$^{-2}$.

**10.** A bullet of mass 20 g enters into a fixed wooden block with a speed of 40 m s$^{-1}$ and stops in it. Find the change in internal energy during the process.

**11.** A 50 kg man is running at a speed of 18 km h$^{-1}$. If all the kinetic energy of the man can be used to increase the temperature of water from 20°C to 30°C, how much water can be heated with this energy ?

**12.** A brick weighing 4·0 kg is dropped into a 1·0 m deep river from a height of 2·0 m. Assuming that 80% of the gravitational potential energy is finally converted into thermal energy, find this thermal energy in calorie.

**13.** A van of mass 1500 kg travelling at a speed of 54 km h$^{-1}$ is stopped in 10 s. Assuming that all the mechanical energy lost appears as thermal energy in the brake mechanism, find the average rate of production of thermal energy in cal s$^{-1}$.

**14.** A block of mass 100 g slides on a rough horizontal surface. If the speed of the block decreases from 10 m s$^{-1}$ to 5 m s$^{-1}$, find the thermal energy developed in the process.

**15.** Two blocks of masses 10 kg and 20 kg moving at speeds of 10 m s$^{-1}$ and 20 m s$^{-1}$ respectively in opposite directions, approach each other and collide. If the collision is completely inelastic, find the thermal energy developed in the process.

**16.** A ball is dropped on a floor from a height of 2·0 m. After the collision it rises up to a height of 1·5 m. Assume that 40% of the mechanical energy lost goes as thermal energy into the ball. Calculate the rise in the temperature of the ball in the collision. Heat capacity of the ball is 800 J K$^{-1}$.

**17.** A copper cube of mass 200 g slides down on a rough inclined plane of inclination 37° at a constant speed. Assume that any loss in mechanical energy goes into the copper block as thermal energy. Find the increase in the temperature of the block as it slides down through 60 cm. Specific heat capacity of copper = 420 J kg$^{-1}$ K$^{-1}$.

**18.** A metal block of density 6000 kg m$^{-3}$ and mass 1·2 kg is suspended through a spring of spring constant 200 N m$^{-1}$. The spring–block system is dipped in water kept in a vessel. The water has a mass of 260 g and the block is at a height 40 cm above the bottom of the vessel. If the support to the spring is broken, what will be the rise in the temperature of the water. Specific heat capacity of the block is 250 J kg$^{-1}$ K$^{-1}$ and that of water is 4200 J kg$^{-1}$ K$^{-1}$. Heat capacities of the vessel and the spring are negligible.

## ANSWERS

### OBJECTIVE I

1. (d) 2. (a) 3. (d) 4. (d) 5. (c) 6. (c)
7. (d)

### OBJECTIVE II

1. (c), (d) 2. (c) 3. (b), (c)
4. (a), (b) 5. (c), (d) 6. (c)

### EXERCISES

1. 25°C
2. 950°C
3. 20·3°C
4. (a) 0°C (b) 25 g
5. 7·7 min
6. 80°C
7. 665 g steam and 1·335 kg water
8. 44 min
9. 315 km
10. 16 J
11. 15 g
12. 23 cal
13. 4000 cal $s^{-1}$
14. 3·75 J
15. 3000 J
16. $2{\cdot}5 \times 10^{-3}$°C
17. $8{\cdot}6 \times 10^{-3}$°C
18. 0·003°C

□

## CHAPTER 26

# LAWS OF THERMODYNAMICS

### 26.1 THE FIRST LAW OF THERMODYNAMICS

We have seen that heat is just a form of energy. A system can be given energy either by supplying heat to it (by placing it in contact with a hotter object) or by doing mechanical work on it. Consider an ideal gas in a cylindrical container fitted with a piston (figure 26.1). Suppose the piston is fixed in its position and the walls of the cylinder are kept at a temperature higher than that of the gas. The gas molecules strike the wall and rebound. The average kinetic energy of a wall molecule is larger than the average kinetic energy of a gas molecule. Thus, on collision, the gas molecules receive energy from the wall molecules. This increased kinetic energy is shared by other molecules of the gas and in this way the total internal energy of the gas increases.

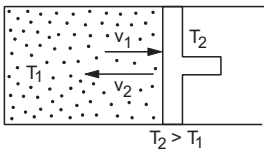



Figure 26.1

Next, consider the same initial situation but now the walls are at the same temperature as the gas. Suppose the piston is pushed slowly to compress the gas. As a gas molecule collides with the piston coming towards it, the speed of the molecule increases on collision (assuming elastic collision, $v_2 = v_1 + 2u$ in figure 26.2). This way the internal energy of the molecules increases as the piston is pushed in.

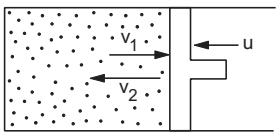



Figure 26.2

We see that the total internal energy of the gas may be increased because of the temperature difference between the walls and the gas (heat transfer) or because of the motion of the piston (work done on the gas).

In a general situation both modes of energy transfer may go together. As an example, consider a gas kept in a cylindrical can fitted with a movable piston. If the can is put on a hot stove, heat is supplied by the hot bottom to the gas and the piston is pushed out to some distance. As the piston moves out, work is done by the gas on it and the gas loses this much amount of energy. Thus, the gas gains energy as heat is supplied to it and it loses energy as work is done by it.

Suppose, in a process, an amount $\Delta Q$ of heat is given to the gas and an amount $\Delta W$ of work is done by it. The total energy of the gas must increase by $\Delta Q - \Delta W$. As a result, the entire gas together with its container may start moving (systematic motion) or the internal energy (random motion of the molecules) of the gas may increase. If the energy does not appear as a systematic motion of the gas then this net energy $\Delta Q - \Delta W$ must go in the form of its internal energy. If we denote the change in internal energy by $\Delta U$, we get

$$\Delta U = \Delta Q - \Delta W$$

or, $$\Delta Q = \Delta U + \Delta W. \quad \ldots (26.1)$$

Equation (26.1) is the statement of the *first law of thermodynamics*. In an ideal monatomic gas, the internal energy of the gas is simply translational kinetic energy of all its molecules. In general, the internal energy may get contributions from the vibrational kinetic energy of molecules, rotational kinetic energy of molecules as well as from the potential energy corresponding to the molecular forces. Equation (26.1) represents a statement of conservation of energy and is applicable to any system, however complicated.

***Example 26.1***

*A gas is contained in a vessel fitted with a movable piston. The container is placed on a hot stove. A total of*

100 cal *of heat is given to the gas and the gas does* 40 J *of work in the expansion resulting from heating. Calculate the increase in internal energy in the process.*

***Solution*** : Heat given to the gas is $\Delta Q = 100 \text{ cal} = 418 \text{ J}$.

Work done by the gas is $\Delta W = 40 \text{ J}$.

The increase in internal energy is

$$\Delta U = \Delta Q - \Delta W$$
$$= 418 \text{ J} - 40 \text{ J} = 378 \text{ J}.$$

First law of thermodynamics may be viewed from different angles. Equation (26.1) tells us that if we take a system from an initial state $i$ to a final state $f$ by several different processes, $\Delta Q - \Delta W$ should be identical in all the processes. This is because $\Delta Q - \Delta W = \Delta U = U_f - U_i$ depends only on the end states $i$ and $f$. Both $\Delta Q$ and $\Delta W$ may be different in different processes, but $\Delta Q - \Delta W$ is the same for all the processes taking the system from $i$ to $f$. Thus, we do not write $\Delta Q = Q_f - Q_i$ or we do not write $\Delta W = W_f - W_i$, but we do write $\Delta U = U_f - U_i$. The first law may be taken as a statement that there exists an internal energy function $U$ that has a fixed value in a given state.

It should be remembered that when work is done *by* the system, $\Delta W$ is positive. If work is done *on* the system, $\Delta W$ is negative. When heat is given *to* the system, $\Delta Q$ is positive. If heat is given *by* the system, $\Delta Q$ is negative. A positive $\Delta W$ decreases the internal energy and a positive $\Delta Q$ increases it.

## 26.2 WORK DONE BY A GAS

Consider a gas contained in a cylinder of cross-sectional area $A$ fitted with a movable piston. Let the pressure of the gas be $p$. The force exerted by the gas on the piston is $pA$ in outward direction. Suppose the gas expands a little and the piston is pushed out by a small distance $\Delta x$. The work done by the gas on the piston is

$$\Delta W = (pA)\,(\Delta x) = p\,\Delta V,$$

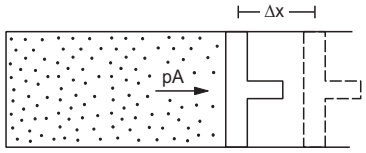



Figure 26.3

where $\Delta V = A\,\Delta x$ is the change in the volume of the gas. For a finite change of volume from $V_1$ to $V_2$, the pressure may not be constant. We can divide the whole process of expansion in small steps and add the work done in each step. Thus, the total work done by the gas in the process is

$$W = \int_{V_1}^{V_2} p\,dV. \qquad \ldots \quad (26.2)$$

If we show the process in a $p$–$V$ diagram, the work done is equal to the area bounded by the $p$–$V$ curve, the $V$-axis and the ordinates $V = V_1$ and $V = V_2$.

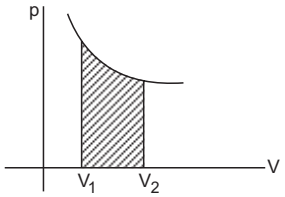



Figure 26.4

Equation (26.2) is derived for a cylindrical vessel only for mathematical simplicity. It is equally true for any shape of the vessel. It is also true for the expansion of solids and liquids or even in phase-changes.

***Example 26.2***

*Calculate the work done by a gas as it is taken from the state a to b, b to c and c to a as shown in figure (26.5).*

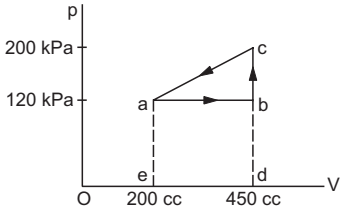



Figure 26.5

***Solution*** : The work done by the gas in the process $a$ to $b$ is the area of *abde*. This is

$$W_{ab} = (120 \text{ kPa})\,(250 \text{ cc})$$
$$= 120 \times 10^{3} \times 250 \times 10^{-6} \text{ J} = 30 \text{ J}.$$

In the process $b$ to $c$ the volume remains constant and the work done is zero.

In the process $c$ to $a$ the gas is compressed. The volume is decreased and the work done by the gas is negative. The magnitude is equal to the area of *caed*. This area is *cab* + *baed*

$$= \frac{1}{2}(80 \text{ kPa})\,(250 \text{ cc}) + 30 \text{ J}$$
$$= 10 \text{ J} + 30 \text{ J} = 40 \text{ J}.$$

Thus, the work done in the process $c$ to $a$ is $-40$ J.

### Work Done in an Isothermal Process on an Ideal Gas

Suppose an ideal gas has initial pressure, volume and temperature as $p_1$, $V_1$ and $T$ respectively. In a process, the temperature is kept constant and its pressure and volume are changed from $p_1$, $V_1$ to $p_2$, $V_2$. The work done by the gas is

$$W = \int_{V_1}^{V_2} p\, dV.$$

As $pV = nRT$, we have $p = \frac{nRT}{V}$.

Thus,
$$W = \int_{V_1}^{V_2} \frac{nRT}{V}\, dV$$
$$= nRT \int_{V_1}^{V_2} \frac{dV}{V}$$
$$= nRT \ln\left(\frac{V_2}{V_1}\right). \quad \ldots (26.3)$$

**Work Done in an Isobaric Process**

Suppose the pressure of a system remains constant at a value $p$ and the volume changes from $V_1$ to $V_2$. The work done by the system is

$$W = \int_{V_1}^{V_2} p\, dV$$
$$= p \int_{V_1}^{V_2} dV = p\,(V_2 - V_1).$$

**Work Done in an Isochoric Process**

In an isochoric process the volume remains constant and no work is done by the system.

## 26.3 HEAT ENGINES

We have seen that when mechanical work is done on a system, its internal energy increases (remember, we assume that the system does not have any systematic motion). The reverse process in which mechanical work is obtained at the expense of internal energy is also possible. Heat engines are devices to perform this task. The basic activity of a heat engine is shown in figure (26.6). It takes some heat from bodies at higher temperature, converts a part of it into the mechanical work and delivers the rest to bodies at lower temperature.

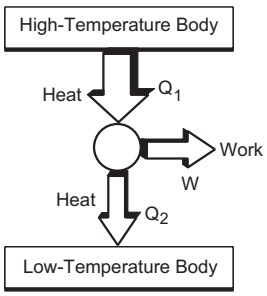



Figure 26.6

The substance inside the engine comes back to the original state. A process in which the final state of a system is the same as its initial state, is called a *cyclic process*. An engine works in cyclic process.

**Efficiency**

Suppose an engine takes an amount $Q_1$ of heat from high-temperature bodies, converts a part $W$ of it into work and rejects an amount $Q_2$ of heat to low-temperature bodies. If the final state of the substance inside the engine is the same as the initial state, there is no change in its internal energy. By first law of thermodynamics, $W = Q_1 - Q_2$.

The efficiency of the engine is defined as

$$\eta = \frac{\text{work done by the engine}}{\text{heat supplied to it}}$$
$$= \frac{W}{Q_1} = \frac{Q_1 - Q_2}{Q_1} = 1 - \frac{Q_2}{Q_1}. \quad \ldots (26.4)$$

We now describe some of the heat engines in use.

**Steam Engine**

A steam engine takes heat from steam and converts a part of it into mechanical motion of a piston which is then used to move heavy objects, such as trains. It is said that James Watt got the idea of steam engine while watching the lid of a kettle being pushed by the steam produced in the kettle in his kitchen. There have been a number of important changes in the design of steam engines, but the essential features remain the same. The main parts of a steam engine are shown in figure (26.7a).

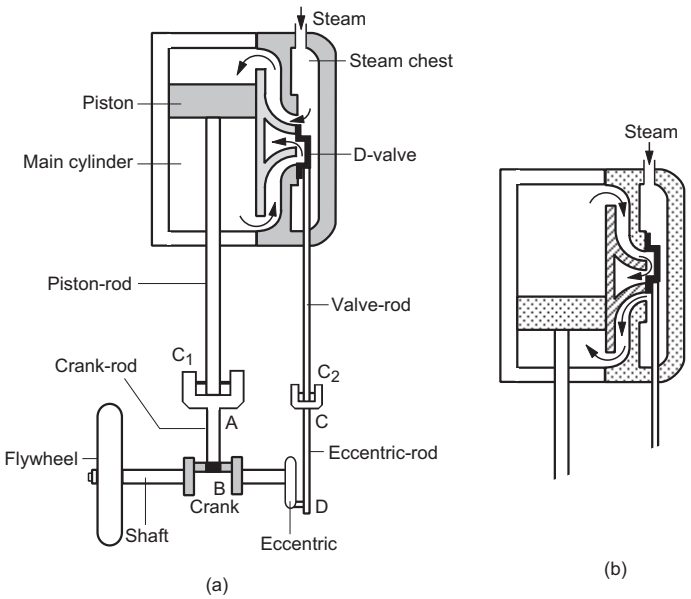



Figure 26.7

**Description**

It consists of a stout cylindrical vessel known as the *main cylinder*. A tight-fitting, movable piston separates the cylinder into two parts: upper part and lower part. The piston is connected to a rod, known as the *piston-rod* which comes out of the main cylinder.

Adjacent to the main cylinder, there is another stout chamber known as the *steam chest*. The steam chest and the cylinder have a common wall in which there are three holes. The upper and the lower holes open in the upper and the lower parts of the cylinder. The middle hole is connected to a *condenser* (not shown in the figure) through a pipe. Condenser is just a vessel in which the steam cools down and condenses.

The steam chest contains a valve of the shape of an open box. It slides on the surface of the common wall between the steam chest and the main cylinder. Together with the wall, the valve forms the shape of "D" and hence it is called a *D-Valve*. Its size is such that at any time it covers either the upper or the lower hole and leaves the other open. The middle hole is always covered. In the situation shown in the figure, the upper hole is open and the middle and the lower holes are covered. The valve is connected to a *valve-rod* which comes out of the steam chest.

A *flywheel* is connected to a *crankshaft*. A *crank* is fixed to this shaft. If a force is applied to the crank in such a way that the force does not intersect the axis of the shaft, a torque acts on the shaft and it rotates. The flywheel and the crank also rotate with it.

A circular disc called *eccentric* is also connected to the shaft. The axis of the disc does not coincide with the axis of the shaft.

The piston-rod is connected to the crank by another rod *AB* known as the *crank-rod*, which is hinged with the piston-rod at the crosshead $C_1$. If the piston moves down, the crosshead $C_1$ also moves down. The crank has to rotate so as to move the end *B* of the crank-rod, farther. Similarly, if the piston-rod moves up, the crank rotates so as to move the end *B* of the crank-rod closer.

Similarly, the valve-rod is connected to the eccentric by another rod *CD* known as the *eccentric-rod*, which is hinged with the valve-rod at the crosshead $C_2$. When the eccentric disc rotates, the corsshead $C_2$ has to move up and down. Correspondingly, the D-valve slides up and down. The arrangement is such that when the piston moves down, the D-valve moves up and when the piston moves up, the D-valve moves down.

**Working**

Water is boiled in a big boiler and the steam so prepared is allowed to go into the steam chest. The flow of steam into the steam chest is controlled by a valve.

In the position shown in figure (26.7a), the piston is near the top end of the cylinder and the D-valve is near its lowest position. Steam is forced into the upper part of the cylinder through the upper hole. The piston is pushed down and the steam in the lower part passes through the lower hole and then through the middle hole to the condenser. As the piston comes down, the crank rotates the shaft and the D-valve slides up. When the piston reaches near the bottom end, the D-valve closes the upper hole and opens the lower hole (figure 26.7b). Thus, steam is forced into the lower part of the cylinder which pushes the piston up. The steam in the upper part passes through the upper hole and then through the middle hole to the condenser. The D-valve slides down. This process is repeated continuously. The piston thus keeps on moving up and down rotating the shaft and the flywheel. The kinetic energy of the piston, flywheel and any part connected to the engine comes from the internal energy of the steam. Thus, a part of the internal energy is converted into mechanical energy.

The efficiency of a steam engine is often measured as the ratio of the mechanical work obtained to the heat that could be produced in burning the fuel (to produce steam in this case). The efficiency of a steam engine is typically of the order of 3 to 10%.

**Internal Combustion Engine**

In a steam engine, there is a separate furnace to boil water. There are engines in which there is no separate furnace and heat is produced in the main cylinder itself. Such engines are called *internal combustion engines*. We will describe two types of internal combustion engines: *petrol engine* and *diesel engine*.

**Petrol Engine**

This engine was designed by Otto in 1876 and hence is also called an *Otto engine*.

In this, petrol from a tank goes to a chamber known as the *carburettor*, in the form of jets. In the carburettor, the petrol is mixed with proper amount of air and the mixture is allowed to go into the main cylinder shown schematically in figure (26.8). The cylinder is made of steel and is fitted with a movable piston just fitting in the cylinder. Two valves $V_1$ and $V_2$ are fixed at the top end of the cylinder. The valve $V_1$ is used for inlet into the cylinder and $V_2$ is used for outlet from the cylinder. Valves are opened

and closed at proper times. The piston is rigidly connected to a piston-rod which is connected to a crankshaft much like that in a steam engine.

A spark plug is placed in the main cylinder and is used to produce electric sparks. These sparks burn the petrol–air mixture.

The working of the engine may be described in four steps known as four strokes. Figure (26.8) is used to explain these strokes.

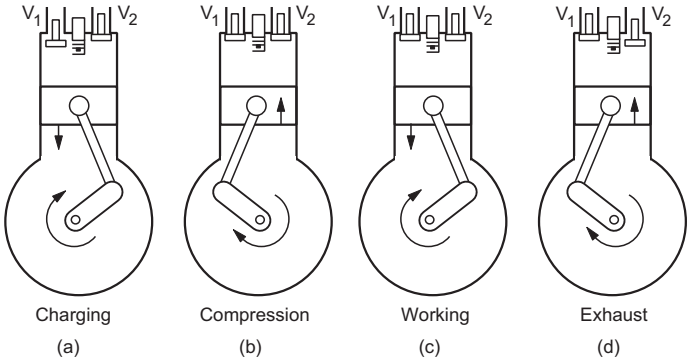



Figure 26.8

**(a) Charging Stroke**

The valve $V_1$ is opened and the mixture of petrol vapour and air enters into the cylinder. The piston goes down.

**(b) Compression Stroke**

Both the inlet and the outlet valves are closed and the piston moves up in the cylinder. This compresses the mixture to a high pressure and the temperature increases to about 500°C.

**(c) Working Stroke**

The spark plug produces a spark at the end of the compression stroke. Both the valves are closed. The fuel mixture ignites. The temperature increases to about 2000°C and the pressure to about 15 atm. The piston is pushed down and this rotates the crankshaft and the flywheel connected to it. This stroke provides a large amount of mechanical energy and, therefore, is called the working stroke.

**(d) Exhaust Stroke**

In this stroke, the valve $V_2$ is opened and the burnt gases are flushed out. The piston moves in and the cycle is completed.

**Diesel Engine**

In a diesel engine, diesel is used as a fuel. Its construction and working is similar to that of a petrol engine.

The main action takes place in a cylinder fitted with an inlet valve, an outlet valve and a valve that allows the fuel to come into the cylinder. No spark plug is used in it. The inlet valve is opened and air is sucked in due to the forward motion of the piston. When the piston moves backward, the inlet valve is closed and the air gets compressed. Due to the compression, the temperature rises to about 1000°C and the pressure to about 36 atm. The fuel valve is opened at this moment and fuel is injected into the cylinder. The fuel ignites readily due to the high temperature. The piston is pushed forward with a great force. This is the working stroke of the engine in which large amount of mechanical energy is obtained. At the end of this stroke the outlet valve opens and the burnt gases are expelled by the backward motion of the piston.

Internal combustion engines have better efficiency than steam engines. They occupy small space and are used with scooters, motorcycles, etc.

## 26.4 THE SECOND LAW OF THERMODYNAMICS

When a body at 100°C is kept in contact with a similar body at 0°C, heat flows from the hotter body to the colder body and both come to 50°C. Is the reverse process possible ? That is, if we put two similar bodies both at 50°C in contact, can heat flow from one body to the other so that one body reaches 0°C and the other 100°C ? A block moving at a speed $v_0$ on a rough table eventually stops and the table and the block warm up. The kinetic energy of the block appears as the internal energy of the table and the block. Can the reverse process be possible ? That is, we heat the block and the table and put the block on the table. Can the bodies cool down and the block start sliding with speed $v_0$ on the table converting the internal energy into kinetic energy ? Consider a container with rigid walls divided in two parts by a partition having a valve. A gas is put in one part and vacuum is created in the other part. The valve is now opened. The gas eventually occupies both the parts of the container. Is the reverse process possible ? We put the gas distributed in both the parts with the valve open. Can the gas go into one part evacuating the other part all by itself ?

The answer to all these questions is NO. Of course, the first law of thermodynamics is not violated in any of these proposed reverse processes. The energy is conserved in the direct process as well as in the reverse process. Still the reverse process is not possible. There must be a law of nature other than the first law which decides, whether a given process, allowed by the first law, will actually take place or not. This law is the second law of thermodynamics. This law may be stated

in various ways. We give here one statement in terms of working of heat engines. We know that a heat engine takes $Q_1$ amount of heat energy from a hot body, converts a part of it into mechanical work and rejects the rest amount $Q_2$ to a cold body. The efficiency of the engine is $1 - Q_2 / Q_1$. The efficiency would be 1, that is, 100% if $Q_2 = 0$. Such an engine would not need any "low-temperature body" to which it needs to reject heat. Hence, it needs only one body at a single temperature, from which it will take heat and convert it completely into mechanical work. This temperature can even be the temperature of the surrounding and hence we will not have to burn any fuel to prepare steam or gases at high temperature to run the engine. A scooter could be run by an engine taking heat from the body of the scooter without needing any petrol. A ship could be run by an engine taking heat from the ocean. However, all attempts to construct such a 100% efficient engine failed. In fact, it is not possible to have such an engine and this is one form of the second law of thermodynamics stated more precisely as follows:

*It is not possible to design a heat engine which works in cyclic process and whose only result is to take heat from a body at a single temperature and convert it completely into mechanical work.*

This statement of the second law is called the *Kelvin–Planck statement.*

One can convert mechanical work completely into heat but one cannot convert heat completely into mechanical work. In this respect, heat and work are not equivalent. We shall now study some other aspects of the second law of thermodynamics.

## 26.5 REVERSIBLE AND IRREVERSIBLE PROCESSES

Consider a sample of an ideal gas kept in an enclosure. The state of the gas is described by specifying its pressure $p$, volume $V$ and temperature $T$. If these parameters can be uniquely specified at a time, we say that the gas is in *thermodynamic equilibrium*. If we put the enclosure on a hot stove, the temperature of various parts of the gas will be different and we will not be able to specify a unique temperature of the gas. The gas is not in thermodynamic equilibrium in such a case.

When we perform a process on a given system, its state is, in general, changed. Suppose the initial state of the system is described by the values $p_1$, $V_1$, $T_1$ and the final state by $p_2$, $V_2$, $T_2$. If the process is performed in such a way that at any instant during the process, the system is very nearly in thermodynamic equilibrium, the process is called *quasi-static*. This means, we can specify the parameters $p$, $V$, $T$ uniquely at any instant during such a process.

Actual processes are not quasi-static. To change the pressure of a gas, we can move a piston inside the enclosure. The gas near the piston is acted upon by greater force as compared to the gas away from the piston. The pressure of the gas may not be uniform everywhere while the piston is moving. However, we can move the piston very slowly to make the process as close to quasi-static as we wish. Thus, a *quasi-static process is an idealised process in which all changes take place infinitely slowly.*

A quasi-static process on a gas can be represented by a curve on a $p$–$V$ diagram (or a $p$–$T$ or a $V$–$T$ diagram). This is because at any instant we have a unique value of $p$ and a unique value of $V$. Suppose the curve in figure (26.9) shows such a quasi-static process taking the system from an initial state $i$ to a final state $f$. Let $AB$ be any arbitrary small part of this process. Suppose in this part the gas takes an amount $\Delta Q$ of heat from its surrounding and performs an amount $\Delta W$ of work on the surrounding. It may be possible to design a reverse quasi-static process which takes the system from the state $f$ to the state $i$ satisfying the following conditions:

(a) the reverse process is represented by the same curve as the direct process, with the arrow inverted,

(b) in the part $BA$, the system gives an amount $\Delta Q$ of heat to the surrounding and an amount $\Delta W$ of work is performed on the system.

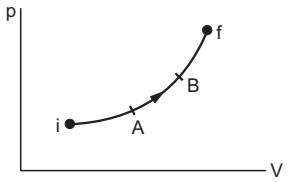



Figure 26.9

If such a reverse process is possible, the original process is called *reversible process*. In the direct process the system has passed through certain equilibrium states in a sequence. When the process is reversed, the system passes through the same states in the reverse sequence. Also, in any small part of the reverse process, it returns the same amount of heat to the surrounding as was taken during the corresponding part in the direct process. Similarly, any work done by the system in the direct process is compensated by the equal work done on the system in the corresponding reverse process.

A process can be reversible if it satisfies two conditions. The process must be quasi-static and it should be nondissipative. This means, friction, viscosity, etc., should be completely absent.

*All processes described in this and the following chapters will be assumed to be reversible unless stated otherwise.*

**Reversible Cycle**

We know that if the state of a system at the end of a process is the same as the state of the system at the beginning, the process is called a cyclic process. If all parts of a cyclic process are reversible, it is called a *reversible cycle*.

## 26.6 ENTROPY

Like pressure, volume, temperature, internal energy, etc., we have another thermodynamic variable of a system, named *entropy*. In a given equilibrium state, the system has a definite value of entropy. If the system has a temperature $T$ (in absolute scale) and a small amount of heat $\Delta Q$ is given to it, we define the change in the entropy of the system as

$$\Delta S = \frac{\Delta Q}{T} \cdot \quad \ldots \quad (26.5)$$

In general, the temperature of the system may change during a process. If the process is reversible, the change in entropy is defined as

$$S_f - S_i = \int_i^f \frac{\Delta Q}{T} \cdot \quad \ldots \quad (26.6)$$

In an adiabatic reversible process, no heat is given to the system. The entropy of the system remains constant in such a process.

Entropy is related to the *disorder* in the system. Thus, if all the molecules in a given sample of a gas are made to move in the same direction with the same velocity, the entropy will be smaller than that in the actual situation in which the molecules move randomly in all directions.

An interesting fact about entropy is that it is not a conserved quantity. More interesting is the fact that entropy can be created but cannot be destroyed. Once some entropy is created in a process, the universe has to carry the burden of that entropy for ever. The second law of thermodynamics may be stated in terms of entropy as follows:

*It is not possible to have a process in which the entropy of an isolated system is decreased.*

## 26.7 CARNOT ENGINE

The French scientist N L Sadi Carnot, in 1824, suggested an idealised engine which we call *Carnot engine* and which has an intimate relation with the second law of thermodynamics. To understand the principle, let us consider an ideal gas taken in a cylinder. The bottom of the cylinder is diathermic whereas rest of it is adiabatic. An adiabatic piston is fitted into the cylinder. Also, suppose we have two large bodies, one at a constant high temperature $T_1$ and the other at a lower temperature $T_2$.

Figure (26.10a) shows the basic process of a Carnot engine on a $p$–$V$ diagram. The other parts of the figure represent the process schematically. Suppose, the cylinder is kept in contact with the high-temperature body at temperature $T_1$ in a compressed state. This state is represented by the point $a$ in the $p$–$V$ diagram. The gas is isothermally expanded to a state $b$ (figure 26.10b). Work is done by the gas and $Q_1$ amount of heat is supplied to it by the body at temperature $T_1$. The cylinder is now kept on an adiabatic platform and the gas is allowed to expand further to the state $c$ (figure 26.10c). This is an adiabatic expansion and the temperature falls from $T_1$ to $T_2$. Work is done by the gas. At this stage, the cylinder is put in contact with the lower temperature body at temperature $T_2$. It is isothermally compressed to a state $d$ (figure 26.10d). Work is done on the gas and the gas rejects an amount $Q_2$ of heat to the body at the lower temperature $T_2$. Finally, it is kept on the adiabatic platform and is further compressed to reach the state $a$ where the temperature is $T_1$ (figure 26.10e).

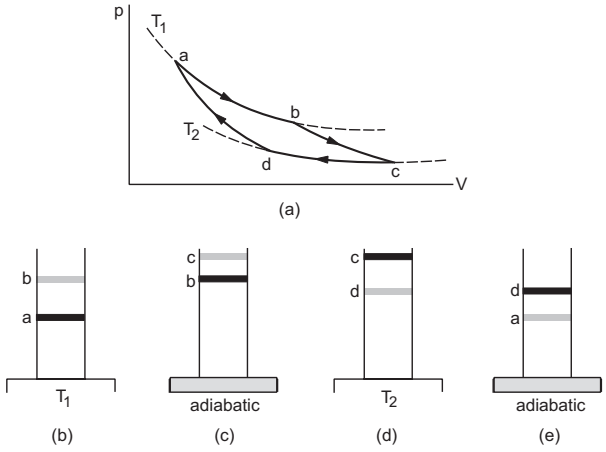



Figure 26.10

The process represented by *abcda* in figure (26.10a) is a cyclic process. If the piston is frictionless and is always moved very slowly, the process is a reversible cyclic process.

**Efficiency of a Carnot Engine**

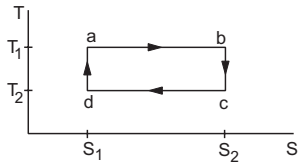



Figure 26.11

The basic process of a Carnot engine, described above, is again shown in figure (26.11) in a $T$–$S$ (temperature–entropy) diagram. The points $a$, $b$, $c$ and

$d$ represent the same states as in figure (26.10a). Let the entropy in state $a$ be $S_1$. An amount $Q_1$ of heat is supplied to the system in the isothermal process $ab$ at the temperature $T_1$. The entropy increases in this part as heat is supplied to the system. Also, by definition,

$$S_2 - S_1 = \frac{Q_1}{T_1}. \quad \ldots \text{(i)}$$

The entropy remains constant in the part $bc$ as it describes an adiabatic process. So the entropy in state $c$ is $S_2$. In the part $cd$, the system gives a heat $Q_2$ at the lower temperature $T_2$ and its entropy is decreased. The part $da$ represents an adiabatic process and the entropy remains constant. As the entropy in state $a$ is $S_1$, the entropy in state $d$ is also $S_1$. Using the definition of change in entropy for the process $cd$,

$$S_1 - S_2 = \frac{-Q_2}{T_2}. \quad \ldots \text{(ii)}$$

From (i) and (ii),

$$\frac{Q_1}{T_1} = \frac{Q_2}{T_2}$$

or,
$$\frac{Q_2}{Q_1} = \frac{T_2}{T_1}.$$

The efficiency of the engine is

$$\eta = \frac{W}{Q_1} = \frac{Q_1 - Q_2}{Q_1}$$

$$= 1 - \frac{Q_2}{Q_1} = 1 - \frac{T_2}{T_1}. \quad \ldots (26.7)$$

Thus, the efficiency of the engine depends only on the temperatures of the hot and cold bodies between which the engine works.

**Carnot's Theorem**

Carnot engine is a reversible engine. It can be proved from the second law of thermodynamics that:

*All reversible engines operating between the same two temperatures have equal efficiency and no engine operating between the same two temperatures can have an efficiency greater than this.*

This theorem is called *Carnot's theorem*. It is a consequence of the second law and puts a theoretical limit $\eta = 1 - \frac{T_2}{T_1}$ to the maximum efficiency of heat engines.

**Refrigerator or Heat Pump**

A heat engine takes heat from a hot body, converts part of it into work and rejects the rest to a cold body. The reverse operation is done by a *refrigerator* also known as a *heat pump*. It takes an amount $Q_2$ of heat from a cold body, an amount $W$ of work is done on it by the surrounding and the total energy $Q_1 = Q_2 + W$ is supplied to a hot body in the form of heat. Thus, heat is passed from the cold body to the hot body. Figure (26.12) shows the process schematically. If the heat is taken at a single low temperature $T_2$, it is rejected at a single high temperature $T_1$ and all the parts of the process are carried out reversibly, we get a *Carnot refrigerator*. If the operating temperatures are fixed, a Carnot refrigerator needs minimum amount of work done to extract a given amount $Q_2$ of heat from the colder body.

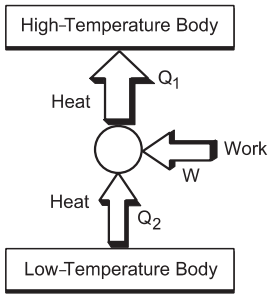



Figure 26.12

In this case,

$$\frac{Q_1}{Q_2} = \frac{T_1}{T_2}$$

or,
$$\frac{Q_2 + W}{Q_2} = \frac{T_1}{T_2}$$

or,
$$W = Q_2\left(\frac{T_1}{T_2} - 1\right).$$

A minimum of this much work has to be done by the surrounding, if we wish to transfer heat $Q_2$ from the low-temperature body to the high-temperature body. This leads to another statement of second law of thermodynamics as follows:

It is not possible to design a refrigerator which works in cyclic process and whose only result is to transfer heat from a body to a hotter body.

This is known as the *Claussius statement* of the second law.

## *Worked Out Examples*

**1.** *A sample of an ideal gas is taken through the cyclic process abca (figure 26-W1). It absorbs* 50 J *of heat during the part ab, no heat during bc and rejects* 70 J *of heat during ca.* 40 J *of work is done on the gas during the part bc. (a) Find the internal energy of the gas at b and c if it is* 1500 J *at a. (b) Calculate the work done by the gas during the part ca.*

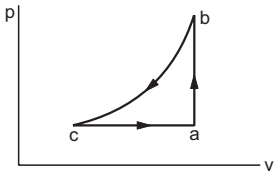



Figure 26-W1

***Solution*** : (a) In the part *ab* the volume remains constant. Thus, the work done by the gas is zero. The heat absorbed by the gas is 50 J. The increase in internal energy from *a* to *b* is

$$\Delta U = \Delta Q = 50 \text{ J}.$$

As the internal energy is 1500 J at *a*, it will be 1550 J at *b*. In the part *bc*, the work done by the gas is $\Delta W = -40$ J and no heat is given to the system. The increase in internal energy from *b* to *c* is

$$\Delta U = -\Delta W = 40 \text{ J}.$$

As the internal energy is 1550 J at *b*, it will be 1590 J at *c*.

(b) The change in internal energy from *c* to *a* is

$$\Delta U = 1500 \text{ J} - 1590 \text{ J} = -90 \text{ J}.$$

The heat given to the system is $\Delta Q = -70$ J.

Using $\Delta Q = \Delta U + \Delta W$,

$$\Delta W = \Delta Q - \Delta U$$
$$= -70 \text{ J} + 90 \text{ J} = 20 \text{ J}.$$

**2.** *A thermodynamic system is taken through the cycle abcda (figure 26-W2). (a) Calculate the work done by the gas during the parts ab, bc, cd and da. (b) Find the total heat rejected by the gas during the process.*

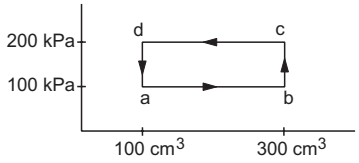



Figure 26-W2

***Solution*** : (a) The work done during the part *ab*,

$$= \int_a^b p\, dV = (100 \text{ kPa}) \int_a^b dV$$
$$= (100 \text{ kPa}) (300 \text{ cm}^3 - 100 \text{ cm}^3)$$
$$= 20 \text{ J}.$$

The work done during *bc* is zero as the volume does not change. The work done during *cd*

$$= \int_c^d p\, dV = (200 \text{ kPa}) (100 \text{ cm}^3 - 300 \text{ cm}^3)$$
$$= -40 \text{ J}.$$

The work done during *da* is zero as the volume does not change.

(b) The total work done by the system during the cycle *abcda*

$$\Delta W = 20 \text{ J} - 40 \text{ J} = -20 \text{ J}.$$

The change in internal energy $\Delta U = 0$ as the initial state is the same as the final state. Thus $\Delta Q = \Delta U + \Delta W = -20$ J. So the system rejects 20 J of heat during the cycle.

**3.** *Calculate the increase in internal energy of* 1 kg *of water at* 100°C *when it is converted into steam at the same temperature and at* 1 atm (100 kPa). *The density of water and steam are* 1000 kg m$^{-3}$ *and* 0·6 kg m$^{-3}$ *respectively. The latent heat of vaporization of water* $= 2{\cdot}25 \times 10^6$ J kg$^{-1}$.

***Solution*** : The volume of 1 kg of water

$$= \frac{1}{1000} \text{ m}^3 \text{ and of 1 kg of steam} = \frac{1}{0{\cdot}6} \text{ m}^3.$$

The increase in volume

$$= \frac{1}{0{\cdot}6} \text{ m}^3 - \frac{1}{1000} \text{ m}^3$$
$$= (1{\cdot}7 - 0{\cdot}001) \text{ m}^3 \approx 1{\cdot}7 \text{ m}^3.$$

The work done by the system is $p\Delta V$

$$= (100 \text{ kPa}) (1{\cdot}7 \text{ m}^3)$$
$$= 1{\cdot}7 \times 10^5 \text{ J}.$$

The heat given to convert 1 kg of water into steam

$$= 2{\cdot}25 \times 10^6 \text{ J}.$$

The change in internal energy is

$$\Delta U = \Delta Q - \Delta W$$
$$= 2{\cdot}25 \times 10^6 \text{ J} - 1{\cdot}7 \times 10^5 \text{ J}$$
$$= 2{\cdot}08 \times 10^6 \text{ J}.$$

**4.** *The internal energy of a monatomic ideal gas is* 1·5 *nRT. One mole of helium is kept in a cylinder of cross section* 8·5 cm$^2$. *The cylinder is closed by a light frictionless piston. The gas is heated slowly in a process during which a total of* 42 J *heat is given to the gas. If the temperature rises through* 2°C, *find the distance moved by the piston. Atmospheric pressure* = 100 kPa.

***Solution*** : The change in internal energy of the gas is

$$\Delta U = 1{\cdot}5\ nR\ (\Delta T)$$
$$= 1{\cdot}5\ (1\ \text{mol})\ (8{\cdot}3\ \text{J K}^{-1}\ \text{mol}^{-1})\ (2\ \text{K})$$
$$= 24{\cdot}9\ \text{J}.$$

The heat given to the gas = 42 J.

The work done by the gas is

$$\Delta W = \Delta Q - \Delta U$$
$$= 42\ \text{J} - 24{\cdot}9\ \text{J} = 17{\cdot}1\ \text{J}.$$

If the distance moved by the piston is $x$, the work done is

$$\Delta W = (100\ \text{kPa})\ (8{\cdot}5\ \text{cm}^2)\ x.$$

Thus,

$$(10^5\ \text{N m}^{-2})\ (8{\cdot}5 \times 10^{-4}\ \text{m}^{-2})\ x = 17{\cdot}1\ \text{J}$$

or, $x = 0{\cdot}2\ \text{m} = 20\ \text{cm}.$

**5.** *A steam engine intakes* 100 g *of steam at* 100°C *per minute and cools it down to* 20°C. *Calculate the heat rejected by the steam engine per minute. Latent heat of vaporization of steam* = 540 cal $g^{-1}$.

***Solution*** : Heat rejected during the condensation of steam in one minute

$$= (100\ \text{g}) \times (540\ \text{cal g}^{-1}) = 5{\cdot}4 \times 10^4\ \text{cal}.$$

Heat rejected during the cooling of water

$$= (100\ \text{g}) \times (1\ \text{cal g}^{-1}\ °\text{C}^{-1})\ (100°\text{C} - 20°\text{C})$$
$$= 8000\ \text{cal}.$$

Thus, heat rejected by the engine per minute

$$= 5{\cdot}4 \times 10^4\ \text{cal} + 0{\cdot}8 \times 10^4\ \text{cal}$$
$$= 6{\cdot}2 \times 10^4\ \text{cal}.$$

**6.** *Figure (26-W3) shows a process ABCA performed on an ideal gas. Find the net heat given to the system during the process.*

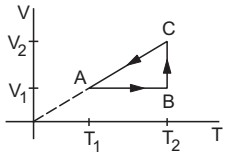



Figure 26-W3

***Solution*** : As the process is cyclic, the change in internal energy is zero. The heat given to the system is then equal to the work done by it.

The work done in part $AB$ is $W_1 = 0$ as the volume remains constant. The part $BC$ represents an isothermal process so that the work done by the gas during this part is

$$W_2 = nR\ T_2 \ln(V_2 / V_1).$$

During the part $CA$,

$$V \propto T.$$

So, $V/T$ is constant and hence,

$$p = \frac{nRT}{V} \text{ is constant.}$$

The work done by the gas during the part $CA$ is

$$W_3 = p\ (V_1 - V_2)$$
$$= nRT_1 - nRT_2$$
$$= -nR\ (T_2 - T_1).$$

The net work done by the gas in the process $ABCA$ is

$$W = W_1 + W_2 + W_3 = nR\ [T_2 \ln\frac{V_2}{V_1} - (T_2 - T_1)].$$

The same amount of heat is given to the gas.

**7.** *Consider the cyclic process ABCA on a sample of* 2·0 mol *of an ideal gas as shown in figure (26-W4). The temperatures of the gas at A and B are* 300 K *and* 500 K *respectively. A total of* 1200 J *heat is withdrawn from the sample in the process. Find the work done by the gas in part BC. Take* $R = 8{\cdot}3\ \text{J K}^{-1}\ \text{mol}^{-1}$.

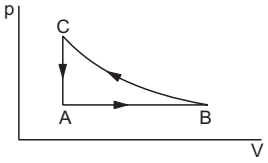



Figure 26-W4

***Solution*** : The change in internal energy during the cyclic process is zero. Hence, the heat supplied to the gas is equal to the work done by it. Hence,

$$W_{AB} + W_{BC} + W_{CA} = -1200\ \text{J}. \quad \ldots \text{(i)}$$

The work done during the process $AB$ is

$$W_{AB} = p_A(V_B - V_A)$$
$$= nR(T_B - T_A)$$
$$= (2{\cdot}0\ \text{mol})\ (8{\cdot}3\ \text{J K}^{-1}\ \text{mol}^{-1})\ (200\ \text{K})$$
$$= 3320\ \text{J}.$$

The work done by the gas during the process $CA$ is zero as the volume remains constant. From (i),

$$3320\ \text{J} + W_{BC} = -1200\ \text{J}$$

or, $W_{BC} = -4520\ \text{J}.$

$$\approx -4500\ \text{J}.$$

**8.** 2·00 mol *of a monatomic ideal gas* ($U = 1{\cdot}5\ nRT$) *is enclosed in an adiabatic, fixed, vertical cylinder fitted with a smooth, light adiabatic piston. The piston is connected to a vertical spring of spring constant* 200 N $m^{-1}$ *as shown in figure (26-W5). The area of cross section of the cylinder is* 20.0 $cm^2$. *Initially, the spring is at its natural length and the temperature of the gas is* 300 K. *The atmospheric pressure is* 100 kPa. *The gas is heated slowly for some time by means of an electric heater so as to move the*

*piston up through* 10 cm. *Find (a) the work done by the gas (b) the final temperature of the gas and (c) the heat supplied by the heater.*

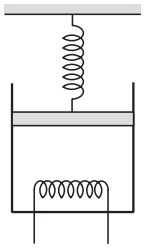


Figure 26-W5

***Solution*** : (a) The force by the gas on the piston is

$$F = p_0 A + kx$$

where $p_0 = 100$ kPa is the atmospheric pressure, $A = 20\ \text{cm}^2$ is the area of cross section, $k = 200\ \text{N m}^{-1}$ is the spring constant and $x$ is the compression of the spring. The work done by the gas as the piston moves through $l = 10$ cm is

$$W = \int_0^l F\,dx$$

$$= p_0 Al + \frac{1}{2} kl^2$$

$$= (100 \times 10^3\ \text{Pa}) \times (20 \times 10^{-4}\ \text{m}^2) \times (10 \times 10^{-2}\ \text{m}) + \frac{1}{2}(200\ \text{N m}^{-1}) \times (100 \times 10^{-4}\ \text{m}^2)$$

$$= 20\ \text{J} + 1\ \text{J} = 21\ \text{J}.$$

(b) The initial temperature is $T_1 = 300$ K. Let the final temperature be $T_2$. We have

$$nRT_1 = p_0 V_0$$

and $$nRT_2 = pV_2 = \left(p_0 + \frac{kl}{A}\right)(V_0 + Al)$$

$$= nRT_1 + p_0 Al + kl^2 + \frac{kl\ nRT_1}{A\,p_0}$$

or, $$T_2 = T_1 + \frac{p_0 Al + kl^2}{nR} + \frac{klT_1}{Ap_0}$$

$$= (300\ \text{K}) + \frac{20\ \text{J} + 2\ \text{J}}{(2{\cdot}0\ \text{mol})\,(8{\cdot}3\ \text{J K}^{-1}\ \text{mol}^{-1})} + \frac{(200\ \text{N m}^{-1}) \times (10 \times 10^{-2}\ \text{m}) \times (300\ \text{K})}{(20 \times 10^{-4}\ \text{m}^2) \times (100 \times 10^3\ \text{Pa})}$$

$$= 300\ \text{K} + 1{\cdot}325\ \text{K} + 30\ \text{K}$$

$$\approx 331\ \text{K}.$$

(c) The internal energy is $U = 1{\cdot}5\ nRT$.

The change in internal energy is

$$\Delta U = 1{\cdot}5\ nR\,\Delta T$$

$$= 1{\cdot}5 \times (2{\cdot}00\ \text{mol}) \times (8{\cdot}3\ \text{J K}^{-1}\ \text{mol}^{-1}) \times (31\ \text{K})$$

$$= 772\ \text{J}.$$

From the first law,

$$\Delta Q = \Delta U + \Delta W$$

$$= 772\ \text{J} + 21\ \text{J} = 793\ \text{J}.$$

**9.** *A sample of an ideal gas has pressure $p_0$, volume $V_0$ and temperature $T_0$. It is isothermally expanded to twice its original volume. It is then compressed at constant pressure to have the original volume $V_0$. Finally, the gas is heated at constant volume to get the original temperature. (a) Show the process in a V–T diagram (b) Calculate the heat absorbed in the process.*

***Solution*** :

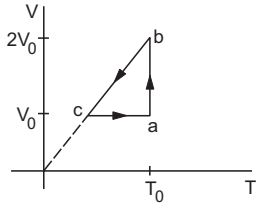



Figure 26-W6

(a) The *V–T* diagram for the process is shown in figure (26-W6). The initial state is represented by the point *a*. In the first step, it is isothermally expanded to a volume $2V_0$. This is shown by *ab*. Then the pressure is kept constant and the gas is compressed to the volume $V_0$. From the ideal gas equation, *V*/*T* is constant at constant pressure. Hence, the process is shown by a line *bc* which passes through the origin. At point *c*, the volume is $V_0$. In the final step, the gas is heated at constant volume to a temperature $T_0$. This is shown by *ca*. The final state is the same as the inital state.

(b) The process is cyclic so that the change in internal energy is zero. The heat supplied is, therefore, equal to the work done by the gas. The work done during *ab* is

$$W_1 = nRT_0 \ln \frac{2V_0}{V_0} = nRT_0 \ln 2 = p_0 V_0 \ln 2.$$

Also from the ideal gas equation,

$$p_a V_a = p_b V_b$$

or, $$p_b = \frac{p_a V_a}{V_b} = \frac{p_0 V_0}{2V_0} = \frac{p_0}{2}.$$

In the step *bc*, the pressure remains constant. Hence, the work done is,

$$W_2 = \frac{p_0}{2}(V_0 - 2V_0) = -\frac{p_0 V_0}{2}.$$

In the step *ca*, the volume remains constant and so the work done is zero. The net work done by the gas in the cyclic process is

$$W = W_1 + W_2$$

$$= p_0 V_0\,[\ln 2 - 0{\cdot}5]$$

$$= 0{\cdot}193\ p_0 V_0.$$

Hence, the heat supplied to the gas is $0{\cdot}193\ p_0 V_0$.

**10.** *A sample of* 100 g *water is slowly heated from* 27°C *to* 87°C. *Calculate the change in the entropy of the water. Specific heat capacity of water* = 4200 J kg$^{-1}$ K$^{-1}$.

***Solution*** : The heat supplied to increase the temperature of the sample from $T$ to $T + \Delta T$ is

$$\Delta Q = ms\,\Delta T,$$

where $m = 100$ g $= 0{\cdot}1$ kg and $C = 4200$ J kg$^{-1}$ K$^{-1}$.

The change in entropy during this process is

$$\Delta S = \frac{\Delta Q}{T} = ms\,\frac{\Delta T}{T}\cdot$$

The total change in entropy as the temperature rises from $T_1$ to $T_2$ is,

$$S_2 - S_1 = \int_{T_1}^{T_2} ms\,\frac{dT}{T}$$

$$= ms \ln \frac{T_2}{T_1}\cdot$$

Putting $T_1 = 27°\text{C} = 300$ K and $T_2 = 87°\text{C} = 360$ K,

$$S_2 - S_1 = (0{\cdot}1 \text{ kg})\,(4200 \text{ J kg}^{-1} \text{ K}^{-1}) \ln \frac{360}{300}$$

$$= 76{\cdot}6 \text{ J K}^{-1}.$$

**11.** *A heat engine operates between a cold reservoir at temperature* $T_2 = 300$ K *and a hot reservoir at temperature* $T_1$. *It takes* 200 J *of heat from the hot reservoir and delivers* 120 J *of heat to the cold reservoir in a cycle. What could be the minimum temperature of the hot reservoir* ?

***Solution*** : The work done by the engine in a cycle is

$$W = 200 \text{ J} - 120 \text{ J} = 80 \text{ J}.$$

The efficiency of the engine is

$$\eta = \frac{W}{Q} = \frac{80 \text{ J}}{200 \text{ J}} = 0{\cdot}40.$$

From Carnot's theorem, no engine can have an efficiency greater than that of a Carnot engine.

Thus, $$0{\cdot}40 \le 1 - \frac{T_2}{T_1} = 1 - \frac{300 \text{ K}}{T_1}$$

or, $$\frac{300 \text{ K}}{T_1} \le 1 - 0{\cdot}40 = 0{\cdot}60$$

or, $$T_1 \ge \frac{300 \text{ K}}{0{\cdot}60}$$

or, $$T_1 \ge 500 \text{ K}.$$

The minimum temperature of the hot reservoir has to be 500 K.

□

## QUESTIONS FOR SHORT ANSWER

**1.** Should the internal energy of a system necessarily increase if heat is added to it ?

**2.** Should the internal energy of a system necessarily increase if its temperature is increased ?

**3.** A cylinder containing a gas is lifted from the first floor to the second floor. What is the amount of work done on the gas ? What is the amount of work done by the gas ? Is the internal energy of the gas increased ? Is the temperature of the gas increased ?

**4.** A force $F$ is applied on a block of mass $M$. The block is displaced through a distance $d$ in the direction of the force. What is the work done by the force on the block ? Does the internal energy change because of this work ?

**5.** The outer surface of a cylinder containing a gas is rubbed vigorously by a polishing machine. The cylinder and its gas become warm. Is the energy transferred to the gas heat or work ?

**6.** When we rub our hands they become warm. Have we supplied heat to the hands ?

**7.** A closed bottle contains some liquid. The bottle is shaken vigorously for 5 minutes. It is found that the temperature of the liquid is increased. Is heat transferred to the liquid ? Is work done on the liquid ? Neglect expansion on heating.

**8.** The final volume of a system is equal to the initial volume in a certain process. Is the work done by the system necessarily zero ? Is it necessarily nonzero ?

**9.** Can work be done by a system without changing its volume ?

**10.** An ideal gas is pumped into a rigid container having diathermic walls so that the temperature remains constant. In a certain time interval, the pressure in the container is doubled. Is the internal energy of the contents of the container also doubled in the interval ?

**11.** When a tyre bursts, the air coming out is cooler than the surrounding air. Explain.

**12.** When we heat an object, it expands. Is work done by the object in this process ? Is heat given to the object equal to the increase in its internal energy ?

**13.** When we stir a liquid vigorously, it becomes warm. Is it a reversible process ?

**14.** What should be the condition for the efficiency of a Carnot engine to be equal to 1 ?

**15.** When an object cools down, heat is withdrawn from it. Does the entropy of the object decrease in this process ? If yes, is it a violation of the second law of thermodynamics stated in terms of increase in entropy ?

## OBJECTIVE I

**1.** The first law of thermodynamics is a statement of
(a) conservation of heat
(b) conservation of work
(c) conservation of momentum
(d) conservation of energy.

**2.** If heat is supplied to an ideal gas in an isothermal process,
(a) the internal energy of the gas will increase
(b) the gas will do positive work
(c) the gas will do negative work
(d) the said process is not possible.

**3.** Figure (26-Q1) shows two processes $A$ and $B$ on a system. Let $\Delta Q_1$ and $\Delta Q_2$ be the heat given to the system in processes $A$ and $B$ respectively. Then
(a) $\Delta Q_1 > \Delta Q_2$ (b) $\Delta Q_1 = \Delta Q_2$ (c) $\Delta Q_1 < \Delta Q_2$ (d) $\Delta Q_1 \leq \Delta Q_2$.

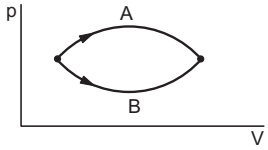



Figure 26-Q1

**4.** Refer to figure (26-Q1). Let $\Delta U_1$ and $\Delta U_2$ be the changes in internal energy of the system in the processes $A$ and $B$. Then
(a) $\Delta U_1 > \Delta U_2$ (b) $\Delta U_1 = \Delta U_2$
(c) $\Delta U_1 < \Delta U_2$ (d) $\Delta U_1 \neq \Delta U_2$.

**5.** Consider the process on a system shown in figure (26-Q2). During the process, the work done by the system
(a) continuously increases
(b) continuously decreases
(c) first increases then decreases
(d) first decreases then increases.

Figure 26-Q2

**6.** Consider the following two statements.
(A) If heat is added to a system, its temperature must increase.
(B) If positive work is done by a system in a thermodynamic process, its volume must increase.
(a) Both A and B are correct.
(b) A is correct but B is wrong.
(c) B is correct but A is wrong.
(d) Both A and B are wrong.

**7.** An ideal gas goes from the state $i$ to the state $f$ as shown in figure (26-Q3). The work done by the gas during the process
(a) is positive (b) is negative (c) is zero
(d) cannot be obtained from this information.

Figure 26-Q3

**8.** Consider two processes on a system as shown in figure (26-Q4).

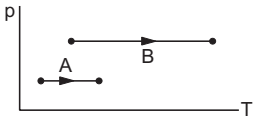



Figure 26-Q4

The volumes in the initial states are the same in the two processes and the volumes in the final states are also the same. Let $\Delta W_1$ and $\Delta W_2$ be the work done by the system in the processes $A$ and $B$ respectively.
(a) $\Delta W_1 > \Delta W_2$. (b) $\Delta W_1 = \Delta W_2$. (c) $\Delta W_1 < \Delta W_2$.
(d) Nothing can be said about the relation between $\Delta W_1$ and $\Delta W_2$.

**9.** A gas is contained in a metallic cylinder fitted with a piston. The piston is suddenly moved in to compress the gas and is maintained at this position. As time passes the pressure of the gas in the cylinder
(a) increases (b) decreases
(c) remains constant
(d) increases or decreases depending on the nature of the gas.

## OBJECTIVE II

**1.** The pressure $p$ and volume $V$ of an ideal gas both increase in a process.
(a) Such a process is not possible.
(b) The work done by the system is positive.
(c) The temperature of the system must increase.
(d) Heat supplied to the gas is equal to the change in internal energy.

**2.** In a process on a system, the initial pressure and volume are equal to the final pressure and volume.
(a) The initial temperature must be equal to the final temperature.

(b) The initial internal energy must be equal to the final internal energy.
(c) The net heat given to the system in the process must be zero.
(d) The net work done by the system in the process must be zero.

**3.** A system can be taken from the initial state $p_1, V_1$ to the final state $p_2, V_2$ by two different methods. Let $\Delta Q$ and $\Delta W$ represent the heat given to the system and the work done by the system. Which of the following must be the same in both the methods ?
(a) $\Delta Q$ (b) $\Delta W$ (c) $\Delta Q + \Delta W$ (d) $\Delta Q - \Delta W$.

**4.** Refer to figure (26-Q5). Let $\Delta U_1$ and $\Delta U_2$ be the change in internal energy in processes $A$ and $B$ respectively, $\Delta Q$ be the net heat given to the system in process $A + B$ and $\Delta W$ be the net work done by the system in the process $A + B$.

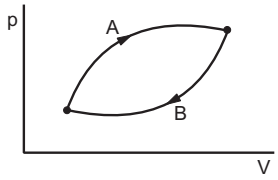



Figure 26-Q5

(a) $\Delta U_1 + \Delta U_2 = 0$. (b) $\Delta U_1 - \Delta U_2 = 0$.
(c) $\Delta Q - \Delta W = 0$. (d) $\Delta Q + \Delta W = 0$.

**5.** The internal energy of an ideal gas decreases by the same amount as the work done by the system.
(a) The process must be adiabatic.
(b) The process must be isothermal.
(c) The process must be isobaric.
(d) The temperature must decrease.

## EXERCISES

**1.** A thermally insulated, closed copper vessel contains water at 15°C. When the vessel is shaken vigorously for 15 minutes, the temperature rises to 17°C. The mass of the vessel is 100 g and that of the water is 200 g. The specific heat capacities of copper and water are 420 J kg$^{-1}$ K$^{-1}$ and 4200 J kg$^{-1}$ K$^{-1}$ respectively. Neglect any thermal expansion. (a) How much heat is transferred to the liquid–vessel system ? (b) How much work has been done on this system ? (c) How much is the increase in internal energy of the system ?

**2.** Figure (26-E1) shows a paddle wheel coupled to a mass of 12 kg through fixed frictionless pulleys. The paddle is immersed in a liquid of heat capacity 4200 J K$^{-1}$ kept in an adiabatic container. Consider a time interval in which the 12 kg block falls slowly through 70 cm. (a) How much heat is given to the liquid ? (b) How much work is done on the liquid ? (c) Calculate the rise in the temperature of the liquid neglecting the heat capacity of the container and the paddle.

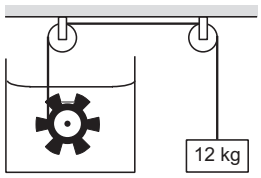



Figure 26-E1

**3.** A 100 kg block is started with a speed of 2·0 m s$^{-1}$ on a long, rough belt kept fixed in a horizontal position. The coefficient of kinetic friction between the block and the belt is 0·20. (a) Calculate the change in the internal energy of the block–belt system as the block comes to a stop on the belt. (b) Consider the situation from a frame of reference moving at 2·0 m s$^{-1}$ along the initial velocity of the block. As seen from this frame, the block is gently put on a moving belt and in due time the block starts moving with the belt at 2·0 m s$^{-1}$. Calculate the increase in the kinetic energy of the block as it stops slipping past the belt. (c) Find the work done in this frame by the external force holding the belt.

**4.** Calculate the change in internal energy of a gas kept in a rigid container when 100 J of heat is supplied to it.

**5.** The pressure of a gas changes linearly with volume from 10 kPa, 200 cc to 50 kPa, 50 cc. (a) Calculate the work done by the gas. (b) If no heat is supplied or extracted from the gas, what is the change in the internal energy of the gas ?

**6.** An ideal gas is taken from an initial state $i$ to a final state $f$ in such a way that the ratio of the pressure to the absolute temperature remains constant. What will be the work done by the gas ?

**7.** Figure (26-E2) shows three paths through which a gas can be taken from the state $A$ to the state $B$. Calculate the work done by the gas in each of the three paths.

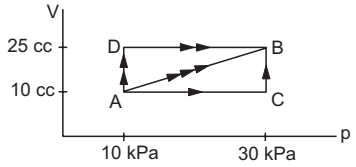



Figure 26-E2

**8.** When a system is taken through the process $abc$ shown in figure (26-E3), 80 J of heat is absorbed by the system and 30 J of work is done by it. If the system does 10 J

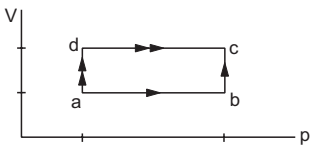



Figure 26-E3

of work during the process *adc*, how much heat flows into it during the process ?

**9.** 50 cal of heat should be supplied to take a system from the state *A* to the state *B* through the path *ACB* as shown in figure (26-E4). Find the quantity of heat to be supplied to take it from *A* to *B* via *ADB*.

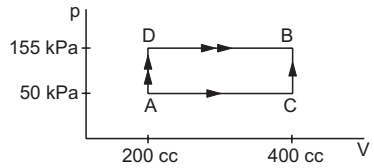



Figure 26-E4

**10.** Calculate the heat absorbed by a system in going through the cyclic process shown in figure (26-E5).

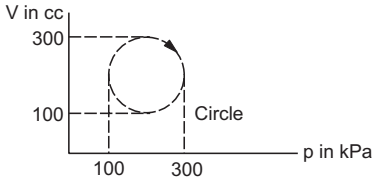



Figure 26-E5

**11.** A gas is taken through a cyclic process *ABCA* as shown in figure (26-E6). If 2·4 cal of heat is given in the process, what is the value of *J* ?

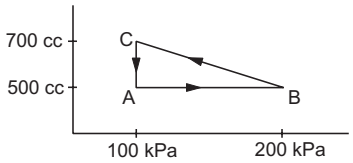



Figure 26-E6

**12.** A substance is taken through the process *abc* as shown in figure (26-E7). If the internal energy of the substance increases by 5000 J and a heat of 2625 cal is given to the system, calculate the value of *J*.

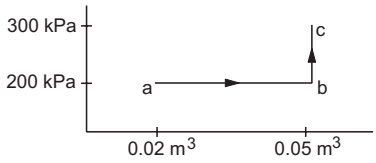



Figure 26-E7

**13.** A gas is taken along the path *AB* as shown in figure (26-E8). If 70 cal of heat is extracted from the gas in the process, calculate the change in the internal energy of the system.

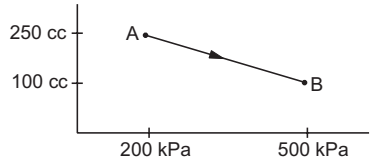



Figure 26-E8

**14.** The internal energy of a gas is given by $U = 1{\cdot}5\,pV$. It expands from 100 $cm^3$ to 200 $cm^3$ against a constant pressure of $1{\cdot}0 \times 10^5$ Pa. Calculate the heat absorbed by the gas in the process.

**15.** A gas is enclosed in a cylindrical vessel fitted with a frictionless piston. The gas is slowly heated for some time. During the process, 10 J of heat is supplied and the piston is found to move out 10 cm. Find the increase in the internal energy of the gas. The area of cross section of the cylinder = 4 $cm^2$ and the atmospheric pressure = 100 kPa.

**16.** A gas is initially at a pressure of 100 kPa and its volume is 2·0 $m^3$. Its pressure is kept constant and the volume is changed from 2·0 $m^3$ to 2·5 $m^3$. Its volume is now kept constant and the pressure is increased from 100 kPa to 200 kPa. The gas is brought back to its initial state, the pressure varying linearly with its volume. (a) Whether the heat is supplied to or extracted from the gas in the complete cycle ? (b) How much heat was supplied or extracted ?

**17.** Consider the cyclic process *ABCA*, shown in figure (26-E9), performed on a sample of 2·0 mol of an ideal gas. A total of 1200 J of heat is withdrawn from the sample in the process. Find the work done by the gas during the part *BC*.

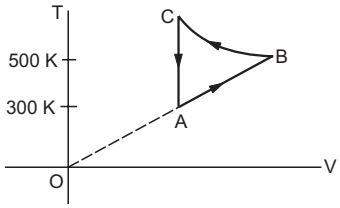



Figure 26-E9

**18.** Figure (26-E10) shows the variation in the internal energy *U* with the volume *V* of 2·0 mol of an ideal gas in a cyclic process *abcda*. The temperatures of the gas at *b* and *c* are 500 K and 300 K respectively. Calculate the heat absorbed by the gas during the process.

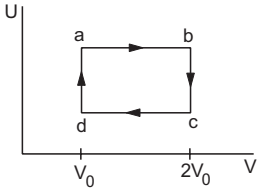



Figure 26-E10

**19.** Find the change in the internal energy of 2 kg of water as it is heated from 0°C to 4°C. The specific heat capacity of water is 4200 J $kg^{-1}$ $K^{-1}$ and its densities at 0°C and 4°C are 999·9 kg $m^{-3}$ and 1000 kg $m^{-3}$ respectively. Atmospheric pressure = $10^5$ Pa.

**20.** Calculate the increase in the internal energy of 10 g of water when it is heated from 0°C to 100°C and converted

into steam at 100 kPa. The density of steam = 0·6 kg $m^{-3}$. Specific heat capacity of water = 4200 J $kg^{-1}$ $°C^{-1}$ and the latent heat of vaporization of water = $2.25 \times 10^6$ J $kg^{-1}$.

**21.** Figure (26-E11) shows a cylindrical tube of volume $V$ with adiabatic walls containing an ideal gas. The internal energy of this ideal gas is given by $1{\cdot}5\,nRT$. The tube is divided into two equal parts by a fixed diathermic wall. Initially, the pressure and the temperature are $p_1$, $T_1$ on the left and $p_2$, $T_2$ on the right. The system is left for sufficient time so that the temperature becomes equal on the two sides. (a) How much work has been done by the gas on the left part ? (b) Find the final pressures on the two sides. (c) Find the final equilibrium temperature. (d) How much heat has flown from the gas on the right to the gas on the left ?

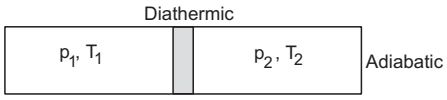



Figure 26-E11

**22.** An adiabatic vessel of total volume $V$ is divided into two equal parts by a conducting separator. The separator is fixed in this position. The part on the left contains one mole of an ideal gas ($U = 1{\cdot}5\,nRT$) and the part on the right contains two moles of the same gas. Initially, the pressure on each side is $p$. The system is left for sufficient time so that a steady state is reached. Find (a) the work done by the gas in the left part during the process, (b) the temperature on the two sides in the beginning, (c) the final common temperature reached by the gases, (d) the heat given to the gas in the right part and (e) the increase in the internal energy of the gas in the left part.

□

## ANSWERS

### OBJECTIVE I

1. (d) 2. (b) 3. (a) 4. (b) 5. (a) 6. (c)
7. (c) 8. (c) 9. (b)

### OBJECTIVE II

1. (b), (c) 2. (a), (b) 3. (d) 4. (a), (c)
5. (a), (d)

### EXERCISES

1. (a) zero (b) 1764 J (c) 1764 J
2. (a) zero (b) 84 J (c) 0·02°C
3. (a) 200 J (b) 200 J (c) 400 J
4. 100 J
5. (a) –4·5 J (b) 4·5 J
6. zero
7. 0·30 J in *AB*, 0·450 J in *ACB* and 0·150 J in *ADB*
8. 60 J
9. 55 cal
10. 31·4 J
11. 4·17 J $cal^{-1}$
12. 4·19 J $cal^{-1}$
13. –241 J
14. 25 J
15. 6 J
16. (a) extracted (b) 25000 J
17. –4520 J
18. 2300 J
19. (33600 – 0·02) J
20. $2{\cdot}5 \times 10^4$ J
21. (a) zero
    (b) $\frac{p_1T_2\,(p_1+p_2)}{\lambda}$ on the left and $\frac{p_2T_1\,(p_1+p_2)}{\lambda}$ on the right
    (c) $\frac{T_1T_2\,(p_1+p_2)}{\lambda}$
    (d) $\frac{3p_1p_2\,(T_2-T_1)\,V}{4\lambda}$ where $\lambda = p_1T_2 + p_2T_1$
22. (a) zero (b) $\frac{pV}{(2\text{ mol})\,R}$, $\frac{pV}{(4\text{ mol})\,R}$
    (c) $\frac{pV}{(3\text{ mol})\,R}$ (d) $\frac{pV}{4}$ (e) $\frac{-pV}{4}$

□

## CHAPTER 27

# SPECIFIC HEAT CAPACITIES OF GASES

### 27.1 TWO KINDS OF SPECIFIC HEAT CAPACITIES OF GASES

The specific heat capacity of a substance is defined as the heat supplied per unit mass of the substance per unit rise in the temperature. If an amount $\Delta Q$ of heat is given to a mass $m$ of the substance and its temperature rises by $\Delta T$, the specific heat capacity $s$ is given by the equation

$$s = \frac{\Delta Q}{m\,\Delta T}. \qquad \ldots \quad (27.1)$$

This definition applies to any mode of the substance, solid, liquid or gas.

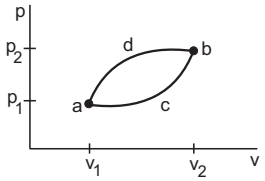



Figure 27.1

Consider a mass $m$ of a gas at pressure $p_1$, volume $V_1$ and temperature $T_1$. We show this state by the point $a$ in the $p$–$V$ diagram of figure (27.1). The point $b$ represents another state in which the pressure is $p_2$, the volume is $V_2$ and the temperature is $T_2$. The change in temperature as the system is taken from $a$ to $b$ is $\Delta T = T_2 - T_1$. The internal energy of the gas at $a$ is $U_1$ and at $b$ it is $U_2$. The change in internal energy as the system is taken from $a$ to $b$ is $\Delta U = U_2 - U_1$. If the work done by the gas in taking it from $a$ to $b$ is $\Delta W$ and the heat supplied is $\Delta Q$, from the first law of thermodynamics,

$$\Delta Q = \Delta U + \Delta W.$$

Now, the work done $\Delta W$ depends on the process by which the gas is taken from $a$ to $b$. For example, the work done is smaller if the gas is taken through the process $acb$ and is larger if the process is $adb$. Accordingly, $\Delta Q$ is smaller in the process $acb$ and is larger in the process $adb$. In both these processes, the change in temperature $\Delta T$ is the same. So the heat $\Delta Q$ given to raise the temperature of a gas by $\Delta T$, depends on the process involved. From equation (27.1), the specific heat capacity $s$ also depends on the process. Thus, to define the specific heat capacity of a gas the process should also be specified.

Suppose, the volume of a gas of mass $m$ is kept constant and heat $\Delta Q$ is given to it. If its temperature rises by $\Delta T$, the specific heat capacity given by equation (27.1) is called the *specific heat capacity at constant volume* and is denoted by the symbol $s_V$. Thus,

$$s_V = \left(\frac{\Delta Q}{m\,\Delta T}\right)_{\text{constant volume}}. \qquad \ldots \quad (27.2)$$

Next suppose, the pressure of a gas of mass $m$ is kept constant and heat $\Delta Q$ is given to it. If the temperature rises by $\Delta T$, the specific heat capacity given by equation (27.1) is called the *specific heat capacity at constant pressure* and is denoted by the symbol $s_p$. Thus,

$$s_p = \left(\frac{\Delta Q}{m\,\Delta T}\right)_{\text{constant pressure}}. \qquad \ldots \quad (27.3)$$

There can be many more processes, but these two are more important and correspondingly two specific heat capacities are defined for gases.

The above discussion is also valid for solids and liquids. However, in these systems the expansion is quite small and hence the work done in a process is small. Thus, the specific heat capacity depends only slightly on the process and the process is generally not mentioned.

The molar heat capacities of a gas are defined as the heat given per mole of the gas per unit rise in the temperature. The *molar heat capacity at constant volume*, denoted by $C_V$, is

$$C_V = \left(\frac{\Delta Q}{n\,\Delta T}\right)_{\text{constant volume}} \qquad \ldots \quad (27.4)$$

and the *molar heat capacity at constant pressure*, denoted by $C_p$, is

$$C_p = \left(\frac{\Delta Q}{n\,\Delta T}\right)_{\text{constant pressure}} \quad \ldots \quad (27.5)$$

where $n$ is the amount of the gas in number of moles. Quite often, the term *specific heat capacity* or *specific heat* is used for molar heat capacity. It is advised that the unit be carefully noted to determine the actual meaning. The unit of specific heat capacity is J kg$^{-1}$ K$^{-1}$ whereas that of molar heat capacity is J K$^{-1}$ mol$^{-1}$.

***Example 27.1***

*0·32 g of oxygen is kept in a rigid container and is heated. Find the amount of heat needed to raise the temperature from 25°C to 35°C. The molar heat capacity of oxygen at constant volume is 20 J K$^{-1}$ mol$^{-1}$.*

***Solution*** : The molecular weight of oxygen = 32 g mol$^{-1}$.

The amount of the gas in moles is

$$n = \frac{0{\cdot}32\text{ g}}{32\text{ g mol}^{-1}} = 0{\cdot}01\text{ mol}.$$

The amount of heat needed is $Q = nC_V\,\Delta T$

$$= (0{\cdot}01\text{ mol})\,(20\text{ J K}^{-1}\text{ mol}^{-1})\,(10\text{ K}) = 2{\cdot}0\text{ J}.$$

***Example 27.2***

*A tank of volume 0·2 m$^3$ contains helium gas at a temperature of 300 K and pressure 1·0 × 10$^5$ N m$^{-2}$. Find the amount of heat required to raise the temperature to 400 K. The molar heat capacity of helium at constant volume is 3·0 cal K$^{-1}$ mol$^{-1}$. Neglect any expansion in the volume of the tank.*

***Solution*** : The amount of the gas in moles is

$$n = \frac{pV}{RT}$$

$$= \frac{(1{\cdot}0 \times 10^5\text{ N m}^{-2})\,(0{\cdot}2\text{ m}^3)}{(8{\cdot}31\text{ J K}^{-1}\text{ mol}^{-1})\,(300\text{ K})} = 8{\cdot}0\text{ mol}.$$

The amount of heat required is

$$\Delta Q = nC_V\,\Delta T$$

$$= (8{\cdot}0\text{ mol})\,(3{\cdot}0\text{ cal mol}^{-1}\text{ K}^{-1})\,(100\text{ K}) = 2400\text{ cal}.$$

## 27.2 RELATION BETWEEN $C_p$ AND $C_V$ FOR AN IDEAL GAS

Suppose, the volume of a gas is kept constant and heat is supplied to it. The work done by the gas is zero. The entire heat supplied goes as internal energy of the gas and is used to increase the temperature. Now consider a process in which the pressure is kept constant. If heat is supplied to the gas, its volume increases. A part of the heat supplied is used by the gas to do work in the expansion. Only the remaining part goes as increase in the internal energy which increases the temperature. Thus, for a given amount $\Delta Q$ of heat, the rise in temperature of a gas at constant pressure is smaller than the rise in temperature at constant volume. Thus, $C_p > C_V$.

Consider an amount $n$ (in moles) of an ideal gas kept in a rigid container at an initial pressure $p$, volume $V$ and temperature $T$. An amount $(dQ)_V$ of heat is supplied to the gas. Its temperature rises from $T$ to $T + dT$ whereas the volume remains constant. The work done by the gas is $dW = 0$. From the first law of thermodynamics, the internal energy changes by $dU$ such that

$$(dQ)_V = dU. \quad \ldots \quad \text{(i)}$$

Now suppose, the same sample of the gas is taken in a vessel with a movable light piston of cross-sectional area $A$ (figure 27.2). The initial pressure, volume and temperature are $p$, $V$ and $T$. The piston is pushed by a constant external force $F = pA$. The pressure inside the gas thus remains constant at the value $p$.

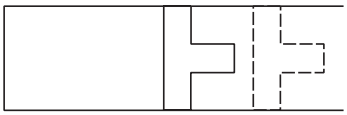


Figure 27.2

The gas is given heat $(dQ)_p$ which raises the temperature by the same amount $dT$. The piston moves out so that the volume increases by $dV$. The work done by the gas is $p\,dV$ and from the first law of thermodynamics,

$$(dQ)_p = dU' + p\,dV. \quad \ldots \quad \text{(ii)}$$

For an ideal gas, $pV = nRT$.

As the pressure remains constant,

$$p(V + dV) = nR(T + dT)$$

and hence,

$$p\,dV = nR\,dT.$$

From (ii),

$$(dQ)_p = dU' + nR\,dT. \quad \ldots \quad \text{(iii)}$$

As the temperature rises by the same amount in the two cases and the internal energy of an ideal gas depends only on its temperature,

$$dU = dU'.$$

Thus, from (i) and (iii),

$$(dQ)_p = (dQ)_V + nR\,dT$$

or,
$$\frac{1}{n}\frac{(dQ)_p}{dT} = \frac{1}{n}\frac{(dQ)_V}{dT} + R. \quad \ldots \quad \text{(iv)}$$

But $(dQ)_p$ is the heat given to increase the temperature of the gas by $dT$ at constant pressure. Thus, by definition,

$$C_p = \frac{1}{n}\frac{(dQ)_p}{dT}.$$

Similarly, $$C_V = \frac{1}{n}\frac{(dQ)_V}{dT}$$

and (iv) becomes

$$C_p = C_V + R$$

or, $$C_p - C_V = R. \quad \ldots (27.6)$$

**Relations of $C_V$ with energy**

From (i), $$(dQ)_V = dU$$

or, $$\frac{1}{n}\frac{(dQ)_V}{dT} = \frac{1}{n}\frac{dU}{dT}$$

or, $$C_V = \frac{1}{n}\frac{dU}{dT} \quad \ldots (27.7)$$

or, $$dU = nC_V\,dT.$$

Taking the energy to be zero at $T = 0$,

$$U = nC_V T. \quad \ldots (27.8)$$

***Example 27.3***

*The molar heat capacity of a gas at constant volume is found to be* 5 cal mol$^{-1}$ K$^{-1}$. *Find the ratio* $\gamma = C_p/C_V$ *for the gas. The gas constant* $R = 2$ cal mol$^{-1}$ K$^{-1}$.

***Solution*** : We have $C_V = 5$ cal mol$^{-1}$ K$^{-1}$.

Thus, $C_p = C_V + R = 5$ cal mol$^{-1}$ K$^{-1}$ + 2 cal mol$^{-1}$ K$^{-1}$

$= 7$ cal mol$^{-1}$ K$^{-1}$

or, $$\frac{C_p}{C_V} = \frac{7}{5} = 1{\cdot}4.$$

## 27.3 DETERMINATION OF $C_p$ OF A GAS

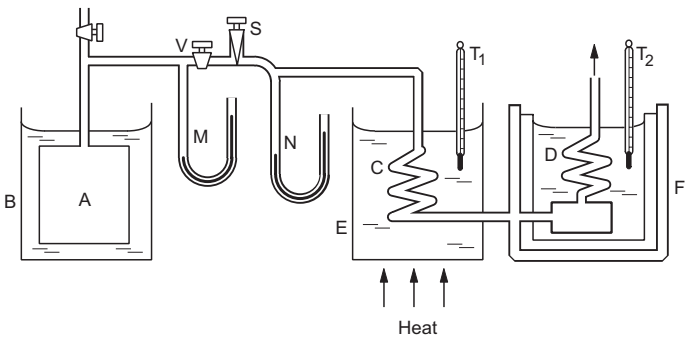



Figure 27.3

Figure (27.3) shows a schematic diagram of the Regnault's apparatus to measure $C_p$ of a gas. The experimental gas is taken at a high pressure in a large tank $A$ immersed in water at a constant temperature. The tank is connected through a tube to two copper coils $C$ and $D$. A valve $V$ in the tube opens or closes the path for the gas to flow. The rate of flow can be increased or decreased by adjusting a screw valve $S$. The first copper coil $C$ is immersed in a hot oil bath $E$ and the second copper coil $D$ is immersed in a calorimeter $F$ containing water. Thermometers $T_1$ and $T_2$ are provided in the bath $E$ and the calorimeter $F$. A manometer $M$ connected close to the tank measures the pressure of the gas in the tank. Another manometer $N$ connected after the screw valve $S$ measures the pressure of the gas flowing in the calorimeter.

To do the experiment, the oil bath is heated with a burner to keep it at a high temperature which is measured by the thermometer $T_1$. A measured mass $m$ of water is taken in the calorimeter. The calorimeter should be almost full with water so that the gas flowing through the coil $D$ gets maximum time to exchange heat with the water. The initial temperature of the water is measured by $T_2$. The difference in the heights of mercury in the two arms of the manometer $M$ is noted. The valve $V$ is opened to allow the gas to flow through the system. As the amount of the gas in the tank reduces, the pressure in it decreases and the rate of flow tends to decrease. The screw valve $S$ is continuously adjusted to keep the rate of flow constant. This is decided by keeping the difference in the levels of mercury in manometer $N$ constant. The pressure of the gas going in the coil then remains constant. The gas is allowed to flow for some time and the final temperature of water and the final difference in the mercury levels in the manometer $M$ are noted.

Let the water equivalent of the calorimeter together with the coil $D$ $= W$

mass of the water $= m$

temperature of the oil bath $= \theta_1$

initial temperature of water $= \theta_2$

final temperature of water $= \theta_3$

and the amount of the gas (in moles) passed through the water $= n$.

The gas at temperature $\theta_1$ enters the coil $D$. In the beginning of the experiment, the gas leaves the coil $D$ at temperature $\theta_2$. This temperature gradually increases and at the end of the experiment it becomes $\theta_3$. The average temperature of the gas leaving the coil $D$ is, therefore, $\frac{\theta_2 + \theta_3}{2}$. The heat lost by the gas is

$$\Delta Q = n\,C_p\left[\theta_1 - \frac{\theta_2 + \theta_3}{2}\right]. \quad \ldots (i)$$

This heat is used to increase the temperature of the calorimeter, the water and the coil $D$ from $\theta_2$ to $\theta_3$. The heat received by them is

$$\Delta Q = (W + m)s\,(\theta_3 - \theta_2) \quad \ldots (ii)$$

where $s$ is the specific heat capacity of water. From (i) and (ii),

$$nC_p\left[\theta_1 - \frac{\theta_2 + \theta_3}{2}\right] = (W + m)s\,(\theta_3 - \theta_2)$$

or, $$C_p = \frac{(W+m)s\,(\theta_3 - \theta_2)}{n\left[\theta_1 - \frac{\theta_2 + \theta_3}{2}\right]}. \quad \ldots \text{(iii)}$$

**Determination of *n***

Suppose the difference in the mercury levels in the manometer $M$ is $h$ and the atmospheric pressure is equal to a height $H$ of mercury. The pressure in the tank is $p$, equal to a height $H + h$ of mercury. By noting the difference $h$ at the beginning and at the end of the experiment, the initial pressure $p_1$ and the final pressure $p_2$ are determined. Assuming the gas to be ideal,

$$p_1 V = n_1 RT$$

and $$p_2 V = n_2 RT.$$

Here $n_1$, $n_2$ are the amounts of the gas (in moles) in the tank at the beginning and at the end respectively, $V$ is the volume of the tank and $T$ is the absolute temperature of the tank. Thus,

$$(p_1 - p_2)V = (n_1 - n_2)RT$$

or, $$n = n_1 - n_2 = \frac{(p_1 - p_2)V}{RT}.$$

Putting in (iii), one can find the value of $C_p$.

## 27.4 DETERMINATION OF $C_V$ OF A GAS

Figure (27.4) shows a schematic diagram of Joly's differential steam calorimeter used to measure $C_v$ of a gas. Two hollow copper spheres $A$ and $B$ are suspended from the pans of a sensitive balance. The spheres are enclosed in a steam chamber. The balance is placed over the steam chamber. Two pans $C$ and $D$ are fitted below the spheres.

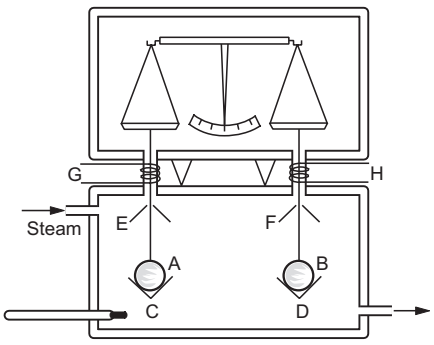



Figure 27.4

Umbrella-like shields $E$, $F$ are provided over the spheres. Plaster of paris tubes are provided at the holes in the steam chamber through which the suspension wires pass. These are heated electrically by the coils $G$, $H$. This ensures that when steam is sent into the steam chamber, no drops are formed on the wires and they move freely as the balance pans oscillate. The shields $E$, $F$ do not allow the drops to fall on the spheres or on the pans. A thermometer is fitted in the steam chamber.

To do the experiment, air is pumped out of the spheres and the spheres are balanced. The experimental gas is filled in one of the spheres, say $B$, and additional weights are put on the balance pan so that the spheres are again balanced. This gives the mass of the gas. The temperature of the steam chamber is noted. This gives the initial temperature of the gas.

Steam is now passed through the steam chamber. The steam condenses on the spheres and the water formed is collected in the pans $C$ and $D$. The temperatures of the spheres rise. More steam condenses on the pan which is below the sphere containing the gas. This is because the steam has to raise the temperature of the sphere as well as of the gas. When the temperature becomes steady, the spheres are again balanced by putting extra weights. This extra weight gives the amount of steam needed to raise the temperature of the gas only. The final temperature in steady state is noted.

Suppose,

| | |
|---|---|
| the mass of the gas taken | $= m_1$ |
| the mass of the extra steam condensed | $= m_2$ |
| initial temperature of the gas | $= \theta_1$ |
| final temperature of the gas | $= \theta_2$ |
| specific latent heat of vaporization of water | $= L$. |

If the molecular weight of the gas is $M$, the amount of the gas in moles is $n = m_1 / M$. The heat lost by the steam is $= m_2 L$ and the heat gained by the gas is $nC_V(\theta_2 - \theta_1)$.

Thus, $$m_2 L = nC_V(\theta_2 - \theta_1) = \frac{m_1}{M} C_V(\theta_2 - \theta_1)$$

or, $$C_V = \frac{Mm_2 L}{m_1(\theta_2 - \theta_1)}.$$

In this we have neglected the increase in the volumes of the spheres as the temperature rises.

## 27.5 ISOTHERMAL AND ADIABATIC PROCESSES

**Isothermal Process**

A process on a system is called isothermal if the temperature of the system remains constant. In case of an ideal gas the internal energy remains constant in such a process. The amount of heat supplied is equal to the work done by the gas. The gas obeys Boyle's law and the work done by the gas is

$$W = \int_{V_i}^{V_f} p\, dV = \int_{V_i}^{V_f} n \frac{RT}{V}\, dV = nRT \ln(V_f / V_i).$$

This is also the amount of heat given as the volume of the gas is changed from $V_i$ to $V_f$. As the change in temperature is zero, the molar heat capacity in such a process is

$$C_{isothermal} = \frac{\Delta Q}{n\, \Delta T} = \text{infinity}.$$

An isothermal process may be achieved by immersing the system in a large reservoir and performing the process very slowly. The temperature of the system then remains equal to the temperature of the reservoir. Heat may be exchanged between the reservoir and the system if necessary.

As an example, if a gas is taken in a metal cylinder (good conductor of heat) fitted with a piston and the piston is moved slowly, the temperature of the gas does not change. It remains equal to the temperature of the surrounding air. If the temperature tends to increase, heat is conducted from the gas to the air through the metallic walls. Similarly, if the temperature tends to decrease, heat is conducted from the surrounding air to the gas through the metallic walls. Here the surrounding air acts as a large reservoir.

**Adiabatic Process**

A process on a system is called adiabatic if no heat is supplied to it or extracted from it. In such a case, the temperature changes without adding any heat. The molar heat capacity in such a process is

$$C_{adiabatic} = \frac{\Delta Q}{n\, \Delta T} = \text{zero}.$$

The work done by the gas in an adiabatic process equals the decrease in its internal energy. Thus, if a gas enclosed in a container with adiabatic walls expands, the internal energy decreases and hence the temperature falls. If the gas is compressed adiabatically, the temperature rises.

## 27.6 RELATIONS BETWEEN *p*, *V*, *T* IN A REVERSIBLE ADIABATIC PROCESS

**Relation between *p* and *V***

Consider an adiabatic process on an ideal gas. During a short part of the process, the pressure, the volume and the temperature change from $p$, $V$, $T$ to $p + dp$, $V + dV$ and $T + dT$ respectively. The internal energy changes from $U$ to $U + dU$. As the amount of heat supplied is zero, the first law of thermodynamics gives

$$0 = dU + p\, dV. \quad \ldots \text{(i)}$$

We have, $C_V = \frac{1}{n} \frac{dU}{dT}$

or, $dU = nC_V dT$.

Thus, from (i),

$$nC_V dT + p\, dV = 0. \quad \ldots \text{(ii)}$$

As the gas is ideal,

$$pV = nRT$$

or, $p\, dV + V\, dp = nR dT$

or, $dT = \frac{p\, dV + V\, dp}{nR}$.

Substituting this expression for $dT$ in (ii),

$$C_V \left( \frac{p\, dV + V\, dp}{R} \right) + p\, dV = 0$$

or, $(C_V + R)\, p\, dV + C_V\, V\, dp = 0$

or, $C_p\, p\, dV + C_V\, V\, dp = 0$

or, $\frac{C_p}{C_V} \frac{dV}{V} + \frac{dp}{p} = 0$

or, $\gamma \frac{dV}{V} = -\frac{dp}{p}$

where $\gamma = C_p / C_V$.

Let the initial pressure and volume be $p_i$ and $V_i$ respectively and the final pressure and volume be $p_f$ and $V_f$ respectively. Then

$$\int_{V_i}^{V_f} \gamma \frac{dV}{V} = -\int_{p_i}^{p_f} \frac{dp}{p}.$$

or, $\gamma \ln \frac{V_f}{V_i} = -\ln \frac{p_f}{p_i}$

or, $\ln \left( \frac{V_f}{V_i} \right)^{\gamma} = \ln \left( \frac{p_i}{p_f} \right)$

or, $\frac{V_f^{\gamma}}{V_i^{\gamma}} = \frac{p_i}{p_f}$

or, $p_i V_i^{\gamma} = p_f V_f^{\gamma}$

or, $pV^{\gamma} = \text{constant}.$ $\quad \ldots \text{(27.9)}$

Thus, $pV^{\gamma}$ remains constant in a reversible adiabatic process.

**Relation between *p* and *T***

We have $pV = nRT$

or, $V = \frac{nRT}{p}$.

Putting in (27.9),

$$p \left( \frac{nRT}{p} \right)^{\gamma} = \text{constant}$$

or, $p^{1-\gamma} T^{\gamma} = \text{constant}$

or, $$\frac{T^{\gamma}}{p^{\gamma-1}} = \text{constant.} \quad \ldots \quad (27.10)$$

**Relation between $V$ and $T$**

We have $$pV = nRT$$

or, $$p = \frac{nRT}{V}.$$

Putting in (27.9),

$$\left(\frac{nRT}{V}\right) V^{\gamma} = \text{constant}$$

or, $$TV^{\gamma-1} = \text{constant.} \quad \ldots \quad (27.11)$$

***Example 27.4***

*Dry air at* 15°C *and* 10 atm *is suddenly released at atmospheric pressure. Find the final temperature of the air* $[C_p / C_V = 1{\cdot}41]$.

***Solution*** : As the air is suddenly released, it does not get time to exchange heat with the surrounding. Thus the process is adiabatic. Assuming the process to be reversible,

$$\frac{T^{\gamma}}{p^{\gamma-1}} = \text{constant}$$

or, $$\frac{T_1^{\gamma}}{p_1^{\gamma-1}} = \frac{T_2^{\gamma}}{p_2^{\gamma-1}}$$

or, $$\left(\frac{T_2}{T_1}\right)^{\gamma} = \left(\frac{p_2}{p_1}\right)^{\gamma-1}$$

or, $$T_2 = T_1 \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}.$$

Taking $p_1 = 10$ atm, $p_2 = 1$ atm, $\gamma = 1{\cdot}41$ and $T_1 = (273 + 15)$ K = 288 K, the final temperature is $T_2 = 148$ K.

## 27.7 WORK DONE IN AN ADIABATIC PROCESS

Suppose a sample of gas has initial pressure $p_1$ and initial volume $V_1$. In an adiabatic process, the pressure and volume change to $p_2$ and $V_2$. We have

$$pV^{\gamma} = p_1 V_1^{\gamma} = p_2 V_2^{\gamma} = K. \quad \ldots \quad (i)$$

Thus, $$p = \frac{K}{V^{\gamma}}.$$

The work done by the gas in the process is

$$W = \int_{V_1}^{V_2} p\, dV = \int_{V_1}^{V_2} \frac{K}{V^{\gamma}}\, dV$$

$$= \frac{1}{1-\gamma}\left[\frac{K}{V_2^{\gamma-1}} - \frac{K}{V_1^{\gamma-1}}\right].$$

From (i), $\frac{K}{V_2^{\gamma}} = p_2$ and $\frac{K}{V_1^{\gamma}} = p_1$.

Thus, $$W = -\frac{1}{\gamma-1}(p_2 V_2 - p_1 V_1)$$

$$= \frac{p_1 V_1 - p_2 V_2}{\gamma - 1}. \quad \ldots \quad (27.12)$$

## 27.8 EQUIPARTITION OF ENERGY

When we assume the molecules of a gas to be like hard spheres of negligible size, the energy of each molecule may be written as

$$E = \frac{1}{2} m v_x^2 + \frac{1}{2} m v_y^2 + \frac{1}{2} m v_z^2.$$

There are three terms in this expression and each may be treated independently. The above picture is suitable for monatomic gases. In diatomic gases the molecules are assumed to be in the shape of dumbbells, two hard spheres of negligible size at a separation. Apart from translational motion, the molecule can rotate about its centre. The energy is the sum of translational kinetic energy and rotational kinetic energy. If the line joining the two particles is taken as the $z$-axis, the rotational kinetic energy may be written as

$$\frac{1}{2} I_x \omega_x^2 + \frac{1}{2} I_y \omega_y^2,$$

the first term describing the energy of rotation about the $x$-axis and the second term describing the energy of rotation about the $y$-axis. As the size of each atom is assumed negligible, the moment of inertia $I_z$ is negligible and no term like $\frac{1}{2} I_z \omega_z^2$ is added. The total energy is then

$$E = \frac{1}{2} m v_x^2 + \frac{1}{2} m v_y^2 + \frac{1}{2} m v_z^2 + \frac{1}{2} I_x \omega_x^2 + \frac{1}{2} I_y \omega_y^2.$$

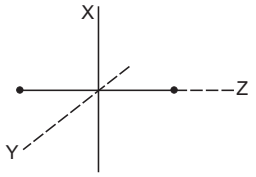



Figure 27.5

The number of independent terms in this expression is 5. If the two atoms of a diatomic molecule vibrate along its length, additional energy results. Such a vibration involves kinetic energy of vibration as well as the potential energy of the pair of atoms. The energy of vibration will be of the form

$$\frac{1}{2} \mu v^2 + \frac{1}{2} k r^2$$

where $r$ is the separation between the atoms and $v = (dr/dt)$. The constant μ is related to the mass of the molecule and $k$ is related to the force constant of the system. The total energy of a molecule is

$$E = \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 + \frac{1}{2}mv_z^2 + \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 + \frac{1}{2}\mu v^2 + \frac{1}{2}kr^2.$$

There are 7 independent terms.

For a polyatomic molecule, the number of terms in the rotational and vibrational energy depends on the actual arrangement of atoms in the molecule.

The number of independent terms in the expression of energy of a molecule is called its *degree of freedom*. Thus, the degree of freedom is 3 for a monatomic gas molecule, it is 5 for a diatomic molecule if the molecule does not vibrate and is 7 if it vibrates.

*Equipartition of energy* states that *the average energy of a molecule in a gas associated with each degree of freedom is* $\frac{1}{2}kT$ *where k is the Boltzmann constant and T is its absolute temperature*.

The theorem may be proved using more advanced techniques of statistical mechanics.

According to the equipartition theorem, the average energy of a molecule in a monatomic gas is $\frac{3}{2}kT$ as the degree of freedom is 3. This is also the result of the kinetic theory of gases.

For diatomic gases, the average energy per molecule is $\frac{5}{2}kT$ if the molecules translate and rotate but do not vibrate, and is $\frac{7}{2}kT$ if they vibrate also.

Now, consider a sample of amount $n$ (in moles) of an ideal gas. The total number of molecules is $nN_A$ where $N_A$ is the Avogadro number. If the gas is monatomic, the internal energy of the gas is

$$U = nN_A\left(\frac{3}{2}kT\right) = n\frac{3}{2}RT.$$

The molar heat capacity at constant volume is

$$C_V = \frac{1}{n}\frac{dU}{dT} = \frac{3}{2}R.$$

The molar heat capacity at constant pressure is

$$C_p = C_V + R = \frac{3}{2}R + R = \frac{5}{2}R.$$

Thus, $$\gamma = \frac{C_p}{C_V} = \frac{5}{3} = 1{\cdot}67.$$

For a sample of a diatomic gas,

$$U = nN_A\left(\frac{5}{2}kT\right) = n\frac{5}{2}RT$$

if the molecules do not vibrate. In this case,

$$C_V = \frac{1}{n}\frac{dU}{dT} = \frac{5}{2}R \text{ and } C_p = C_V + R = \frac{5}{2}R + R = \frac{7}{2}R.$$

Thus, $\gamma = \frac{C_p}{C_V} = \frac{7}{5} = 1{\cdot}40.$

If the molecules do vibrate, $U = n\frac{7}{2}RT$

so that, $C_V = \frac{7}{2}R$, $C_p = \frac{9}{2}R$ and $\gamma = \frac{C_p}{C_V} = \frac{9}{7} = 1{\cdot}29.$

Our expectations about $C_V$, $C_p$ and γ are summarized in table (27.1a) and the experimental values for a number of gases are given in table (27.1b).

**Table 27.1a:** ***Expected values of $C_V$, $C_p$ and γ***

| Nature of the gas | $C_V$ (J K$^{-1}$ mol$^{-1}$) | $C_p$ (J K$^{-1}$ mol$^{-1}$) | $C_p - C_V$ (J K$^{-1}$ mol$^{-1}$) | γ |
|---|---|---|---|---|
| Monatomic | 12·5 | 20·8 | 8·31 | 1·67 |
| Diatomic, assuming no vibrations | 20·8 | 29·1 | 8·31 | 1·40 |
| Diatomic, assuming vibrations | 29·1 | 37·4 | 8·31 | 1·29 |

**Table 27.1b:** ***Experimental values of $C_V$, $C_p$ and γ***

| Gas (15°C) | $C_V$ (J K$^{-1}$ mol$^{-1}$) | $C_p$ (J K$^{-1}$ mol$^{-1}$) | $C_p - C_V$ (J K$^{-1}$ mol$^{-1}$) | γ |
|---|---|---|---|---|
| He | 12·5 | 20·8 | 8·30 | 1·66 |
| Ne | 12·7 | 20·8 | 8·12 | 1·64 |
| Ar | 12·5 | 20·8 | 8·30 | 1·67 |
| Kr | 12·3 | 20·8 | 8·49 | 1·69 |
| Xe | 12·6 | 21·0 | 8·36 | 1·67 |
| $H_2$ | 20·4 | 28·8 | 8·45 | 1·41 |
| $O_2$ | 21·0 | 29·3 | 8·32 | 1·40 |
| $N_2$ | 20·8 | 29·1 | 8·32 | 1·40 |
| CO | 20·6 | 29·0 | 8·45 | 1·41 |
| HCl | 21·0 | 29·6 | 8·61 | 1·41 |
| $CO_2$ | 28·2 | 36·5 | 8·32 | 1·30 |
| $H_2O$ (200°C) | 27·0 | 35·4 | 8·35 | 1·31 |
| $CH_4$ | 27·1 | 35·4 | 8·36 | 1·31 |

We find excellent agreement between the prediction of the equipartition theorem and the experimental values. The results also suggest that diatomic molecules do not vibrate at ordinary temperatures. The table suggests that the degree of freedom of polyatomic molecules $CO_2$, $H_2O$ and $CH_4$ is 6.

According to the equipartition theorem, the molar heat capacities should be independent of temperature. However, variations in $C_V$ and $C_p$ are observed as the temperature changes. At very high temperatures, vibrations are also important and that affects the values of $C_V$ and $C_p$ for diatomic and polyatomic gases.

While $H_2O$ and $CH_4$ are nonlinear molecules, $CO_2$ is linear. Because of this the degree of freedom of $CO_2$ corresponding to rotation should be 2 just like $H_2O$ and $CH_4$. Then how can the specific heat

capacities of $CO_2$ be similar to $CH_4$ and $H_2O$? For $CO_2$, some fraction of molecules get into vibrations even at room temperature or so. This extra contribution from vibrations increases the specific heats and it looks like having degree of freedom 6.

***Example 27.5***

*Calculate the internal energy of* 1 g *of oxygen at STP.*

***Solution*** : Oxygen is a diatomic gas. The average energy per molecule is, therefore, $\frac{5}{2}kT$ and the average energy per mole is $\frac{5}{2}RT$. As the molecular weight of oxygen is 32 g mol$^{-1}$, 1 g of oxygen has

$$n = \frac{1\text{ g}}{32\text{ g mol}^{-1}} = \frac{1}{32}\text{ mol}.$$

The temperature of oxygen is 273 K. Thus, the internal energy is

$$U = n\left(\frac{5}{2}RT\right)$$
$$= \left(\frac{1}{32}\text{ mol}\right)\left(\frac{5}{2}\right)(8{\cdot}31\text{ J K}^{-1}\text{ mol}^{-1})\,(273\text{ K})$$
$$= 177\text{ J}.$$

## *Worked Out Examples*

**1.** *Calculate the value of mechanical equivalent of heat from the following data. Specific heat capacity of air at constant volume* $= 170\text{ cal kg}^{-1}\text{ K}^{-1}$, $\gamma = C_p/C_V = 1{\cdot}4$ *and the density of air at STP is* $1{\cdot}29\text{ kg m}^{-3}$. *Gas constant* $R = 8{\cdot}3\text{ J K}^{-1}\text{ mol}^{-1}$.

***Solution*** : Using $pV = nRT$, the volume of 1 mole of air at STP is

$$V = \frac{nRT}{p} = \frac{(1\text{ mol}) \times (8{\cdot}3\text{ J K}^{-1}\text{ mol}^{-1}) \times (273\text{ K})}{1{\cdot}0 \times 10^5\text{ N m}^{-2}}$$
$$= 0{\cdot}0224\text{ m}^3.$$

The mass of 1 mole is, therefore,

$$(1{\cdot}29\text{ kg m}^{-3}) \times (0{\cdot}0224\text{ m}^3) = 0{\cdot}029\text{ kg}.$$

The number of moles in 1 kg is $\frac{1}{0{\cdot}029}$. The molar heat capacity at constant volume is

$$C_V = \frac{170\text{ cal}}{(1/0{\cdot}029)\text{ mol K}^{-1}}$$
$$= 4{\cdot}93\text{ cal K}^{-1}\text{ mol}^{-1}.$$

Hence, $C_p = \gamma C_V = 1{\cdot}4 \times 4{\cdot}93\text{ cal K}^{-1}\text{ mol}^{-1}$

or, $C_p - C_V = 0{\cdot}4 \times 4{\cdot}93\text{ cal K}^{-1}\text{ mol}^{-1}$

$$= 1{\cdot}97\text{ cal K}^{-1}\text{ mol}^{-1}.$$

Also,

$$C_p - C_V = R = 8{\cdot}3\text{ J K}^{-1}\text{ mol}^{-1}.$$

Thus, $8{\cdot}3\text{ J} = 1{\cdot}97\text{ cal}.$

The mechanical equivalent of heat is

$$\frac{8{\cdot}3\text{ J}}{1{\cdot}97\text{ cal}} = 4{\cdot}2\text{ J cal}^{-1}.$$

**2.** *An ideal gas has a molar heat capacity at constant pressure* $C_p = 2{\cdot}5\,R$. *The gas is kept in a closed vessel of volume* $0{\cdot}0083\text{ m}^3$, *at a temperature of* 300 K *and a pressure of* $1{\cdot}6 \times 10^6\text{ N m}^{-2}$. *An amount* $2{\cdot}49 \times 10^4\text{ J}$ *of heat energy is supplied to the gas. Calculate the final temperature and pressure of the gas.*

***Solution*** : We have

$$C_V = C_p - R = 2{\cdot}5\,R - R = 1{\cdot}5\,R.$$

The amount of the gas (in moles) is $n = \frac{pV}{RT}$

$$= \frac{(1{\cdot}6 \times 10^6\text{ N m}^{-2}) \times (0{\cdot}0083\text{ m}^3)}{(8{\cdot}3\text{ J K}^{-1}\text{ mol}^{-1})\,(300\text{ K})} = 5{\cdot}3\text{ mol}.$$

As the gas is kept in a closed vessel, its volume is constant. Thus, we have

$$\Delta Q = n\,C_V\,\Delta T$$

or, $\Delta T = \frac{\Delta Q}{nC_V}$

$$= \frac{2{\cdot}49 \times 10^4\text{ J}}{(5{\cdot}3\text{ mol})\,(1{\cdot}5 \times 8{\cdot}3\text{ J K}^{-1}\text{ mol}^{-1})} = 377\text{ K}.$$

The final temperature is 300 K + 377 K = 677 K.

We have,

$$\frac{p_1V_1}{T_1} = \frac{p_2V_2}{T_2}.$$

Here $V_1 = V_2$. Thus,

$$p_2 = \frac{T_2}{T_1}p_1 = \frac{677}{300} \times 1{\cdot}6 \times 10^6\text{ N m}^{-2}$$
$$= 3{\cdot}6 \times 10^6\text{ N m}^{-2}.$$

**3.** *A sample of ideal gas* ($\gamma = 1{\cdot}4$) *is heated at constant pressure. If an amount* 140 J *of heat is supplied to the gas, find (a) the change in internal energy of the gas, (b) the work done by the gas.*

***Solution*** : Suppose the sample contains $n$ moles. Also, suppose the volume changes from $V_1$ to $V_2$ and the temperature changes from $T_1$ to $T_2$.

The heat supplied is

$$\Delta Q = nC_p(T_2 - T_1).$$

(a) The change in internal energy is

$$\Delta U = n C_V (T_2 - T_1) = \frac{C_V}{C_p} n C_p (T_2 - T_1)$$

$$= \frac{C_V}{C_p} \Delta Q = \frac{140 \text{ J}}{1{\cdot}4} = 100 \text{ J}.$$

(b) The work done by the gas is

$$\Delta W = \Delta Q - \Delta U$$

$$= 140 \text{ J} - 100 \text{ J} = 40 \text{ J}.$$

**4.** *An experiment is performed to measure the molar heat capacity of a gas at constant pressure using Regnault's method. The gas is initially contained in a cubical reservoir of size* $40 \text{ cm} \times 40 \text{ cm} \times 40 \text{ cm}$ *at* 600 kPa *at* 27°C. *A part of the gas is brought out, heated to* 100°C *and is passed through a calorimeter at constant pressure. The water equivalent of the calorimeter and its contents is* 100 g. *The temperature of the calorimeter and its contents increases from* 20°C *to* 30°C *during the experiment and the pressure in the reservoir decreases to* 525 kPa. *Specific heat capacity of water* $= 4200 \text{ J kg}^{-1} \text{K}^{-1}$. *Calculate the molar heat capacity* $C_p$ *from these data.*

***Solution*** : We have $pV = nRT$ or, $n = \frac{pV}{RT}$. The amount of the gas in the reservoir is $n_1 = \frac{p_1 V}{RT}$ before the gas is taken out and $n_2 = \frac{p_2 V}{RT}$ after the gas is taken out. The amount taken out is

$$\Delta n = n_1 - n_2 = (p_1 - p_2) \frac{V}{RT}$$

$$= \frac{(600 - 525) \times 10^3 \text{ N m}^{-2} \times (40 \times 10^{-2} \text{ m})^3}{(8{\cdot}3 \text{ J K}^{-1} \text{ mol}^{-1}) \times (300 \text{ K})}$$

$$= 1{\cdot}925 \text{ mol}.$$

The gas is heated to 100°C and cools down as it passes through the calorimeter. The average final temperature of the gas is $\frac{20°\text{C} + 30°\text{C}}{2} = 25°\text{C}$. Thus, the average decrease in temperature of the gas is

$$\Delta T = (100°\text{C} - 25°\text{C}) = 75°\text{C}$$

or, $$\Delta T = 75 \text{ K}.$$

The heat lost by the gas is

$$\Delta Q = \Delta n \, C_p \, \Delta T.$$

The heat gained by the calorimeter and its contents is

$(100 \text{ g}) (4200 \text{ J kg}^{-1} \text{K}^{-1}) (30 - 20)°\text{C} = 4200 \text{ J}$.

Thus, $\Delta n \, C_p \, \Delta T = 4200 \text{ J}$

or, $$C_p = \frac{4200 \text{ J}}{(1{\cdot}925 \text{ mol}) (75 \text{ K})} = 29 \text{ J K}^{-1} \text{ mol}^{-1}.$$

**5.** *A quantity of air is kept in a container having walls which are slightly conducting. The initial temperature and volume are* 27°C *(equal to the temperature of the surrounding) and* $800 \text{ cm}^3$ *respectively. Find the rise in the temperature if the gas is compressed to* $200 \text{ cm}^3$ *(a) in a short time (b) in a long time. Take* $\gamma = 1{\cdot}4$.

***Solution*** : (a) When the gas is compressed in a short time, the process is adiabatic. Thus,

$$T_2 V_2^{\gamma - 1} = T_1 V_1^{\gamma - 1}$$

or, $$T_2 = T_1 \left( \frac{V_1}{V_2} \right)^{\gamma - 1}$$

$$= (300 \text{ K}) \times \left[ \frac{800}{200} \right]^{0{\cdot}4} = 522 \text{ K}.$$

Rise in temperature $= T_2 - T_1 = 222$ K.

(b) When the gas is compressed in a long time, the process is isothermal. Thus, the temperature remains equal to the temperature of the surrounding that is 27°C. The rise in temperature = 0.

**6.** *A sample of gas* ($\gamma = 1{\cdot}5$) *is taken through an adiabatic process in which the volume is compressed from* $1600 \text{ cm}^3$ *to* $400 \text{ cm}^3$. *If the initial pressure is* 150 kPa, *(a) what is the final pressure and (b) how much work is done by the gas in the process* ?

***Solution*** : (a) For an adiabatic process,

$$p_1 V_1^{\gamma} = p_2 V_2^{\gamma}.$$

Thus,

$$p_2 = p_1 \left( \frac{V_1}{V_2} \right)^{\gamma}$$

$$= (150 \text{ kPa}) \left( \frac{1600 \text{ cm}^3}{400 \text{ cm}^3} \right)^{3/2} = 1200 \text{ kPa}.$$

(b) Work done by the gas in an adiabatic process is

$$W = \frac{p_1 V_1 - p_2 V_2}{\gamma - 1}$$

$$= \frac{(150 \text{ kPa}) (1600 \text{ cm}^3) - (1200 \text{ kPa}) (400 \text{ cm}^3)}{1{\cdot}5 - 1}$$

$$= \frac{240 \text{ J} - 480 \text{ J}}{0{\cdot}5} = -480 \text{ J}.$$

**7.** *Two moles of helium gas* ($\gamma = 5/3$) *are initially at* 27°C *and occupy a volume of* 20 litres. *The gas is first expanded at constant pressure until the volume is doubled. Then it undergoes an adiabatic change until the temperature returns to its initial value. (a) Sketch the process in a p–V diagram. (b) What is the final volume and pressure of the gas* ? *(c) What is the work done by the gas* ?

***Solution*** : (a) The process is shown in figure (27-W1).

During the part *ab*, the pressure is constant.

We have,

$$\frac{p_a V_a}{T_a} = \frac{p_b V_b}{T_b}$$

or, $$T_b = \frac{V_b}{V_a} T_a = 2T_a = 600 \text{ K}.$$

During the part *bc*, the gas is adiabatically returned to the temperature $T_a$. The point *a* and the point *c* are on the same isotherm. Thus, we draw an adiabatic curve from *b* and an isotherm from *a* and look for the point of intersection *c*. That is the final state.

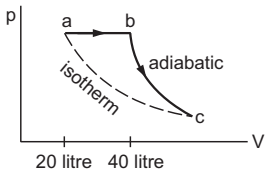



Figure 27-W1

(b) From the isotherm *ac*,

$$p_a V_a = p_c V_c \quad \ldots \text{(i)}$$

and from the adiabatic curve *bc*,

$$p_b V_b^{\gamma} = p_c V_c^{\gamma}$$

or, $$p_a (2V_a)^{\gamma} = p_c V_c^{\gamma}. \quad \ldots \text{(ii)}$$

Dividing (ii) by (i),

$$2^{\gamma} (V_a)^{\gamma - 1} = (V_c)^{\gamma - 1}$$

or, $$V_c = 2^{\gamma/(\gamma-1)} V_a = 4\sqrt{2} V_a = 113 \text{ litres}.$$

From (i), $$p_c = \frac{p_a V_a}{V_c} = \frac{nRT_a}{V_c}$$

$$= \frac{2 \text{ mol} \times (8{\cdot}3 \text{ J K}^{-1} \text{ mol}^{-1}) \times (300 \text{ K})}{113 \times 10^{-3} \text{ m}^3}$$

$$= 4{\cdot}4 \times 10^4 \text{ Pa}.$$

(c) Work done by the gas in the part *ab*

$$= p_a (V_b - V_a)$$
$$= p_b V_b - p_a V_a$$
$$= nRT_2 - nRT_1$$
$$= 2 \text{ mol} \times (8{\cdot}3 \text{ J K}^{-1} \text{ mol}^{-1}) \times (600 \text{ K} - 300 \text{ K})$$
$$= 4980 \text{ J}.$$

The work done in the adiabatic part *bc*

$$= \frac{p_b V_b - p_c V_c}{\gamma - 1}$$
$$= \frac{nR (T_2 - T_1)}{\gamma - 1}$$
$$= \frac{4980 \text{ J}}{5/3 - 1} = 7470 \text{ J}.$$

The net work done by the gas

$$= 4980 \text{ J} + 7470 \text{ J} = 12450 \text{ J}.$$

**8.** *An ideal gas enclosed in a vertical cylindrical container supports a freely moving piston of mass M. The piston and the cylinder have equal cross-sectional area A. When the piston is in equilibrium, the volume of the gas is $V_0$ and its pressure is $p_0$. The piston is slightly displaced from the equilibrium position and released. Assuming that the system is completely isolated from its surrounding, show that the piston executes simple harmonic motion and find the frequency of oscillations.*

***Solution*** : Suppose the piston is displaced through a distance *x* above the equilibrium position. The volume of the gas increases by $\Delta V = Ax$. As the system is completely isolated from its surrounding, the process is adiabatic. Thus,

$$pV^{\gamma} = \text{constant}$$

or, $$\ln p + \gamma \ln V = \text{constant}$$

or, $$\frac{\Delta p}{p} + \gamma \frac{\Delta V}{V} = 0$$

or, $$\Delta p = -\frac{\gamma p}{V} \Delta V.$$

As the piston is only slightly pushed, we can write

$$\Delta p = -\frac{\gamma p_0}{V_0} \Delta V.$$

The resultant force acting on the piston in this position is

$$F = A \Delta p = -A \frac{\gamma p_0}{V_0} \Delta V$$

$$= -\frac{A^2 \gamma p_0}{V_0} x = -kx$$

where $k = \frac{A^2 \gamma p_0}{V_0}$.

Thus, the motion of the piston is simple harmonic. The angular frequency ω is given by

$$\omega = \sqrt{\frac{k}{M}} = \sqrt{\frac{A^2 \gamma p_0}{M V_0}}$$

and the frequency is $\nu = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{A^2 \gamma p_0}{M V_0}}$.

**9.** *Two vessels of volumes $V_1$ and $V_2$ contain the same ideal gas. The pressures in the vessels are $p_1$ and $p_2$ and the temperatures are $T_1$ and $T_2$ respectively. The two vessels are now connected to each other through a narrow tube. Assuming that no heat is exchanged between the surrounding and the vessels, find the common pressure and temperature attained after the connection.*

***Solution*** :

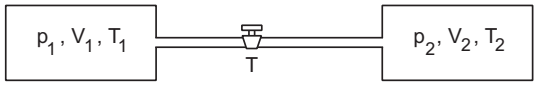



Figure 27-W2

The amount of the gas in vessel 1 is $n_1 = \frac{p_1 V_1}{RT_1}$

and that in vessel 2 is $n_2 = \frac{p_2V_2}{RT_2}$.

If $p'$ and $T'$ be the common pressure and temperature after the connection is made, the amounts are

$$n_1' = \frac{p'V_1}{RT'} \text{ and } n_2' = \frac{p'V_2}{RT'}.$$

We have $n_1 + n_2 = n_1' + n_2'$

or, $$\frac{p_1V_1}{RT_1} + \frac{p_2V_2}{RT_2} = \frac{p'V_1}{RT'} + \frac{p'V_2}{RT'}$$

or, $$\frac{p'}{T'} = \frac{1}{V_1 + V_2}\left(\frac{p_1V_1}{T_1} + \frac{p_2V_2}{T_2}\right)$$

or, $$\frac{T'}{p'} = \frac{T_1T_2(V_1 + V_2)}{p_1V_1T_2 + p_2V_2T_1}. \quad \ldots \text{(i)}$$

As the vessels have fixed volume, no work is done by the gas plus the vessels system. Also, no heat is exchanged with the surrounding. Thus, the internal energy of the total system remains constant.

The internal energy of an ideal gas is

$$U = nC_VT = C_V\frac{pV}{R}.$$

The internal energy of the gases before the connection

$$= \frac{C_Vp_1V_1}{R} + \frac{C_Vp_2V_2}{R}$$

and after the connection

$$= \frac{C_Vp'(V_1 + V_2)}{R}.$$

Neglecting the change in internal energy of the vessels (the heat capacity of the vessels is assumed negligible),

$$\frac{C_Vp_1V_1}{R} + \frac{C_Vp_2V_2}{R} = \frac{C_Vp'(V_1 + V_2)}{R}$$

or, $$p' = \frac{p_1V_1 + p_2V_2}{V_1 + V_2}.$$

From (i), $$T' = \frac{T_1T_2(p_1V_1 + p_2V_2)}{p_1V_1T_2 + p_2V_2T_1}.$$

**10.** 4 mol *of an ideal gas having* $\gamma = 1{\cdot}67$ *are mixed with* 2 mol *of another ideal gas having* $\gamma = 1{\cdot}4$. *Find the equivalent value of* $\gamma$ *for the mixture.*

***Solution*** : Let,

$C'_V$ = molar heat capacity of the first gas,

$C''_V$ = molar heat capacity of the second gas,

$C_V$ = molar heat capacity of the mixture

and similar symbols for other quantities. Then,

$$\gamma = \frac{C'_p}{C'_V} = 1{\cdot}67$$

and $$C'_p = C'_V + R.$$

This gives $$C'_V = \frac{3}{2}R \text{ and } C'_p = \frac{5}{2}R.$$

Similarly, $\gamma = 1{\cdot}4$ gives $C''_V = \frac{5}{2}R$ and $C''_p = \frac{7}{2}R$.

Suppose the temperature of the mixture is increased by $dT$. The increase in the internal energy of the first gas $= n_1C'_VdT$. The increase in internal energy of the second gas $= n_2C''_VdT$ and the increase in internal energy of the mixture $= (n_1 + n_2)C_VdT$. Thus,

$$(n_1 + n_2)\,C_VdT = n_1C'_VdT + n_2C''_VdT$$

or, $$C_V = \frac{n_1C'_V + n_2C''_V}{n_1 + n_2}. \quad \ldots \text{(i)}$$

$$C_p = C_V + R = \frac{n_1C'_V + n_2C''_V}{n_1 + n_2} + R$$

$$= \frac{n_1(C'_V + R) + n_2(C''_V + R)}{n_1 + n_2}$$

$$= \frac{n_1C'_p + n_2C''_p}{n_1 + n_2}. \quad \ldots \text{(ii)}$$

From (i) and (ii), $$\gamma = \frac{C_p}{C_V} = \frac{n_1C'_p + n_2C''_p}{n_1C'_V + n_2C''_V}$$

$$= \frac{4 \times \frac{5}{2}R + 2 \times \frac{7}{2}R}{4 \times \frac{3}{2}R + 2 \times \frac{5}{2}R} = 1{\cdot}54.$$

**11.** *A diatomic gas* ($\gamma = 1{\cdot}4$) *does* 200 J *of work when it is expanded isobarically. Find the heat given to the gas in the process.*

***Solution*** : For a diatomic gas, $C_V = \frac{5}{2}R$ and $C_p = \frac{7}{2}R$. The work done in an isobaric process is

$$W = p(V_2 - V_1) = nRT_2 - nRT_1$$

or, $$T_2 - T_1 = \frac{W}{nR}.$$

The heat given in an isobaric process is

$$Q = nC_p(T_2 - T_1) = nC_p\frac{W}{nR} = \frac{7}{2}W$$

$$= \frac{7}{2} \times 200 \text{ J} = 700 \text{ J}.$$

**12.** *Calculate the ratio* $C_p/C_V$ *of oxygen from the following data. Speed of sound in oxygen at* $0°\text{C} = 315 \text{ m s}^{-1}$, *molecular weight of oxygen* $= 32 \text{ g mol}^{-1}$ *and the gas constant* $R = 8{\cdot}3 \text{ J K}^{-1} \text{ mol}^{-1}$.

***Solution*** : The speed of sound in a gas is given by

$$v = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{\gamma RT}{M}}$$

or, $$\gamma = \frac{Mv^2}{RT}$$

$$= \frac{(32 \times 10^{-3} \text{ kg mol}^{-1})\,(315 \text{ m s}^{-1})^2}{(8{\cdot}3 \text{ J K}^{-1} \text{ mol}^{-1})\,(273 \text{ K})} = 1{\cdot}4.$$

□

## QUESTIONS FOR SHORT ANSWER

1. Does a gas have just two specific heat capacities or more than two ? Is the number of specific heat capacities of a gas countable ?
2. Can we define specific heat capacity at constant temperature ?
3. Can we define specific heat capacity for an adiabatic process ?
4. Does a solid also have two kinds of molar heat capacities $C_p$ and $C_V$ ? If yes, do we have $C_p > C_V$ ? Is $C_p - C_V = R$ ?
5. In a real gas the internal energy depends on temperature and also on volume. The energy increases when the gas expands isothermally. Looking into the derivation of $C_p - C_V = R$, find whether $C_p - C_V$ will be more than $R$, less than $R$ or equal to $R$ for a real gas.
6. Can a process on an ideal gas be both adiabatic and isothermal ?
7. Show that the slope of $p$–$V$ diagram is greater for an adiabatic process as compared to an isothermal process.
8. Is a slow process always isothermal ? Is a quick process always adiabatic ?
9. Can two states of an ideal gas be connected by an isothermal process as well as an adiabatic process ?
10. The ratio $C_p / C_V$ for a gas is 1·29. What is the degree of freedom of the molecules of this gas ?

## OBJECTIVE I

1. Work done by a sample of an ideal gas in a process $A$ is double the work done in another process $B$. The temperature rises through the same amount in the two processes. If $C_A$ and $C_B$ be the molar heat capacities for the two processes,
   (a) $C_A = C_B$ (b) $C_A < C_B$
   (c) $C_A > C_B$ (d) $C_A$ and $C_B$ cannot be defined.
2. For a solid with a small expansion coefficient,
   (a) $C_p - C_V = R$ (b) $C_p = C_V$
   (c) $C_p$ is slightly greater than $C_V$
   (d) $C_p$ is slightly less than $C_V$.
3. The value of $C_p - C_V$ is $1{\cdot}00\,R$ for a gas sample in state $A$ and is $1{\cdot}08\,R$ in state $B$. Let $p_A, p_B$ denote the pressures and $T_A$ and $T_B$ denote the temperatures of the states $A$ and $B$ respectively. Most likely
   (a) $p_A < p_B$ and $T_A > T_B$ (b) $p_A > p_B$ and $T_A < T_B$
   (c) $p_A = p_B$ and $T_A < T_B$ (d) $p_A > p_B$ and $T_A = T_B$
4. Let $C_V$ and $C_p$ denote the molar heat capacities of an ideal gas at constant volume and constant pressure respectively. Which of the following is a universal constant ?
   (a) $\dfrac{C_p}{C_V}$ (b) $C_pC_V$ (c) $C_p - C_V$ (d) $C_p + C_V$
5. 70 calories of heat is required to raise the temperature of 2 mole of an ideal gas at constant pressure from 30°C to 35°C. The amount of heat required to raise the temperature of the same gas through the same range at constant volume is
   (a) 30 calories (b) 50 calories
   (c) 70 calories (d) 90 calories.

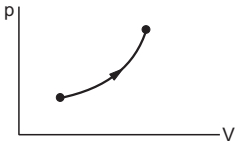



Figure 27-Q1

6. Figure (27-Q1) shows a process on a gas in which pressure and volume both change. The molar heat capacity for this process is $C$.
   (a) $C = 0$ (b) $C = C_V$ (c) $C > C_V$ (d) $C < C_V$
7. The molar heat capacity for the process shown in figure (27-Q2) is
   (a) $C = C_p$ (b) $C = C_V$ (c) $C > C_V$ (d) $C = 0$.

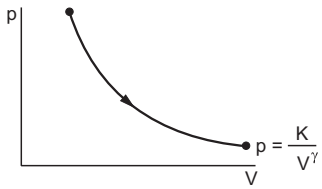



Figure 27-Q2

8. In an isothermal process on an ideal gas, the pressure increases by 0·5%. The volume decreases by about
   (a) 0·25% (b) 0·5% (c) 0·7% (d) 1%.
9. In an adiabatic process on a gas with $\gamma = 1{\cdot}4$, the pressure is increased by 0·5%. The volume decreases by about
   (a) 0·36% (b) 0·5% (c) 0·7% (d) 1%.
10. Two samples $A$ and $B$ are initially kept in the same state. The sample $A$ is expanded through an adiabatic process and the sample $B$ through an isothermal process. The final volumes of the samples are the same. The final pressures in $A$ and $B$ are $p_A$ and $p_B$ respectively.
    (a) $p_A > p_B$ (b) $p_A = p_B$ (c) $p_A < p_B$
    (d) The relation between $p_A$ and $p_B$ cannot be deduced.
11. Let $T_a$ and $T_b$ be the final temperatures of the samples $A$ and $B$ respectively in the previous question.
    (a) $T_a < T_b$ (b) $T_a = T_b$ (c) $T_a > T_b$
    (d) The relation between $T_a$ and $T_b$ cannot be deduced.
12. Let $\Delta W_a$ and $\Delta W_b$ be the work done by the systems $A$ and $B$ respectively in the previous question.
    (a) $\Delta W_a > \Delta W_b$ (b) $\Delta W_a = \Delta W_b$ (c) $\Delta W_a < \Delta W_b$

(d) The relation between $\Delta W_a$ and $\Delta W_b$ cannot be deduced.

**13.** The molar heat capacity of oxygen gas at STP is nearly $2{\cdot}5\,R$. As the temperature is increased, it gradually increases and approaches $3{\cdot}5\,R$. The most appropriate reason for this behaviour is that at high temperatures
(a) oxygen does not behave as an ideal gas
(b) oxygen molecules dissociate in atoms
(c) the molecules collide more frequently
(d) molecular vibrations gradually become effective.

## OBJECTIVE II

**1.** A gas kept in a container of finite conductivity is suddenly compressed. The process
(a) must be very nearly adiabatic
(b) must be very nearly isothermal
(c) may be very nearly adiabatic
(d) may be very nearly isothermal.

**2.** Let $Q$ and $W$ denote the amount of heat given to an ideal gas and the work done by it in an isothermal process.
(a) $Q = 0$ (b) $W = 0$ (c) $Q \neq W$ (d) $Q = W$

**3.** Let $Q$ and $W$ denote the amount of heat given to an ideal gas and the work done by it in an adiabatic process.
(a) $Q = 0$ (b) $W = 0$ (c) $Q = W$ (d) $Q \neq W$

**4.** Consider the processes $A$ and $B$ shown in figure (27-Q3). It is possible that

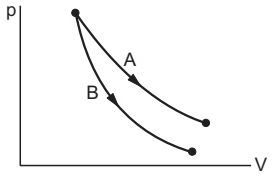



Figure 27-Q3

(a) both the processes are isothermal
(b) both the processes are adiabatic
(c) $A$ is isothermal and $B$ is adiabatic
(d) $A$ is adiabatic and $B$ is isothermal.

**5.** Three identical adiabatic containers $A$, $B$ and $C$ contain helium, neon and oxygen respectively at equal pressure. The gases are pushed to half their original volumes.
(a) The final temperatures in the three containers will be the same.
(b) The final pressures in the three containers will be the same.
(c) The pressures of helium and neon will be the same but that of oxygen will be different.
(d) The temperatures of helium and neon will be the same but that of oxygen will be different.

**6.** A rigid container of negligible heat capacity contains one mole of an ideal gas. The temperature of the gas increases by 1°C if 3·0 cal of heat is added to it. The gas may be
(a) helium (b) argon (c) oxygen (d) carbon dioxide.

**7.** Four cylinders contain equal number of moles of argon, hydrogen, nitrogen and carbon dioxide at the same temperature. The energy is minimum in
(a) argon (b) hydrogen (c) nitrogen (d) carbon dioxide.

## EXERCISES

**1.** A vessel containing one mole of a monatomic ideal gas (molecular weight = 20 g mol$^{-1}$) is moving on a floor at a speed of 50 m s$^{-1}$. The vessel is stopped suddenly. Assuming that the mechanical energy lost has gone into the internal energy of the gas, find the rise in its temperature.

**2.** 5 g of a gas is contained in a rigid container and is heated from 15°C to 25°C. Specific heat capacity of the gas at constant volume is 0·172 cal g$^{-1}$ °C$^{-1}$ and the mechanical equivalent of heat is 4·2 J cal$^{-1}$. Calculate the change in the internal energy of the gas.

**3.** Figure (27-E1) shows a cylindrical container containing oxygen ($\gamma = 1{\cdot}4$) and closed by a 50 kg frictionless piston. The area of cross section is 100 cm$^2$, atmospheric pressure is 100 kPa and $g$ is 10 m s$^{-2}$. The cylinder is slowly heated for some time. Find the amount of heat supplied to the gas if the piston moves out through a distance of 20 cm.

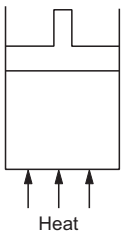



Figure 27-E1

**4.** The specific heat capacities of hydrogen at constant volume and at constant pressure are 2·4 cal g$^{-1}$ °C$^{-1}$ and 3·4 cal g$^{-1}$ °C$^{-1}$ respectively. The molecular weight of hydrogen is 2 g mol$^{-1}$ and the gas constant $R = 8{\cdot}3 \times 10^7$ erg °C$^{-1}$ mol$^{-1}$. Calculate the value of $J$.

**5.** The ratio of the molar heat capacities of an ideal gas is $C_p/C_V = 7/6$. Calculate the change in internal energy of 1·0 mole of the gas when its temperature is raised by

50 K (a) keeping the pressure constant, (b) keeping the volume constant and (c) adiabatically.

**6.** A sample of air weighing 1·18 g occupies $1{\cdot}0 \times 10^3$ cm$^3$ when kept at 300 K and $1{\cdot}0 \times 10^5$ Pa. When 2·0 cal of heat is added to it at constant volume, its temperature increases by 1°C. Calculate the amount of heat needed to increase the temperature of air by 1°C at constant pressure if the mechanical equivalent of heat is $4.2 \times 10^7$ erg cal$^{-1}$. Assume that air behaves as an ideal gas.

**7.** An ideal gas expands from 100 cm$^3$ to 200 cm$^3$ at a constant pressure of $2{\cdot}0 \times 10^5$ Pa when 50 J of heat is supplied to it. Calculate (a) the change in internal energy of the gas, (b) the number of moles in the gas if the initial temperature is 300 K, (c) the molar heat capacity $C_p$ at constant pressure and (d) the molar heat capacity $C_V$ at constant volume.

**8.** An amount $Q$ of heat is added to a monatomic ideal gas in a process in which the gas performs a work $Q/2$ on its surrounding. Find the molar heat capacity for the process.

**9.** An ideal gas is taken through a process in which the pressure and the volume are changed according to the equation $p = kV$. Show that the molar heat capacity of the gas for the process is given by $C = C_V + \frac{R}{2}$.

**10.** An ideal gas ($C_p / C_V = \gamma$) is taken through a process in which the pressure and the volume vary as $p = aV^b$. Find the value of $b$ for which the specific heat capacity in the process is zero.

**11.** Two ideal gases have the same value of $C_p / C_V = \gamma$. What will be the value of this ratio for a mixture of the two gases in the ratio 1 : 2 ?

**12.** A mixture contains 1 mole of helium ($C_p = 2{\cdot}5\,R$, $C_V = 1{\cdot}5\,R$) and 1 mole of hydrogen ($C_p = 3{\cdot}5\,R$, $C_V = 2{\cdot}5\,R$). Calculate the values of $C_p$, $C_V$ and $\gamma$ for the mixture.

**13.** Half mole of an ideal gas ($\gamma = 5/3$) is taken through the cycle *abcda* as shown in figure (27-E2). Take $R = \frac{25}{3}$ J K$^{-1}$ mol$^{-1}$. (a) Find the temperature of the gas in the states *a, b, c* and *d*. (b) Find the amount of heat supplied in the processes *ab* and *bc*. (c) Find the amount of heat liberated in the processes *cd* and *da*.

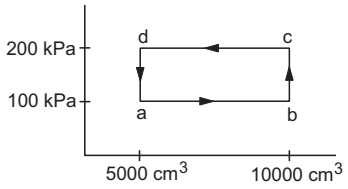



Figure 27-E2

**14.** An ideal gas ($\gamma = 1{\cdot}67$) is taken through the process *abc* shown in figure (27-E3). The temperature at the point *a* is 300 K. Calculate (a) the temperatures at *b* and *c*, (b) the work done in the process, (c) the amount of heat supplied in the path *ab* and in the path *bc* and (d) the change in the internal energy of the gas in the process.

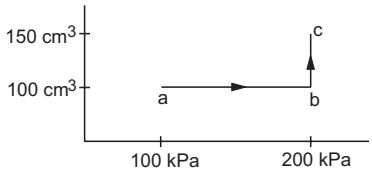



Figure 27-E3

**15.** In Joly's differential steam calorimeter, 3 g of an ideal gas is contained in a rigid closed sphere at 20°C. The sphere is heated by steam at 100°C and it is found that an extra 0·095 g of steam has condensed into water as the temperature of the gas becomes constant. Calculate the specific heat capacity of the gas in J g$^{-1}$ K$^{-1}$. The latent heat of vaporization of water = 540 cal g$^{-1}$.

**16.** The volume of an ideal gas ($\gamma = 1{\cdot}5$) is changed adiabatically from 4·00 litres to 3·00 litres. Find the ratio of (a) the final pressure to the initial pressure and (b) the final temperature to the initial temperature.

**17.** An ideal gas at pressure $2{\cdot}5 \times 10^5$ Pa and temperature 300 K occupies 100 cc. It is adiabatically compressed to half its original volume. Calculate (a) the final pressure, (b) the final temperature and (c) the work done by the gas in the process. Take $\gamma = 1{\cdot}5$.

**18.** Air ($\gamma = 1{\cdot}4$) is pumped at 2 atm pressure in a motor tyre at 20°C. If the tyre suddenly bursts, what would be the temperature of the air coming out of the tyre. Neglect any mixing with the atmospheric air.

**19.** A gas is enclosed in a cylindrical can fitted with a piston. The walls of the can and the piston are adiabatic. The initial pressure, volume and temperature of the gas are 100 kPa, 400 cm$^3$ and 300 K respectively. The ratio of the specific heat capacities of the gas is $C_p / C_V = 1{\cdot}5$. Find the pressure and the temperature of the gas if it is (a) suddenly compressed (b) slowly compressed to 100 cm$^3$.

**20.** The initial pressure and volume of a given mass of a gas ($C_p / C_V = \gamma$) are $p_0$ and $V_0$. The gas can exchange heat with the surrounding. (a) It is slowly compressed to a volume $V_0/2$ and then suddenly compressed to $V_0/4$. Find the final pressure. (b) If the gas is suddenly compressed from the volume $V_0$ to $V_0/2$ and then slowly compressed to $V_0/4$, what will be the final pressure ?

**21.** Consider a given sample of an ideal gas ($C_p / C_V = \gamma$) having initial pressure $p_0$ and volume $V_0$. (a) The gas is isothermally taken to a pressure $p_0/2$ and from there adiabatically to a pressure $p_0/4$. Find the final volume. (b) The gas is brought back to its initial state. It is adiabatically taken to a pressure $p_0/2$ and from there isothermally to a pressure $p_0/4$. Find the final volume.

**22.** A sample of an ideal gas ($\gamma = 1{\cdot}5$) is compressed adiabatically from a volume of 150 cm$^3$ to 50 cm$^3$. The initial pressure and the initial temperature are 150 kPa and 300 K. Find (a) the number of moles of the gas in the sample, (b) the molar heat capacity at constant volume, (c) the final pressure and temperature, (d) the

work done by the gas in the process and (e) the change in internal energy of the gas.

**23.** Three samples $A$, $B$ and $C$ of the same gas ($\gamma = 1{\cdot}5$) have equal volumes and temperatures. The volume of each sample is doubled, the process being isothermal for $A$, adiabatic for $B$ and isobaric for $C$. If the final pressures are equal for the three samples, find the ratio of the initial pressures.

**24.** Two samples $A$ and $B$ of the same gas have equal volumes and pressures. The gas in sample $A$ is expanded isothermally to double its volume and the gas in $B$ is expanded adiabatically to double its volume. If the work done by the gas is the same for the two cases, show that $\gamma$ satisfies the equation $1 - 2^{1-\gamma} = (\gamma - 1)\ln 2$.

**25.** 1 litre of an ideal gas ($\gamma = 1{\cdot}5$) at 300 K is suddenly compressed to half its original volume. (a) Find the ratio of the final pressure to the initial pressure. (b) If the original pressure is 100 kPa, find the work done by the gas in the process. (c) What is the change in internal energy ? (d) What is the final temperature ? (e) The gas is now cooled to 300 K keeping its pressure constant. Calculate the work done during the process. (f) The gas is now expanded isothermally to achieve its original volume of 1 litre. Calculate the work done by the gas. (g) Calculate the total work done in the cycle.

**26.** Figure (27-E4) shows a cylindrical tube with adiabatic walls and fitted with an adiabatic separator. The separator can be slid into the tube by an external mechanism. An ideal gas ($\gamma = 1{\cdot}5$) is injected in the two sides at equal pressures and temperatures. The separator remains in equilibrium at the middle. It is now slid to a position where it divides the tube in the ratio 1 : 3. Find the ratio of the temperatures in the two parts of the vessel.

Figure 27-E4

**27.** Figure (27-E5) shows two rigid vessels $A$ and $B$, each of volume 200 $cm^3$ containing an ideal gas ($C_V = 12{\cdot}5$ J $K^{-1}$ $mol^{-1}$). The vessels are connected to a manometer tube containing mercury. The pressure in both the vessels is 75 cm of mercury and the temperature is 300 K. (a) Find the number of moles of the gas in each vessel. (b) 5·0 J of heat is supplied to the gas in the vessel $A$ and 10 J to the gas in the vessel $B$. Assuming no appreciable transfer of heat from $A$ to $B$ calculate the difference in the heights of mercury in the two sides of the manometer. Gas constant $R = 8{\cdot}3$ J $K^{-1}$ $mol^{-1}$.

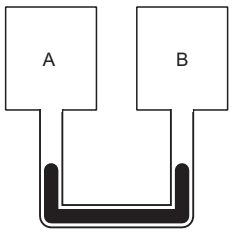



Figure 27-E5

**28.** Figure (27-E6) shows two vessels with adiabatic walls, one containing 0·1 g of helium ($\gamma = 1{\cdot}67$, $M = 4$ g $mol^{-1}$) and the other containing some amount of hydrogen ($\gamma = 1{\cdot}4$, $M = 2$ g $mol^{-1}$). Initially, the temperatures of the two gases are equal. The gases are electrically heated for some time during which equal amounts of heat are given to the two gases. It is found that the temperatures rise through the same amount in the two vessels. Calculate the mass of hydrogen.

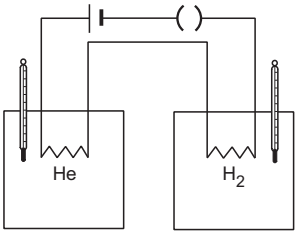



Figure 27-E6

**29.** Two vessels $A$ and $B$ of equal volume $V_0$ are connected by a narrow tube which can be closed by a valve. The vessels are fitted with pistons which can be moved to change the volumes. Initially, the valve is open and the vessels contain an ideal gas ($C_p / C_V = \gamma$) at atmospheric pressure $p_0$ and atmospheric temperature $T_0$. The walls of the vessel $A$ are diathermic and those of $B$ are adiabatic. The valve is now closed and the pistons are slowly pulled out to increase the volumes of the vessels to double the original value. (a) Find the temperatures and pressures in the two vessels. (b) The valve is now opened for sufficient time so that the gases acquire a common temperature and pressure. Find the new values of the temperature and the pressure.

**30.** Figure (27-E7) shows an adiabatic cylindrical tube of volume $V_0$ divided in two parts by a frictionless adiabatic separator. Initially, the separator is kept in the middle, an ideal gas at pressure $p_1$ and temperature $T_1$ is injected into the left part and another ideal gas at pressure $p_2$ and temperature $T_2$ is injected into the right part. $C_p / C_V = \gamma$ is the same for both the gases. The separator is slid slowly and is released at a position where it can stay in equilibrium. Find (a) the volumes of the two parts, (b) the heat given to the gas in the left part and (c) the final common pressure of the gases.

Figure 27-E7

**31.** An adiabatic cylindrical tube of cross-sectional area 1 $cm^2$ is closed at one end and fitted with a piston at the other end. The tube contains 0·03 g of an ideal gas. At 1 atm pressure and at the temperature of the surrounding, the length of the gas column is 40 cm. The piston is suddenly pulled out to double the length of the column. The pressure of the gas falls to 0·355 atm. Find the speed of sound in the gas at atmospheric temperature.

**32.** The speed of sound in hydrogen at 0°C is $1280\ \text{m s}^{-1}$. The density of hydrogen at STP is $0{\cdot}089\ \text{kg m}^{-3}$. Calculate the molar heat capacities $C_p$ and $C_V$ of hydrogen.

**33.** 4·0 g of helium occupies $22400\ \text{cm}^3$ at STP. The specific heat capacity of helium at constant pressure is $5{\cdot}0\ \text{cal K}^{-1}\ \text{mol}^{-1}$. Calculate the speed of sound in helium at STP.

**34.** An ideal gas having density $1{\cdot}7 \times 10^{-3}\ \text{g cm}^{-3}$ at a pressure $1{\cdot}5 \times 10^5$ Pa is filled in a Kundt tube. When the gas is resonated at a frequency of 3·0 kHz, nodes are formed at a separation of 6·0 cm. Calculate the molar heat capacities $C_p$ and $C_V$ of the gas.

**35.** Standing waves of frequency 5·0 kHz are produced in a tube filled with oxygen at 300 K. The separation between the consecutive nodes is 3·3 cm. Calculate the specific heat capacities $C_p$ and $C_V$ of the gas.

□

## ANSWERS

### OBJECTIVE I

1. (c) 2. (c) 3. (a) 4. (c) 5. (b) 6. (c)
7. (d) 8. (b) 9. (a) 10. (c) 11. (a) 12. (c)
13. (d)

### OBJECTIVE II

1. (c), (d) 2. (d) 3. (a), (d)
4. (c) 5. (c), (d) 6. (a), (b)
7. (a)

### EXERCISES

1. 2·0 K
2. 36 J
3. 1050 J
4. $4{\cdot}15 \times 10^7\ \text{erg cal}^{-1}$
5. 2490 J in all cases
6. 2·08 cal
7. (a) 30 J (b) 0·008
   (c) $20{\cdot}8\ \text{J K}^{-1}\ \text{mol}^{-1}$ (d) $12{\cdot}5\ \text{J K}^{-1}\ \text{mol}^{-1}$
8. $3R$
10. $-\gamma$
11. $\gamma$
12. $3R, 2R, 1{\cdot}5$
13. (a) 120 K, 240 K, 480 K, 240 K
    (b) 1250 J, 1500 J (c) 2500 J, 750 J
14. (a) 600 K, 900 K (b) 10 J
    (c) 14·9 J, 24·9 J (d) 29·8 J
15. $0{\cdot}90\ \text{J g}^{-1}\ \text{K}^{-1}$
16. (a) 1·54 (b) 1·15
17. (a) $7{\cdot}1 \times 10^5$ Pa (b) 424 K (c) – 21 J
18. 240 K
19. 800 kPa, 600 K in both cases
20. $2^{\gamma+1}\ p_0$ in both cases
21. $2^{(\gamma+1)/\gamma}\ V_0$ in each case
22. (a) 0·009 (b) $2R = 16{\cdot}6\ \text{J K}^{-1}\ \text{mol}^{-1}$
    (c) 780 kPa, 520 K (d) –33 J (e) 33 J
23. $2 : 2\sqrt{2} : 1$
25. (a) $2\sqrt{2}$ (b) –82 J
    (c) 82 J (d) 424 K
    (e) –41·4 J (f) 103 J
    (g) –23·4 J
26. $\sqrt{3} : 1$
27. (a) 0·008 (b) 12·5 cm
28. 0·03 g
29. (a) $T_0$, $\frac{p_0}{2}$ in vessel $A$ and $T_0/2^{\gamma-1}$, $p_0/2^{\gamma}$ in vessel $B$
    (b) $T_0$, $p_0/2$
30. (a) $\frac{{p_1}^{1/\gamma}V_0}{A}$, $\frac{{p_2}^{1/\gamma}V_0}{A}$ (b) zero
    (c) $(A/2)^{\gamma}$ where $A = {p_1}^{1/\gamma} + {p_2}^{1/\gamma}$
31. $447\ \text{m s}^{-1}$
32. $18{\cdot}0\ \text{J K}^{-1}\ \text{mol}^{-1}$, $26{\cdot}3\ \text{J K}^{-1}\ \text{mol}^{-1}$
33. $960\ \text{m s}^{-1}$
34. $26\ \text{J K}^{-1}\ \text{mol}^{-1}$, $17{\cdot}7\ \text{J K}^{-1}\ \text{mol}^{-1}$
35. $29{\cdot}0\ \text{J K}^{-1}\ \text{mol}^{-1}$, $20{\cdot}7\ \text{J K}^{-1}\ \text{mol}^{-1}$

□

## CHAPTER 28

# HEAT TRANSFER

Heat can be transferred from one place to another by three different methods, namely, conduction, convection and radiation. Conduction usually takes place in solids, convection in liquids and gases, and no medium is required for radiation.

### 28.1 THERMAL CONDUCTION

If one end of a metal rod is placed in a stove, the temperature of the other end gradually increases. Heat is transferred from one end of the rod to the other end. This transfer takes place due to molecular collisions and the process is called *heat conduction*. The molecules at one end of the rod gain heat from the stove and their average kinetic energy increases. As these molecules collide with the neighbouring molecules having less kinetic energy, the energy is shared between these two groups. The kinetic energy of these neighbouring molecules increases. As they collide with their neighbours on the colder side, they transfer energy to them. This way, heat is passed along the rod from molecule to molecule. The average position of a molecule does not change and hence, there is no mass movement of matter.

#### Thermal Conductivity

The ability of a material to conduct heat is measured by *thermal conductivity* (defined below) of the material.

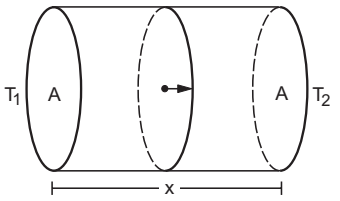



Figure 28.1

Consider a slab of uniform cross section $A$ and length $x$. Let one face of the slab be maintained at temperature $T_1$ and the other at $T_2$. Also, suppose the remaining surface is covered with a nonconducting material so that no heat is transferred through the sides. After sufficient time, steady state is reached and the temperature at any point remains unchanged as time passes. In such a case, the amount of heat crossing per unit time through any cross section of the slab is equal. If $\Delta Q$ amount of heat crosses through any cross section in time $\Delta t$, $\Delta Q/\Delta t$ is called the *heat current*. It is found that in steady state the heat current is proportional to the area of cross section $A$, proportional to the temperature difference $(T_1 - T_2)$ between the ends and inversely proportional to the length $x$. Thus,

$$\frac{\Delta Q}{\Delta t} = K \frac{A(T_1 - T_2)}{x} \qquad \ldots \quad (28.1)$$

where $K$ is a constant for the material of the slab and is called the *thermal conductivity* of the material.

If the area of cross section is not uniform or if the steady-state conditions are not reached, the equation can only be applied to a thin layer of material perpendicular to the heat flow. If $A$ be the area of cross section at a place, $dx$ be a small thickness along the direction of heat flow, and $dT$ be the temperature difference across the layer of thickness $dx$, the heat current through this cross section is

$$\frac{\Delta Q}{\Delta t} = -KA \frac{dT}{dx} \cdot \qquad \ldots \quad (28.2)$$

The quantity $dT/dx$ is called the *temperature gradient*. The minus sign indicates that $dT/dx$ is negative along the direction of the heat flow.

The unit of thermal conductivity can be easily worked out using equation (28.1) or (28.2). The SI unit is $Js^{-1}m^{-1}K^{-1}$ or $Wm^{-1}K^{-1}$. As a change of 1 K and a change of 1°C are the same, the unit may also be written as $Wm^{-1}{}^{\circ}C^{-1}$.

***Example 28.1***

*One face of a copper cube of edge* 10 cm *is maintained at* 100°C *and the opposite face is maintained at* 0°C. *All other surfaces are covered with an insulating material.*

*Find the amount of heat flowing per second through the cube. Thermal conductivity of copper is* 385 $Wm^{-1}°C^{-1}$.

***Solution*** : The heat flows from the hotter face towards the colder face. The area of cross section perpendicular to the heat flow is

$$A = (10 \text{ cm})^2.$$

The amount of heat flowing per second is

$$\frac{\Delta Q}{\Delta t} = KA\frac{T_1 - T_2}{x}$$

$$= (385 \text{ Wm}^{-1}°\text{C}^{-1}) \times (0{\cdot}1 \text{ m})^2 \times \frac{(100°\text{C} - 0°\text{C})}{0{\cdot}1 \text{ m}}$$

$$= 3850 \text{ W}.$$

In general, solids are better conductors than liquids and liquids are better conductors than gases. Metals are much better conductors than nonmetals. This is because, in metals we have a large number of "free electrons" which can move freely anywhere in the body of the metal. These free electrons help in carrying the thermal energy from one place to another in a metal.

Table (28.1) gives thermal conductivities of some materials.

**Table 28.1 :** ***Thermal conductivities***

| Material | $K(\text{Wm}^{-1}\text{K}^{-1})$ |
|---|---|
| Aluminium | 209 |
| Brass | 109 |
| Copper | 385 |
| Silver | 414 |
| Steel | 46·0 |
| Water | 0·585 |
| Transformer oil | 0·176 |
| Air | 0·0238 |
| Hydrogen | 0·167 |
| Oxygen | 0·0242 |
| Brick | 0·711 |
| Celotex (Sugarcane fibre) | 0·50 |
| Concrete | 1·30 |
| Glass | 0·669 |
| Masonite | 0·046 |
| Rock & Galls wool | 0·042 |
| Wood (oak) | 0·146 |
| Wood (pine) | 0·117 |

We can now understand why cooking utensils are made of metals whereas their handles are made of plastic or wood. When a rug is placed in bright sun on a tiled floor, both the rug and the floor acquire the same temperature. But it is much more difficult to stay bare foot on the tiles than to stay on the rug. This is because the conductivity of the rug is lesser than the tiles, and hence, the heat current going into the foot is smaller.

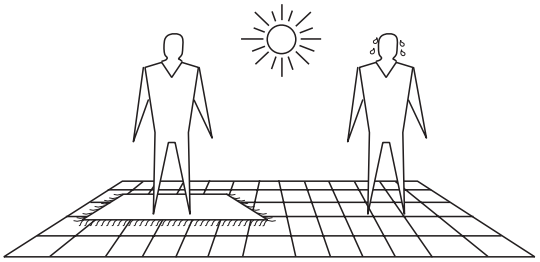


Figure 28.2

**Thermal Resistance**

The quantity $\frac{x}{KA}$ in equation (28.1) is called the *thermal resistance* $R$. Writing the heat current $\Delta Q/\Delta t$ as $i$, we have

$$i = \frac{T_1 - T_2}{R}. \qquad \ldots \quad (28.3)$$

This is mathematically equivalent to Ohm's law to be introduced in a later chapter. Many results derived from Ohm's law are also valid for thermal conduction.

***Example 28.2***

*Find the thermal resistance of an aluminium rod of length* 20 cm *and area of cross section* 1 $\text{cm}^2$. *The heat current is along the length of the rod. Thermal conductivity of aluminium* = 200 $\text{Wm}^{-1}\text{K}^{-1}$.

***Solution*** : The thermal resistance is

$$R = \frac{x}{KA} = \frac{20 \times 10^{-2} \text{ m}}{(200 \text{ Wm}^{-1}\text{K}^{-1})\,(1 \times 10^{-4} \text{ m}^2)} = 10 \text{ KW}^{-1}.$$

## 28.2 SERIES AND PARALLEL CONNECTION OF RODS

### (a) Series Connection

Figure 28.3

Consider two rods of thermal resistances $R_1$ and $R_2$ joined one after the other as shown in figure (28.3). The free ends are kept at temperatures $T_1$ and $T_2$ with $T_1 > T_2$. In steady state, any heat that goes through the first rod also goes through the second rod. Thus, the same heat current passes through the two rods. Such a connection of rods is called a *series connection*. Suppose, the temperature of the junction is $T$. From

equation (28.3), the heat current through the first rod is

$$i = \frac{\Delta Q}{\Delta t} = \frac{T_1 - T}{R_1}$$

or, $T_1 - T = R_1 i$ … (i)

and that through the second rod is

$$i = \frac{\Delta Q}{\Delta t} = \frac{T - T_2}{R_2}$$

or, $T - T_2 = R_2 i.$ … (ii)

Adding (i) and (ii),

$$T_1 - T_2 = (R_1 + R_2)\, i$$

or, $$i = \frac{T_1 - T_2}{R_1 + R_2}.$$

Thus, the two rods together is equivalent to a single rod of thermal resistance $R_1 + R_2$.

If more than two rods are joined in series, the equivalent thermal resistance is given by

$$R = R_1 + R_2 + R_3 + \ldots$$

**(b) Parallel Connection**

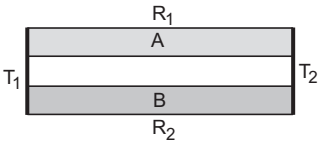



Figure 28.4

Now, suppose the two rods are joined at their ends as shown in figure (28.4). The left ends of both the rods are kept at temperature $T_1$ and the right ends are kept at temperature $T_2$. So the same temperature difference is maintained between the ends of each rod. Such a connection of rods is called a *parallel connection*. The heat current going through the first rod is

$$i_1 = \frac{\Delta Q_1}{\Delta t} = \frac{T_1 - T_2}{R_1}$$

and that through the second rod is

$$i_2 = \frac{\Delta Q_2}{\Delta t} = \frac{T_1 - T_2}{R_2}.$$

The total heat current going through the left end is

$$i = i_1 + i_2$$

$$= (T_1 - T_2)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

or, $$i = \frac{T_1 - T_2}{R}$$

where $$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$ … (i)

Thus, the system of the two rods is equivalent to a single rod of thermal resistance $R$ given by (i).

If more than two rods are joined in parallel, the equivalent thermal resistance $R$ is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots$$

## 28.3 MEASUREMENT OF THERMAL CONDUCTIVITY OF A SOLID

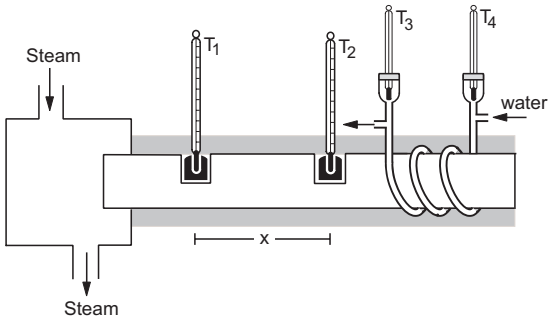



Figure 28.5

Figure (28.5) shows Searle's apparatus to measure the thermal conductivity of a solid. The solid is taken in the form of a cylindrical rod. One end of the rod goes into a steam chamber. A copper tube is coiled around the rod near the other end of the rod. A steady flow of water is maintained in the copper tube. Water enters the tube at the end away from the steam chamber and it leaves at the end nearer to it. Thermometers $T_3$ and $T_4$ are provided to measure the temperatures of the outgoing and incoming water. Two holes are drilled in the rod and mercury is filled in these holes to measure the temperature of the rod at these places with the help of thermometers $T_1$ and $T_2$. The whole apparatus is covered properly with layers of an insulating material like wool or felt so as to prevent any loss of heat from the sides.

Steam is passed in the steam chamber and a steady flow of water is maintained. The temperatures of all the four thermometers rise initially and ultimately become constant in time as the steady state is reached. The readings $\theta_1$, $\theta_2$, $\theta_3$ and $\theta_4$ are noted in steady state.

A beaker is weighed and the water coming out of the copper tube is collected in it for a fixed time $t$ measured by a stop clock. The beaker together with the water is weighed. The mass $m$ of the water collected is then calculated. The area of cross section of the rod is calculated by measuring its radius with a slide calipers. The distance between the holes in the rod is measured with the help of a divider and a metre scale.

Let the length of the rod between the holes be $x$ and the area of cross section of the rod be $A$. If the thermal conductivity of the material of the rod is $K$, the rate of heat flow (heat current) from the steam chamber to the rod is

$$\frac{\Delta Q}{\Delta t} = K\frac{A(\theta_1 - \theta_2)}{x}.$$

In a time $t$, the chamber supplies a heat

$$Q = K\frac{A(\theta_1 - \theta_2)}{x}t. \quad \ldots \text{(i)}$$

As the mass of the water collected in time $t$ is $m$, the heat taken by the water is

$$Q = ms(\theta_3 - \theta_4) \quad \ldots \text{(ii)}$$

where $s$ is the specific heat capacity of water.

As the entire rod is covered with an insulating material and the temperature of the rod does not change with time at any point, any heat given by the steam chamber must go into the flowing water. Hence, the same $Q$ is used in (i) and (ii). Thus,

$$K\frac{A(\theta_1 - \theta_2)}{x}t = ms(\theta_3 - \theta_4)$$

or,
$$K = \frac{x\,ms(\theta_3 - \theta_4)}{A(\theta_1 - \theta_2)t}. \quad \ldots \text{(28.4)}$$

## 28.4 CONVECTION

In convection, heat is transferred from one place to the other by the actual motion of heated material. For example, in a hot air blower, air is heated by a heating element and is blown by a fan. The air carries the heat wherever it goes. When water is kept in a vessel and heated on a stove, the water at the bottom gets heat due to conduction through the vessel's bottom. Its density decreases and consequently it rises. Thus, the heat is carried from the bottom to the top by the actual movement of the parts of the water. If the heated material is forced to move, say by a blower or a pump, the process of heat transfer is called *forced convection*. If the material moves due to difference in density, it is called *natural* or *free convection*.

Natural convection and the anomalous expansion of water play important roles in saving the lives of aquatic animals like fishes when the atmospheric temperature goes below 0°C. As the water at the surface is cooled, it becomes denser and goes down. The less cold water from the bottom rises up to the surface and gets cooled. This way the entire water is cooled to 4°C. As the water at the surface is further cooled, it expands and the density decreases. Thus, it remains at the surface and gets further cooled. Finally, it starts freezing. Heat is now lost to the atmosphere by the water only due to conduction through the ice. As ice is a poor conductor of heat, the further freezing is very slow. The temperature of the water at the bottom remains constant at 4°C for a large period of time. The atmospheric temperature ultimately improves and the animals are saved.

The main mechanism for heat transfer inside a human body is forced convection. Heart serves as the pump and blood as the circulating fluid. Heat is lost to the atmosphere through all the three processes conduction, convection and radiation. The rate of loss depends on the clothing, the tiredness and perspiration, atmospheric temperature, air current, humidity and several other factors. The system, however, transports the just required amount of heat and hence maintains a remarkably constant body temperature.

## 28.5 RADIATION

The process of radiation does not need any material medium for heat transfer. Energy is emitted by a body and this energy travels in the space just like light. When it falls on a material body, a part is absorbed and the thermal energy of the receiving body is increased. The energy emitted by a body in this way is called *radiant energy, thermal radiation* or simply *radiation*. Thus, the word "radiation" is used in two meanings. It refers to the process by which the energy is emitted by a body, is transmitted in space and falls on another body. It also refers to the energy itself which is being transmitted in space. The heat from the sun reaches the earth by this process, travelling millions of kilometres of empty space.

## 28.6 PRÉVOST THEORY OF EXCHANGE

Way back in 1792, Pierre Prévost put forward the theory of radiation in a systematic way now known as the *theory of exchange*. According to this theory, all bodies radiate thermal radiation at all temperatures. The amount of thermal radiation radiated per unit time depends on the nature of the emitting surface, its area and its temperature. The rate is faster at higher temperatures. Besides, a body also absorbs part of the thermal radiation emitted by the surrounding bodies when this radiation falls on it. If a body radiates more than what it absorbs, its temperature falls. If a body radiates less than what it absorbs, its temperature rises.

Now, consider a body kept in a room for a long time. One finds that the temperature of the body remains constant and is equal to the room temperature. The body is still radiating thermal radiation. But it is also absorbing part of the radiation emitted by the surrounding objects, walls, etc. We thus conclude that when the temperature of a body is equal to the temperature of its surroundings, it radiates at

the same rate as it absorbs. If we place a hotter body in the room, it radiates at a faster rate than the rate at which it absorbs. Thus, the body suffers a net loss of thermal energy in any given time and its temperature decreases. Similarly, if a colder body is kept in a warm surrounding, it radiates less to the surrounding than what it absorbs from the surrounding. Consequently, there is a net increase in the thermal energy of the body and the temperature rises.

## 28.7 BLACKBODY RADIATION

Consider two bodies of equal surface areas suspended in a room. One of the bodies has polished surface and the other is painted black. After sufficient time, the temperature of both the bodies will be equal to the room temperature. As the surface areas of the bodies are the same, equal amount of radiation falls on the two surfaces. The polished surface reflects a large part of it and absorbs a little, while the black-painted surface reflects a little and absorbs a large part of it. As the temperature of each body remains constant, we conclude that the polished surface radiates at a slower rate and the black-painted surface radiates at a faster rate. So, good absorbers of radiation are also good emitters.

A body that absorbs all the radiation falling on it is called a *blackbody*. Such a body will emit radiation at the fastest rate. The radiation emitted by a blackbody is called *blackbody radiation*. The radiation inside an enclosure with its inner walls maintained at a constant temperature has the same properties as the blackbody radiation and is also called blackbody radiation. A blackbody is also called an *ideal radiator*. A perfect blackbody, absorbing 100% of the radiation falling on it, is only an ideal concept. Among the materials, lampblack is close to a blackbody. It reflects only about 1% of the radiation falling on it. If an enclosure is painted black from inside and a small hole is made in the wall (figure 28.6) the hole acts as a very good blackbody.

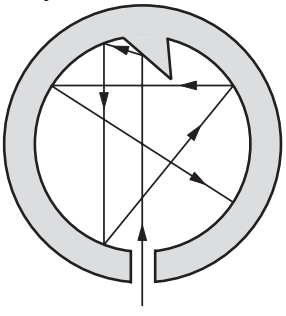


Figure 28.6

Any radiation that falls on the hole goes inside. This radiation has little chance to come out of the hole again and it gets absorbed after multiple reflections. The cone directly opposite to the hole (figure 28.6) ensures that the incoming radiation is not directly reflected back to the hole.

## 28.8 KIRCHHOFF'S LAW

We have learnt that good absorbers of radiation are also good radiators. This aspect is described quantitatively by Kirchhoff's law of radiation. Before stating the law let us define certain terms.

### Emissive Power

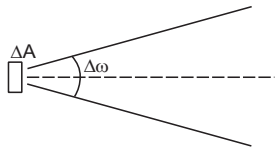



Figure 28.7

Consider a small area $\Delta A$ of a body emitting thermal radiation. Consider a small solid angle $\Delta\omega$ (see the chapter "Gauss's Law") about the normal to the radiating surface. Let the energy radiated by the area $\Delta A$ of the surface in the solid angle $\Delta\omega$ in time $\Delta t$ be $\Delta U$. We define *emissive power* of the body as

$$E = \frac{\Delta U}{(\Delta A)\,(\Delta\omega)\,(\Delta t)}\,.$$

Thus, emissive power denotes the energy radiated per unit area per unit time per unit solid angle along the normal to the area.

### Absorptive Power

*Absorptive power* of a body is defined as the fraction of the incident radiation that is absorbed by the body. If we denote the absorptive power by $a$,

$$a = \frac{\text{energy absorbed}}{\text{energy incident}}\,.$$

As all the radiation incident on a blackbody is absorbed, the absorptive power of a blackbody is unity.

Note that the absorptive power is a dimensionless quantity but the emissive power is not.

### Kirchhoff's Law

*The ratio of emissive power to absorptive power is the same for all bodies at a given temperature and is equal to the emissive power of a blackbody at that temperature.* Thus,

$$\frac{E(\text{body})}{a(\text{body})} = E(\text{blackbody})\,.$$

Kirchhoff's law tells that if a body has high emissive power, it should also have high absorptive

power to have the ratio $E/a$ same. Similarly, a body having low emissive power should have low absorptive power. Kirchhoff's law may be easily proved by a simple argument as described below.

Figure 28.8

Consider two bodies $A$ and $B$ of identical geometrical shapes placed in an enclosure. Suppose $A$ is an arbitrary body and $B$ is a blackbody. In thermal equilibrium, both the bodies will have the same temperature as the temperature of the enclosure. Suppose an amount $\Delta U$ of radiation falls on the body $A$ in a given time $\Delta t$. As $A$ and $B$ have the same geometrical shapes, the radiation falling on the blackbody $B$ is also $\Delta U$. The blackbody absorbs all of this $\Delta U$. As the temperature of the blackbody remains constant, it also emits an amount $\Delta U$ of radiation in that time. If the emissive power of the blackbody is $E_0$, we have

$$\Delta U \propto E_0 \text{ or } \Delta U = kE_0 \quad \ldots \text{ (i)}$$

where $k$ is a constant.

Let the absorptive power of $A$ be $a$. Thus, it absorbs an amount $a\Delta U$ of the radiation falling on it in time $\Delta t$. As its temperature remains constant, it must also emit the same amount $a\Delta U$ in that time. If the emissive power of the body $A$ is $E$, we have

$$a\Delta U = kE. \quad \ldots \text{ (ii)}$$

The same proportionality constant $k$ is used in (i) and (ii) because the two bodies have identical geometrical shapes and radiation emitted in the same time $\Delta t$ is considered.

From (i) and (ii),

$$a = \frac{E}{E_0}$$

or,
$$\frac{E}{a} = E_0$$

or,
$$\frac{E(\text{body})}{a(\text{body})} = E(\text{blackbody}).$$

This proves Kirchhoff's law.

## 28.9 NATURE OF THERMAL RADIATION

Thermal radiation, once emitted, is an electromagnetic wave like light. It, therefore, obeys all the laws of wave theory. The wavelengths are still small compared to the dimensions of usual obstacles encountered, so the rules of geometrical optics are valid, i.e., it travels in a straight line, casts shadow, is reflected and refracted at the change of medium, etc. The radiation emitted by a body is a mixture of waves of different wavelengths. However, only a small range of wavelength has significant contribution in the total radiation. The radiation emitted by a body at a temperature of 300 K (room temperature) has significant contribution from wavelengths around 9550 nm which is in long infrared region (visible light has a range of about 380–780 nm). As the temperature of the emitter increases, this dominant wavelength decreases. At around 1100 K, the radiation has a good contribution from red region of wavelengths and the object appears red. At temperatures around 3000 K, the radiation contains enough shorter wavelengths and the object appears white. Even at such a high temperature most significant contributions come from wavelengths around 950 nm.

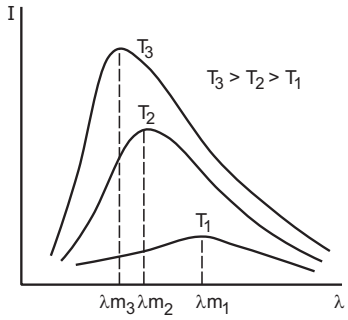



Figure 28.9

The relative importance of different wavelengths in a thermal radiation can be studied qualitatively from figure (28.9). Here the intensity of radiation near a given wavelength is plotted against the wavelength for different temperatures. We see that as the temperature is increased, the wavelength corresponding to the highest intensity decreases. In fact, this wavelength $\lambda_m$ is inversely proportional to the absolute temperature of the emitter. So,

$$\lambda_m T = b \quad \ldots \text{ (28.5)}$$

where $b$ is a constant.

This equation is known as the *Wien's displacement law*. For a blackbody, the constant $b$ appearing in equation (28.5) is measured to be 0·288 cmK and is known as the *Wien constant*.

***Example 28.3***

*The light from the sun is found to have a maximum intensity near the wavelength of* 470 nm. *Assuming that the surface of the sun emits as a blackbody, calculate the temperature of the surface of the sun.*

***Solution*** : For a blackbody, $\lambda_m T = 0{\cdot}288$ cmK.

Thus, $T = \dfrac{0{\cdot}288 \text{ cmK}}{470 \text{ nm}} = 6130$ K.

Distribution of radiant energy among different wavelengths has played a very significant role in the development of quantum mechanics and in our understanding of nature in a new way. Classical physics had predicted a very different and unrealistic wavelength distribution. Planck put forward a bold hypothesis that radiation can be emitted or absorbed only in discrete steps, each step involving an amount of energy given by $E = nh\nu$ where $\nu$ is the frequency of the radiation and $n$ is an integer. A new fundamental constant $h$ named as *Planck constant* was introduced in physics. This opened the gateway of modern physics through which we look into the atomic and subatomic world.

## 28.10 STEFAN–BOLTZMANN LAW

*The energy of thermal radiation emitted per unit time by a blackbody of surface area A is given by*

$$u = \sigma A T^4 \quad \ldots (28.6)$$

*where* $\sigma$ *is a universal constant known as Stefan–Boltzmann constant and T is its temperature on absolute scale.* The measured value of $\sigma$ is $5{\cdot}67 \times 10^{-8}$ $\mathrm{Wm^{-2}K^{-4}}$. Equation (28.6) itself is called the *Stefan–Boltzmann law*. Stefan had suggested this law from experimental data available on radiation and Boltzmann derived it from thermodynamical considerations. The law is also quoted as *Stefan's law* and the constant $\sigma$ as *Stefan constant*.

A body which is not a blackbody, emits less radiation than given by equation (28.6). It is, however, proportional to $T^4$. The energy emitted by such a body per unit time is written as

$$u = e\sigma A T^4 \quad \ldots (28.7)$$

where $e$ is a constant for the given surface having a value between 0 and 1. This constant is called the *emissivity* of the surface. It is zero for completely reflecting surface and is unity for a blackbody.

Using Kirchhoff's law,

$$\frac{E(\text{body})}{E(\text{blackbody})} = a \quad \ldots (i)$$

where $a$ is the absorptive power of the body. The emissive power $E$ is proportional to the energy radiated per unit time, that is, proportional to $u$. Using equations (28.6) and (28.7) in (i),

$$\frac{e\sigma A T^4}{\sigma A T^4} = a \text{ or } e = a.$$

Thus, emissivity and absorptive power have the same value.

Consider a body of emissivity $e$ kept in thermal equilibrium in a room at temperature $T_0$. The energy of radiation absorbed by it per unit time should be equal to the energy emitted by it per unit time. This is because the temperature remains constant. Thus, the energy of the radiation absorbed per unit time is

$$u = e\sigma A T_0^4.$$

Now, suppose the temperature of the body is changed to $T$ but the room temperature remains $T_0$. The energy of the thermal radiation emitted by the body per unit time is

$$u = e\sigma A T^4.$$

The energy absorbed per unit time by the body is

$$u_0 = e\sigma A T_0^4.$$

Thus, the net loss of thermal energy per unit time is

$$\Delta u = u - u_0 = e\sigma A(T^4 - T_0^4). \quad \ldots (28.8)$$

***Example 28.4***

*A blackbody of surface area* $10\ \mathrm{cm^2}$ *is heated to* 127°C *and is suspended in a room at temperature* 27°C. *Calculate the initial rate of loss of heat from the body to the room.*

***Solution*** : For a blackbody at temperature $T$, the rate of emission is $u = \sigma A T^4$. When it is kept in a room at temperature $T_0$, the rate of absorption is $u_0 = \sigma A T_0^4$.

The net rate of loss of heat is $u - u_0 = \sigma A(T^4 - T_0^4)$.

Here $A = 10 \times 10^{-4}\ \mathrm{m^2}$, $T = 400$ K and $T_0 = 300$ K . Thus,

$u - u_0$

$= (5{\cdot}67 \times 10^{-8}\ \mathrm{Wm^{-2}K^{-4}})\ (10 \times 10^{-4}\ \mathrm{m^2})\ (400^4 - 300^4)\ \mathrm{K^4}$

$= 0{\cdot}99$ W.

## 28.11 NEWTON'S LAW OF COOLING

Suppose, a body of surface area $A$ at an absolute temperature $T$ is kept in a surrounding having a lower temperature $T_0$. The net rate of loss of thermal energy from the body due to radiation is

$$\Delta u_1 = e\sigma A(T^4 - T_0^4).$$

If the temperature difference is small, we can write

$$T = T_0 + \Delta T$$

or, $$T^4 - T_0^4 = (T_0 + \Delta T)^4 - T_0^4$$

$$= T_0^4\left(1 + \frac{\Delta T}{T_0}\right)^4 - T_0^4$$

$$= T_0^4\left[1 + 4\,\frac{\Delta T}{T_0} + \text{higher powers of } \frac{\Delta T}{T_0}\right] - T_0^4$$

$$\approx 4\,T_0^3 \Delta T = 4\,T_0^3(T - T_0).$$

Thus, $$\Delta u_1 = 4\,e\sigma A T_0^3 (T - T_0)$$

$$= b_1 A(T - T_0).$$

The body may also lose thermal energy due to convection in the surrounding air. For small temperature difference, the rate of loss of heat due to convection is also proportional to the temperature difference and the area of the surface. This rate may, therefore, be written as

$$\Delta u_2 = b_2 A(T - T_0).$$

The net rate of loss of thermal energy due to convection and radiation is

$$\Delta u = \Delta u_1 + \Delta u_2 = (b_1 + b_2)A(T - T_0).$$

If $s$ be the specific heat capacity of the body and $m$ its mass, the rate of fall of temperature is

$$\frac{-dT}{dt} = \frac{\Delta u}{ms} = \frac{b_1 + b_2}{ms} A(T - T_0)$$
$$= bA(T - T_0).$$

Thus, for small temperature difference between a body and its surrounding, the rate of cooling of the body is directly proportional to the temperature difference and the surface area exposed. We can write

$$\frac{dT}{dt} = -bA(T - T_0). \quad \ldots (28.9)$$

This is known as *Newton's law of cooling*. The constant $b$ depends on the nature of the surface involved and the surrounding conditions. The minus sign indicates that if $T > T_0$, $dT/dt$ is negative, that is, the temperature decreases with time. As the difference in temperature is the same for absolute and Celsius scale, equation (28.9) may also be written as

$$\frac{d\theta}{dt} = -bA(\theta - \theta_0) = -k(\theta - \theta_0)$$

where $\theta$ refers to temperature in Celsius scale.

***Example 28.5***

*A liquid cools from* 70°C *to* 60°C *in* 5 minutes . *Calculate the time taken by the liquid to cool from* 60°C *to* 50°C, *if the temperature of the surrounding is constant at* 30°C.

***Solution*** : The average temperature of the liquid in the first case is

$$\theta_1 = \frac{70°\text{C} + 60°\text{C}}{2} = 65°\text{C}.$$

The average temperature difference from the surrounding is

$$\theta_1 - \theta_0 = 65°\text{C} - 30°\text{C} = 35°\text{C}.$$

The rate of fall of temperature is

$$-\frac{d\theta_1}{dt} = \frac{70°\text{C} - 60°\text{C}}{5 \text{ min}} = 2°\text{C min}^{-1}.$$

From Newton's law of cooling,

$$2°\text{C min}^{-1} = bA(35°\text{C})$$

or,
$$bA = \frac{2}{35 \text{ min}}. \quad \ldots (i)$$

In the second case, the average temperature of the liquid is

$$\theta_2 = \frac{60°\text{C} + 50°\text{C}}{2} = 55°\text{C}$$

so that, $\theta_2 - \theta_0 = 55°\text{C} - 30°\text{C} = 25°\text{C}.$

If it takes a time $t$ to cool down from 60°C to 50°C, the rate of fall of temperature is

$$-\frac{d\theta_2}{dt} = \frac{60°\text{C} - 50°\text{C}}{t} = \frac{10°\text{C}}{t}.$$

From Newton's law of cooling and (i),

$$\frac{10°\text{C}}{t} = \frac{2}{35 \text{ min}} \times 25°\text{C}$$

or,
$$t = 7 \text{ min}.$$

## 28.12 DETECTION AND MEASUREMENT OF RADIATION

Several instruments are used to detect and measure the amount of thermal radiation. We describe two of them here, a bolometer and a thermopile.

**Bolometer**

The bolometer is based on the theory of Wheatstone bridge which was introduced while discussing resistance thermometer. A thin (a small fraction of a millimetre) foil of platinum is taken and strips are cut from it to leave a grid-type structure as shown in figure (28.10a). Four such grids, identical in all respect, are prepared and joined with a battery and a galvanometer to form a Wheatstone bridge (figure 28.10b). Grid 1 faces the radiation to be detected or measured. The particular arrangement of the four grids ensures that the radiation passing through the empty spaces in grid 1 falls on the strips of grid 4. Grids 2 and 3 are protected from the radiation.

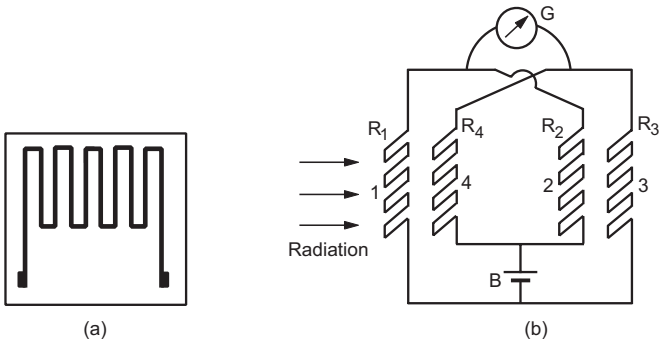



Figure 28.10

When no radiation falls on the bolometer, all the grids have the same resistance so that

$R_1 = R_2 = R_3 = R_4$. Thus, $\frac{R_1}{R_2} = \frac{R_3}{R_4}$ and the bridge is balanced. There is no deflection in the galvanometer. When radiation falls on the bolometer, grids 1 and 4 get heated. As the temperature increases, the resistances $R_1$ and $R_4$ increase and hence the product $R_1 R_4$ increases. On the other hand, $R_2 R_3$ remains unchanged. Thus,

$$R_1 R_4 > R_2 R_3$$

or, $$\frac{R_1}{R_2} > \frac{R_3}{R_4}.$$

The bridge becomes unbalanced and there is a deflection in the galvanometer which indicates the presence of radiation. The magnitude of deflection is related to the amount of radiation falling on the bolometer.

The bolometer is usually enclosed in a glass bulb evacuated to low pressures. This increases the sensitivity.

**Thermopile**

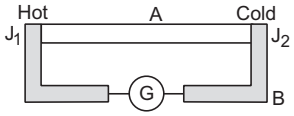



Figure 28.11

A thermopile is based on the principle of *Seebeck effect*. Figure (28.11) illustrates the principle. Two dissimilar metals $A$ and $B$ are joined to form two junctions $J_1$ and $J_2$. A galvanometer is connected between the junctions through the metal $B$. If the junctions are at the same temperature, there is no deflection in the galvanometer. But if the temperatures of the junctions are different, the galvanometer deflects. Such an instrument is called a *thermocouple*.

In a thermopile, a number of thermocouples are joined in series. The thermocouples are made from antimony and bismuth metals. The free ends are joined to a galvanometer (figure 28.12). The series connection of thermocouples increases the sensitivity of the system.

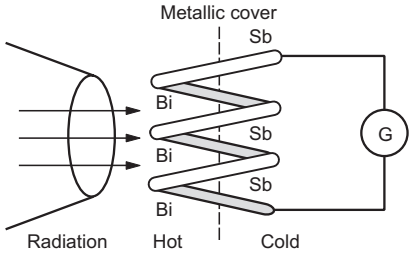



Figure 28.12

The junctions are arranged in such a way that all the hot junctions lie on a plane face and all the cold junctions lie on the opposite plane face. The face of the hot junctions is blackened so that it may absorb large fraction of radiation falling on it. This face is exposed to the radiation and the other face is protected from it by a metallic cover. A metallic cone generally concentrates the radiation on the hot face.

The radiation is detected and measured by observing the deflection in the galvanometer.

## *Worked Out Examples*

**1.** *The lower surface of a slab of stone of face-area* 3600 $cm^2$ *and thickness* 10 cm *is exposed to steam at* 100°C. *A block of ice at* 0°C *rests on the upper surface of the slab.* 4·8 g *of ice melts in one hour. Calculate the thermal conductivity of the stone. Latent heat of fusion of ice* $= 3{\cdot}36 \times 10^5$ J $kg^{-1}$.

***Solution*** : The amount of heat transferred through the slab to the ice in one hour is

$$Q = (4{\cdot}8 \times 10^{-3}\ \text{kg}) \times (3{\cdot}36 \times 10^5\ \text{J kg}^{-1})$$

$$= 4{\cdot}8 \times 336\ \text{J}.$$

Using the equation

$$Q = \frac{KA(\theta_1 - \theta_2)t}{x},$$

$$4{\cdot}8 \times 336\ \text{J} = \frac{K(3600\ \text{cm}^2)\ (100°\text{C}) \times (3600\ \text{s})}{10\ \text{cm}}$$

or, $$K = 1{\cdot}24 \times 10^{-3}\ \text{W m}^{-1}°\text{C}^{-1}.$$

**2.** *An icebox made of* 1·5 cm *thick styrofoam has dimensions* 60 cm × 60 cm × 30 cm. *It contains ice at* 0°C *and is kept in a room at* 40°C. *Find the rate at which the ice is melting. Latent heat of fusion of ice* $= 3{\cdot}36 \times 10^5$ J $kg^{-1}$. *and thermal conductivity of styrofoam* $= 0{\cdot}04$ W $m^{-1}°C^{-1}$.

***Solution*** : The total surface area of the walls

$= 2(60\ \text{cm} \times 60\ \text{cm} + 60\ \text{cm} \times 30\ \text{cm} + 60\ \text{cm} \times 30\ \text{cm})$

$= 1{\cdot}44\ \text{m}^2$.

The thickness of the walls = 1·5 cm = 0·015 m.

The rate of heat flow into the box is

$$\frac{\Delta Q}{\Delta t} = \frac{KA(\theta_1 - \theta_2)}{x}$$

$$= \frac{(0{\cdot}04\ \text{W m}^{-1}°\text{C}^{-1})\ (1{\cdot}44\ \text{m}^2)\ (40°\text{C})}{0{\cdot}015\ \text{m}} = 154\ \text{W}.$$

The rate at which the ice melts is

$$= \frac{154\ \text{W}}{3{\cdot}36 \times 10^5\ \text{J kg}^{-1}} = 0{\cdot}46\ \text{g s}^{-1}.$$

**3.** *A closed cubical box is made of perfectly insulating material and the only way for heat to enter or leave the box is through two solid cylindrical metal plugs, each of cross sectional area* 12 cm$^2$ *and length* 8 cm *fixed in the opposite walls of the box. The outer surface of one plug is kept at a temperature of* 100°C *while the outer surface of the other plug is maintained at a temperature of* 4°C. *The thermal conductivity of the material of the plug is* 2·0 Wm$^{-1}$°C$^{-1}$. *A source of energy generating* 13 W *is enclosed inside the box. Find the equilibrium temperature of the inner surface of the box assuming that it is the same at all points on the inner surface.*

***Solution*** :

Figure 28-W1

The situation is shown in figure (28-W1). Let the temperature inside the box be θ. The rate at which heat enters the box through the left plug is

$$\frac{\Delta Q_1}{\Delta t} = \frac{KA(\theta_1 - \theta)}{x}.$$

The rate of heat generation in the box = 13 W. The rate at which heat flows out of the box through the right plug is

$$\frac{\Delta Q_2}{\Delta t} = \frac{KA(\theta - \theta_2)}{x}.$$

In the steady state

$$\frac{\Delta Q_1}{\Delta t} + 13\ \text{W} = \frac{\Delta Q_2}{\Delta t}$$

or, $$\frac{KA}{x}(\theta_1 - \theta) + 13\ \text{W} = \frac{KA}{x}(\theta - \theta_2)$$

or, $$2\frac{KA}{x}\theta = \frac{KA}{x}(\theta_1 + \theta_2) + 13\ \text{W}$$

or, $$\theta = \frac{\theta_1 + \theta_2}{2} + \frac{(13\ \text{W})\,x}{2KA}$$

$$= \frac{100°\text{C} + 4°\text{C}}{2} + \frac{(13\ \text{W}) \times 0{\cdot}08\ \text{m}}{2 \times (2{\cdot}0\ \text{W m}^{-1}\text{°C}^{-1})\,(12 \times 10^{-4}\ \text{m}^2)}$$

$$= 52°\text{C} + 216{\cdot}67°\text{C} \approx 269°\text{C}.$$

**4.** *A bar of copper of length* 75 cm *and a bar of steel of length* 125 cm *are joined together end to end. Both are of circular cross section with diameter* 2 cm. *The free ends of the copper and the steel bars are maintained at* 100°C *and* 0°C *respectively. The curved surfaces of the bars are thermally insulated. What is the temperature of the copper–steel junction? What is the amount of heat transmitted per unit time across the junction? Thermal conductivity of copper is* 386 J s$^{-1}$ m$^{-1}$°C$^{-1}$ *and that of steel is* 46 J s$^{-1}$ m$^{-1}$°C$^{-1}$.

***Solution*** :

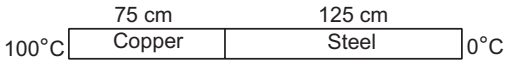



Figure 28-W2

The situation is shown in figure (28-W2). Let the temperature at the junction be θ (on Celsius scale). The same heat current passes through the copper and the steel rods. Thus,

$$\frac{\Delta Q}{\Delta t} = \frac{K_{cu}\,A(100°\text{C} - \theta)}{75\ \text{cm}} = \frac{K_{steel}\,A\theta}{125\ \text{cm}}$$

or, $$\frac{K_{cu}\,(100°\text{C} - \theta)}{75} = \frac{K_{steel}\,\theta}{125}$$

or, $$\frac{100°\text{C} - \theta}{\theta} = \frac{75\,K_{steel}}{125\,K_{cu}} = \frac{3}{5} \times \frac{46}{386}$$

or, $$\theta = 93°\text{C}.$$

The rate of heat flow is

$$\frac{\Delta Q}{\Delta t} = \frac{K_{steel} A\theta}{125\ \text{cm}}$$

$$= \frac{(46\ \text{J s}^{-1}\ \text{m}^{-1}\text{°C}^{-1})\,(\pi \times 1\ \text{cm}^2) \times 93°\text{C}}{125\ \text{cm}}$$

$$= 1{\cdot}07\ \text{J s}^{-1}.$$

**5.** *Two parallel plates A and B are joined together to form a compound plate (figure 28-W3). The thicknesses of the plates are* 4·0 cm *and* 2·5 cm *respectively and the area of cross section is* 100 cm$^2$ *for each plate. The thermal conductivities are* $K_A = 200$ W m$^{-1}$°C$^{-1}$ *for the plate A and* $K_B = 400$ W m$^{-1}$°C$^{-1}$ *for the plate B. The outer surface of the plate A is maintained at* 100°C *and the outer surface of the plate B is maintained at* 0°C. *Find (a) the rate of heat flow through any cross section, (b) the temperature at the interface and (c) the equivalent thermal conductivity of the compound plate.*

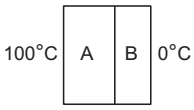



Figure 28-W3

***Solution*** : (a) Let the temperature of the interface be θ. The area of cross section of each plate is $A = 100\ \text{cm}^2 = 0{\cdot}01\ \text{m}^2$. The thicknesses are $x_A = 0{\cdot}04$ m and $x_B = 0{\cdot}025$ m.

The thermal resistance of the plate $A$ is

$$R_1 = \frac{1}{K_A}\frac{x_A}{A}$$

and that of the plate $B$ is

$$R_2 = \frac{1}{K_B}\frac{x_B}{A}.$$

The equivalent thermal resistance is

$$R = R_1 + R_2 = \frac{1}{A}\left(\frac{x_A}{K_A} + \frac{x_B}{K_B}\right). \quad \ldots \text{(i)}$$

Thus, $\frac{\Delta Q}{\Delta t} = \frac{\theta_1 - \theta_2}{R}$

$$= \frac{A(\theta_1 - \theta_2)}{x_A/K_A + x_B/K_B}$$

$$= \frac{(0{\cdot}01\ \text{m}^2)\ (100°\text{C})}{(0{\cdot}04\ \text{m})/(200\ \text{W m}^{-1}°\text{C}^{-1}) + (0{\cdot}025\ \text{m})/(400\ \text{W m}^{-1}°\text{C}^{-1})}$$

$= 3810$ W.

(b) We have $\frac{\Delta Q}{\Delta t} = \frac{A(\theta - \theta_2)}{x_B/K_B}$

or, $3810\ \text{W} = \frac{(0{\cdot}01\ \text{m}^2)\ (\theta - 0°\text{C})}{(0{\cdot}025\ \text{m})/(400\ \text{W m}^{-1}°\text{C}^{-1})}$

or, $\theta = 24°\text{C}$.

(c) If $K$ is the equivalent thermal conductivity of the compound plate, its thermal resistance is

$$R = \frac{1}{A}\frac{x_A + x_B}{K}.$$

Comparing with (i),

$$\frac{x_A + x_B}{K} = \frac{x_A}{K_A} + \frac{x_B}{K_B}$$

or, $$K = \frac{x_A + x_B}{x_A/K_A + x_B/K_B}$$

$$= 248\ \text{W m}^{-1}°\text{C}^{-1}.$$

**6.** *A room has a* $4\ \text{m} \times 4\ \text{m} \times 10\ \text{cm}$ *concrete roof* ($K = 1{\cdot}26\ \text{Wm}^{-1}°\text{C}^{-1}$). *At some instant, the temperature outside is* 46°C *and that inside is* 32°C. *(a) Neglecting convection, calculate the amount of heat flowing per second into the room through the roof. (b) Bricks* ($K = 0{\cdot}65\ \text{Wm}^{-1}°\text{C}^{-1}$) *of thickness* 7·5 cm *are laid down on the roof. Calculate the new rate of heat flow under the same temperature conditions.*

***Solution*** : The area of the roof

$$= 4\ \text{m} \times 4\ \text{m} = 16\ \text{m}^2.$$

The thickness $x = 10\ \text{cm} = 0{\cdot}10\ \text{m}$.

(a) The thermal resistance of the roof is

$$R_1 = \frac{1}{K}\frac{x}{A} = \frac{1}{1{\cdot}26\ \text{W m}^{-1}°\text{C}^{-1}}\frac{0{\cdot}10\ \text{m}}{16\ \text{m}^2}$$

$$= 4{\cdot}96 \times 10^{-3}\ °\text{C W}^{-1}.$$

The heat current is

$$\frac{\Delta Q}{\Delta t} = \frac{\theta_1 - \theta_2}{R_1} = \frac{46°\text{C} - 32°\text{C}}{4{\cdot}96 \times 10^{-3}\ °\text{C W}^{-1}}$$

$$= 2822\ \text{W}.$$

(b) The thermal resistance of the brick layer is

$$R_2 = \frac{1}{K}\frac{x}{A} = \frac{1}{0{\cdot}65\ \text{W m}^{-1}°\text{C}^{-1}}\frac{7{\cdot}5 \times 10^{-2}\ \text{m}}{16\ \text{m}^2}$$

$$= 7{\cdot}2 \times 10^{-3}\ °\text{C W}^{-1}.$$

The equivalent thermal resistance is

$$R = R_1 + R_2 = (4{\cdot}96 + 7{\cdot}2) \times 10^{-3}\ °\text{C W}^{-1}$$

$$= 1{\cdot}216 \times 10^{-2}\ °\text{C W}^{-1}.$$

The heat current is

$$\frac{\Delta Q}{\Delta t} = \frac{\theta_1 - \theta_2}{R} = \frac{46°\text{C} - 32°\text{C}}{1{\cdot}216 \times 10^{-2}\ °\text{C W}^{-1}}.$$

$$= 1152\ \text{W}.$$

**7.** *An electric heater is used in a room of total wall area* $137\ \text{m}^2$ *to maintain a temperature of* 20°C *inside it, when the outside temperature is* – 10°C. *The walls have three different layers of materials. The innermost layer is of wood of thickness* 2·5 cm, *the middle layer is of cement of thickness* 1·0 cm *and the outermost layer is of brick of thickness* 25·0 cm. *Find the power of the electric heater. Assume that there is no heat loss through the floor and the ceiling. The thermal conductivities of wood, cement and brick are* $0{\cdot}125\ \text{Wm}^{-1}°\text{C}^{-1}$, $1{\cdot}5\ \text{Wm}^{-1}°\text{C}^{-1}$ *and* $1{\cdot}0\ \text{Wm}^{-1}°\text{C}^{-1}$ *respectively.*

***Solution*** :

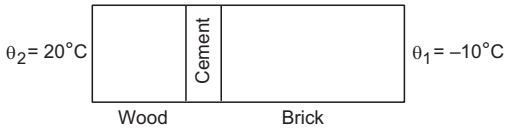



Figure 28-W4

The situation is shown in figure (28-W4).

The thermal resistances of the wood, the cement and the brick layers are

$$R_W = \frac{1}{K}\frac{x}{A}$$

$$= \frac{1}{0{\cdot}125\ \text{W m}^{-1}°\text{C}^{-1}}\frac{2{\cdot}5 \times 10^{-2}\ \text{m}}{137\ \text{m}^2}$$

$$= \frac{0{\cdot}20}{137}\ °\text{C W}^{-1},$$

$$R_C = \frac{1}{1{\cdot}5\ \text{W m}^{-1}°\text{C}^{-1}}\frac{1{\cdot}0 \times 10^{-2}\ \text{m}}{137\ \text{m}^2}$$

$$= \frac{0{\cdot}0067}{137}\ °\text{C W}^{-1}$$

and $$R_B = \frac{1}{1{\cdot}0\ \text{W m}^{-1}°\text{C}^{-1}}\frac{25{\cdot}0 \times 10^{-2}\ \text{m}}{137\ \text{m}^2}$$

$$= \frac{0{\cdot}25}{137}\ °\text{C W}^{-1}.$$

As the layers are connected in series, the equivalent thermal resistance is

$$R = R_W + R_C + R_B$$

$$= \frac{0{\cdot}20 + 0{\cdot}0067 + 0{\cdot}25}{137} \,^\circ\text{C W}^{-1}$$

$$= 3{\cdot}33 \times 10^{-3} \,^\circ\text{C W}^{-1}.$$

The heat current is

$$i = \frac{\theta_1 - \theta_2}{R}$$

$$= \frac{20^\circ\text{C} - (-10^\circ\text{C})}{3{\cdot}33 \times 10^{-3} \,^\circ\text{C W}^{-1}} \approx 9000 \text{ W}.$$

The heater must supply 9000 W to compensate the outflow of heat.

**8.** *Three rods of material x and three of material y are connected as shown in figure (28-W5). All the rods are identical in length and cross sectional area. If the end A is maintained at* 60°C *and the junction E at* 10°C, *calculate the temperature of the junction B. The thermal conductivity of x is* 800 W m$^{-1}$°C$^{-1}$ *and that of y is* 400 W m$^{-1}$°C$^{-1}$.

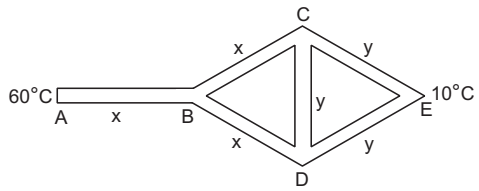



Figure 28-W5

***Solution*** : It is clear from the symmetry of the figure that the points *C* and *D* are equivalent in all respect and hence, they are at the same temperature, say θ. No heat will flow through the rod *CD*. We can, therefore, neglect this rod in further analysis.

Let *l* and *A* be the length and the area of cross section of each rod. The thermal resistances of *AB*, *BC* and *BD* are equal. Each has a value

$$R_1 = \frac{1}{K_x}\frac{l}{A}. \quad \ldots \text{(i)}$$

Similarly, thermal resistances of *CE* and *DE* are equal, each having a value

$$R_2 = \frac{1}{K_y}\frac{l}{A}. \quad \ldots \text{(ii)}$$

As the rod *CD* has no effect, we can say that the rods *BC* and *CE* are joined in series. Their equivalent thermal resistance is

$$R_3 = R_{BC} + R_{CE} = R_1 + R_2.$$

Also, the rods *BD* and *DE* together have an equivalent thermal resistance $R_4 = R_{BD} + R_{DE} = R_1 + R_2$.

The resistances $R_3$ and $R_4$ are joined in parallel and hence their equivalent thermal resistance is given by

$$\frac{1}{R_5} = \frac{1}{R_3} + \frac{1}{R_4} = \frac{2}{R_3}$$

or, $$R_5 = \frac{R_3}{2} = \frac{R_1 + R_2}{2}.$$

This resistance $R_5$ is connected in series with *AB*. Thus, the total arrangement is equivalent to a thermal resistance

$$R = R_{AB} + R_5 = R_1 + \frac{R_1 + R_2}{2} = \frac{3R_1 + R_2}{2}.$$

Figure (28-W6) shows the successive steps in this reduction.

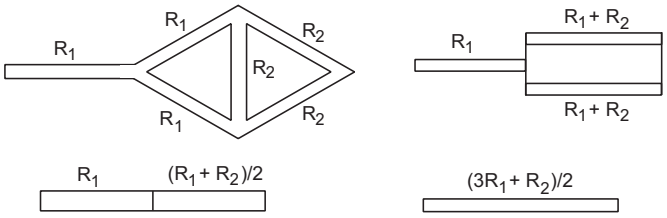



Figure 28-W6

The heat current through *A* is

$$i = \frac{\theta_A - \theta_E}{R} = \frac{2(\theta_A - \theta_E)}{3R_1 + R_2}.$$

This current passes through the rod *AB*. We have

$$i = \frac{\theta_A - \theta_B}{R_{AB}}$$

or, $$\theta_A - \theta_B = (R_{AB})i$$

$$= R_1 \frac{2(\theta_A - \theta_E)}{3R_1 + R_2}.$$

Putting from (i) and (ii),

$$\theta_A - \theta_B = \frac{2K_y(\theta_A - \theta_E)}{K_x + 3K_y}$$

$$= \frac{2 \times 400}{800 + 3 \times 400} \times 50^\circ\text{C} = 20^\circ\text{C}$$

or, $$\theta_B = \theta_A - 20^\circ\text{C} = 40^\circ\text{C}.$$

**9.** *A rod CD of thermal resistance* 5·0 K W$^{-1}$ *is joined at the middle of an identical rod AB as shown in figure (28-W7). The ends A, B and D are maintained at* 100°C, 0°C *and* 25°C *respectively. Find the heat current in CD.*

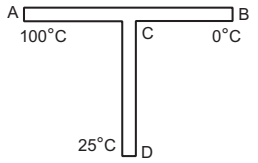



Figure 28-W7

***Solution*** : The thermal resistance of *AC* is equal to that of *CB* and is equal to 2·5 K W$^{-1}$. Suppose, the temperature at *C* is θ. The heat currents through *AC*, *CB* and *CD* are

$$\frac{\Delta Q_1}{\Delta t} = \frac{100^\circ\text{C} - \theta}{2{\cdot}5 \text{ K W}^{-1}},$$

$$\frac{\Delta Q_2}{\Delta t} = \frac{\theta - 0°C}{2{\cdot}5 \text{ K W}^{-1}}$$

and $$\frac{\Delta Q_3}{\Delta t} = \frac{\theta - 25°C}{5{\cdot}0 \text{ K W}^{-1}}.$$

We also have

$$\frac{\Delta Q_1}{\Delta t} = \frac{\Delta Q_2}{\Delta t} + \frac{\Delta Q_3}{\Delta t}$$

or, $$\frac{100°C - \theta}{2{\cdot}5} = \frac{\theta - 0°C}{2{\cdot}5} + \frac{\theta - 25°C}{5}$$

or, $$225°C = 5\theta$$

or, $$\theta = 45°C.$$

Thus, $$\frac{\Delta Q_3}{\Delta t} = \frac{45°C - 25°C}{5{\cdot}0 \text{ K W}^{-1}} = \frac{20 \text{ K}}{5{\cdot}0 \text{ K W}^{-1}}$$

$$= 4{\cdot}0 \text{ W}.$$

**10.** *Two thin metallic spherical shells of radii $r_1$ and $r_2$ ($r_1 < r_2$) are placed with their centres coinciding. A material of thermal conductivity K is filled in the space between the shells. The inner shell is maintained at temperature $\theta_1$ and the outer shell at temperature $\theta_2$ ($\theta_1 < \theta_2$). Calculate the rate at which heat flows radially through the material.*

***Solution*** :

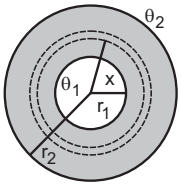



Figure 28-W8

Let us draw two spherical shells of radii $x$ and $x + dx$ concentric with the given system. Let the temperatures at these shells be $\theta$ and $\theta + d\theta$ respectively. The amount of heat flowing radially inward through the material between $x$ and $x + dx$ is

$$\frac{\Delta Q}{\Delta t} = \frac{K\, 4\pi x^2\, d\theta}{dx}.$$

Thus,

$$K\, 4\pi \int_{\theta_1}^{\theta_2} d\theta = \frac{\Delta Q}{\Delta t} \int_{r_1}^{r_2} \frac{dx}{x^2}$$

or, $$K\, 4\pi(\theta_2 - \theta_1) = \frac{\Delta Q}{\Delta t}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

or, $$\frac{\Delta Q}{\Delta t} = \frac{4\pi K r_1 r_2 (\theta_2 - \theta_1)}{r_2 - r_1}.$$

**11.** *On a cold winter day, the atmospheric temperature is $-\theta$ (on Celsius scale) which is below 0°C. A cylindrical drum of height h made of a bad conductor is completely filled with water at 0°C and is kept outside without any lid. Calculate the time taken for the whole mass of water to freeze. Thermal conductivity of ice is K and its latent heat of fusion is L. Neglect expansion of water on freezing.*

***Solution*** :

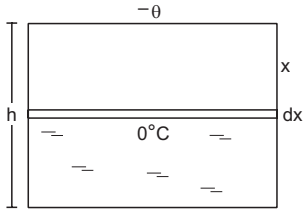



Figure 28-W9

Suppose, the ice starts forming at time $t = 0$ and a thickness $x$ is formed at time $t$. The amount of heat flown from the water to the surrounding in the time interval $t$ to $t + dt$ is

$$\Delta Q = \frac{KA\theta}{x}\, dt.$$

The mass of the ice formed due to the loss of this amount of heat is

$$dm = \frac{\Delta Q}{L} = \frac{KA\theta}{xL}\, dt.$$

The thickness $dx$ of ice formed in time $dt$ is

$$dx = \frac{dm}{A\rho} = \frac{K\theta}{\rho x L}\, dt$$

or, $$dt = \frac{\rho L}{K\theta}\, x\, dx.$$

Thus, the time $T$ taken for the whole mass of water to freeze is given by

$$\int_0^T dt = \frac{\rho L}{K\theta} \int_0^h x dx$$

or, $$T = \frac{\rho L h^2}{2K\theta}.$$

**12.** *Figure (28-W10) shows a large tank of water at a constant temperature $\theta_0$ and a small vessel containing a mass m of water at an initial temperature $\theta_1 (< \theta_0)$. A metal rod of length L, area of cross section A and thermal conductivity K connects the two vessels. Find the time taken for the temperature of the water in the smaller vessel to become $\theta_2 (\theta_1 < \theta_2 < \theta_0)$. Specific heat capacity of water is s and all other heat capacities are negligible.*

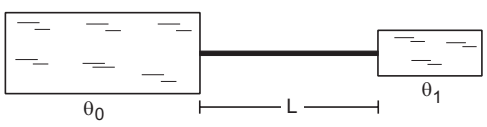



Figure 28-W10

***Solution*** : Suppose, the temperature of the water in the smaller vessel is θ at time $t$. In the next time interval $dt$, a heat $\Delta Q$ is transferred to it where

$$\Delta Q = \frac{KA}{L}(\theta_0 - \theta)\, dt. \quad \ldots \text{(i)}$$

This heat increases the temperature of the water of mass $m$ to $\theta + d\theta$ where

$$\Delta Q = ms\, d\theta. \quad \ldots \text{(ii)}$$

From (i) and (ii),

$$\frac{KA}{L}(\theta_0 - \theta)\, dt = ms\, d\theta$$

or, $$dt = \frac{Lms}{KA}\,\frac{d\theta}{\theta_0 - \theta}$$

or, $$\int_0^T dt = \frac{Lms}{KA}\int_{\theta_1}^{\theta_2}\frac{d\theta}{\theta_0 - \theta}$$

where $T$ is the time required for the temperature of the water to become $\theta_2$.

Thus, $$T = \frac{Lms}{KA}\ln\frac{\theta_0 - \theta_1}{\theta_0 - \theta_2}.$$

**13.** *One mole of an ideal monatomic gas is kept in a rigid vessel. The vessel is kept inside a steam chamber whose tempreature is* 97°C. *Initially, the temperature of the gas is* 5·0°C. *The walls of the vessel have an inner surface of area* 800 $cm^2$ *and thickness* 1·0 cm. *If the temperature of the gas increases to* 9·0°C *in* 5·0 seconds, *find the thermal conductivity of the material of the walls.*

***Solution*** : The initial temperature difference is 97°C – 5°C = 92°C and at 5·0 s the temperature difference becomes 97°C – 9°C = 88°C. As the change in the temperature difference is small, we work with the average temperature difference

$$\frac{92°C + 88°C}{2} = 90°C = 90\ K.$$

The rise in the temperature of the gas is

$$9·0°C - 5·0°C = 4°C = 4\ K.$$

The heat supplied to the gas in 5·0 s is

$$\Delta Q = nC_v\,\Delta T$$

$$= (1\ mol)\times\left(\frac{3}{2}\times 8·3\ JK^{-1}\ mol^{-1}\right)\times(4\ K)$$

$$= 49·8\ J.$$

If the thermal conductivity is $K$,

$$49·8\ J = \frac{K(800\times10^{-4}\ m^2)\times(90\ K)}{1·0\times10^{-2}\ m}\times 5·0\ s$$

or, $$K = \frac{49·8\ J}{3600\ msK} = 0·014\ J\ s^{-1}\ m^{-1}\ K^{-1}.$$

**14.** *A monatomic ideal gas is contained in a rigid container of volume V with walls of total inner surface area A, thickness x and thermal conductivity K. The gas is at an initial temperature $T_0$ and pressure $p_0$. Find the pressure of the gas as a function of time if the temperature of the surrounding air is $T_s$. All temperatures are in absolute scale.*

***Solution*** : As the volume of the gas is constant, a heat $\Delta Q$ given to the gas increases its temperature by $\Delta T = \Delta Q/C_v$. Also, for a monatomic gas, $C_v = \frac{3}{2}R$. If the temperature of the gas at time $t$ is $T$, the heat current into the gas is

$$\frac{\Delta Q}{\Delta t} = \frac{KA(T_s - T)}{x}$$

or, $$\frac{\Delta T}{\Delta t} = \frac{2\ KA}{3\ xR}(T_s - T)$$

or, $$\int_{T_0}^{T}\frac{dT}{T_s - T} = \int_0^t \frac{2\ KA}{3\ xR}\,dt$$

or, $$\ln\frac{T_s - T_0}{T_s - T} = \frac{2\ KA}{3\ xR}\,t$$

or, $$T_s - T = (T_s - T_0)\,e^{-\frac{2\ KA}{3xR}t}$$

or, $$T = T_s - (T_s - T_0)\,e^{-\frac{2\ KA}{3xR}t}.$$

As the volume remains constant,

$$\frac{p}{T} = \frac{p_0}{T_0}$$

or, $$p = \frac{p_0}{T_0}T$$

$$= \frac{p_0}{T_0}\left[T_s - (T_s - T_0)\,e^{-\frac{2\ KA}{3\ xR}t}\right].$$

**15.** *Consider a cubical vessel of edge a having a small hole in one of its walls. The total thermal resistance of the walls is r. At time $t = 0$, it contains air at atmospheric pressure $p_a$ and temperature $T_0$. The temperature of the surrounding air is $T_a (> T_0)$. Find the amount of the gas (in moles) in the vessel at time t. Take $C_v$ of air to be* 5 $R$/2.

***Solution*** : As the gas can leak out of the hole, the pressure inside the vessel will be equal to the atmospheric pressure $p_a$. Let $n$ be the amount of the gas (moles) in the vessel at time $t$. Suppose an amount $\Delta Q$ of heat is given to the gas in time $dt$. Its temperature increases by $dT$ where

$$\Delta Q = nC_p dT.$$

If the temperature of the gas is $T$ at time $t$, we have

$$\frac{\Delta Q}{dt} = \frac{T_a - T}{r}$$

or, $$(C_p r)n\ dT = (T_a - T)dt. \quad \ldots \text{(i)}$$

We have, $$p_a\, a^3 = nRT$$

or, $$n\ dT + T\ dn = 0$$

or, $$n\ dT = -\ T\ dn. \quad \ldots \text{(ii)}$$

Also, $$T = \frac{p_a a^3}{nR}. \quad \ldots \text{(iii)}$$

Using (ii) and (iii) in (i),

$$\frac{-C_p r p_a a^3}{nR} dn = \left(T_a - \frac{p_a a^3}{nR}\right) dt$$

or, $$\frac{dn}{nR\left(T_a - \frac{p_a a^3}{nR}\right)} = -\frac{dt}{C_p r p_a a^3}$$

or, $$\int_{n_0}^{n} \frac{dn}{nRT_a - p_a a^3} = -\int_0^t \frac{dt}{C_p r p_a a^3}$$

where $n_0 = \frac{p_a a^3}{RT_0}$ is the initial amount of the gas in the vessel. Thus,

$$\frac{1}{RT_a} \ln \frac{nRT_a - p_a a^3}{n_0 RT_a - p_a a^3} = -\frac{t}{C_p r p_a a^3}$$

or, $$nRT_a - p_a a^3 = (n_0 RT_a - p_a a^3)\, e^{-\frac{RT_a}{C_p r p_a a^3} t}.$$

Writing $$n_0 = \frac{p_a a^3}{RT_0} \text{ and } C_p = C_v + R = \frac{7R}{2},$$

$$n = \frac{p_a a^3}{RT_a}\left[1 + \left(\frac{T_a}{T_0} - 1\right) e^{-\frac{2T_a}{7 r p_a a^3} t}\right].$$

**16** *A blackbody of surface area* $1\ \text{cm}^2$ *is placed inside an enclosure. The enclosure has a constant temperature* 27°C *and the blackbody is maintained at* 327°C *by heating it electrically. What electric power is needed to maintain the temperature ?* $\sigma = 6{\cdot}0 \times 10^{-8}\ \text{W m}^{-2}\ \text{K}^{-4}$.

***Solution*** : The area of the blackbody is $A = 10^{-4}\ \text{m}^2$, its temperature is $T_1 = 327°\text{C} = 600\ \text{K}$ and the temperature of the enclosure is $T_2 = 27°\text{C} = 300\ \text{K}$. The blackbody emits radiation at the rate of $A\sigma T_1^4$. The radiation falls on it (and gets absorbed) at the rate of $A\sigma T_2^4$. The net rate of loss of energy is $A\sigma(T_1^4 - T_2^4)$. The heater must supply this much of power. Thus, the power needed is

$A\sigma(T_1^4 - T_2^4)$

$= (10^{-4}\ \text{m}^2)(6{\cdot}0 \times 10^{-8}\ \text{W m}^{-2}\ \text{K}^{-4})[(600\ \text{K})^4 - (300\ \text{K})^4]$

$= 0{\cdot}73\ \text{W}.$

**17.** *An electric heater emits* 1000 W *of thermal radiation. The coil has a surface area of* $0{\cdot}020\ \text{m}^2$. *Assuming that the coil radiates like a blackbody, find its temperature.* $\sigma = 6{\cdot}00 \times 10^{-8}\ \text{W m}^{-2}\ \text{K}^{-4}$.

***Solution*** : Let the temperature of the coil be $T$. The coil will emit radiation at a rate $A\sigma T^4$. Thus,

$$1000\ \text{W} = (0{\cdot}020\ \text{m}^2) \times (6{\cdot}0 \times 10^{-8}\ \text{W m}^{-2}\ \text{K}^{-4}) \times T^4$$

or, $$T^4 = \frac{1000}{0{\cdot}020 \times 6{\cdot}00 \times 10^{-8}}\ \text{K}^4$$

$$= 8{\cdot}33 \times 10^{11}\ \text{K}^4$$

or, $$T = 955\ \text{K}.$$

**18.** *The earth receives solar radiation at a rate of* $8{\cdot}2\ \text{J cm}^{-2}\ \text{min}^{-1}$. *Assuming that the sun radiates like a blackbody, calculate the surface temperature of the sun. The angle subtended by the sun on the earth is* 0·53° *and the Stefan constant* $\sigma = 5{\cdot}67 \times 10^{-8}\ \text{W m}^{-2}\ \text{K}^{-4}$.

***Solution*** :

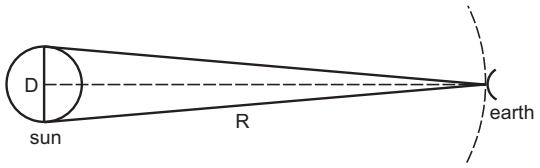



Figure 28-W11

Let the diameter of the sun be $D$ and its distance from the earth be $R$. From the question,

$$\frac{D}{R} \approx 0{\cdot}53 \times \frac{\pi}{180}$$

$$= 9{\cdot}25 \times 10^{-3}. \quad \ldots \text{(i)}$$

The radiation emitted by the surface of the sun per unit time is

$$4\pi\left(\frac{D}{2}\right)^2 \sigma T^4 = \pi D^2 \sigma T^4.$$

At distance $R$, this radiation falls on an area $4\pi R^2$ in unit time. The radiation received at the earth's surface per unit time per unit area is, therefore,

$$\frac{\pi D^2 \sigma T^4}{4\pi R^2} = \frac{\sigma T^4}{4}\left(\frac{D}{R}\right)^2.$$

Thus, $$\frac{\sigma T^4}{4}\left(\frac{D}{R}\right)^2 = 8{\cdot}2\ \text{J cm}^{-2}\ \text{min}^{-1}$$

or, $$\frac{1}{4} \times (5{\cdot}67 \times 10^{-8}\ \text{W m}^{-2}\ \text{K}^{-4})\, T^4 \times (9{\cdot}25 \times 10^{-3})^2$$

$$= \frac{8{\cdot}2}{10^{-4} \times 60}\ \text{W m}^{-2}$$

or, $$T = 5794\ \text{K} \approx 5800\ \text{K}.$$

**19.** *The temperature of a body falls from* 40°C *to* 36°C *in* 5 *minutes when placed in a surrounding of constant temperature* 16°C. *Find the time taken for the temperature of the body to become* 32°C.

***Solution*** : As the temperature differences are small, we can use Newton's law of cooling.

$$\frac{d\theta}{dt} = -k(\theta - \theta_0)$$

or, $$\frac{d\theta}{\theta - \theta_0} = -k\,dt \quad \ldots \text{(i)}$$

where $k$ is a constant, $\theta$ is the temperature of the body at time $t$ and $\theta_0 = 16°\text{C}$ is the temperature of the surrounding. We have,

$$\int_{40°C}^{36°C} \frac{d\theta}{\theta - \theta_0} = -k(5 \text{ min})$$

or, $$\ln \frac{36°C - 16°C}{40°C - 16°C} = -k(5 \text{ min})$$

or, $$k = -\frac{\ln(5/6)}{5 \text{ min}}.$$

If $t$ be the time required for the temperature to fall from 36°C to 32°C then by (i),

$$\int_{36°C}^{32°C} \frac{d\theta}{\theta - \theta_0} = -kt$$

or, $$\ln \frac{32°C - 16°C}{36°C - 16°C} = \frac{\ln(5/6)t}{5 \text{ min}}$$

or, $$t = \frac{\ln (4/5)}{\ln (5/6)} \times 5 \text{ min}$$

$$= 6{\cdot}1 \text{ min}.$$

**Alternative method**

The mean temperature of the body as it cools from 40°C to 36°C is $\frac{40°C + 36°C}{2} = 38°C$. The rate of decrease of temperature is $\frac{40°C - 36°C}{5 \text{ min}} = 0{\cdot}80°C \text{ min}^{-1}$.

Newton's law of cooling is

$$\frac{d\theta}{dt} = -k(\theta - \theta_0)$$

or, $$-0{\cdot}8°C \text{ min}^{-1} = -k(38°C - 16°C) = -k(22°C)$$

or, $$k = \frac{0{\cdot}8}{22} \text{ min}^{-1}.$$

Let the time taken for the temperature to become 32°C be $t$.

During this period,

$$\frac{d\theta}{dt} = -\frac{36°C - 32°C}{t} = -\frac{4°C}{t}.$$

The mean temperature is $\frac{36°C + 32°C}{2} = 34°C$.

Now,

$$\frac{d\theta}{dt} = -k(\theta - \theta_0)$$

or, $$\frac{-4°C}{t} = -\frac{0{\cdot}8}{22} \times (34°C - 16°C) \text{ min}^{-1}$$

or, $$t = \frac{22 \times 4}{0{\cdot}8 \times 18} \text{ min} = 6{\cdot}1 \text{ min}.$$

□

**20.** *A hot body placed in air is cooled down according to Newton's law of cooling, the rate of decrease of temperature being $k$ times the temperature difference from the surrounding. Starting from $t = 0$, find the time in which the body will lose half the maximum heat it can lose.*

***Solution*** : We have,

$$\frac{d\theta}{dt} = -k(\theta - \theta_0)$$

where $\theta_0$ is the temperature of the surrounding and $\theta$ is the temperature of the body at time $t$. Suppose $\theta = \theta_1$ at $t = 0$.

Then,

$$\int_{\theta_1}^{\theta} \frac{d\theta}{\theta - \theta_0} = -k \int_0^t dt$$

or, $$\ln \frac{\theta - \theta_0}{\theta_1 - \theta_0} = -kt$$

or, $$\theta - \theta_0 = (\theta_1 - \theta_0)\, e^{-kt}. \quad \ldots \text{(i)}$$

The body continues to lose heat till its temperature becomes equal to that of the surrounding. The loss of heat in this entire period is

$$\Delta Q_m = ms(\theta_1 - \theta_0).$$

This is the maximum heat the body can lose. If the body loses half this heat, the decrease in its temperature will be,

$$\frac{\Delta Q_m}{2\, ms} = \frac{\theta_1 - \theta_0}{2}.$$

If the body loses this heat in time $t_1$, the temperature at $t_1$ will be

$$\theta_1 - \frac{\theta_1 - \theta_0}{2} = \frac{\theta_1 + \theta_0}{2}.$$

Putting these values of time and temperature in (i),

$$\frac{\theta_1 + \theta_0}{2} - \theta_0 = (\theta_1 - \theta_0)\, e^{-kt_1}$$

or, $$e^{-kt_1} = \frac{1}{2}$$

or, $$t_1 = \frac{\ln 2}{k}.$$

## QUESTIONS FOR SHORT ANSWER

**1.** The heat current is written as $\frac{\Delta Q}{\Delta t}$. Why don't we write $\frac{dQ}{dt}$?

**2.** Does a body at 20°C radiate in a room, where the room temperature is 30°C ? If yes, why does its temperature not fall further ?

**3.** Why does blowing over a spoonful of hot tea cools it ? Does evaporation play a role ? Does radiation play a role ?

**4.** On a hot summer day we want to cool our room by opening the refrigerator door and closing all the windows and doors. Will the process work ?

**5.** On a cold winter night you are asked to sit on a chair. Would you like to choose a metal chair or a wooden chair ? Both are kept in the same lawn and are at the same temperature.

**6.** Two identical metal balls one at $T_1 = 300$ K and the other at $T_2 = 600$ K are kept at a distance of 1 m in vacuum. Will the temperatures equalise by radiation ? Will the rate of heat gained by the colder sphere be proportional to $T_2^4 - T_1^4$ as may be expected from the Stefan's law ?

**7.** An ordinary electric fan does not cool the air, still it gives comfort in summer. Explain.

**8.** The temperature of the atmosphere at a high altitude is around 500°C. Yet an animal there would freeze to death and not boil. Explain.

**9.** Standing in the sun is more pleasant on a cold winter day than standing in shade. Is the temperature of air in the sun considerably higher than that of the air in shade ?

**10.** Cloudy nights are warmer than the nights with clean sky. Explain.

**11.** Why is a white dress more comfortable than a dark dress in summer ?

## OBJECTIVE I

**1.** The thermal conductivity of a rod depends on
(a) length (b) mass
(c) area of cross section (d) material of the rod.

**2.** In a room containing air, heat can go from one place to another
(a) by conduction only (b) by convection only
(c) by radiation only (d) by all the three modes.

**3.** A solid at temperature $T_1$ is kept in an evacuated chamber at temperature $T_2 > T_1$. The rate of increase of temperature of the body is proportional to
(a) $T_2 - T_1$ (b) $T_2^2 - T_1^2$
(c) $T_2^3 - T_1^3$ (d) $T_2^4 - T_1^4$.

**4.** The thermal radiation emitted by a body is proportional to $T^n$ where $T$ is its absolute temperature. The value of $n$ is exactly 4 for
(a) a blackbody (b) all bodies
(c) bodies painted black only (d) polished bodies only.

**5.** Two bodies $A$ and $B$ having equal surface areas are maintained at temperatures 10°C and 20°C. The thermal radiation emitted in a given time by $A$ and $B$ are in the ratio
(a) 1 : 1·15 (b) 1 : 2
(c) 1 : 4 (d) 1 : 16.

**6.** One end of a metal rod is kept in a furnace. In steady state, the temperature of the rod
(a) increases (b) decreases
(c) remains constant (d) is nonuniform.

**7.** Newton's law of cooling is a special case of
(a) Wien's displacement law (b) Kirchhoff's law
(c) Stefan's law (d) Planck's law.

**8.** A hot liquid is kept in a big room. Its temperature is plotted as a function of time. Which of the following curves may represent the plot ?

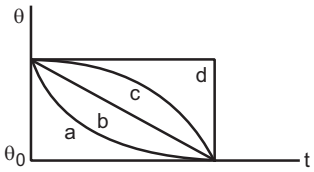



Figure 28-Q1

**9.** A hot liquid is kept in a big room. The logarithm of the numerical value of the temperature difference between the liquid and the room is plotted against time. The plot will be very nearly
(a) a straight line (b) a circular arc
(c) a parabola (d) an ellipse.

**10.** A body cools down from 65°C to 60°C in 5 minutes. It will cool down from 60°C to 55°C in
(a) 5 minutes (b) less than 5 minutes
(c) more than 5 minutes
(d) less than or more than 5 minutes depending on whether its mass is more than or less than 1 kg.

## OBJECTIVE II

**1.** One end of a metal rod is dipped in boiling water and the other is dipped in melting ice.
(a) All parts of the rod are in thermal equilibrium with each other.
(b) We can assign a temperature to the rod.
(c) We can assign a temperature to the rod after steady state is reached.
(d) The state of the rod does not change after steady state is reached.

**2.** A blackbody does not
(a) emit radiation (b) absorb radiation
(c) reflect radiation (d) refract radiation.

**3.** In summer, a mild wind is often found on the shore of a calm river. This is caused due to
(a) difference in thermal conductivity of water and soil
(b) convection currents
(c) conduction between air and the soil
(d) radiation from the soil.

**4.** A piece of charcoal and a piece of shining steel of the same surface area are kept for a long time in an open lawn in bright sun.
(a) The steel will absorb more heat than the charcoal.
(b) The temperature of the steel will be higher than that of the charcoal.
(c) If both are picked up by bare hands, the steel will be felt hotter than the charcoal.
(d) If the two are picked up from the lawn and kept in a cold chamber, the charcoal will lose heat at a faster rate than the steel.

**5.** A heated body emits radiation which has maximum intensity near the frequency $\nu_0$. The emissivity of the material is 0·5. If the absolute temperature of the body is doubled,
(a) the maximum intensity of radiation will be near the frequency $2\nu_0$
(b) the maximum intensity of radiation will be near the frequency $\nu_0/2$
(c) the total energy emitted will increase by a factor of 16
(d) the total energy emitted will increase by a factor of 8.

**6.** A solid sphere and a hollow sphere of the same material and of equal radii are heated to the same temperature.
(a) Both will emit equal amount of radiation per unit time in the biginning.
(b) Both will absorb equal amount of radiation from the surrounding in the biginning.
(c) The initial rate of cooling ($dT/dt$) will be the same for the two spheres.
(d) The two spheres will have equal temperatures at any instant.

## EXERCISES

**1.** A uniform slab of dimension $10\text{ cm} \times 10\text{ cm} \times 1\text{ cm}$ is kept between two heat reservoirs at temperatures 10°C and 90°C. The larger surface areas touch the reservoirs. The thermal conductivity of the material is $0{\cdot}80\text{ W m}^{-1}\text{°C}^{-1}$. Find the amount of heat flowing through the slab per minute.

**2.** A liquid-nitrogen container is made of a 1-cm thick styrofoam sheet having thermal conductivity $0{\cdot}025\text{ J s}^{-1}\text{ m}^{-1}\text{°C}^{-1}$. Liquid nitrogen at 80 K is kept in it. A total area of $0{\cdot}80\text{ m}^2$ is in contact with the liquid nitrogen. The atmospheric temperature is 300 K. Calculate the rate of heat flow from the atmosphere to the liquid nitrogen.

**3.** The normal body-temperature of a person is 97°F. Calculate the rate at which heat is flowing out of his body through the clothes assuming the following values. Room temperature = 47°F, surface of the body under clothes = $1{\cdot}6\text{ m}^2$, conductivity of the cloth = $0{\cdot}04\text{ J s}^{-1}\text{ m}^{-1}\text{°C}^{-1}$, thickness of the cloth = 0·5 cm.

**4.** Water is boiled in a container having a bottom of surface area $25\text{ cm}^2$, thickness 1·0 mm and thermal conductivity $50\text{ W m}^{-1}\text{°C}^{-1}$. 100 g of water is converted into steam per minute in the steady state after the boiling starts. Assuming that no heat is lost to the atmosphere, calculate the temperature of the lower surface of the bottom. Latent heat of vaporization of water = $2{\cdot}26 \times 10^6\text{ J kg}^{-1}$.

**5.** One end of a steel rod ($K = 46\text{ J s}^{-1}\text{ m}^{-1}\text{°C}^{-1}$) of length 1·0 m is kept in ice at 0°C and the other end is kept in boiling water at 100°C. The area of cross section of the rod is $0{\cdot}04\text{ cm}^2$. Assuming no heat loss to the atmosphere, find the mass of the ice melting per second. Latent heat of fusion of ice = $3{\cdot}36 \times 10^5\text{ J kg}^{-1}$.

**6.** An icebox almost completely filled with ice at 0°C is dipped into a large volume of water at 20°C. The box has walls of surface area $2400\text{ cm}^2$, thickness 2·0 mm and thermal conductivity $0{\cdot}06\text{ W m}^{-1}\text{°C}^{-1}$. Calculate the rate at which the ice melts in the box. Latent heat of fusion of ice = $3{\cdot}4 \times 10^5\text{ J kg}^{-1}$.

**7.** A pitcher with 1-mm thick porous walls contains 10 kg of water. Water comes to its outer surface and evaporates at the rate of $0{\cdot}1\text{ g s}^{-1}$. The surface area of the pitcher (one side) = $200\text{ cm}^2$. The room temperature = 42°C, latent heat of vaporization = $2{\cdot}27 \times 10^6\text{ J kg}^{-1}$, and the thermal conductivity of the porous walls = $0{\cdot}80\text{ J s}^{-1}\text{ m}^{-1}\text{°C}^{-1}$. Calculate the temperature of water in the pitcher when it attains a constant value.

**8.** A steel frame ($K = 45\text{ W m}^{-1}\text{°C}^{-1}$) of total length 60 cm and cross sectional area $0{\cdot}20\text{ cm}^2$, forms three sides of a square. The free ends are maintained at 20°C and 40°C. Find the rate of heat flow through a cross section of the frame.

**9.** Water at 50°C is filled in a closed cylindrical vessel of height 10 cm and cross sectional area $10\text{ cm}^2$. The walls of the vessel are adiabatic but the flat parts are made of 1-mm thick aluminium ($K = 200\text{ J s}^{-1}\text{ m}^{-1}\text{°C}^{-1}$). Assume that the outside temperature is 20°C. The density of water is $1000\text{ kg m}^{-3}$, and the specific heat capacity of water = $4200\text{ J k}^{-1}\text{°C}^{-1}$. Estimate the time taken for the temperature to fall by 1·0°C. Make any simplifying assumptions you need but specify them.

**10.** The left end of a copper rod (length = 20 cm, area of cross section = $0{\cdot}20\text{ cm}^2$) is maintained at 20°C and the right end is maintained at 80°C. Neglecting any loss of heat through radiation, find (a) the temperature at a point 11 cm from the left end and (b) the heat current through the rod. Thermal conductivity of copper = $385\text{ W m}^{-1}\text{°C}^{-1}$.

**11.** The ends of a metre stick are maintained at 100°C and 0°C. One end of a rod is maintained at 25°C. Where should its other end be touched on the metre stick so that there is no heat current in the rod in steady state ?

**12.** A cubical box of volume 216 $cm^3$ is made up of 0·1 cm thick wood. The inside is heated electrically by a 100 W heater. It is found that the temperature difference between the inside and the outside surface is 5°C in steady state. Assuming that the entire electrical energy spent appears as heat, find the thermal conductivity of the material of the box.

**13.** Figure (28-E1) shows water in a container having 2·0-mm thick walls made of a material of thermal conductivity 0·50 W $m^{-1}°C^{-1}$. The container is kept in a melting-ice bath at 0°C. The total surface area in contact with water is 0·05 $m^2$. A wheel is clamped inside the water and is coupled to a block of mass $M$ as shown in the figure. As the block goes down, the wheel rotates. It is found that after some time a steady state is reached in which the block goes down with a constant speed of 10 cm $s^{-1}$ and the temperature of the water remains constant at 1·0°C. Find the mass $M$ of the block. Assume that the heat flows out of the water only through the walls in contact. Take $g = 10$ m $s^{-2}$.

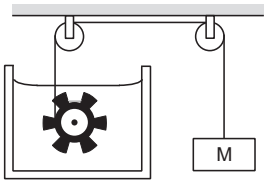



Figure 28-E1

**14.** On a winter day when the atmospheric temperature drops to –10°C, ice forms on the surface of a lake. (a) Calculate the rate of increase of thickness of the ice when 10 cm of ice is already formed. (b) Calculate the total time taken in forming 10 cm of ice. Assume that the temperature of the entire water reaches 0°C before the ice starts forming. Density of water = 1000 kg $m^{-3}$, latent heat of fusion of ice = $3{\cdot}36 \times 10^5$ J $kg^{-1}$ and thermal conductivity of ice = 1·7 W $m^{-1}°C^{-1}$. Neglect the expansion of water on freezing.

**15.** Consider the situation of the previous problem. Assume that the temperature of the water at the bottom of the lake remains constant at 4°C as the ice forms on the surface (the heat required to maintain the temperature of the bottom layer may come from the bed of the lake). The depth of the lake is 1·0 m. Show that the thickness of the ice formed attains a steady state maximum value. Find this value. The thermal conductivity of water = 0·50 W $m^{-1}°C^{-1}$. Take other relevant data from the previous problem.

**16.** Three rods of lengths 20 cm each and area of cross section 1 $cm^2$ are joined to form a triangle $ABC$. The conductivities of the rods are $K_{AB} = 50$ J $s^{-1}$ $m^{-1}°C^{-1}$, $K_{BC} = 200$ J $s^{-1}$ $m^{-1}°C^{-1}$ and $K_{AC} = 400$ J $s^{-1}$ $m^{-1}°C^{-1}$. The junctions $A$, $B$ and $C$ are maintained at 40°C, 80°C and 80°C respectively. Find the rate of heat flowing through the rods $AB$, $AC$ and $BC$.

**17.** A semicircular rod is joined at its end to a straight rod of the same material and the same cross-sectional area. The straight rod forms a diameter of the other rod. The junctions are maintained at different temperatures. Find the ratio of the heat transferred through a cross section of the semicircular rod to the heat transferred through a cross section of the straight rod in a given time.

**18.** A metal rod of cross sectional area 1·0 $cm^2$ is being heated at one end. At one time, the temperature gradient is 5·0°C $cm^{-1}$ at cross section $A$ and is 2·5°C$cm^{-1}$ at cross section $B$. Calculate the rate at which the temperature is increasing in the part $AB$ of the rod. The heat capacity of the part $AB$ = 0·40 J$°C^{-1}$, thermal conductivity of the material of the rod = 200 W $m^{-1}°C^{-1}$. Neglect any loss of heat to the atmosphere.

**19.** Steam at 120°C is continuously passed through a 50-cm long rubber tube of inner and outer radii 1·0 cm and 1·2 cm. The room temperature is 30°C. Calculate the rate of heat flow through the walls of the tube. Thermal conductivity of rubber = 0·15 J $s^{-1}$ $m^{-1}°C^{-1}$.

**20.** A hole of radius $r_1$ is made centrally in a uniform circular disc of thickness $d$ and radius $r_2$. The inner surface ( a cylinder of length $d$ and radius $r_1$) is maintained at a temperature $\theta_1$ and the outer surface (a cylinder of length $d$ and radius $r_2$) is maintained at a temperature $\theta_2 (\theta_1 > \theta_2)$. The thermal conductivity of the material of the disc is $K$. Calculate the heat flowing per unit time through the disc.

**21.** A hollow tube has a length $l$, inner radius $R_1$ and outer radius $R_2$. The material has a thermal conductivity $K$. Find the heat flowing through the walls of the tube if (a) the flat ends are maintained at temperatures $T_1$ and $T_2 (T_2 > T_1)$ (b) the inside of the tube is maintained at temperature $T_1$ and the outside is maintained at $T_2$.

**22.** A composite slab is prepared by pasting two plates of thicknesses $L_1$ and $L_2$ and thermal conductivities $K_1$ and $K_2$. The slabs have equal cross-sectional area. Find the equivalent conductivity of the composite slab.

**23.** Figure (28-E2) shows a copper rod joined to a steel rod. The rods have equal length and equal cross sectional area. The free end of the copper rod is kept at 0°C and that of the steel rod is kept at 100°C. Find the temperature at the junction of the rods. Conductivity of copper = 390 W $m^{-1}°C^{-1}$ and that of steel = 46 W $m^{-1}°C^{-1}$.

Figure 28-E2

**24.** An aluminium rod and a copper rod of equal length 1·0 m and cross-sectional area 1 $cm^2$ are welded together as shown in figure (28-E3). One end is kept at a temperature of 20°C and the other at 60°C. Calculate the amount of heat taken out per second from the hot end. Thermal conductivity of aluminium = 200 W$m^{-1}°C^{-1}$ and of copper = 390 W $m^{-1}°C^{-1}$.

Figure 28-E3

**25.** Figure (28-E4) shows an aluminium rod joined to a copper rod. Each of the rods has a length of 20 cm and area of cross section 0·20 $cm^2$. The junction is maintained at a constant temperature 40°C and the two ends are maintained at 80°C. Calculate the amount of heat taken out from the cold junction in one minute after the steady state is reached. The conductivities are $K_{Al} = 200\ W\ m^{-1}°C^{-1}$ and $K_{Cu} = 400\ W\ m^{-1}°C^{-1}$.

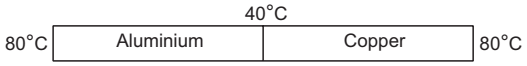



Figure 28-E4

**26.** Consider the situation shown in figure (28-E5). The frame is made of the same material and has a uniform cross-sectional area everywhere. Calculate the amount of heat flowing per second through a cross section of the bent part if the total heat taken out per second from the end at 100°C is 130 J.

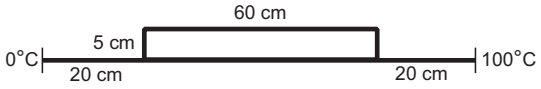



Figure 28-E5

**27.** Suppose the bent part of the frame of the previous problem has a thermal conductivity of 780 $J\ s^{-1}\ m^{-1}°C^{-1}$ whereas it is 390 $J\ s^{-1}\ m^{-1}°C^{-1}$ for the straight part. Calculate the ratio of the rate of heat flow through the bent part to the rate of heat flow through the straight part.

**28.** A room has a window fitted with a single 1·0 m × 2·0 m glass of thickness 2 mm. (a) Calculate the rate of heat flow through the closed window when the temperature inside the room is 32°C and that outside is 40°C. (b) The glass is now replaced by two glasspanes, each having a thickness of 1 mm and separated by a distance of 1 mm. Calculate the rate of heat flow under the same conditions of temperature. Thermal conductivity of window glass = 1·0 $J\ s^{-1}\ m^{-1}°C^{-1}$ and that of air = 0·025 $J\ s^{-1}\ m^{-1}°C^{-1}$.

**29.** The two rods shown in figure (28-E6) have identical geometrical dimensions. They are in contact with two heat baths at temperatures 100°C and 0°C. The temperature of the junction is 70°C. Find the temperature of the junction if the rods are interchanged.

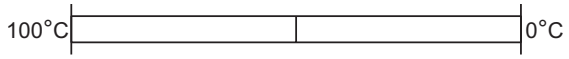



Figure 28-E6

**30.** The three rods shown in figure (28-E7) have identical geometrical dimensions. Heat flows from the hot end at a rate of 40 W in the arrangement (a). Find the rates of heat flow when the rods are joined as in arrangement (b) and in (c). Thermal conductivities of aluminium and copper are 200 $W\ m^{-1}°C^{-1}$ and 400 $W\ m^{-1}°C^{-1}$ respectively.

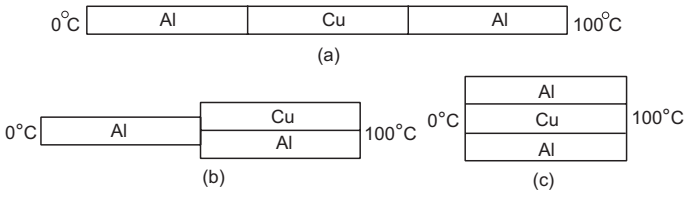



Figure 28-E7

**31.** Four identical rods *AB*, *CD*, *CF* and *DE* are joined as shown in figure (28-E8). The length, cross-sectional area and thermal conductivity of each rod are *l*, *A* and *K* respectively. The ends *A*, *E* and *F* are maintained at temperatures $T_1$, $T_2$ and $T_3$ respectively. Assuming no loss of heat to the atmosphere, find the temperature at *B*.

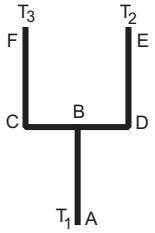



Figure 28-E8

**32.** Seven rods *A*, *B*, *C*, *D*, *E*, *F* and *G* are joined as shown in figure (28-E9). All the rods have equal cross-sectional area *A* and length *l*. The thermal conductivities of the rods are $K_A = K_C = K_0$, $K_B = K_D = 2K_0$, $K_E = 3K_0$, $K_F = 4K_0$ and $K_G = 5K_0$. The rod *E* is kept at a constant temperature $T_1$ and the rod *G* is kept at a constant temperature $T_2 (T_2 > T_1)$. (a) Show that the rod *F* has a uniform temperature $T = (T_1 + 2T_2)/3$. (b) Find the rate of heat flowing from the source which maintains the temperature $T_2$.

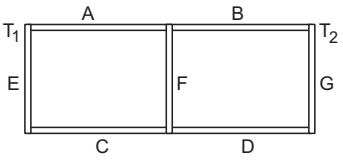



Figure 28-E9

**33.** Find the rate of heat flow through a cross section of the rod shown in figure (28-E10) $(\theta_2 > \theta_1)$. Thermal conductivity of the material of the rod is *K*.

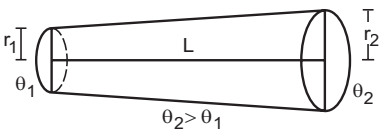



Figure 28-E10

**34.** A rod of negligible heat capacity has length 20 cm, area of cross section 1·0 $cm^2$ and thermal conductivity 200 W $m^{-1}°C^{-1}$. The temperature of one end is maintained at 0°C and that of the other end is slowly and linearly varied from 0°C to 60°C in 10 minutes. Assuming no loss of heat through the sides, find the total heat transmitted through the rod in these 10 minutes.

**35.** A hollow metallic sphere of radius 20 cm surrounds a concentric metallic sphere of radius 5 cm. The space between the two spheres is filled with a nonmetallic material. The inner and outer spheres are maintained at 50°C and 10°C respectively and it is found that 100 J of heat passes from the inner sphere to the outer sphere per second. Find the thermal conductivity of the material between the spheres.

**36.** Figure (28-E11) shows two adiabatic vessels, each containing a mass $m$ of water at different temperatures. The ends of a metal rod of length $L$, area of cross section $A$ and thermal conductivity $K$, are inserted in the water as shown in the figure. Find the time taken for the difference between the temperatures in the vessels to become half of the original value. The specific heat capacity of water is $s$. Neglect the heat capacity of the rod and the container and any loss of heat to the atmosphere.

Figure 28-E11

**37.** Two bodies of masses $m_1$ and $m_2$ and specific heat capacities $s_1$ and $s_2$ are connected by a rod of length $l$, cross-sectional area $A$, thermal conductivity $K$ and negligible heat capacity. The whole system is thermally insulated. At time $t = 0$, the temperature of the first body is $T_1$ and the temperature of the second body is $T_2$ ($T_2 > T_1$). Find the temperature difference between the two bodies at time $t$.

**38.** An amount $n$ (in moles) of a monatomic gas at an initial temperature $T_0$ is enclosed in a cylindrical vessel fitted with a light piston. The surrounding air has a temperature $T_s (> T_0)$ and the atmospheric pressure is $p_a$. Heat may be conducted between the surrounding and the gas through the bottom of the cylinder. The bottom has a surface area $A$, thickness $x$ and thermal conductivity $K$. Assuming all changes to be slow, find the distance moved by the piston in time $t$.

**39.** Assume that the total surface area of a human body is 1·6 $m^2$ and that it radiates like an ideal radiator. Calculate the amount of energy radiated per second by the body if the body temperature is 37°C. Stefan constant $\sigma$ is $6{\cdot}0 \times 10^{-8}$ W $m^{-2}$ $K^{-4}$.

**40.** Calculate the amount of heat radiated per second by a body of surface area 12 $cm^2$ kept in thermal equilibrium in a room at temperature 20°C. The emissivity of the surface = 0·80 and $\sigma = 6{\cdot}0 \times 10^{-8}$ W $m^{-2}$ $K^{-4}$.

**41.** A solid aluminium sphere and a solid copper sphere of twice the radius are heated to the same temperature and are allowed to cool under identical surrounding temperatures. Assume that the emissivity of both the spheres is the same. Find the ratio of (a) the rate of heat loss from the aluminium sphere to the rate of heat loss from the copper sphere and (b) the rate of fall of temperature of the aluminium sphere to the rate of fall of temperature of the copper sphere. The specific heat capacity of aluminium = 900 J $kg^{-1}°C^{-1}$ and that of copper = 390 J $kg^{-1}°C^{-1}$. The density of copper = 3·4 times the density of aluminium.

**42.** A 100 W bulb has tungsten filament of total length 1·0 m and radius $4 \times 10^{-5}$ m. The emissivity of the filament is 0·8 and $\sigma = 6{\cdot}0 \times 10^{-8}$ W $m^{-2}$ $K^4$. Calculate the temperature of the filament when the bulb is operating at correct wattage.

**43.** A spherical ball of surface area 20 $cm^2$ absorbs any radiation that falls on it. It is suspended in a closed box maintained at 57°C. (a) Find the amount of radiation falling on the ball per second. (b) Find the net rate of heat flow to or from the ball at an instant when its temperature is 200°C. Stefan constant = $6{\cdot}0 \times 10^{-8}$ W $m^{-2}$ $K^{-4}$.

**44.** A spherical tungsten piece of radius 1·0 cm is suspended in an evacuated chamber maintained at 300 K. The piece is maintained at 1000 K by heating it electrically. Find the rate at which the electrical energy must be supplied. The emissivity of tungsten is 0·30 and the Stefan constant $\sigma$ is $6{\cdot}0 \times 10^{-8}$ W $m^{-2}$ $K^{-4}$.

**45.** A cubical block of mass 1·0 kg and edge 5·0 cm is heated to 227°C. It is kept in an evacuated chamber maintained at 27°C. Assuming that the block emits radiation like a blackbody, find the rate at which the temperature of the block will decrease. Specific heat capacity of the material of the block is 400 J $kg^{-1}$ $K^{-1}$.

**46.** A copper sphere is suspended in an evacuated chamber maintained at 300 K. The sphere is maintained at a constant temperature of 500 K by heating it electrically. A total of 210 W of electric power is needed to do it. When the surface of the copper sphere is completely blackened, 700 W is needed to maintain the same temperature of the sphere. Calculate the emissivity of copper.

**47.** A spherical ball $A$ of surface area 20 $cm^2$ is kept at the centre of a hollow spherical shell $B$ of area 80 $cm^2$. The surface of $A$ and the inner surface of $B$ emit as blackbodies. Both $A$ and $B$ are at 300 K. (a) How much is the radiation energy emitted per second by the ball $A$ ? (b) How much is the radiation energy emitted per second by the inner surface of $B$ ? (c) How much of the energy emitted by the inner surface of $B$ falls back on this surface itself ?

**48.** A cylindrical rod of length 50 cm and cross sectional area 1 $cm^2$ is fitted between a large ice chamber at 0°C and an evacuated chamber maintained at 27°C as shown in figure (28-E12). Only small portions of the rod are inside the chambers and the rest is thermally insulated from the surrounding. The cross section going into the

evacuated chamber is blackened so that it completely absorbs any radiation falling on it. The temperature of the blackened end is 17°C when steady state is reached. Stefan constant $\sigma = 6 \times 10^{-8}$ W m$^{-2}$ K$^{-4}$. Find the thermal conductivity of the material of the rod.

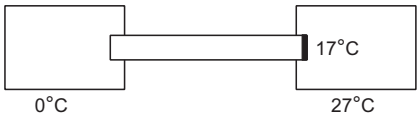



Figure 28-E12

**49.** One end of a rod of length 20 cm is inserted in a furnace at 800 K. The sides of the rod are covered with an insulating material and the other end emits radiation like a blackbody. The temperature of this end is 750 K in the steady state. The temperature of the surrounding air is 300 K. Assuming radiation to be the only important mode of energy transfer between the surrounding and the open end of the rod, find the thermal conductivity of the rod. Stefan constant $\sigma = 6{\cdot}0 \times 10^{-8}$ W m$^{-2}$ K$^{-4}$.

**50.** A calorimeter of negligible heat capacity contains 100 cc of water at 40°C. The water cools to 35°C in 5 minutes. The water is now replaced by K-oil of equal volume at 40°C. Find the time taken for the temperature to become 35°C under similar conditions. Specific heat capacities of water and K-oil are 4200 J kg$^{-1}$ K$^{-1}$ and 2100 J kg$^{-1}$ K$^{-1}$ respectively. Density of K-oil = 800 kg m$^{-3}$.

**51.** A body cools down from 50°C to 45°C in 5 minutes and to 40°C in another 8 minutes. Find the temperature of the surrounding.

**52.** A calorimeter contains 50 g of water at 50°C. The temperature falls to 45°C in 10 minutes. When the calorimeter contains 100 g of water at 50°C, it takes 18 minutes for the temperature to become 45°C. Find the water equivalent of the calorimeter.

**53.** A metal ball of mass 1 kg is heated by means of a 20 W heater in a room at 20°C. The temperature of the ball becomes steady at 50°C. (a) Find the rate of loss of heat to the surrounding when the ball is at 50°C. (b) Assuming Newton's law of cooling, calculate the rate of loss of heat to the surrounding when the ball is at 30°C. (c) Assume that the temperature of the ball rises uniformly from 20°C to 30°C in 5 minutes. Find the total loss of heat to the surrounding during this period. (d) Calculate the specific heat capacity of the metal.

**54.** A metal block of heat capacity 80 J°C$^{-1}$ placed in a room at 20°C is heated electrically. The heater is switched off when the temperature reaches 30°C. The temperature of the block rises at the rate of 2 °C s$^{-1}$ just after the heater is switched on and falls at the rate of 0·2 °C s$^{-1}$ just after the heater is switched off. Assume Newton's law of cooling to hold. (a) Find the power of the heater. (b) Find the power radiated by the block just after the heater is switched off. (c) Find the power radiated by the block when the temperature of the block is 25°C. (d) Assuming that the power radiated at 25°C represents the average value in the heating process, find the time for which the heater was kept on.

**55.** A hot body placed in a surrounding of temperature $\theta_0$ obeys Newton's law of cooling $\frac{d\theta}{dt} = -k(\theta - \theta_0)$. Its temperature at $t = 0$ is $\theta_1$. The specific heat capacity of the body is $s$ and its mass is $m$. Find (a) the maximum heat that the body can lose and (b) the time starting from $t = 0$ in which it will lose 90% of this maximum heat.

□

## ANSWERS

### OBJECTIVE I

1. (d) 2. (d) 3. (d) 4. (b) 5. (a) 6. (d)
7. (c) 8. (a) 9. (a) 10. (c)

### OBJECTIVE II

1. (d) 2. (c), (d) 3. (b)
4. (c), (d) 5. (a), (c) 6. (a), (b)

### EXERCISES

1. 3840 J
2. 440 W
3. 356 J s$^{-1}$
4. 130°C
5. 5·5 × 10$^{-5}$ g
6. 1·5 kg h$^{-1}$
7. 28°C
8. 0·03 W
9. 0·035 s
10. (a) 53°C (b) 2·31 J s$^{-1}$
11. 25 cm from the cold end
12. 0·92 W m$^{-1}$°C$^{-1}$
13. 12·5 kg

14. (a) $5{\cdot}0 \times 10^{-7}$ m s$^{-1}$ (b) 27·5 hours
15. 89 cm
16. 1 W, 8 W, zero
17. 2 : π
18. 12·5°C s$^{-1}$
19. 233 J s$^{-1}$
20. $\dfrac{2\pi Kd(\theta_1 - \theta_2)}{\ln(r_2/r_1)}$
21. (a) $\dfrac{K\pi(R_2^2 - R_1^2)\,(T_2 - T_1)}{l}$ (b) $\dfrac{2\pi Kl(T_2 - T_1)}{\ln(R_2/R_1)}$
22. $\dfrac{K_1K_2(L_1 + L_2)}{L_1K_2 + L_2K_1}$
23. 10·6°C
24. 2·36 J
25. 144 J
26. 60 J
27. 12 : 7
28. (a) 8000 J s$^{-1}$ (b) 381 J s$^{-1}$
29. 30°C
30. 75 W, 400 W
31. $\dfrac{3\,T_1 + 2(T_2 + T_3)}{7}$
32. (b) $\dfrac{4\,K_0\,A(T_2 - T_1)}{3l}$
33. $\dfrac{K\pi r_1 r_2(\theta_2 - \theta_1)}{L}$
34. 1800 J
35. 3·0 W m$^{-1}$°C$^{-1}$
36. $\dfrac{Lms}{2KA}\ln 2$
37. $(T_2 - T_1)\,e^{-\lambda t}$ where $\lambda = \dfrac{KA(m_1s_1 + m_2s_2)}{lm_1m_2s_1s_2}$
38. $\dfrac{nR}{P_aA}(T_s - T_0)\,(1 - e^{-2KAt/5Rnx})$
39. 887 J
40. 0·42 J
41. (a) 1 : 4 (b) 2·9 : 1
42. 1700 K
43. (a) 1·4 J (b) 4·58 W from the ball
44. 22 W
45. 0·12°C s$^{-1}$
46. 0·3
47. (a) 0.94 J (b) 3.8 J (c) 2.8 J
48. 1·8 W m$^{-1}$°C$^{-1}$
49. 74 W m$^{-1}$ K$^{-1}$
50. 2 min
51. 34°C
52. 12·5 g
53. (a) 20 W (b) $\dfrac{20}{3}$ W (c) 1000 J (d) 500 J kg$^{-1}$ K$^{-1}$
54. (a) 160 W (b) 16 W (c) 8 W (d) 5·2 s
55. (a) $ms(\theta_1 - \theta_0)$ (b) $\dfrac{\ln 10}{k}$

□

CHAPTER 29

# ELECTRIC FIELD AND POTENTIAL

## 29.1 WHAT IS ELECTRIC CHARGE ?

Matter is made of certain elementary particles. With the advancement in technology, we have discovered hundreds of elementary particles. Many of them are rare and of no concern to us in the present course. The three most common elementary particles are electrons, protons and neutrons having masses $m_e = 9{\cdot}10940 \times 10^{-31}$ kg, $m_p = 1{\cdot}67262 \times 10^{-27}$ kg and $m_n = 1{\cdot}67493 \times 10^{-27}$ kg. Because of their mass these particles attract each other by gravitational forces. Thus, an electron attracts another electron, placed 1 cm away, with a gravitational force

$$F = \frac{Gm_1m_2}{r^2}$$

$$= \frac{(6{\cdot}67 \times 10^{-11}\ \text{N m}^2\ \text{kg}^{-2}) \times (9{\cdot}1 \times 10^{-31}\ \text{kg})^2}{(10^{-2}\ \text{m})^2}$$

$$= 5{\cdot}5 \times 10^{-67}\ \text{N}.$$

However, an electron is found to repel another electron at 1 cm with a force of $2{\cdot}3 \times 10^{-24}$ N. This extra force is called the *electric force*. The electric force is very large as compared to the gravitational force. The electrons must have some additional property, apart from their mass, which is responsible for the electric force. We call this property *charge*. Just as masses are responsible for the gravitational force, charges are responsible for the electric force. Two protons placed at a distance of 1 cm also repel each other with a force of $2{\cdot}3 \times 10^{-24}$ N. Thus, protons also have charge. Two neutrons placed at a distance of 1 cm attract each other with a force of $1{\cdot}9 \times 10^{-60}$ N which is equal to $\frac{Gm_1m_2}{r^2}$. Thus, neutrons exert only gravitational force on each other and experience no electric force. The neutrons have mass but no charge.

### Two Kinds of Charges

As mentioned above, the electric force between two electrons is the same as the electric force between two protons placed at the same separation. We may guess that the amount of charge on an electron is the same as that on a proton. However, if a proton and an electron are placed 1 cm apart, they attract each other with a force of $2{\cdot}3 \times 10^{-24}$ N. Certainly this force is electric, but it is attractive and not repulsive. The charge on an electron repels the charge on another electron but attracts the charge on a proton. Thus, although the charge on an electron and that on a proton have the same strength, they are of two different nature. Also, if we pack a proton and an electron together in a small volume, the combination does not attract or repel another electron or proton placed at a distance. The net charge on the proton–electron system seems to be zero. It is, therefore, convenient to define one charge as positive and the other as negative. We arbitrarily call the charge on a proton as positive and that on an electron as negative. This assignment of positive and negative signs to the proton charge and the electron charge is purely a convention. It does not mean that the charge on an electron is "less" than the charge on a proton.

### Unit of Charge

The above discussion suggests that charge is a basic property associated with the elementary particles and its definition is as difficult as the definition of mass or time or length. We can measure the charge on a system by comparing it with the charge on a standard body but we do not know what exactly it is that we intend to measure. The SI unit of charge is coulomb abbreviated as C. 1 coulomb is the charge flowing through a wire in 1 *s* if the electric current in it is 1 A. The charge on a proton is

$$e = 1{\cdot}60218 \times 10^{-19}\ \text{C}.$$

The charge on an electron is the negative of this value.

### Charge is Quantized

If protons and electrons are the only charge carriers in the universe, all observable charges must

be integral multiples of $e$. If an object contains $n_1$ protons and $n_2$ electrons, the net charge on the object is

$$n_1(e) + n_2(-e) = (n_1 - n_2)e.$$

Indeed, there are elementary particles other than protons and electrons, which carry charge. However, they all carry charges which are integral multiples of $e$. Thus, the charge on any object is always an integral multiple of $e$ and can be changed only in steps of $e$, i.e., charge is quantized.

The step size $e$ is usually so small that we can easily neglect the quantization. If we rub a glass rod with a silk cloth, typically charges of the order of a microcoulomb appear on the rubbed objects. Now, 1 μC contains $n$ units of basic charge $e$ where

$$n = \frac{1\ \mu\text{C}}{1{\cdot}6 \times 10^{-19}\ \text{C}} \approx 6 \times 10^{12}.$$

The step size is thus very small as compared to the charges usually found and in many cases we can assume a continuous charge variation.

**Charge is Conserved**

The charge of an isolated system is conserved. It is possible to create or destroy charged particles but it is not possible to create or destroy *net charge*. In a beta decay process, a neutron converts itself into a proton and a fresh electron is created. The charge however, remains zero before and after the event.

**Frictional Electricity : Induction**

The simplest way to experience electric charges is to rub certain solid bodies against each other. Long ago, around 600 BC, the Greeks knew that when amber is rubbed with wool, it acquires the property of attracting light objects such as small pieces of paper. This is because amber becomes electrically charged. If we pass a comb through dry hair, the comb becomes electrically charged and can attract small pieces of paper. An automobile becomes charged when it travels through the air. A paper sheet becomes charged when it passes through a printing machine. A gramophone record becomes charged when cleaned with a dry cloth.

The explanation of appearance of electric charge on rubbing is simple. All material bodies contain large number of electrons and equal number of protons in their normal state. When rubbed against each other, some electrons from one body may pass on to the other body. The body that receives the extra electrons, becomes negatively charged. The body that donates the electrons, becomes positively charged because it has more protons than electrons. Thus, when a glass rod is rubbed with a silk cloth, electrons are transferred from the glass rod to the silk cloth. The glass rod becomes positively charged and the silk cloth becomes negatively charged.

If we take a positively charged glass rod near small pieces of paper, the rod attracts the pieces. Why does the rod attract paper pieces which are uncharged ? This is because the positively charged rod attracts the electrons of a paper piece towards itself. Some of the electrons accumulate at that edge of the paper piece which is closer to the rod. At the farther end of the piece there is a deficiency of electrons and hence positive charge appears there. Such a redistribution of charge in a material, due to the presence of a nearby charged body, is called *induction*. The rod exerts larger attraction on the negative charges of the paper piece as compared to the repulsion on the positive charges. This is because the negative charges are closer to the rod. Hence, there is a net attraction between the rod and the paper piece.

## 29.2 COULOMB'S LAW

The experiments of Coulomb and others established that the force exerted by a charged particle on the other is given by

$$F = \frac{kq_1q_2}{r^2}, \qquad \ldots (29.1)$$

where $q_1$ and $q_2$ are the charges on the particles, $r$ is the separation between them and $k$ is a constant. The force is attractive if the charges are of opposite signs and is repulsive if they are of the same sign. We can write Coulomb's law as

$$\vec{F} = \frac{kq_1q_2\,\vec{r}}{r^3},$$

where $\vec{r}$ is the position vector of the force-experiencing particle with respect to the force-exerting particle. In this form, the equation includes the direction of the force.

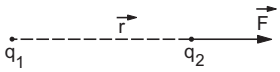



Figure 29.1

As $F$, $q_1$, $q_2$ and $r$ are all independently defined quantities, the constant $k$ can be measured experimentally. In SI units, the constant $k$ is measured to be $8{\cdot}98755 \times 10^9\ \text{N m}^2\ \text{C}^{-2}$.

The constant $k$ is often written as $\frac{1}{4\pi\varepsilon_0}$ so that equation (29.1) becomes

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r^2}. \qquad \ldots (29.2)$$

The constant $\varepsilon_0$ is called the *permittivity of free space* and its value is

$$\varepsilon_0 = \frac{1}{4\pi k} = 8{\cdot}85419 \times 10^{-12}\ \text{C}^2\ \text{N}^{-1}\ \text{m}^{-2}.$$

### 29.3 ELECTRIC FIELD

We have already discussed in the chapter on gravitation that a particle cannot directly interact with another particle kept at a distance. A particle creates a gravitational field around it and this field exerts force on another particle placed in it. The electric force between two charged particles is also seen as a two-step process. A charge produces something called an *electric field* in the space around it and this electric field exerts a force on any charge (except the source charge itself) placed in it. The electric field has its own existence and is present even if there is no additional charge to experience the force. The field takes finite time to propagate. Thus, if a charge is displaced from its position, the field at a distance $r$ will change after a time $t = r/c$, where $c$ is the speed of light. We define the *intensity of electric field* at a point as follows:

Bring a charge $q$ at the given point without disturbing any other charge that has produced the field. If the charge $q$ experiences an electric force $\vec{F}$, we define the intensity of electric field at the given point as

$$\vec{E} = \frac{\vec{F}}{q}. \qquad \ldots (29.3)$$

The charge $q$ used to define $\vec{E}$ is called a *test charge*.

One way to ensure that the test charge $q$ does not disturb other charges is to keep its magnitude very small. If this magnitude is not small, the positions of the other charges may change. Equation (29.3) then gives the electric field due to the charges in the changed positions. The intensity of electric field is often abbreviated as *electric field*.

The electric field at a point is a vector quantity. Suppose, $\vec{E}_1$ is the field at a point due to a charge $Q_1$ and $\vec{E}_2$ is the field at the same point due to a charge $Q_2$. The resultant field when both the charges are present, is $\vec{E} = \vec{E}_1 + \vec{E}_2$.

#### Electric Field due to a Point Charge

Consider a point charge $Q$ placed at a point $A$ (figure 29.2). We are interested in the electric field $\vec{E}$ at a point $P$ at a distance $r$ from $Q$. Let us imagine a test charge $q$ placed at $P$. The charge $Q$ creates a field $\vec{E}$ at $P$ and this field exerts a force $\vec{F} = q\vec{E}$ on the charge $q$. But, from Coulomb's law the force on the charge $q$ in the given situation is

$$F = \frac{Qq}{4\pi\varepsilon_0 r^2}$$

along $AP$. The electric field at $P$ is, therefore,

$$E = \frac{F}{q} = \frac{Q}{4\pi\varepsilon_0 r^2} \qquad \ldots (29.4)$$

along $AP$.

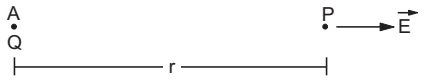



Figure 29.2

The electric field due to a set of charges may be obtained by finding the fields due to each individual charge and then adding these fields according to the rules of vector addition.

***Example 29.1***

*Two charges* 10 μC *and* −10 μC *are placed at points A and B separated by a distance of* 10 cm. *Find the electric field at a point P on the perpendicular bisector of AB at a distance of* 12 cm *from its middle point.*

***Solution*** :

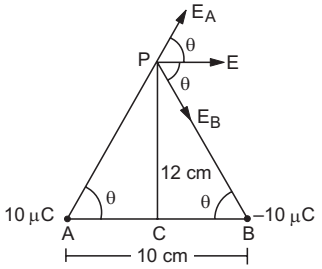



Figure 29.3

The situation is shown in figure (29.3). The distance $AP = BP = \sqrt{(5\text{ cm})^2 + (12\text{ cm})^2} = 13$ cm.

The field at the point $P$ due to the charge 10 μC is

$$E_A = \frac{10\ \mu\text{C}}{4\pi\varepsilon_0 (13\text{ cm})^2} = \frac{(10 \times 10^{-6}\text{ C}) \times (9 \times 10^9\text{ N m}^2\text{ C}^{-2})}{169 \times 10^{-4}\text{ m}^2}$$

$$= 5{\cdot}3 \times 10^6\text{ N C}^{-1}.$$

This field is along $AP$. The field due to −10 μC at $P$ is $E_B = 5{\cdot}3 \times 10^6$ N C$^{-1}$ along $PB$. As $E_A$ and $E_B$ are equal in magnitude, the resultant will bisect the angle between the two. The geometry of the figure shows that this resultant is parallel to the base $AB$. The magnitude of the resultant field is

$$E = E_A \cos\theta + E_B \cos\theta$$

$$= 2 \times (5{\cdot}3 \times 10^6\text{ N C}^{-1}) \times \frac{5}{13}$$

$$= 4{\cdot}1 \times 10^6\text{ N C}^{-1}.$$

If a given charge distribution is continuous, we can use the technique of integration to find the resultant electric field at a point. A small element $dQ$ is chosen in the distribution and the field $d\vec{E}$ due to $dQ$ is

calculated. The resultant field is then calculated by integrating the components of $d\vec{E}$ under proper limits.

***Example 29.2***

*A ring of radius a contains a charge q distributed uniformly over its length. Find the electric field at a point on the axis of the ring at a distance x from the centre.*

***Solution :***

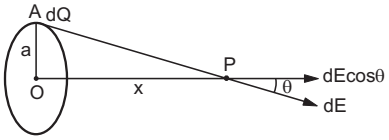



Figure 29.4

Figure (29.4) shows the situation. Let us consider a small element of the ring at the point $A$ having a charge $dQ$. The field at $P$ due to this element is

$$dE = \frac{dQ}{4\pi\varepsilon_0 (AP)^2}.$$

By symmetry, the field at $P$ will be along the axis $OP$. The component of $dE$ along this direction is

$$dE\cos\theta = \frac{dQ}{4\pi\varepsilon_0 (AP)^2}\left(\frac{OP}{AP}\right)$$

$$= \frac{x\,dQ}{4\pi\varepsilon_0 (a^2 + x^2)^{3/2}}.$$

The net field at $P$ is

$$E = \int dE\cos\theta = \int \frac{x\,dQ}{4\pi\varepsilon_0 (a^2 + x^2)^{3/2}}$$

$$= \frac{x}{4\pi\varepsilon_0 (a^2 + x^2)^{3/2}} \int dQ = \frac{xQ}{4\pi\varepsilon_0 (a^2 + x^2)^{3/2}}.$$

## 29.4 LINES OF ELECTRIC FORCE

The electric field in a region can be graphically represented by drawing certain curves known as *lines of electric force* or *electric field lines*. Lines of force are drawn in such a way that the tangent to a line of force gives the direction of the resultant electric field there. Thus, the electric field due to a positive point charge is represented by straight lines originating from the charge (figure 29.5a). The electric field due to a negative point charge is represented by straight lines terminating at the charge (figure 29.5b). If we draw the lines isotropically (the lines are drawn uniformly in all directions, originating from the point charge), we can compare the intensities of the field at two points by just looking at the distribution of the lines of force.

Consider two points $P_1$ and $P_2$ in figure (29.5). Draw equal small areas through $P_1$ and $P_2$ perpendicular to the lines. More number of lines pass through the area at $P_1$ and less number of lines pass through the area at $P_2$. Also, the intensity of electric field is more at $P_1$ than at $P_2$. In fact, the electric field is proportional to the lines per unit area if the lines originate isotropically from the charge.

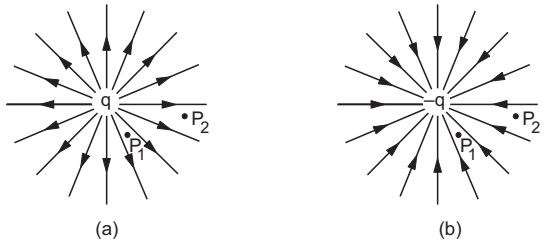



Figure 29.5

We can draw the lines of force for a charge distribution containing more than one charge. From each charge we can draw the lines isotropically. The lines may not be straight as one moves away from a charge. Figure (29.6) shows the shapes of these lines for some charge distributions.

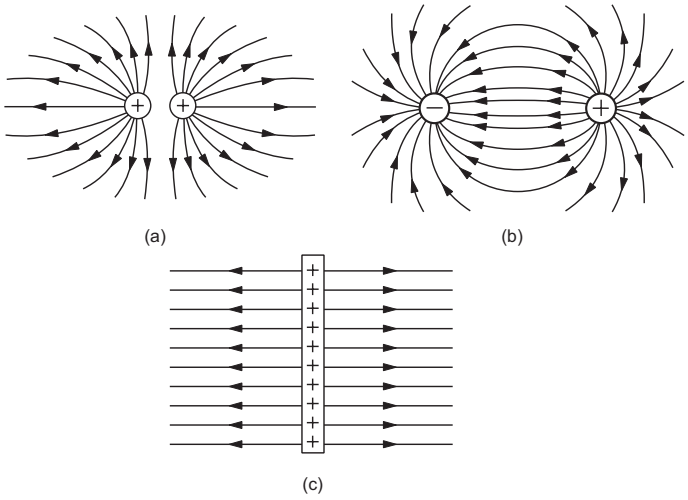



Figure 29.6

The lines of force are purely a geometrical construction which help us to visualise the nature of electric field in a region. They have no physical existence.

## 29.5 ELECTRIC POTENTIAL ENERGY

Consider a system of charges. The charges of the system exert electric forces on each other. If the position of one or more charges is changed, work may be done by these electric forces. We define *change in electric potential energy* of the system as negative of the work done by the electric forces as the configuration of the system changes.

Consider a system of two charges $q_1$ and $q_2$. Suppose, the charge $q_1$ is fixed at a point $A$ and the charge $q_2$ is taken from a point $B$ to a point $C$ along the line $ABC$ (figure 29.7).

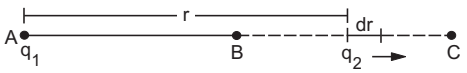



Figure 29.7

Let the distance $AB = r_1$ and the distance $AC = r_2$.

Consider a small displacement of the charge $q_2$ in which its distance from $q_1$ changes from $r$ to $r + dr$. The electric force on the charge $q_2$ is

$$F = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2} \text{ towards } \vec{AB}.$$

The work done by this force in the small displacement $dr$ is

$$dW = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2} dr.$$

The total work done as the charge $q_2$ moves from $B$ to $C$ is

$$W = \int_{r_1}^{r_2} \frac{q_1 q_2}{4\pi\varepsilon_0 r^2} dr = \frac{q_1 q_2}{4\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

No work is done by the electric force on the charge $q_1$ as it is kept fixed. The change in potential energy $U(r_2) - U(r_1)$ is, therefore,

$$U(r_2) - U(r_1) = -W = \frac{q_1 q_2}{4\pi\varepsilon_0}\left(\frac{1}{r_2} - \frac{1}{r_1}\right). \quad \ldots (29.5)$$

We choose the potential energy of the two-charge system to be zero when they have infinite separation (that means when they are widely separated). This means $U(\infty) = 0$. The potential energy when the separation is $r$ is

$$U(r) = U(r) - U(\infty)$$
$$= \frac{q_1 q_2}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{\infty}\right) = \frac{q_1 q_2}{4\pi\varepsilon_0 r}. \quad \ldots (29.6)$$

The above equation is derived by assuming that one of the charges is fixed and the other is displaced. However, the potential energy depends essentially on the separation between the charges and is independent of the spatial location of the charged particles. Equations (29.5) and (29.6) are, therefore, general.

Equation (29.6) gives the electric potential energy of a pair of charges. If there are three charges $q_1$, $q_2$ and $q_3$, there are three pairs. Similarly for an $N$-particle system, the potential energy of the system is equal to the sum of the potential energies of all the pairs of the charged particles.

***Example 29.3***

*Three particles, each having a charge of* 10 μC*, are placed at the vertices of an equilateral triangle of side* 10 cm. *Find the work done by a person in pulling them apart to infinite separations.*

***Solution*** : The potential energy of the system in the initial condition is

$$U = \frac{3 \times (10\ \mu\text{C}) \times (10\ \mu\text{C})}{4\pi\varepsilon_0\ (10\ \text{cm})}$$
$$= \frac{(3 \times 10^{-10}\ \text{C}^2) \times (9 \times 10^9\ \text{N m}^2\ \text{C}^{-2})}{0{\cdot}1\ \text{m}} = 27\ \text{J}.$$

When the charges are infinitely separated, the potential energy is reduced to zero. If we assume that the charges do not get kinetic energy in the process, the total mechanical energy of the system decreases by 27 J. Thus, the work done by the person on the system is –27 J.

## 29.6 ELECTRIC POTENTIAL

The electric field in a region of space is described by assigning a vector quantity $\vec{E}$ at each point. The same field can also be described by assigning a scalar quantity $V$ at each point. We now define this scalar quantity known as *electric potential*.

Suppose, a test charge $q$ is moved in an electric field from a point $A$ to a point $B$ while all the other charges in question remain fixed. If the electric potential energy changes by $U_B - U_A$ due to this displacement, we define the *potential difference* between the point $A$ and the point $B$ as

$$V_B - V_A = \frac{U_B - U_A}{q}. \quad \ldots (29.7)$$

Conversely, if a charge $q$ is taken through a potential difference $V_B - V_A$, the electric potential energy is increased by $U_B - U_A = q(V_B - V_A)$. This equation defines potential difference between any two points in an electric field. We can define absolute electric potential at any point by choosing a reference point $P$ and saying that the potential at this point is zero. The electric potential at a point $A$ is then given by (equation 29.7)

$$V_A = V_A - V_P = \frac{U_A - U_P}{q}. \quad \ldots (29.8)$$

So, *the potential at a point A is equal to the change in electric potential energy per unit test charge when it is moved from the reference point to the point A*.

Suppose, the test charge is moved in an electric field without changing its kinetic energy. The total work done on the charge should be zero from the work–energy theorem. If $W_{ext}$ and $W_{el}$ be the work done by the external agent and by the electric field as the charge moves, we have,

$$W_{ext} + W_{el} = 0$$

or, $$W_{ext} = -W_{el} = \Delta U,$$

where $\Delta U$ is the change in electric potential energy. Using this equation and equation (29.8), the potential at a point $A$ may also be defined as follows:

*The potential at a point A is equal to the work done per unit test charge by an external agent in moving the test charge from the reference point to the point A (without changing its kinetic energy).*

The choice of reference point is purely ours. Generally, a point widely separated from all charges in question is taken as the reference point. Such a point is assumed to be at infinity.

As potential energy is a scalar quantity, potential is also a scalar quantity. Thus, if $V_1$ is the potential at a given point due to a charge $q_1$ and $V_2$ is the potential at the same point due to a charge $q_2$, the potential due to both the charges is $V_1 + V_2$.

## 29.7 ELECTRIC POTENTIAL DUE TO A POINT CHARGE

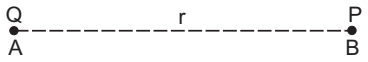



Figure 29.8

Consider a point charge $Q$ placed at a point $A$ (figure 29.8). We have to find the electric potential at a point $P$ where $AP = r$. Let us take the reference point at $r = \infty$. Suppose, a test charge $q$ is moved from $r = \infty$ to the point $P$. The change in electric potential energy of the system is, from equation (29.6),

$$U_P - U_\infty = \frac{Qq}{4\pi\varepsilon_0 r}.$$

The potential at $P$ is, from equation (29.8),

$$V_P = \frac{U_P - U_\infty}{q} = \frac{Q}{4\pi\varepsilon_0 r}. \quad \ldots (29.9)$$

The electric potential due to a system of charges may be obtained by finding potentials due to the individual charges using equation (29.9) and then adding them. Thus,

$$V = \frac{1}{4\pi\varepsilon_0}\sum \frac{Q_i}{r_i}.$$

***Example 29.4***

*Two charges* $+10\,\mu C$ *and* $+20\,\mu C$ *are placed at a separation of* 2 cm. *Find the electric potential due to the pair at the middle point of the line joining the two charges.*

***Solution*** : Using the equation $V = \frac{Q}{4\pi\varepsilon_0 r}$, the potential due to $+10\,\mu C$ is

$$V_1 = \frac{(10\times10^{-6}\text{ C})\times(9\times10^{9}\text{ N m}^2\text{C}^{-2})}{1\times10^{-2}\text{ m}} = 9\text{ MV}.$$

The potential due to $+20\,\mu C$ is

$$V_2 = \frac{(20\times10^{-6}\text{ C})\times(9\times10^{9}\text{ N m}^2\text{C}^{-2})}{1\times10^{-2}\text{ m}} = 18\text{ MV}.$$

The net potential at the given point is

$$9\text{ MV} + 18\text{ MV} = 27\text{ MV}.$$

If the charge distribution is continuous, we may use the technique of integration to find the electric potential.

## 29.8 RELATION BETWEEN ELECTRIC FIELD AND POTENTIAL

Suppose, the electric field at a point $\vec{r}$ due to a charge distribution is $\vec{E}$ and the electric potential at the same point is $V$. Suppose, a point charge $q$ is displaced slightly from the point $\vec{r}$ to $\vec{r} + d\vec{r}$. The force on the charge is

$$\vec{F} = q\vec{E}$$

and the work done by the electric field during the displacement is

$$dW = \vec{F}\cdot d\vec{r} = q\vec{E}\cdot d\vec{r}.$$

The change in potential energy is

$$dU = -dW = -q\vec{E}\cdot d\vec{r}.$$

The change in potential is

$$dV = \frac{dU}{q}$$

or,
$$dV = -\vec{E}\cdot d\vec{r}. \quad \ldots (29.10)$$

Integrating between the points $\vec{r}_1$ and $\vec{r}_2$, we get

$$V_2 - V_1 = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{E}\cdot d\vec{r} \quad \ldots (29.11)$$

where $V_1$ and $V_2$ are the potentials at $\vec{r}_1$ and $\vec{r}_2$ respectively. If we choose $\vec{r}_1$, at the reference point (say at infinity) and $\vec{r}_2$ at $\vec{r}$, equation (29.11) becomes

$$V(\vec{r}) = -\int_{\infty}^{\vec{r}} \vec{E}\cdot d\vec{r}. \quad \ldots (29.12)$$

***Example 29.5***

*Figure (29.9) shows two metallic plates A and B placed parallel to each other at a separation d. A uniform electric field E exists between the plates in the direction from plate B to plate A. Find the potential difference between the plates.*

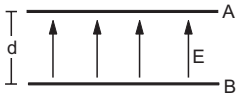



Figure 29.9

***Solution*** : Let us take the origin at plate $A$ and $x$-axis along the direction from plate $A$ to plate $B$. We have

$$V_B - V_A = -\int_{\vec{r}_A}^{\vec{r}_B} \vec{E}\cdot d\vec{r} = -\int_0^d -E\,dx = Ed.$$

If we work in Cartesian coordinate system

$$E = E_x\,\vec{i} + E_y\,\vec{j} + E_z\,\vec{k}$$

and $$d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}.$$

Thus, from (29.10)

$$dV = -E_x\,dx - E_y\,dy - E_z\,dz. \quad \ldots \text{(i)}$$

If we change $x$ to $x + dx$ keeping $y$ and $z$ constant, $dy = dz = 0$ and from (i),

$$E_x = -\frac{\partial V}{\partial x}\cdot$$

Similarly, $$E_y = -\frac{\partial V}{\partial y} \quad \ldots \text{(29.13)}$$

and $$E_z = -\frac{\partial V}{\partial z}\cdot$$

The symbols $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, etc., are used to indicate that while differentiating with respect to one coordinate, the others are kept constant.

If we know the electric field in a region, we can find the electric potential using equation (29.12) and if we know the electric potential in a region, we can find the electric field using (29.13).

Equation (29.10) may also be written as

$$dV = -E\,dr\cos\theta$$

where θ is the angle between the field $\vec{E}$ and the small displacement $d\vec{r}$. Thus,

$$-\frac{dV}{dr} = E\cos\theta. \quad \ldots \text{(29.14)}$$

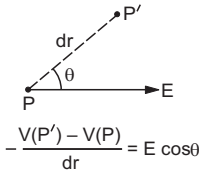



Figure 29.10

We see that, $-\frac{dV}{dr}$ gives the component of the electric field in the direction of displacement $d\vec{r}$. In figure (29.10), we show a small displacement $PP' = dr$. The electric field is $E$ making an angle θ with $PP'$. We have

$$dV = V(P') - V(P)$$

so that $$\frac{V(P) - V(P')}{dr} = E\cos\theta.$$

This gives us a method to get the component of the electric field in any given direction if we know the potential. Move a small distance $dr$ in the given direction and see the change $dV$ in the potential. The component of electric field along that direction is $-\frac{dV}{dr}$.

If we move a distance $dr$ in the direction of the field, θ is zero and $-\frac{dV}{dr} = E$ is maximum. Thus, *the electric field is along the direction in which the potential decreases at the maximum rate.*

If a small displacement $d\vec{r}$ perpendicular to the electric field is considered, $\theta = 90°$ and $dV = -\vec{E}\cdot d\vec{r} = 0$. The potential does not vary in a direction perpendicular to the electric field.

**Equipotential Surfaces**

If we draw a surface in such a way that the electric potential is the same at all the points of the surface, it is called an *equipotential surface.* The component of electric field parallel to an equipotential surface is zero, as the potential does not change in this direction. Thus, the electric field is perpendicular to the equipotential surface at each point of the surface. For a point charge, the electric field is radial and the equipotential surfaces are concentric spheres with centres at the charge (figure 29.11).

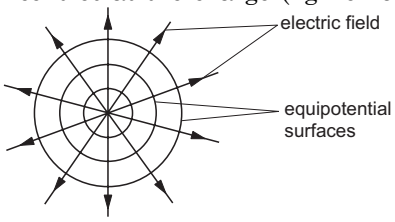



Figure 29.11

## 29.9 ELECTRIC DIPOLE

A combination of two charges $+q$ and $-q$ separated by a small distance $d$ constitutes an *electric dipole*. The *electric dipole moment* of this combination is defined as a vector

$$\vec{p} = q\vec{d}, \quad \ldots \text{(29.15)}$$

where $\vec{d}$ is the vector joining the negative charge to the positive charge. The line along the direction of the dipole moment is called the *axis of the dipole.*

**Electric Potential due to a Dipole**

Suppose, the negative charge $-q$ is placed at a point $A$ and the positive charge $q$ is placed at a point

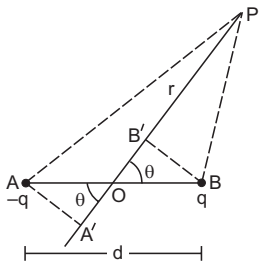



Figure 29.12

$B$ (figure 29.12), the separation $AB = d$. The middle point of $AB$ is $O$. The potential is to be evaluated at a point $P$ where $OP = r$ and $\angle POB = \theta$. Also, let $r >> d$.

Let $AA'$ be the perpendicular from $A$ to $PO$ and $BB'$ be the perpendicular from $B$ to $PO$. As $d$ is very small compared to $r$,

$$AP \approx A'P = OP + OA'$$
$$= OP + AO\cos\theta = r + \frac{d}{2}\cos\theta.$$

Similarly, $BP \approx B'P = OP - OB'$

$$= r - \frac{d}{2}\cos\theta.$$

The potential at $P$ due to the charge $-q$ is

$$V_1 = -\frac{1}{4\pi\varepsilon_0}\frac{q}{AP} \approx -\frac{1}{4\pi\varepsilon_0}\frac{q}{r + \frac{d}{2}\cos\theta}$$

and that due to the charge $+q$ is

$$V_2 = \frac{1}{4\pi\varepsilon_0}\frac{q}{BP} \approx \frac{1}{4\pi\varepsilon_0}\frac{q}{r - \frac{d}{2}\cos\theta}.$$

The net potential at $P$ due to the dipole is

$$V = V_1 + V_2$$
$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{r - \frac{d}{2}\cos\theta} - \frac{q}{r + \frac{d}{2}\cos\theta}\right]$$
$$= \frac{1}{4\pi\varepsilon_0}\frac{q\,d\cos\theta}{r^2 - \frac{d^2}{4}\cos^2\theta}$$
$$\approx \frac{1}{4\pi\varepsilon_0}\frac{q\,d\cos\theta}{r^2}$$

or, $$V = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}. \quad \ldots (29.16)$$

**Generalised Definition of Electric Dipole**

The potential at a distance $r$ from a point charge $q$ is given by

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}.$$

It is inversely proportional to $r$ and is independent of direction. The potential due to a dipole is inversely proportional to $r^2$ and depends on direction as shown by the term $\cos\theta$ in equation (29.16). In general, any charge distribution that produces electric potential given by

$$V = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}$$

is called an electric dipole. The constant $p$ is called its dipole moment and the direction from which the angle $\theta$ is measured to get the above equation is called the direction of the dipole moment.

**Electric Field due to a Dipole**

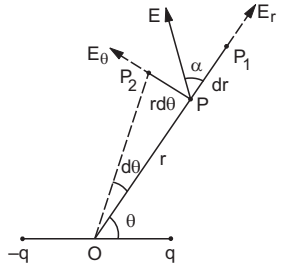



Figure 29.13

We can find the electric field due to an electric dipole using the expression (29.16) for the electric potential. In figure (29.13), $PP_1$ is a small displacement in the direction of $OP$ and $PP_2$ is a small displacement perpendicular to $OP$. Thus, $PP_1$ is in radial direction and $PP_2$ is in transverse direction. In going from $P$ to $P_1$, the angle $\theta$ does not change and the distance $OP$ changes from $r$ to $r + dr$. Thus, $PP_1 = dr$. In going from $P$ to $P_2$, the angle $\theta$ changes from $\theta$ to $\theta + d\theta$ while the distance $r$ remains almost constant. Thus, $PP_2 = r\,d\theta$. From equation (29.14), the component of the electric field at $P$ in the radial direction $PP_1$ is

$$E_r = -\frac{dV}{PP_1} = -\frac{dV}{dr} = -\frac{\partial V}{\partial r}. \quad \ldots \text{(i)}$$

The symbol $\partial$ specifies that $\theta$ should be treated as constant while differentiating with respect to $r$.

Similarly, the component of the electric field at $P$ in the transverse direction $PP_2$ is

$$E_\theta = -\frac{dV}{PP_2} = -\frac{dV}{r\,d\theta} = -\frac{1}{r}\frac{\partial V}{\partial\theta}. \quad \ldots \text{(ii)}$$

As $$V = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2},$$

$$E_r = -\frac{\partial V}{\partial r} = -\frac{1}{4\pi\varepsilon_0}\frac{\partial}{\partial r}\left(\frac{p\cos\theta}{r^2}\right)$$
$$= -\frac{1}{4\pi\varepsilon_0}(p\cos\theta)\frac{d}{dr}\left(\frac{1}{r^2}\right)$$
$$= \frac{1}{4\pi\varepsilon_0}\frac{2p\cos\theta}{r^3} \quad \ldots \text{(iii)}$$

and $$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta} = -\frac{1}{r}\frac{1}{4\pi\varepsilon_0}\frac{\partial}{\partial\theta}\left(\frac{p\cos\theta}{r^2}\right)$$
$$= -\frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}\frac{d}{d\theta}(\cos\theta)$$
$$= \frac{1}{4\pi\varepsilon_0}\frac{p\sin\theta}{r^3}. \quad \ldots \text{(iv)}$$

The resultant electric field at $P$ (figure 29.13) is

$$E = \sqrt{E_r^2 + E_\theta^2}$$
$$= \frac{1}{4\pi\varepsilon_0}\sqrt{\left(\frac{2p\cos\theta}{r^3}\right)^2 + \left(\frac{p\sin\theta}{r^3}\right)^2}$$
$$= \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}\sqrt{3\cos^2\theta + 1}. \quad \ldots (29.17)$$

If the resultant field makes an angle $\alpha$ with the radial direction $OP$, we have

$$\tan\alpha = \frac{E_\theta}{E_r} = \frac{p\sin\theta/r^3}{2p\cos\theta/r^3} = \frac{1}{2}\tan\theta$$

or, $$\alpha = \tan^{-1}\left(\frac{1}{2}\tan\theta\right). \quad \ldots (29.18)$$

**Special Cases**

(a) $\theta = 0$

In this case, the point $P$ is on the axis of the dipole. From equation (29.16), the electric potential is $V = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^2}$.

The field at such a point is, from equation (29.17), $E = \frac{1}{4\pi\varepsilon_0}\frac{2p}{r^3}$ along the axis. Such a position of the point $P$ is called an *end-on position*.

(b) $\theta = 90°$

In this case the point $P$ is on the perpendicular bisector of the dipole. The potential here is zero while the field is, from equation (29.17), $E = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}$.

The angle $\alpha$ is given by

$$\tan\alpha = \frac{\tan\theta}{2} = \infty$$

or, $$\alpha = 90°.$$

The field is antiparallel to the dipole axis. Such a position of the point $P$ is called a *broadside-on position*.

## 29.10 TORQUE ON AN ELECTRIC DIPOLE PLACED IN AN ELECTRIC FIELD

Consider an electric dipole placed in a uniform electric field $\vec{E}$. The dipole consists of charges $-q$ placed at $A$ and $+q$ placed at $B$ (figure 29.14). The mid-point of $AB$ is $O$ and the length $AB = d$. Suppose the axis of the dipole $AB$ makes an angle $\theta$ with the electric field at a certain instant.

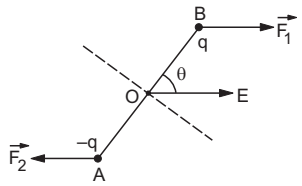



Figure 29.14

The force on the charge $+q$ is $\vec{F}_1 = q\vec{E}$ and the force on the charge $-q$ is $\vec{F}_2 = -q\vec{E}$. Let us calculate the torques $(\vec{r}\times\vec{F})$ of these forces about $O$.

The torque of $\vec{F}_1$ about $O$ is

$$\vec{\Gamma}_1 = \vec{OB}\times\vec{F}_1 = q(\vec{OB}\times\vec{E})$$

and the torque of $\vec{F}_2$ about $O$ is

$$\vec{\Gamma}_2 = \vec{OA}\times\vec{F}_2 = -q(\vec{OA}\times\vec{E}) = q(\vec{AO}\times\vec{E}).$$

The net torque acting on the dipole is

$$\vec{\Gamma} = \vec{\Gamma}_1 + \vec{\Gamma}_2$$
$$= q(\vec{OB}\times\vec{E}) + q(\vec{AO}\times\vec{E})$$
$$= q(\vec{OB} + \vec{AO})\times\vec{E}$$
$$= q\,\vec{AB}\times\vec{E} = \vec{p}\times\vec{E}. \quad \ldots (29.19)$$

The direction of the torque is perpendicular to the plane containing the dipole axis and the electric field. In figure (29.14), this is perpendicular to the plane of paper and is going into the page. The magnitude is $\Gamma = |\vec{\Gamma}| = pE\sin\theta$.

## 29.11 POTENTIAL ENERGY OF A DIPOLE PLACED IN A UNIFORM ELECTRIC FIELD

When an electric dipole is placed in an electric field $\vec{E}$, a torque $\vec{\Gamma} = \vec{p}\times\vec{E}$ acts on it (figure 29.14). If we rotate the dipole through a small angle $d\theta$, the work done by the torque is

$$dW = \Gamma d\theta$$
$$= -pE\sin\theta\, d\theta.$$

The work is negative as the rotation $d\theta$ is opposite to the torque.

The change in electric potential energy of the dipole is, therefore,

$$dU = -dW = pE\sin\theta\, d\theta.$$

If the angle $\theta$ is changed from $90°$ to $\theta$, the change in potential energy is

$$U(\theta) - U(90°) = \int_{90°}^{\theta} pE\sin\theta\, d\theta$$
$$= pE\,[-\cos\theta]_{90°}^{\theta}$$
$$= -pE\cos\theta = -\vec{p}\cdot\vec{E}.$$

If we choose the potential energy of the dipole to be zero when $\theta = 90°$ (dipole axis is perpendicular to the field), $U(90°) = 0$ and the above equation becomes

$$U(\theta) = -\vec{p}\cdot\vec{E}. \quad \ldots (29.20)$$

## 29.12 CONDUCTORS, INSULATORS AND SEMICONDUCTORS

Any piece of matter of moderate size contains millions and millions of atoms or molecules. Each atom contains a positively charged nucleus and several electrons going round it.

In gases, the atoms or molecules almost do not interact with each other. In solids and liquids, the interaction is comparatively stronger. It turns out that the materials may be broadly divided into three categories according to their behaviour when they are placed in an electric field.

In some materials, the outer electrons of each atom or molecule are only weakly bound to it. These electrons are almost free to move throughout the body of the material and are called *free electrons*. They are also known as *conduction electrons*. When such a material is placed in an electric field, the free electrons move in a direction opposite to the field. Such materials are called *conductors*.

Another class of materials is called *insulators* in which all the electrons are tightly bound to their respective atoms or molecules. Effectively, there are no *free electrons*. When such a material is placed in an electric field, the electrons may slightly shift opposite to the field but they can't leave their parent atoms or molecules and hence can't move through long distances. Such materials are also called *dielectrics*.

In *semiconductors*, the behaviour is like an insulator at the temperature 0 K. But at higher temperatures, a small number of electrons are able to free themselves and they respond to the applied electric field. As the number of free electrons in a semiconductor is much smaller than that in a conductor, its behaviour is in between a conductor and an insulator and hence, the name semiconductor. A freed electron in a semiconductor leaves a vacancy in its normal bound position. These vacancies also help in conduction.

We shall learn more about conductivity in later chapters. At the moment we accept the simple approximate model described above. The conductors have large number of free electrons everywhere in the material whereas the insulators have none. The discussion of semiconductors is deferred to a separate chapter.

Roughly speaking, the metals are conductors and the nonmetals are insulators. The above discussion may be extended to liquids and gases. Some of the liquids, such as mercury, and ionized gases are conductors.

## 29.13 THE ELECTRIC FIELD INSIDE A CONDUCTOR

Consider a conducting plate placed in a region. Initially, there is no electric field and the conduction electrons are almost uniformly distributed within the plate (shown by dots in figure 29.15a). In any small volume (which contains several thousand molecules) the number of electrons is equal to the number of protons in the nuclei. The net charge in the volume is zero.

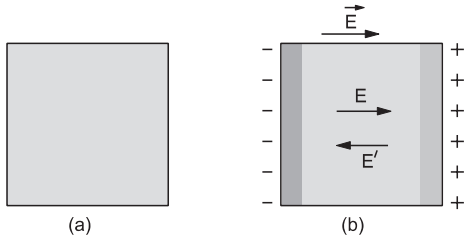



Figure 29.15

Now, suppose an electric field $\vec{E}$ is created in the direction left to right (figure 29.15b). This field exerts force on the free electrons from right to left. The electrons move towards left, the number of electrons on the left face increases and the number on the right face decreases. The left face becomes negatively charged and the right face becomes positively charged. These extra charges produce an extra electric field $E'$ inside the plate from right to left. The electrons continue to drift and the internal field $\vec{E}'$ becomes stronger and stronger. A situation comes when the field $\vec{E}'$ due to the redistribution of free electrons becomes equal in magnitude to $\vec{E}$. The net electric field inside the plate is then zero. The free electrons there do not experience any net force and the process of further drifting stops. Thus, a steady state is reached in which some positive and negative charges appear at the surface of the plate and there is no electric field inside the plate.

Whenever a conductor is placed in an electric field some of the free electrons redistribute themselves on the surface of the conductor. The redistribution takes place in such a way that the electric field is zero at all the points inside the conductor. The redistribution takes a time which is, in general, less than a millisecond. Thus, *there can be no electric field inside a conductor in electrostatics.*

### *Worked Out Examples*

**1.** *Charges* $5{\cdot}0\times10^{-7}$ C, $-2{\cdot}5\times10^{-7}$ C *and* $1{\cdot}0\times10^{-7}$ C *are held fixed at the three corners A, B, C of an equilateral triangle of side* 5·0 cm. *Find the electric force on the charge at C due to the rest two.*

***Solution*** :

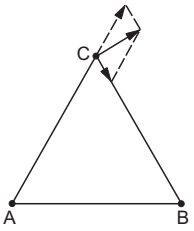



Figure 29-W1

The force on $C$ due to $A$

$$=\frac{1}{4\pi\varepsilon_0}\frac{(5\times10^{-7}\text{ C})\,(1\times10^{-7}\text{ C})}{(0{\cdot}05\text{ m})^2}$$

$$=9\times10^{9}\text{ Nm}^2\text{C}^{-2}\times\frac{5\times10^{-14}\text{ C}^2}{25\times10^{-4}\text{ m}^2}=0{\cdot}18\text{ N}.$$

This force acts along $AC$. The force on $C$ due to $B$

$$=\frac{1}{4\pi\varepsilon_0}\frac{(2{\cdot}5\times10^{-7}\text{ C})\,(1\times10^{-7}\text{ C})}{(0{\cdot}05\text{ m})^2}=0{\cdot}09\text{ N}.$$

This attractive force acts along $CB$. As the triangle is equilateral, the angle between these two forces is 120°. The resultant electric force on $C$ is

$$[(0{\cdot}18\text{ N})^2+(0{\cdot}09\text{ N})^2+2(0{\cdot}18\text{ N})\,(0{\cdot}09\text{ N})\,(\cos120°)]^{1/2}$$

$$=0{\cdot}16\text{ N}.$$

The angle made by this resultant with $CB$ is

$$\tan^{-1}\frac{0{\cdot}18\sin120°}{0{\cdot}09+0{\cdot}18\cos120°}=90°.$$

**2.** *Two particles A and B having charges* $8{\cdot}0\times10^{-6}$ C *and* $-2{\cdot}0\times10^{-6}$ C *respectively are held fixed with a separation of* 20 cm. *Where should a third charged particle be placed so that it does not experience a net electric force* ?

***Solution*** : As the net electric force on $C$ should be equal to zero, the force due to $A$ and $B$ must be opposite in direction. Hence, the particle should be placed on the line $AB$. As $A$ and $B$ have charges of opposite signs, $C$ cannot be between $A$ and $B$. Also, $A$ has larger magnitude of charge than $B$. Hence, $C$ should be placed closer to $B$ than $A$. The situation is shown in figure (29-W2).

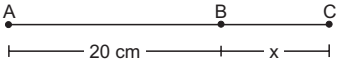



Figure 29-W2

Suppose $BC=x$ and the charge on $C$ is $Q$.

The force due to $A=\dfrac{(8{\cdot}0\times10^{-6}\text{ C})Q}{4\pi\varepsilon_0(20\text{ cm}+x)^2}$.

The force due to $B=\dfrac{(2{\cdot}0\times10^{-6}\text{ C})Q}{4\pi\varepsilon_0 x^2}$.

They are oppositely directed and to have a zero resultant, they should be equal in magnitude. Thus,

$$\frac{8}{(20\text{ cm}+x)^2}=\frac{2}{x^2}$$

or, $$\frac{20\text{ cm}+x}{x}=2,\ \text{giving}\ x=20\text{ cm}.$$

**3.** *Three equal charges, each having a magnitude of* $2{\cdot}0\times10^{-6}$ C, *are placed at the three corners of a right-angled triangle of sides* 3 cm, 4 cm *and* 5 cm. *Find the force on the charge at the right-angle corner.*

***Solution*** :

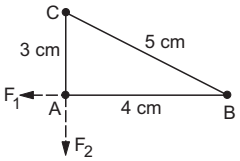



Figure 29-W3

The situation is shown in figure (29-W3). The force on $A$ due to $B$ is

$$F_1=\frac{(2{\cdot}0\times10^{-6}\text{ C})\,(2{\cdot}0\times10^{-6}\text{ C})}{4\pi\varepsilon_0\,(4\text{ cm})^2}$$

$$=9\times10^{9}\text{ N m}^2\text{ C}^{-2}\times4\times10^{-12}\text{ C}^2\times\frac{1}{16\times10^{-4}\text{ m}^2}$$

$$=22{\cdot}5\text{ N}.$$

This force acts along $BA$. Similarly, the force on $A$ due to $C$ is $F_2=40$ N in the direction of $CA$. Thus, the net electric force on $A$ is

$$F=\sqrt{F_1^2+F_2^2}$$

$$=\sqrt{(22{\cdot}5\text{ N})^2+(40\text{ N})^2}=45{\cdot}9\text{ N}.$$

This resultant makes an angle θ with $BA$ where

$$\tan\theta=\frac{40}{22{\cdot}5}=\frac{16}{9}.$$

**4.** *Two small iron particles, each of mass* 280 mg, *are placed at a distance* 10 cm *apart. If* 0·01% *of the electrons of one particle are transferred to the other, find the electric force between them. Atomic weight of iron is* 56 g mol$^{-1}$ *and there are* 26 *electrons in each atom of iron.*

***Solution*** : The atomic weight of iron is 56 g mol$^{-1}$. Thus, 56 g of iron contains $6\times10^{23}$ atoms and each atom contains 26 electrons. Hence, 280 mg of iron contains

$$\frac{280\ \text{mg} \times 6 \times 10^{23} \times 26}{56\ \text{g}} = 7{\cdot}8 \times 10^{22}\ \text{electrons.}$$

The number of electrons transferred from one particle to another

$$= \frac{0{\cdot}01}{100} \times 7{\cdot}8 \times 10^{22} = 7{\cdot}8 \times 10^{18}.$$

The charge transferred is, therefore,

$$1{\cdot}6 \times 10^{-19}\ \text{C} \times 7{\cdot}8 \times 10^{18} = 1{\cdot}2\ \text{C}.$$

The electric force between the particles is

$$(9 \times 10^{9}\ \text{N m}^2\ \text{C}^{-2}) \frac{(1{\cdot}2\ \text{C})^2}{(10 \times 10^{-2}\ \text{m})^2}$$

$$= 1{\cdot}3 \times 10^{12}\ \text{N}.$$

This equals the load of approximately 2000 million grown-up persons !

**5.** *A charge $Q$ is to be divided on two objects. What should be the values of the charges on the objects so that the force between the objects can be maximum* ?

***Solution*** : Suppose one object receives a charge $q$ and the other $Q - q$. The force between the objects is

$$F = \frac{q(Q-q)}{4\pi\varepsilon_0 d^2},$$

where $d$ is the separation between them. For $F$ to be maximum, the quantity

$$y = q(Q-q) = Qq - q^2$$

should be maximum. This is the case when,

$$\frac{dy}{dq} = 0 \text{ or, } Q - 2q = 0 \text{ or, } q = Q/2.$$

Thus, the charge should be divided equally on the two objects.

**6.** *Two particles, each having a mass of* 5 g *and charge* $1{\cdot}0 \times 10^{-7}$ C*, stay in limiting equilibrium on a horizontal table with a separation of* 10 cm *between them. The coefficient of friction between each particle and the table is the same. Find the value of this coefficient.*

***Solution*** : The electric force on one of the particles due to the other is

$$F = (9 \times 10^{9}\ \text{N m}^2\ \text{C}^{-2}) \times (1{\cdot}0 \times 10^{-7}\ \text{C})^2 \times \frac{1}{(0{\cdot}10\ \text{m})^2}$$

$$= 0{\cdot}009\ \text{N}.$$

The frictional force in limiting equilibrium

$$f = \mu \times (5 \times 10^{-3}\ \text{kg}) \times 9{\cdot}8\ \text{m s}^{-2}$$

$$= (0{\cdot}049\ \mu)\ \text{N}.$$

As these two forces balance each other,

$$0{\cdot}049\ \mu = 0{\cdot}009$$

or, $$\mu = 0{\cdot}18.$$

**7.** *A vertical electric field of magnitude* $4{\cdot}00 \times 10^{5}\ \text{N C}^{-1}$ *just prevents a water droplet of mass* $1{\cdot}00 \times 10^{-4}$ kg *from falling. Find the charge on the droplet.*

***Solution*** : The forces acting on the droplet are

(i) the electric force $q\vec{E}$ and

(ii) the force of gravity $m\vec{g}$.

To just prevent from falling, these two forces should be equal and opposite. Thus,

$$q(4{\cdot}00 \times 10^{5}\ \text{N C}^{-1}) = (1{\cdot}00 \times 10^{-4}\ \text{kg}) \times (9{\cdot}8\ \text{m s}^{-2})$$

or, $$q = 2{\cdot}45 \times 10^{-9}\ \text{C}.$$

**8.** *Three charges, each equal to q, are placed at the three corners of a square of side a. Find the electric field at the fourth corner.*

***Solution*** :

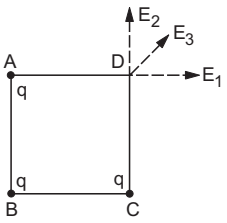



Figure 29-W4

Let the charges be placed at the corners $A$, $B$ and $C$ (figure 29-W4). We shall calculate the electric field at the fourth corner $D$. The field $E_1$ due to the charge at $A$ will have the magnitude $\frac{q}{4\pi\varepsilon_0 a^2}$ and will be along $AD$. The field $E_2$ due to the charge at $C$ will have the same magnitude and will be along $CD$.

The field $E_3$ due to the charge at $B$ will have the magnitude $\frac{q}{4\pi\varepsilon_0 (\sqrt{2}a)^2}$ and will be along $BD$. As $E_1$ and $E_2$ are equal in magnitude, their resultant will be along the bisector of the angle between $E_1$, $E_2$ and hence along $E_3$. The magnitude of this resultant is $\sqrt{E_1^2 + E_2^2}$ as the angle between $E_1$ and $E_2$ is $\pi/2$. The resultant electric field at $D$ is, therefore, along $E_3$ and has magnitude

$$\sqrt{E_1^2 + E_2^2} + E_3$$

$$= \sqrt{\left(\frac{q}{4\pi\varepsilon_0 a^2}\right)^2 + \left(\frac{q}{4\pi\varepsilon_0 a^2}\right)^2} + \frac{q}{4\pi\varepsilon_0 (\sqrt{2}a)^2}$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{\sqrt{2}}{a^2} + \frac{1}{2a^2}\right] = (2\sqrt{2}+1)\frac{q}{8\pi\varepsilon_0 a^2}.$$

**9.** *A charged particle of mass* 1·0 g *is suspended through a silk thread of length* 40 cm *in a horizontal electric field of* $4{\cdot}0 \times 10^{4}\ \text{N C}^{-1}$. *If the particle stays at a distance of* 24 cm *from the wall in equilibrium, find the charge on the particle.*

***Solution*** :

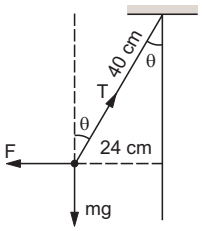



Figure 29-W5

The situation is shown in figure (29-W5).

The forces acting on the particle are

(i) the electric force $F = qE$ horizontally,

(ii) the force of gravity $mg$ downward and

(iii) the tension $T$ along the thread.

As the particle is at rest, these forces should add to zero. Taking components along horizontal and vertical,

$$T\cos\theta = mg \text{ and } T\sin\theta = F$$

or, $$F = mg\tan\theta \quad \ldots \text{(i)}$$

From the figure,

$$\sin\theta = \frac{24}{40} = \frac{3}{5}\cdot$$

Thus, $\tan\theta = \frac{3}{4}\cdot$ From (i),

$$q(4{\cdot}0\times10^{4}\text{ N C}^{-1}) = (1{\cdot}0\times10^{-3}\text{ kg})(9{\cdot}8\text{ m s}^{-2})\frac{3}{4},$$

giving $$q = 1{\cdot}8\times10^{-7}\text{ C}.$$

**10.** *A particle A having a charge of* $5{\cdot}0\times10^{-7}$ C *is fixed in a vertical wall. A second particle B of mass* 100 g *and having equal charge is suspended by a silk thread of length* 30 cm *from the wall. The point of suspension is* 30 cm *above the particle A. Find the angle of the thread with the vertical when it stays in equilibrium.*

***Solution*** :

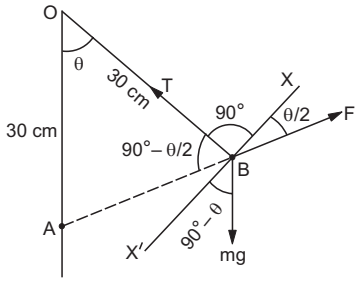



Figure 29-W6

The situation is shown in figure (29-W6). Suppose the point of suspension is $O$ and let θ be the angle between the thread and the vertical. Forces on the particle $B$ are

(i) weight $mg$ downward

(ii) tension $T$ along the thread and

(iii) electric force of repulsion $F$ along $AB$.

For equilibrium, these forces should add to zero. Let $X'BX$ be the line perpendicular to $OB$. We shall take the components of the forces along $BX$. This will give a relation between $F$, $mg$ and θ.

The various angles are shown in the figure. As

$$OA = OB,\ \angle OBA = \angle OAB = 90° - \frac{\theta}{2}\cdot$$

The other angles can be written down directly.

Taking components along $BX$, we get

$$F\cos\frac{\theta}{2} = mg\cos(90° - \theta)$$

$$= 2\,mg\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

or, $$\sin\frac{\theta}{2} = \frac{F}{2\,mg}\cdot \quad \ldots \text{(i)}$$

Now, $$F = (9\times10^{9}\text{ N m}^{2}\text{ C}^{-2})\times(5{\cdot}0\times10^{-7}\text{ C})^{2}\times\frac{1}{AB^{2}}$$

and $$AB = 2(OA)\sin\frac{\theta}{2}\cdot$$

Thus, $$F = \frac{9\times10^{9}\times25\times10^{-14}}{4\times(30\times10^{-2})^{2}\times\sin^{2}\frac{\theta}{2}}\text{ N}. \quad \ldots \text{(ii)}$$

From (i) and (ii),

$$\sin\frac{\theta}{2} = \frac{F}{2\,mg} = \frac{9\times10^{9}\times25\times10^{-14}\text{ N}}{4\times(30\times10^{-2})^{2}\times\sin^{2}\frac{\theta}{2}}\cdot\frac{1}{2\,mg}$$

or,

$$\sin^{3}\frac{\theta}{2} = \frac{9\times10^{9}\times25\times10^{-14}\text{ N}}{4\times9\times10^{-2}\times2\times(100\times10^{-3}\text{ kg})\times9{\cdot}8\text{ m s}^{-2}}$$

$$= 0{\cdot}0032$$

or, $$\sin\frac{\theta}{2} = 0{\cdot}15,\ \text{giving}\ \theta = 17°.$$

**11.** *Four particles, each having a charge q, are placed on the four vertices of a regular pentagon. The distance of each corner from the centre is a. Find the electric field at the centre of the pentagon.*

***Solution*** :

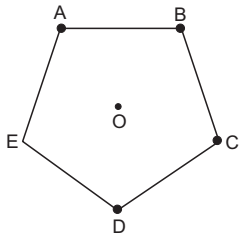



Figure 29-W7

Let the charges be placed at the vertices $A$, $B$, $C$ and $D$ of the pentagon $ABCDE$. If we put a charge $q$ at the corner $E$ also, the field at $O$ will be zero by symmetry. Thus, the field at the centre due to the charges at $A$, $B$,

$C$ and $D$ is equal and opposite to the field due to the charge $q$ at $E$ alone.

The field at $O$ due to the charge $q$ at $E$ is

$$\frac{q}{4\pi\varepsilon_0 a^2} \text{ along } EO.$$

Thus, the field at $O$ due to the given system of charges is $\frac{q}{4\pi\varepsilon_0 a^2}$ along $OE$.

**12.** *Find the electric field at a point P on the perpendicular bisector of a uniformly charged rod. The length of the rod is L, the charge on it is Q and the distance of P from the centre of the rod is a.*

***Solution :***

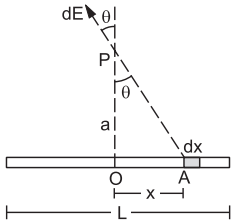



Figure 29-W8

Let us take an element of length $dx$ at a distance $x$ from the centre of the rod (figure 29-W8). The charge on this element is

$$dQ = \frac{Q}{L}dx.$$

The electric field at $P$ due to this element is

$$dE = \frac{dQ}{4\pi\varepsilon_0 (AP)^2}.$$

By symmetry, the resultant field at $P$ will be along $OP$ (if the charge is positive). The component of $dE$ along $OP$ is

$$dE\cos\theta = \frac{dQ}{4\pi\varepsilon_0 (AP)^2}\cdot\frac{OP}{AP} = \frac{a\,Q\,dx}{4\pi\varepsilon_0 L(a^2+x^2)^{3/2}}$$

Thus, the resultant field at $P$ is

$$E = \int dE\cos\theta$$

$$= \frac{aQ}{4\pi\varepsilon_0 L}\int_{-L/2}^{L/2}\frac{dx}{(a^2+x^2)^{3/2}}. \quad \ldots \text{(i)}$$

We have $x = a\tan\theta$ or $dx = a\sec^2\theta\, d\theta$.

Thus, $\int \frac{dx}{(a^2+x^2)^{3/2}} = \int \frac{a\sec^2\theta\, d\theta}{a^3\sec^3\theta}$

$$= \frac{1}{a^2}\int\cos\theta\, d\theta = \frac{1}{a^2}\sin\theta = \frac{1}{a^2}\frac{x}{(x^2+a^2)^{1/2}}.$$

From (i),

$$E = \frac{aQ}{4\pi\varepsilon_0 La^2}\left[\frac{x}{(x^2+a^2)^{1/2}}\right]_{-L/2}^{L/2}$$

$$= \frac{aQ}{4\pi\varepsilon_0 La^2}\left[\frac{2L}{(L^2+4a^2)^{1/2}}\right]$$

$$= \frac{Q}{2\pi\varepsilon_0 a\sqrt{L^2+4a^2}}.$$

**13.** *A uniform electric field E is created between two parallel, charged plates as shown in figure (29-W9). An electron enters the field symmetrically between the plates with a speed $v_0$. The length of each plate is l. Find the angle of deviation of the path of the electron as it comes out of the field.*

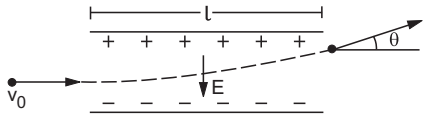



Figure 29-W9

***Solution :*** The acceleration of the electron is $a = \frac{eE}{m}$ in the upward direction. The horizontal velocity remains $v_0$ as there is no acceleration in this direction. Thus, the time taken in crossing the field is

$$t = \frac{l}{v_0}. \quad \ldots \text{(i)}$$

The upward component of the velocity of the electron as it emerges from the field region is

$$v_y = a\,t = \frac{eEl}{mv_0}.$$

The horizontal component of the velocity remains

$$v_x = v_0.$$

The angle θ made by the resultant velocity with the original direction is given by

$$\tan\theta = \frac{v_y}{v_x} = \frac{eEl}{mv_0^2}.$$

Thus, the electron deviates by an angle

$$\theta = \tan^{-1}\frac{eEl}{mv_0^2}.$$

**14.** *In a circuit,* 10 C *of charge is passed through a battery in a given time. The plates of the battery are maintained at a potential difference of* 12 V. *How much work is done by the battery ?*

***Solution :*** By definition, the work done to transport a charge $q$ through a potential difference $V$ is $qV$. Thus, work done by the battery

$$= 10\text{ C} \times 12\text{ V} = 120\text{ J}.$$

**15.** *Charges* $2{\cdot}0\times10^{-6}$ C *and* $1{\cdot}0\times10^{-6}$ C *are placed at corners A and B of a square of side* 5·0 cm *as shown in figure (29-W10). How much work will be done against the electric field in moving a charge of* $1{\cdot}0\times10^{-6}$ C *from C to D ?*

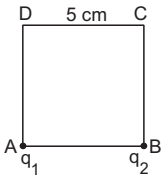



Figure 29-W10

***Solution*** : The electric potential at $C$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{AC}+\frac{q_2}{BC}\right)$$

$$= 9\times10^{9}\ \text{Nm}^{-2}\text{C}^{-2}\left(\frac{2{\cdot}0\times10^{-6}\ \text{C}}{\sqrt{2}\times0{\cdot}05\ \text{m}}+\frac{1{\cdot}0\times10^{-6}\ \text{C}}{0{\cdot}05\ \text{m}}\right)$$

$$= (9000\ \text{V})\left(\frac{2+\sqrt{2}}{\sqrt{2}\times0{\cdot}05}\right).$$

The electric potential at $D$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{AD}+\frac{q_2}{BD}\right)$$

$$= 9\times10^{9}\ \text{Nm}^{-2}\text{C}^{-2}\left(\frac{2{\cdot}0\times10^{-6}\ \text{C}}{0{\cdot}05\ \text{m}}+\frac{1{\cdot}0\times10^{-6}\ \text{C}}{\sqrt{2}\times0{\cdot}05\ \text{m}}\right)$$

$$= (9000\ \text{V})\left(\frac{2\sqrt{2}+1}{\sqrt{2}\times0{\cdot}05}\right).$$

The work done against the electric field in moving the charge $1{\cdot}0\times10^{-6}$ C from $C$ to $D$ is $q(V_D - V_C)$

$$= (1{\cdot}0\times10^{-6}\ \text{C})\ (9000\ \text{V})\left(\frac{2\sqrt{2}+1-2-\sqrt{2}}{\sqrt{2}\times0{\cdot}05}\right)$$

$$= 0{\cdot}053\ \text{J}.$$

**16.** *The electric field in a region is given by* $\vec{E} = (A/x^3)\,\vec{i}$. *Write a suitable SI unit for A. Write an expression for the potential in the region assuming the potential at infinity to be zero.*

***Solution*** : The SI unit of electric field is $\text{N C}^{-1}$ or $\text{V m}^{-1}$. Thus, the unit of $A$ is $\text{N m}^3\ \text{C}^{-1}$ or $\text{V m}^{-2}$.

$$V(x,\ y,\ z) = -\int_{\infty}^{(x,\ y,\ z)} \vec{E}\cdot d\vec{r}$$

$$= -\int_{\infty}^{(x,\ y,\ z)} \frac{A\,dx}{x^3} = \frac{A}{2x^2}.$$

**17.** *Three point charges q, 2q and 8q are to be placed on a* 9 cm *long straight line. Find the positions where the charges should be placed such that the potential energy of this system is minimum. In this situation, what is the electric field at the charge q due to the other two charges* ?

***Solution*** : The maximum contribution may come from the charge $8q$ forming pairs with others. To reduce its effect, it should be placed at a corner and the smallest charge $q$ in the middle. This arrangement shown in figure (29-W11) ensures that the charges in the strongest pair $2q$, $8q$ are at the largest separation.

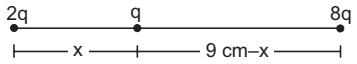



Figure 29-W11

The potential energy is

$$U = \frac{q^2}{4\pi\varepsilon_0}\left[\frac{2}{x}+\frac{16}{9\ \text{cm}}+\frac{8}{9\ \text{cm}-x}\right].$$

This will be minimum if

$$A = \frac{2}{x}+\frac{8}{9\ \text{cm}-x}\ \text{ is minimum.}$$

For this, $\dfrac{dA}{dx} = -\dfrac{2}{x^2}+\dfrac{8}{(9\ \text{cm}-x)^2} = 0$ … (i)

or, $9\ \text{cm} - x = 2x$ or, $x = 3$ cm.

The electric field at the position of charge $q$ is

$$\frac{q}{4\pi\varepsilon_0}\left(\frac{2}{x^2}-\frac{8}{(9\ \text{cm}-x)^2}\right)$$

$$= 0 \qquad \text{from (i).}$$

**18.** *An* HCl *molecule has a dipole moment of* $3{\cdot}4\times10^{-30}$ Cm. *Assuming that equal and opposite charges lie on the two atoms to form a dipole, what is the magnitude of this charge* ? *The separation between the two atoms of* HCl *is* $1{\cdot}0\times10^{-10}$ m.

***Solution*** : If the charges on the two atoms are $q$, $-q$,

$$q(1{\cdot}0\times10^{-10}\ \text{m}) = 3{\cdot}4\times10^{-30}\ \text{Cm}$$

or, $$q = 3{\cdot}4\times10^{-20}\ \text{C}.$$

Note that this is less than the charge of a proton. Can you explain, how such a charge can appear on an atom?

**19.** *Figure (29-W12) shows an electric dipole formed by two particles fixed at the ends of a light rod of length l. The mass of each particle is m and the charges are –q and +q. The system is placed in such a way that the dipole axis is parallel to a uniform electric field E that exists in the region. The dipole is slightly rotated about its centre and released. Show that for small angular displacement, the motion is angular simple harmonic and find its time period.*

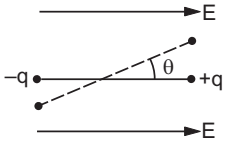



Figure 29-W12

***Solution*** : Suppose, the dipole axis makes an angle θ with the electric field at an instant. The magnitude of the torque on it is

$$| \tau | = | \vec{p} \times \vec{E} |$$

$$= qlE \sin\theta.$$

This torque will tend to rotate the dipole back towards the electric field. Also, for small angular displacement $\sin\theta \approx \theta$ so that

$$\tau = - qlE\theta.$$

The moment of inertia of the system about the axis of rotation is

$$I = 2 \times m\left(\frac{l}{2}\right)^2 = \frac{ml^2}{2}.$$

Thus, the angular acceleration is

$$\alpha = \frac{\tau}{I} = - \frac{2qE}{ml}\theta = - \omega^2\theta$$

where $\omega^2 = \frac{2qE}{ml}$.

Thus, the motion is angular simple harmonic and the time period is $T = 2\pi\sqrt{\frac{ml}{2qE}}$.

□

## QUESTIONS FOR SHORT ANSWER

**1.** The charge on a proton is $+1{\cdot}6 \times 10^{-19}$ C and that on an electron is $-1{\cdot}6 \times 10^{-19}$ C. Does it mean that the electron has a charge $3{\cdot}2 \times 10^{-19}$ C less than the charge of a proton ?

**2.** Is there any lower limit to the electric force between two particles placed at a separation of 1 cm ?

**3.** Consider two particles $A$ and $B$ having equal charges and placed at some distance. The particle $A$ is slightly displaced towards $B$. Does the force on $B$ increase as soon as the particle $A$ is displaced ? Does the force on the particle $A$ increase as soon as it is displaced ?

**4.** Can a gravitational field be added vectorially to an electric field to get a total field ?

**5.** Why does a phonograph-record attract dust particles just after it is cleaned ?

**6.** Does the force on a charge due to another charge depend on the charges present nearby ?

**7.** In some old texts it is mentioned that $4\pi$ lines of force originate from each unit positive charge. Comment on the statement in view of the fact that $4\pi$ is not an integer.

**8.** Can two equipotential surfaces cut each other ?

**9.** If a charge is placed at rest in an electric field, will its path be along a line of force ? Discuss the situation when the lines of force are straight and when they are curved.

**10.** Consider the situation shown in figure (29-Q1). What are the signs of $q_1$ and $q_2$ ? If the lines are drawn in proportion to the charge, what is the ratio $q_1/q_2$ ?

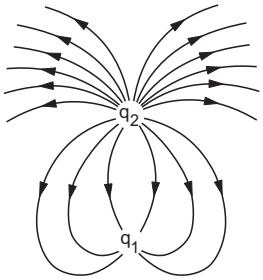



Figure 29-Q1

**11.** A point charge is taken from a point $A$ to a point $B$ in an electric field. Does the work done by the electric field depend on the path of the charge ?

**12.** It is said that the separation between the two charges forming an electric dipole should be small. Small compared to what ?

**13.** The number of electrons in an insulator is of the same order as the number of electrons in a conductor. What is then the basic difference between a conductor and an insulator ?

**14.** When a charged comb is brought near a small piece of paper, it attracts the piece. Does the paper become charged when the comb is brought near it ?

## OBJECTIVE I

**1.** Figure (29-Q2) shows some of the electric field lines corresponding to an electric field. The figure suggests that

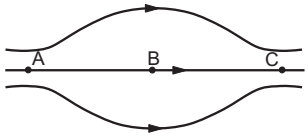



Figure 29-Q2

(a) $E_A > E_B > E_C$ (b) $E_A = E_B = E_C$
(c) $E_A = E_C > E_B$ (d) $E_A = E_C < E_B$.

**2.** When the separation between two charges is increased, the electric potential energy of the charges
(a) increases (b) decreases
(c) remains the same (d) may increase or decrease.

**3.** If a positive charge is shifted from a low-potential region to a high-potential region, the electric potential energy

(a) increases (b) decreases
(c) remains the same (d) may increase or decrease.

**4.** Two equal positive charges are kept at points $A$ and $B$. The electric potential at the points between $A$ and $B$ (excluding these points) is studied while moving from $A$ to $B$. The potential
(a) continuously increases
(b) continuously decreases
(c) increases then decreases
(d) decreases then increases.

**5.** The electric field at the origin is along the positive $x$-axis. A small circle is drawn with the centre at the origin cutting the axes at points $A$, $B$, $C$ and $D$ having coordinates $(a, 0)$, $(0, a)$, $(-a, 0)$, $(0, -a)$ respectively. Out of the points on the periphery of the circle, the potential is minimum at
(a) $A$ (b) $B$ (c) $C$ (d) $D$.

**6.** If a body is charged by rubbing it, its weight
(a) remains precisely constant
(b) increases slightly
(c) decreases slightly
(d) may increase slightly or may decrease slightly.

**7.** An electric dipole is placed in a uniform electric field. The net electric force on the dipole
(a) is always zero
(b) depends on the orientation of the dipole
(c) can never be zero
(d) depends on the strength of the dipole.

**8.** Consider the situation of figure (29-Q3). The work done in taking a point charge from $P$ to $A$ is $W_A$, from $P$ to $B$ is $W_B$ and from $P$ to $C$ is $W_C$.
(a) $W_A < W_B < W_C$ (b) $W_A > W_B > W_C$
(c) $W_A = W_B = W_C$ (d) None of these

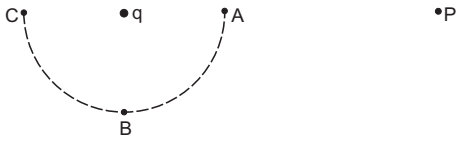



Figure 29-Q3

**9.** A point charge $q$ is rotated along a circle in the electric field generated by another point charge $Q$. The work done by the electric field on the rotating charge in one complete revolution is
(a) zero (b) positive (c) negative
(d) zero if the charge $Q$ is at the centre and nonzero otherwise.

## OBJECTIVE II

**1.** Mark out the correct options.
(a) The total charge of the universe is constant.
(b) The total positive charge of the universe is constant.
(c) The total negative charge of the universe is constant.
(d) The total number of charged particles in the universe is constant.

**2.** A point charge is brought in an electric field. The electric field at a nearby point
(a) will increase if the charge is positive
(b) will decrease if the charge is negative
(c) may increase if the charge is positive
(d) may decrease if the charge is negative.

**3.** The electric field and the electric potential at a point are $E$ and $V$ respectively.
(a) If $E = 0$, $V$ must be zero.
(b) If $V = 0$, $E$ must be zero.
(c) If $E \neq 0$, $V$ cannot be zero.
(d) If $V \neq 0$, $E$ cannot be zero.

**4.** The electric potential decreases uniformly from 120 V to 80 V as one moves on the $x$-axis from $x = -1$ cm to $x = +1$ cm. The electric field at the origin
(a) must be equal to 20 $\text{Vcm}^{-1}$
(b) may be equal to 20 $\text{Vcm}^{-1}$
(c) may be greater than 20 $\text{Vcm}^{-1}$
(d) may be less than 20 $\text{Vcm}^{-1}$.

**5.** Which of the following quantities do not depend on the choice of zero potential or zero potential energy ?
(a) Potential at a point
(b) Potential difference between two points
(c) Potential energy of a two-charge system
(d) Change in potential energy of a two-charge system.

**6.** An electric dipole is placed in an electric field generated by a point charge.
(a) The net electric force on the dipole must be zero.
(b) The net electric force on the dipole may be zero.
(c) The torque on the dipole due to the field must be zero.
(d) The torque on the dipole due to the field may be zero.

**7.** A proton and an electron are placed in a uniform electric field.
(a) The electric forces acting on them will be equal.
(b) The magnitudes of the forces will be equal.
(c) Their accelerations will be equal.
(d) The magnitudes of their accelerations will be equal.

**8.** The electric field in a region is directed outward and is proportional to the distance $r$ from the origin. Taking the electric potential at the origin to be zero,
(a) it is uniform in the region
(b) it is proportional to $r$
(c) it is proportional to $r^2$
(d) it increases as one goes away from the origin.

## EXERCISES

**1.** Find the dimensional formula of $\varepsilon_0$.

**2.** A charge of 1·0 C is placed at the top of your college building and another equal charge at the top of your house. Take the separation between the two charges to be 2·0 km. Find the force exerted by the charges on each other. How many times of your weight is this force ?

**3.** At what separation should two equal charges, 1·0 C each, be placed so that the force between them equals the weight of a 50 kg person ?

**4.** Two equal charges are placed at a separation of 1·0 m. What should be the magnitude of the charges so that the force between them equals the weight of a 50 kg person ?

**5.** Find the electric force between two protons separated by a distance of 1 fermi (1 fermi = $10^{-15}$ m). The protons in a nucleus remain at a separation of this order.

**6.** Two charges $2{\cdot}0 \times 10^{-6}$ C and $1{\cdot}0 \times 10^{-6}$ C are placed at a separation of 10 cm. Where should a third charge be placed such that it experiences no net force due to these charges ?

**7.** Suppose the second charge in the previous problem is $-1{\cdot}0 \times 10^{-6}$ C. Locate the position where a third charge will not experience a net force.

**8.** Two charged particles are placed at a distance 1·0 cm apart. What is the minimum possible magnitude of the electric force acting on each charge ?

**9.** Estimate the number of electrons in 100 g of water. How much is the total negative charge on these electrons ?

**10.** Suppose all the electrons of 100 g water are lumped together to form a negatively charged particle and all the nuclei are lumped together to form a positively charged particle. If these two particles are placed 10·0 cm away from each other, find the force of attraction between them. Compare it with your weight.

**11.** Consider a gold nucleus to be a sphere of radius 6·9 fermi in which protons and neutrons are distributed. Find the force of repulsion between two protons situated at largest separation. Why do these protons not fly apart under this repulsion ?

**12.** Two insulating small spheres are rubbed against each other and placed 1 cm apart. If they attract each other with a force of 0·1 N, how many electrons were transferred from one sphere to the other during rubbing ?

**13.** NaCl molecule is bound due to the electric force between the sodium and the chlorine ions when one electron of sodium is transferred to chlorine. Taking the separation between the ions to be $2{\cdot}75 \times 10^{-8}$ cm, find the force of attraction between them. State the assumptions (if any) that you have made.

**14.** Find the ratio of the electric and gravitational forces between two protons.

**15.** Suppose an attractive nuclear force acts between two protons which may be written as $F = Ce^{-kr}/r^2$. (a) Write down the dimensional formulae and appropriate SI units of $C$ and $k$. (b) Suppose that $k = 1$ fermi$^{-1}$ and that the repulsive electric force between the protons is just balanced by the attractive nuclear force when the separation is 5 fermi. Find the value of $C$.

**16.** Three equal charges, $2{\cdot}0 \times 10^{-6}$ C each, are held fixed at the three corners of an equilateral triangle of side 5 cm. Find the Coulomb force experienced by one of the charges due to the rest two.

**17.** Four equal charges $2{\cdot}0 \times 10^{-6}$ C each are fixed at the four corners of a square of side 5 cm. Find the Coulomb force experienced by one of the charges due to the rest three.

**18.** A hydrogen atom contains one proton and one electron. It may be assumed that the electron revolves in a circle of radius 0·53 angstrom (1 angstrom = $10^{-10}$ m and is abbreviated as Å) with the proton at the centre. The hydrogen atom is said to be in the ground state in this case. Find the magnitude of the electric force between the proton and the electron of a hydrogen atom in its ground state.

**19.** Find the speed of the electron in the ground state of a hydrogen atom. The description of ground state is given in the previous problem.

**20.** Ten positively charged particles are kept fixed on the $x$-axis at points $x = 10$ cm, 20 cm, 30 cm, …, 100 cm. The first particle has a charge $1{\cdot}0 \times 10^{-8}$ C, the second $8 \times 10^{-8}$ C, the third $27 \times 10^{-8}$ C and so on. The tenth particle has a charge $1000 \times 10^{-8}$ C. Find the magnitude of the electric force acting on a 1 C charge placed at the origin.

**21.** Two charged particles having charge $2{\cdot}0 \times 10^{-8}$ C each are joined by an insulating string of length 1 m and the system is kept on a smooth horizontal table. Find the tension in the string.

**22.** Two identical balls, each having a charge of $2{\cdot}00 \times 10^{-7}$ C and a mass of 100 g, are suspended from a common point by two insulating strings each 50 cm long. The balls are held at a separation 5·0 cm apart and then released. Find (a) the electric force on one of the charged balls (b) the components of the resultant force on it along and perpendicular to the string (c) the tension in the string (d) the acceleration of one of the balls. Answers are to be obtained only for the instant just after the release.

**23.** Two identical pith balls are charged by rubbing against each other. They are suspended from a horizontal rod through two strings of length 20 cm each, the separation between the suspension points being 5 cm. In equilibrium, the separation between the balls is 3 cm. Find the mass of each ball and the tension in the strings. The charge on each ball has a magnitude $2{\cdot}0 \times 10^{-8}$ C.

**24.** Two small spheres, each having a mass of 20 g, are suspended from a common point by two insulating strings of length 40 cm each. The spheres are identically charged and the separation between the balls at

equilibrium is found to be 4 cm. Find the charge on each sphere.

**25.** Two identical pith balls, each carrying a charge $q$, are suspended from a common point by two strings of equal length $l$. Find the mass of each ball if the angle between the strings is $2\theta$ in equilibrium.

**26.** A particle having a charge of $2{\cdot}0 \times 10^{-4}$ C is placed directly below and at a separation of 10 cm from the bob of a simple pendulum at rest. The mass of the bob is 100 g. What charge should the bob be given so that the string becomes loose?

**27.** Two particles $A$ and $B$ having charges $q$ and $2q$ respectively are placed on a smooth table with a separation $d$. A third particle $C$ is to be clamped on the table in such a way that the particles $A$ and $B$ remain at rest on the table under electrical forces. What should be the charge on $C$ and where should it be clamped ?

**28.** Two identically charged particles are fastened to the two ends of a spring of spring constant $100\ \text{N m}^{-1}$ and natural length 10 cm. The system rests on a smooth horizontal table. If the charge on each particle is $2{\cdot}0 \times 10^{-8}$ C, find the extension in the length of the spring. Assume that the extension is small as compared to the natural length. Justify this assumption after you solve the problem.

**29.** A particle $A$ having a charge of $2{\cdot}0 \times 10^{-6}$ C is held fixed on a horizontal table. A second charged particle of mass 80 g stays in equilibrium on the table at a distance of 10 cm from the first charge. The coefficient of friction between the table and this second particle is $\mu = 0{\cdot}2$. Find the range within which the charge of this second particle may lie.

**30.** A particle $A$ having a charge of $2{\cdot}0 \times 10^{-6}$ C and a mass of 100 g is placed at the bottom of a smooth inclined plane of inclination 30°. Where should another particle $B$, having same charge and mass, be placed on the incline so that it may remain in equilibrium ?

**31.** Two particles $A$ and $B$, each having a charge $Q$, are placed a distance $d$ apart. Where should a particle of charge $q$ be placed on the perpendicular bisector of $AB$ so that it experiences maximum force ? What is the magnitude of this maximum force ?

**32.** Two particles $A$ and $B$, each carrying a charge $Q$, are held fixed with a separation $d$ between them. A particle $C$ having mass $m$ and charge $q$ is kept at the middle point of the line $AB$. (a) If it is displaced through a distance $x$ perpendicular to $AB$, what would be the electric force experienced by it. (b) Assuming $x << d$, show that this force is proportional to $x$. (c) Under what conditions will the particle $C$ execute simple harmonic motion if it is released after such a small displacement ? Find the time period of the oscillations if these conditions are satisfied.

**33.** Repeat the previous problem if the particle $C$ is displaced through a distance $x$ along the line $AB$.

**34.** The electric force experienced by a charge of $1{\cdot}0 \times 10^{-6}$ C is $1{\cdot}5 \times 10^{-3}$ N. Find the magnitude of the electric field at the position of the charge.

**35.** Two particles $A$ and $B$ having charges of $+2{\cdot}00 \times 10^{-6}$ C and of $-4{\cdot}00 \times 10^{-6}$ C respectively are held fixed at a separation of 20·0 cm. Locate the point(s) on the line $AB$ where (a) the electric field is zero (b) the electric potential is zero.

**36.** A point charge produces an electric field of magnitude $5{\cdot}0\ \text{N C}^{-1}$ at a distance of 40 cm from it. What is the magnitude of the charge ?

**37.** A water particle of mass 10·0 mg and having a charge of $1{\cdot}50 \times 10^{-6}$ C stays suspended in a room. What is the magnitude of electric field in the room ? What is its direction ?

**38.** Three identical charges, each having a value $1{\cdot}0 \times 10^{-8}$ C, are placed at the corners of an equilateral triangle of side 20 cm. Find the electric field and potential at the centre of the triangle.

**39.** Positive charge $Q$ is distributed uniformly over a circular ring of radius $R$. A particle having a mass $m$ and a negative charge $q$, is placed on its axis at a distance $x$ from the centre. Find the force on the particle. Assuming $x << R$, find the time period of oscillation of the particle if it is released from there.

**40.** A rod of length $L$ has a total charge $Q$ distributed uniformly along its length. It is bent in the shape of a semicircle. Find the magnitude of the electric field at the centre of curvature of the semicircle.

**41.** A 10-cm long rod carries a charge of +50 μC distributed uniformly along its length. Find the magnitude of the electric field at a point 10 cm from both the ends of the rod.

**42.** Consider a uniformly charged ring of radius $R$. Find the point on the axis where the electric field is maximum.

**43.** A wire is bent in the form of a regular hexagon and a total charge $q$ is distributed uniformly on it. What is the electric field at the centre ? You may answer this part without making any numerical calculations.

**44.** A circular wire-loop of radius $a$ carries a total charge $Q$ distributed uniformly over its length. A small length $dL$ of the wire is cut off. Find the electric field at the centre due to the remaining wire.

**45.** A positive charge $q$ is placed in front of a conducting solid cube at a distance $d$ from its centre. Find the electric field at the centre of the cube due to the charges appearing on its surface.

**46.** A pendulum bob of mass 80 mg and carrying a charge of $2 \times 10^{-8}$ C is at rest in a uniform, horizontal electric field of $20\ \text{kVm}^{-1}$. Find the tension in the thread.

**47.** A particle of mass $m$ and charge $q$ is thrown at a speed $u$ against a uniform electric field $E$. How much distance will it travel before coming to momentary rest ?

**48.** A particle of mass 1 g and charge $2{\cdot}5 \times 10^{-4}$ C is released from rest in an electric field of $1{\cdot}2 \times 10^{4}\ \text{N C}^{-1}$. (a) Find the electric force and the force of gravity acting on this particle. Can one of these forces be neglected in comparison with the other for approximate analysis ? (b) How long will it take for the particle to travel a distance of 40 cm ? (c) What will be the speed of the particle after travelling this distance ? (d) How much is the work done by the electric force on the particle during this period ?

**49.** A ball of mass 100 g and having a charge of $4{\cdot}9 \times 10^{-5}$ C is released from rest in a region where a horizontal electric field of $2{\cdot}0 \times 10^{4}$ N C$^{-1}$ exists. (a) Find the resultant force acting on the ball. (b) What will be the path of the ball ? (c) Where will the ball be at the end of 2 s ?

**50.** The bob of a simple pendulum has a mass of 40 g and a positive charge of $4{\cdot}0 \times 10^{-6}$ C. It makes 20 oscillations in 45 s. A vertical electric field pointing upward and of magnitude $2{\cdot}5 \times 10^{4}$ N C$^{-1}$ is switched on. How much time will it now take to complete 20 oscillations ?

**51.** A block of mass $m$ having a charge $q$ is placed on a smooth horizontal table and is connected to a wall through an unstressed spring of spring constant $k$ as shown in figure (29-E1). A horizontal electric field $E$ parallel to the spring is switched on. Find the amplitude of the resulting SHM of the block.

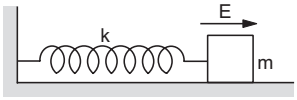



Figure 29-E1

**52.** A block of mass $m$ containing a net positive charge $q$ is placed on a smooth horizontal table which terminates in a vertical wall as shown in figure (29-E2). The distance of the block from the wall is $d$. A horizontal electric field $E$ towards right is switched on. Assuming elastic collisions (if any) find the time period of the resulting oscillatory motion. Is it a simple harmonic motion ?

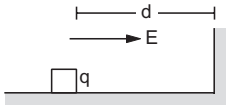



Figure 29-E2

**53.** A uniform electric field of 10 N C$^{-1}$ exists in the vertically downward direction. Find the increase in the electric potential as one goes up through a height of 50 cm.

**54.** 12 J of work has to be done against an existing electric field to take a charge of 0·01 C from $A$ to $B$. How much is the potential difference $V_B - V_A$ ?

**55.** Two equal charges, $2{\cdot}0 \times 10^{-7}$ C each, are held fixed at a separation of 20 cm. A third charge of equal magnitude is placed midway between the two charges. It is now moved to a point 20 cm from both the charges. How much work is done by the electric field during the process ?

**56.** An electric field of 20 N C$^{-1}$ exists along the $x$-axis in space. Calculate the potential difference $V_B - V_A$ where the points $A$ and $B$ are given by,
(a) $A = (0,\ 0)$; $B = (4\text{ m},\ 2\text{ m})$
(b) $A = (4\text{ m},\ 2\text{ m})$; $B = (6\text{ m},\ 5\text{ m})$
(c) $A = (0,\ 0)$; $B = (6\text{ m},\ 5\text{ m})$.
Do you find any relation between the answers of parts (a), (b) and (c) ?

**57.** Consider the situation of the previous problem. A charge of $-2{\cdot}0 \times 10^{-4}$ C is moved from the point $A$ to the point $B$. Find the change in electrical potential energy $U_B - U_A$ for the cases (a), (b) and (c).

**58.** An electric field $\vec{E} = (\vec{i}\,20 + \vec{j}\,30)$ N C$^{-1}$ exists in the space. If the potential at the origin is taken to be zero, find the potential at (2 m, 2 m).

**59.** An electric field $\vec{E} = \vec{i}\,Ax$ exists in the space, where $A = 10$ V m$^{-2}$. Take the potential at (10 m, 20 m) to be zero. Find the potential at the origin.

**60.** The electric potential existing in space is $V(x,\ y,\ z) = A(xy + yz + zx)$. (a) Write the dimensional formula of $A$. (b) Find the expression for the electric field. (c) If $A$ is 10 SI units, find the magnitude of the electric field at (1 m, 1 m, 1 m).

**61.** Two charged particles, having equal charges of $2{\cdot}0 \times 10^{-5}$ C each, are brought from infinity to within a separation of 10 cm. Find the increase in the electric potential energy during the process.

**62.** Some equipotential surfaces are shown in figure (29-E3). What can you say about the magnitude and the direction of the electric field ?

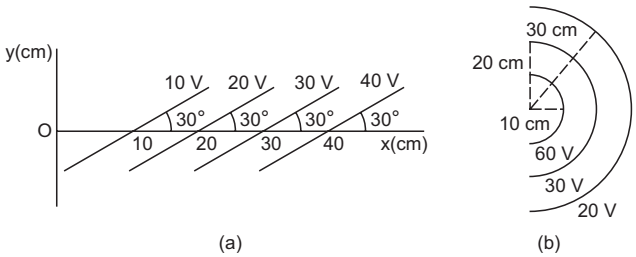



Figure 29-E3

**63.** Consider a circular ring of radius $r$, uniformly charged with linear charge density λ. Find the electric potential at a point on the axis at a distance $x$ from the centre of the ring. Using this expression for the potential, find the electric field at this point.

**64.** An electric field of magnitude 1000 N C$^{-1}$ is produced between two parallel plates having a separation of 2·0 cm as shown in figure (29-E4). (a) What is the potential difference between the plates ? (b) With what minimum speed should an electron be projected from the lower plate in the direction of the field so that it may reach the upper plate ? (c) Suppose the electron is projected from the lower plate with the speed calculated in part (b). The direction of projection makes an angle of 60° with the field. Find the maximum height reached by the electron.

Figure 29-E4

**65.** A uniform field of $2{\cdot}0\ \text{N C}^{-1}$ exists in space in $x$-direction. (a) Taking the potential at the origin to be zero, write an expression for the potential at a general point $(x, y, z)$. (b) At which points, the potential is 25 V ? (c) If the potential at the origin is taken to be 100 V, what will be the expression for the potential at a general point ? (d) What will be the potential at the origin if the potential at infinity is taken to be zero ? Is it practical to choose the potential at infinity to be zero ?

**66.** How much work has to be done in assembling three charged particles at the vertices of an equilateral triangle as shown in figure (29-E5) ?

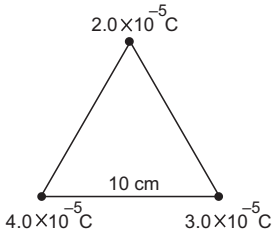



Figure 29-E5

**67.** The kinetic energy of a charged particle decreases by 10 J as it moves from a point at potential 100 V to a point at potential 200 V. Find the charge on the particle.

**68.** Two identical particles, each having a charge of $2{\cdot}0 \times 10^{-4}$ C and mass of 10 g, are kept at a separation of 10 cm and then released. What would be the speeds of the particles when the separation becomes large ?

**69.** Two particles have equal masses of 5·0 g each and opposite charges of $+4{\cdot}0 \times 10^{-5}$ C and $-4{\cdot}0 \times 10^{-5}$ C. They are released from rest with a separation of 1·0 m between them. Find the speeds of the particles when the separation is reduced to 50 cm.

**70.** A sample of HCl gas is placed in an electric field of $2{\cdot}5 \times 10^{4}\ \text{N C}^{-1}$. The dipole moment of each HCl molecule is $3{\cdot}4 \times 10^{-30}$ Cm. Find the maximum torque that can act on a molecule.

**71.** Two particles $A$ and $B$, having opposite charges $2{\cdot}0 \times 10^{-6}$ C and $-2{\cdot}0 \times 10^{-6}$ C, are placed at a separation of 1·0 cm. (a) Write down the electric dipole moment of this pair. (b) Calculate the electric field at a point on the axis of the dipole 1·0 m away from the centre. (c) Calculate the electric field at a point on the perpendicular bisector of the dipole and 1·0 m away from the centre.

**72.** Three charges are arranged on the vertices of an equilateral triangle as shown in figure (29-E6). Find the dipole moment of the combination.

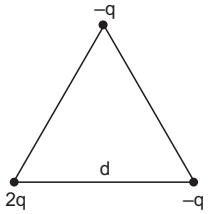



Figure 29-E6

**73.** Find the magnitude of the electric field at the point $P$ in the configuration shown in figure (29-E7) for $d >> a$. Take $2qa = p$.

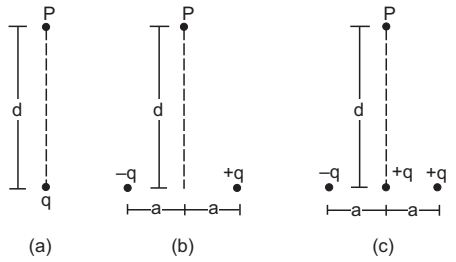



Figure 29-E7

**74.** Two particles, carrying charges $-q$ and $+q$ and having equal masses $m$ each, are fixed at the ends of a light rod of length $a$ to form a dipole. The rod is clamped at an end and is placed in a uniform electric field $E$ with the axis of the dipole along the electric field. The rod is slightly tilted and then released. Neglecting gravity find the time period of small oscillations.

**75.** Assume that each atom in a copper wire contributes one free electron. Estimate the number of free electrons in a copper wire having a mass of 6·4 g (take the atomic weight of copper to be 64 g $\text{mol}^{-1}$).

□

## ANSWERS

### OBJECTIVE I

1. (c) 2. (d) 3. (a) 4. (d) 5. (a) 6. (d)
7. (a) 8. (c) 9. (a)

### OBJECTIVE II

1. (a) 2. (c), (d) 3. none
4. (b), (c) 5. (b), (d) 6. (d)
7. (b) 8. (c)

## EXERCISES

1. $I^2M^{-1}L^{-3}T^4$
2. $2{\cdot}25 \times 10^3$ N
3. $4{\cdot}3 \times 10^3$ m
4. $2{\cdot}3 \times 10^{-4}$ C
5. 230 N
6. 5·9 cm from the larger charge in between the two charges
7. 34·1 cm from the larger charge on the line joining the charge in the side of the smaller charge
8. $2{\cdot}3 \times 10^{-24}$ N
9. $3{\cdot}35 \times 10^{25}$, $5{\cdot}35 \times 10^6$ C
10. $2{\cdot}56 \times 10^{25}$ N
11. 1·2 N
12. $2 \times 10^{11}$
13. $3{\cdot}05 \times 10^{-9}$ N
14. $1{\cdot}23 \times 10^{36}$
15. (a) $ML^3T^{-2}$, $L^{-1}$, N m$^2$, m$^{-1}$ (b) $3{\cdot}4 \times 10^{-26}$ N m$^2$
16. 24·9 N at 30° with the extended sides from the charge under consideration
17. 27·5 N at 45° with the extended sides of the square from the charge under consideration
18. $8{\cdot}2 \times 10^{-8}$ N
19. $2{\cdot}18 \times 10^6$ m s$^{-1}$
20. $4{\cdot}95 \times 10^5$ N
21. $3{\cdot}6 \times 10^{-6}$ N
22. (a) 0·144 N
    (b) zero, 0·095 N away from the other charge
    (c) 0·986 N and (d) 0·95 m s$^{-2}$ perpendicular to the string and going away from the other charge
23. 8·2 g, $8{\cdot}2 \times 10^{-2}$ N
24. $4{\cdot}17 \times 10^{-8}$ C
25. $\dfrac{q^2 \cot\theta}{16\pi\varepsilon_0 g l^2 \sin^2\theta}$
26. $5{\cdot}4 \times 10^{-9}$ C
27. $-(6 - 4\sqrt{2})\,q$, between $q$ and $2q$ at a distance of $(\sqrt{2} - 1)\,d$ from $q$
28. $3{\cdot}6 \times 10^{-6}$ m
29. between $\pm 8{\cdot}71 \times 10^{-8}$ C
30. 27 cm from the bottom
31. $d/2\sqrt{2}$, $3{\cdot}08\,\dfrac{Qq}{4\pi\varepsilon_0 d^2}$
32. (a) $\dfrac{Qqx}{2\pi\varepsilon_0\left(x^2 + \dfrac{d^2}{4}\right)^{3/2}}$ (c) $\left[\dfrac{m\pi^3\varepsilon_0 d^3}{Qq}\right]^{\frac{1}{2}}$
33. time period $= \left[\dfrac{\pi^3\varepsilon_0 md^3}{2Qq}\right]^{\frac{1}{2}}$
34. $1{\cdot}5 \times 10^3$ N C$^{-1}$
35. (a) 48·3 cm from $A$ along $BA$
    (b) 20 cm from $A$ along $BA$ and $\dfrac{20}{3}$ cm from $A$ along $AB$
36. $8{\cdot}9 \times 10^{-11}$ C
37. 65·3 N C$^{-1}$, upward
38. zero, $2{\cdot}3 \times 10^3$ V
39. $\left[\dfrac{16\pi^3\varepsilon_0 mR^3}{Qq}\right]^{1/2}$
40. $\dfrac{Q}{2\varepsilon_0 L^2}$
41. $5{\cdot}2 \times 10^7$ N C$^{-1}$
42. $R/\sqrt{2}$
43. zero
44. $\dfrac{QdL}{8\pi^2\varepsilon_0 a^3}$
45. $\dfrac{q}{4\pi\varepsilon_0 d^2}$ towards the charge $q$
46. $8{\cdot}8 \times 10^{-4}$ N
47. $\dfrac{mu^2}{2qE}$
48. (a) 3·0 N, $9{\cdot}8 \times 10^{-3}$ N (b) $1{\cdot}63 \times 10^{-2}$ s
    (c) 49·0 m s$^{-1}$ (d) 1·20 J
49. (a) 1·4 N making an angle of 45° with $\vec{g}$ and $\vec{E}$
    (b) straight line along the resultant force
    (c) 28 m from the starting point on the line of motion
50. 52 s
51. $qE/k$
52. $\sqrt{\dfrac{8\,md}{qE}}$
53. 5 V
54. 1200 volts
55. $3{\cdot}6 \times 10^{-3}$ J
56. (a) –80 V (b) –40 V (c) –120 V
57. 0·016 J, 0·008 J, 0·024 J
58. –100 V
59. 500 V
60. (a) MT$^{-3}$I$^{-1}$ (b) $-A\{\vec{i}(y+z) + \vec{j}(z+x) + \vec{k}(x+y)\}$
    (c) 35 N C$^{-1}$
61. 36 J
62. (a) 200 V m$^{-1}$ making an angle 120° with the $x$-axis
    (b) radially outward, decreasing with distance as $E = \dfrac{6\ \text{Vm}}{r^2}$.
63. $\dfrac{r\lambda}{2\varepsilon_0 (r^2 + x^2)^{1/2}}$, $\dfrac{r\lambda x}{2\varepsilon_0 (r^2 + x^2)^{3/2}}$

64. (a) 20 V (b) $2{\cdot}65 \times 10^{6}$ m s$^{-1}$ (c) 0·50 cm

65. (a) $-(2{\cdot}0 \text{ V m}^{-1})\,x$

(b) points on the plane $x = -12{\cdot}5$ m

(c) $100 \text{ V} - (2{\cdot}0 \text{ V m}^{-1})\,x$

(d) infinity

66. 234 J

67. 0·1 C

68. 600 m s$^{-1}$

69. 54 m s$^{-1}$ for each particle

70. $8{\cdot}5 \times 10^{-26}$ Nm

71. (a) $2{\cdot}0 \times 10^{-8}$ Cm (b) 360 N C$^{-1}$ (c) 180 N C$^{-1}$

72. $qd\sqrt{3}$, along the bisector of the angle at $2q$, away from the triangle

73. (a) $\dfrac{q}{4\pi\varepsilon_0 d^2}$ (b) $\dfrac{p}{4\pi\varepsilon_0 d^3}$ (c) $\dfrac{1}{4\pi\varepsilon_0 d^3}\sqrt{q^2 d^2 + p^2}$

74. $2\pi\sqrt{\dfrac{ma}{qE}}$

75. $6 \times 10^{22}$

□

## CHAPTER 30

# GAUSS'S LAW

Gauss's law is one of the fundamental laws of physics. It relates the electric field to the charge distribution which has produced this field. In section (30.1) we define the flux of an electric field and in the next section we discuss the concept of a solid angle. These will be needed to state and understand Gauss's law.

### 30.1 FLUX OF AN ELECTRIC FIELD THROUGH A SURFACE

Consider a hypothetical plane surface of area $\Delta S$ and suppose a uniform electric field $\vec{E}$ exists in the space (figure 30.1). Draw a line perpendicular to the surface and call one side of it, the positive normal to the surface. Suppose, the electric field $\vec{E}$ makes an angle $\theta$ with the positive normal.

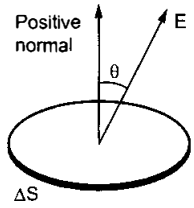



Figure 30.1

The quantity

$$\Delta\Phi = E\,\Delta S\cos\theta$$

is called *the flux of the electric field through the chosen surface*. If we draw a vector of magnitude $\Delta S$ along the positive normal, it is called the *area-vector* $\Delta\vec{S}$ corresponding to the area $\Delta S$. One can then write

$$\Delta\Phi = \vec{E}\cdot\Delta\vec{S}.$$

Remember, the direction of the area-vector is always along the normal to the corresponding surface. If the field $\vec{E}$ is perpendicular to the surface, it is parallel to the area-vector. If $\vec{E}$ is along the positive normal, $\theta = 0$, and $\Delta\Phi = E\,\Delta S$. If it is opposite to the positive normal, $\theta = \pi$, and $\Delta\Phi = -E\,\Delta S$. If the electric field is parallel to the surface, $\theta = \pi/2$, and $\Delta\Phi = 0$.

Flux is a scalar quantity and may be added using the rules of scalar addition. Thus, if the surface $\Delta S$ has two parts $\Delta S_1$ and $\Delta S_2$, the flux through $\Delta S$ equals the flux through $\Delta S_1$ plus the flux through $\Delta S_2$. This gives us a clue to define the flux through surfaces which are not plane, as well as the flux when the field is not uniform. We divide the given surface into smaller parts so that each part is approximately plane and the variation of electric field over each part can be neglected. We calculate the flux through each part separately, using the relation $\Delta\Phi = \vec{E}\cdot\Delta\vec{S}$ and then add the flux through all the parts. Using the techniques of integration, the flux is

$$\Phi = \int \vec{E}\cdot d\vec{S}$$

where integration has to be performed over the entire surface through which the flux is required.

The surface under consideration may be a closed one, enclosing a volume, such as a spherical surface. A hemispherical surface is an open surface. A cylindrical surface is also open. A cylindrical surface plus two plane surfaces perpendicular to the axis enclose a volume and these three taken together form a closed surface. When flux through a closed surface is required, we use a small circular sign on the integration symbol;

$$\Phi = \oint \vec{E}\cdot d\vec{S}.$$

It is customary to take the outward normal as positive in this case.

***Example 30.1***

*A square frame of edge* 10 cm *is placed with its positive normal making an angle of* 60° *with a uniform electric field of* 20 V m$^{-1}$. *Find the flux of the electric field through the surface bounded by the frame.*

***Solution*** : The surface considered is plane and the electric field is uniform (figure 30.2). Hence, the flux is

$$\Delta\Phi = \vec{E}\cdot\Delta\vec{S}$$
$$= E\,\Delta S\cos 60°$$

$$= (20 \text{ V m}^{-1})\,(0{\cdot}01 \text{ m}^2)\left(\frac{1}{2}\right) = 0{\cdot}1 \text{ Vm}.$$

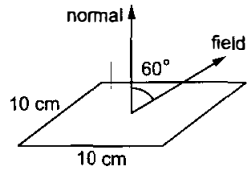



Figure 30.2

***Example 30.2***

*A charge q is placed at the centre of a sphere. Taking outward normal as positive, find the flux of the electric field through the surface of the sphere due to the enclosed charge.*

***Solution :***

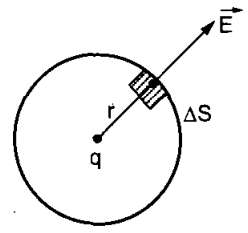



Figure 30.3

Let us take a small element $\Delta S$ on the surface of the sphere (Figure 30.3). The electric field here is radially outward and has the magnitude

$$\frac{q}{4\pi\varepsilon_0 r^2},$$

where $r$ is the radius of the sphere. As the positive normal is also outward, $\theta = 0$ and the flux through this part is

$$\Delta\Phi = \vec{E}\cdot\Delta\vec{S} = \frac{q}{4\pi\varepsilon_0 r^2}\Delta S.$$

Summing over all the parts of the spherical surface,

$$\Phi = \sum \Delta\Phi = \frac{q}{4\pi\varepsilon_0 r^2}\sum \Delta S = \frac{q}{4\pi\varepsilon_0 r^2}\,4\pi r^2 = \frac{q}{\varepsilon_0}$$

***Example 30.3***

*A uniform electric field exists in space. Find the flux of this field through a cylindrical surface with the axis parallel to the field.*

***Solution :***

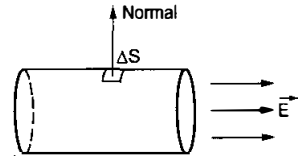



Figure 30.4

Consider figure (30.4) and take a small area $\Delta S$ on the cylindrical surface. The normal to this area will be perpendicular to the axis of the cylinder. But the electric field is parallel to the axis and hence

$$\Delta\Phi = \vec{E}\cdot\Delta\vec{S} = E\,\Delta S\cos(\pi/2) = 0.$$

This is true for each small part of the cylindrical surface. Summing over the entire surface, the total flux is zero.

## 30.2 SOLID ANGLE

Solid angle is a generalisation of the plane angle. In figure (30.5a) we show a plane curve $AB$. The end points $A$ and $B$ are joined to the point $O$. We say that the curve $AB$ subtends an *angle* or a *plane angle* at $O$. An angle is formed at $O$ by the two lines $OA$ and $OB$ passing through $O$.

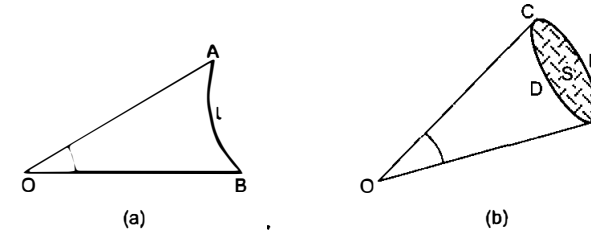



Figure 30.5

To construct a solid angle, we start with a surface $S$ (figure 30.5b) and join all the points on the periphery such as $A$, $B$, $C$, $D$, etc., with the given point $O$. We then say that a *solid angle* is formed at $O$ and that the surface $S$ has subtended the solid angle. The solid angle is formed by the lines joining the points on the periphery with $O$. The whole figure looks like a cone. As a typical example, think of the paper containers used by Moongfaliwalas.

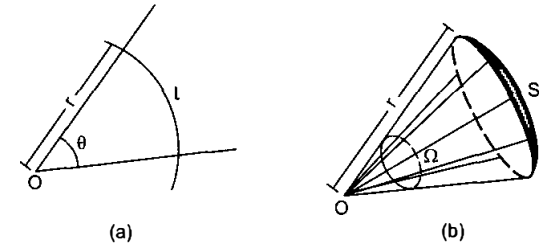



Figure 30.6

How do we measure a solid angle ? Let us consider how do we measure a plane angle. See figure (30.6a). We draw a circle of any radius $r$ with the centre at $O$ and measure the length $l$ of the arc intercepted by the angle. The angle $\theta$ is then defined as $\theta = l/r$. In order to measure a solid angle at the point $O$ (figure 30.6b), we draw a sphere of any radius $r$ with $O$ as the centre and measure the area $S$ of the part of the sphere intercepted by the cone. The solid angle $\Omega$ is then defined as

$$\Omega = S/r^2.$$

Note that this definition makes the solid angle a dimensionless quantity. It is independent of the radius of the sphere drawn.

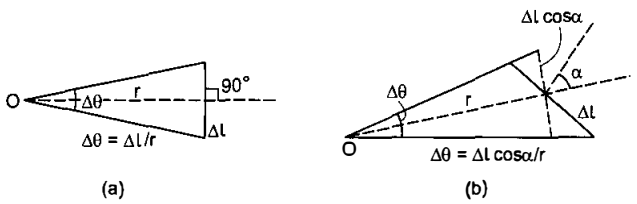



Figure 30.7

Next, consider a plane angle subtended at a point $O$ by a small line segment $\Delta l$ (figure 30.7a). Suppose, the line joining $O$ to the middle point of $\Delta l$ is perpendicular to $\Delta l$. As the segment is small, we can approximately write

$$\Delta\theta = \frac{\Delta l}{r}.$$

As $\Delta l$ gets smaller, the approximation becomes better. Now suppose, the line joining $O$ to $\Delta l$ is not perpendicluar to $\Delta l$ (figure 30.7b). Suppose, this line makes an angle $\alpha$ with the perpendicular to $\Delta l$. The angle subtended by $\Delta l$ at $O$ is

$$\Delta\theta = \frac{\Delta l \cos\alpha}{r}.$$

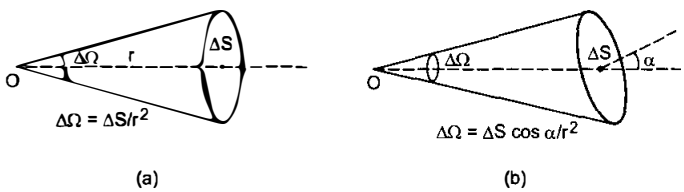



Figure 30.8

Similarly, if a small plane area $\Delta S$ (figure 30.8a) subtends a solid angle $\Delta\Omega$ at $O$ in such a way that the line joining $O$ to $\Delta S$ is normal to $\Delta S$, we can write $\Delta\Omega = \Delta S/r^2$. But if the line joining $O$ to $\Delta S$ makes an angle $\alpha$ with the normal to $\Delta S$ (figure 30.8b), we should write

$$\Delta\Omega = \frac{\Delta S \cos\alpha}{r^2}.$$

A complete circle subtends an angle

$$\theta = \frac{l}{r} = \frac{2\pi r}{r} = 2\pi$$

at the centre. In fact, any closed curve subtends an angle $2\pi$ at any of the internal points. Similarly, a complete sphere subtends a solid angle

$$\Omega = \frac{S}{r^2} = \frac{4\pi r^2}{r^2} = 4\pi$$

at the centre. Also, any closed surface subtends a solid angle $4\pi$ at any internal point.

How much is the angle subtended by a closed plane curve at an external point ?

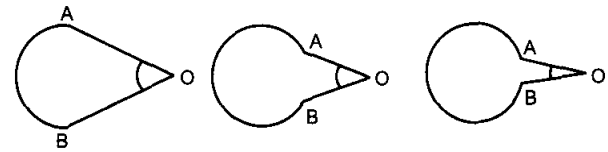



Figure 30.9

See figure (30.9). As we gradually close the curve, the angle finally diminishes to zero. A closed curve subtends zero angle at an external point. Similarly, a closed surface subtends zero solid angle at an external point.

## 30.3 GAUSS'S LAW AND ITS DERIVATION FROM COULOMB'S LAW

The statement of the Gauss's law may be written as follows:

*The flux of the net electric field through a closed surface equals the net charge enclosed by the surface divided by* $\varepsilon_0$. In symbols,

$$\oint \vec{E}\cdot d\vec{S} = \frac{q_{in}}{\varepsilon_0} \qquad \ldots (30.1)$$

where $q_{in}$ is the net charge enclosed by the surface through which the flux is calculated.

It should be carefully noted that the electric field on the left-hand side of equation (30.1) is the resultant electric field due to all the charges existing in the space, whereas, the charge appearing on the right-hand side includes only those which are inside the closed surface.

Gauss's law is taken as a fundamental law of nature, a law whose validity is shown by experiments. However, historically Coulomb's law was discovered before Gauss's law and it is possible to derive Gauss's law from Coulomb's law.

**Proof of Gauss's Law (Assuming Coulomb's Law)**

**Flux due to an internal charge**

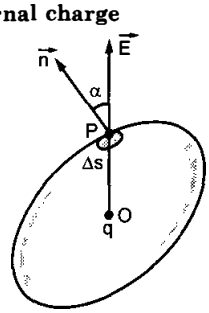



Figure 30.10

Suppose a charge $q$ is placed at a point $O$ inside a "closed" surface (figure 30.10). Take a point $P$ on the surface and consider a small area $\Delta S$ on the surface around $P$. Let $OP = r$. The electric field at $P$ due to the

charge $q$ is

$$E = \frac{q}{4\pi\varepsilon_0 r^2}$$

along the line $OP$. Suppose this line $OP$ makes an angle $\alpha$ with the outward normal to $\Delta S$. The flux of the electric field through $\Delta S$ is

$$\Delta\Phi = \vec{E}\cdot\Delta\vec{S} = E\,\Delta S\cos\alpha$$

$$= \frac{q}{4\pi\varepsilon_0 r^2}\Delta S\cos\alpha$$

$$= \frac{q}{4\pi\varepsilon_0}\Delta\Omega$$

where $\Delta\Omega = \frac{\Delta S\cos\alpha}{r^2}$ is the solid angle subtended by $\Delta S$ at $O$. The flux through the entire surface is

$$\Phi = \Sigma\,\frac{q}{4\pi\varepsilon_0}\,\Delta\Omega = \frac{q}{4\pi\varepsilon_0}\,\Sigma\,\Delta\Omega.$$

The sum over $\Delta\Omega$ is the total solid angle subtended by the closed surface at the internal point $O$ and hence is equal to $4\pi$.

The total flux of the electric field due to the internal charge $q$ through the closed surface is, therefore,

$$\Phi = \frac{q}{4\pi\varepsilon_0}\,4\pi = \frac{q}{\varepsilon_0}. \qquad \ldots \text{(i)}$$

**Flux due to an external charge**

Now, suppose a charge $q$ is placed at a point $O$ outside the closed surface. The flux of the electric field due to $q$ through the small area $\Delta S$ is again

$$\Delta\Phi = \frac{q}{4\pi\varepsilon_0}\,\frac{\Delta S\cos\alpha}{r^2} = \frac{q}{4\pi\varepsilon_0}\Delta\Omega.$$

When we sum over all the small area elements of the closed surface we get $\Sigma\Delta\Omega = 0$ as this is the total solid angle subtended by the closed surface at an external point. Hence,

$$\Phi = 0. \qquad \ldots \text{(ii)}$$

**Flux due to a combination of charges**

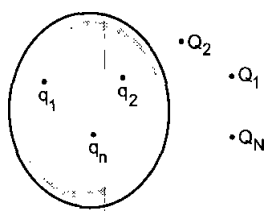



Figure 30.11

Finally, consider a general situation (figure 30.11) where charges $q_1, q_2, \ldots, q_n$ are inside a closed surface and charges $Q_1, Q_2, \ldots, Q_N$ are outside it. The resultant electric field at any point is

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \ldots + \vec{E}_n + \vec{E}'_1 + \vec{E}'_2 + \ldots + \vec{E}'_N$$

where $E_i$ and $E'_i$ are the fields due to $q_i$ and $Q_i$ respectively. Thus, the flux of the resultant electric field through the closed surface is

$$\Phi = \oint \vec{E}\cdot d\vec{S} = \oint \vec{E}_1\cdot d\vec{S} + \oint \vec{E}_2\cdot d\vec{S} + \ldots + \oint \vec{E}_n\cdot d\vec{S}$$

$$+ \oint \vec{E}'_1\cdot d\vec{S} + \oint \vec{E}'_2\cdot d\vec{S} + \ldots + \oint \vec{E}'_N\cdot d\vec{S}. \qquad \ldots \text{(iii)}$$

Now, $\oint \vec{E}_i\cdot d\vec{S}$ is the flux of the electric field due to the charge $q_i$ only. As this charge is inside the closed surface, from (i), it is equal to $q_i/\varepsilon_0$. Also, $\oint \vec{E}'_i\cdot d\vec{S}$ is the flux of the electric field due to the charge $Q_i$ which is outside the closed surface. This flux is, therefore, zero from (ii). Using these results in (iii),

$$\Phi = \frac{q_1}{\varepsilon_0} + \frac{q_2}{\varepsilon_0} + \ldots + \frac{q_n}{\varepsilon_0} + 0 + \ldots + 0$$

or, $$\Phi = \frac{1}{\varepsilon_0}\,\Sigma\,q_i$$

or, $$\oint \vec{E}\cdot d\vec{S} = \frac{q_{in}}{\varepsilon_0}.$$

This completes the derivation of Gauss's law (equation 30.1).

We once again emphasise that the electric field appearing in the Gauss's law is the resultant electric field due to all the charges present inside as well as outside the given closed surface. On the other hand, the charge $q_{in}$ appearing in the law is only the charge contained within the closed surface. The contribution of the charges outside the closed surface in producing the flux is zero. A surface on which Gauss's law is applied, is sometimes called the *Gaussian surface.*

***Example 30.4***

*A charge Q is distributed uniformly on a ring of radius r. A sphere of equal radius r is constructed with its centre at the periphery of the ring (figure 30.12). Find the flux of the electric field through the surface of the sphere.*

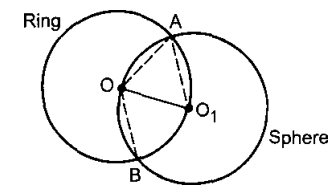



Figure 30.12

***Solution*** : From the geometry of the figure, $OA = OO_1$ and $O_1A = O_1O$. Thus, $OAO_1$ is an equilateral triangle. Hence $\angle AOO_1 = 60°$ or $\angle AOB = 120°$.

The arc $AO_1B$ of the ring subtends an angle 120° at the centre $O$. Thus, one third of the ring is inside the sphere.

The charge enclosed by the sphere $= \frac{Q}{3}$. From Gauss's law, the flux of the electric field through the surface of the sphere is $\frac{Q}{3\varepsilon_0}$.

## 30.4 APPLICATIONS OF GAUSS'S LAW

### (A) Charged Conductor

As discussed earlier, an electric conductor has a large number of free electrons and when placed in an electric field, these electrons redistribute themselves to make the field zero at all the points inside the conductor.

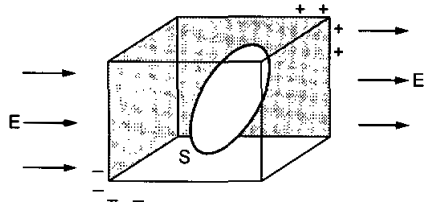



Figure 30.13

Consider a charged conductor placed in an electric field (figure 30.13). We assume that the redistribution of free electrons (if any) is complete. Draw a closed surface $S$ going through the interior points only. As the electric field at all the points of this surface is zero (they are interior points), the flux $\oint \vec{E}\cdot d\vec{S}$ is also zero. But from Gauss's law, this equals the charge contained inside the surface divided by $\varepsilon_0$. Hence, the charge enclosed by the surface is zero. This shows that any volume completely inside a conductor is electrically neutral. If a charge is injected anywhere in the conductor, it must come over to the surface of the conductor so that the interior is always charge free. Also, if the conductor has a cavity, the charge must come over to the outer surface.

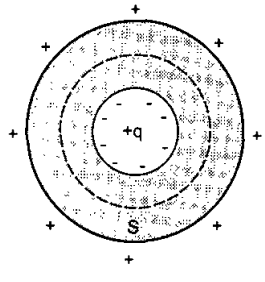



Figure 30.14

However, if a charge is placed within the cavity as in figure (30.14), the inner surface cannot be charge free. Taking the Gaussian surface $S$ as shown, $\vec{E} = 0$ at all the points of this surface and hence $\oint \vec{E}\cdot d\vec{S} = 0$. This ensures that the charge contained in $S$ is zero and if a charge $+q$ is placed in the cavity, there must be a charge $-q$ on the inner surface of the conductor. If the conductor is neutral, i.e., no charge is placed on it, a charge $+q$ will appear on the outer surface.

If there is a cavity in the conductor and no charge is placed in the cavity, the electric field at all the points in the cavity is zero. This can be proved using a little more advanced mathematics.

### (B) Electric Field due to a Uniformly Charged Sphere

Suppose a total charge $Q$ is uniformly distributed in a spherical volume of radius $R$ and we are required to find the electric field at a point $P$ which is at a distance $r$ from the centre of the charge distribution.

**Field at an outside point**

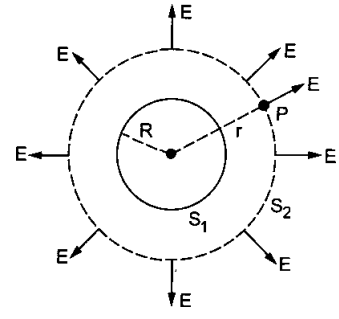



Figure 30.15

For a point $P$ outside the charge distribution (figure 30.15), we have $r > R$. Draw a spherical surface passing through the point $P$ and concentric with the charge distribution. Take this to be the Gaussian surface. The electric field is radial by symmetry and if $Q$ is positive the field is outward. Also, its magnitude at all the points of the Gaussian surface must be equal. Let this magnitude be $E$. This is also the magnitude of the field at $P$. As the field $\vec{E}$ is normal to the surface element everywhere, $\vec{E}\cdot d\vec{S} = E\,dS$ for each element. The flux of the electric field through this closed surface is

$$\Phi = \oint \vec{E}\cdot d\vec{S}$$
$$= \oint E\,dS = E \oint dS = E\,4\pi r^2.$$

This should be equal to the charge contained inside the Gaussian surface divided by $\varepsilon_0$. As the entire charge $Q$ is contained inside the Gaussian surface, we get

$$E\,4\pi r^2 = Q/\varepsilon_0$$

or,
$$E = \frac{Q}{4\pi\varepsilon_0 r^2}. \qquad \ldots (30.2)$$

*The electric field due to a uniformly charged sphere at a point outside it, is identical with the field due to an equal point charge placed at the centre.*

Notice the use of the argument of symmetry. All the points of the sphere through $P$ are equivalent. No point on this surface has any special property which a different point does not have. That is why we could say that the field has the same magnitude $E$ at all these points. Also, the field is radial at all the points. We have wisely chosen the Gaussian surface which has these properties. We could then easily evaluate the flux $\oint \vec{E}\cdot d\vec{S} = E\,4\pi r^2$.

**Field at an internal point**

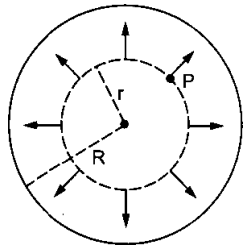



Figure 30.16

Suppose, we wish to find the electric field at a point $P$ inside the spherical charge distribution (figure 30.16). We draw a spherical surface passing through $P$ and concentric with the given charge distribution. The radius of this sphere will be $r$. All the points of this sphere are equivalent. By symmetry, the field is radial at all the points of this surface and has a constant magnitude $E$. The flux through this spherical surface is

$$\Phi = \oint \vec{E}\cdot d\vec{S}$$

$$= \oint E\,dS = E \oint dS = E\,4\pi r^2. \qquad \ldots \text{(i)}$$

Let us now calculate the total charge contained inside this spherical surface. As the charge is uniformly distributed within the given spherical volume, the charge per unit volume is $\frac{Q}{\frac{4}{3}\pi R^3}$. The volume enclosed by the Gaussian surface, through which the flux is calculated, is $\frac{4}{3}\pi r^3$. Hence, the charge enclosed is

$$\frac{Q}{\frac{4}{3}\pi R^3} \cdot \frac{4}{3}\pi r^3 = \frac{Q r^3}{R^3}.$$

Using Gauss's law and (i),

$$E\,4\pi r^2 = \frac{Q r^3}{\varepsilon_0 R^3}$$

or,
$$E = \frac{Qr}{4\pi\varepsilon_0 R^3} \qquad \ldots (30.3)$$

***The electric field due to a uniformly charged sphere at an internal point is $\frac{Qr}{4\pi\varepsilon_0 R^3}$ in radial direction.***

At the centre, $r = 0$ and hence $E = 0$. This is clear from the symmetry arguments as well. At the centre, all directions are equivalent. If the electric field is not zero, what can be its direction ? You cannot choose a unique direction. The field has to be zero. It is proportional to the distance $r$ from the centre for the internal points. Equations (30.2) and (30.3) give the same value of the field at the surface, where $r = R$.

**(C) Electric Field due to a Linear**

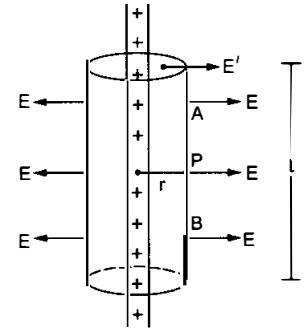



Figure 30.17

**Charge Distribution**

Consider a long line charge with a linear charge density (that is, charge per unit length) $\lambda$. We have to calculate the electric field at a point $P$ which is at a distance $r$ from the line charge (figure 30.17). What can be the direction of the electric field at $P$ ? Can it be along $PA$ ? If yes, then why not along $PB$ ? $PA$ and $PB$ are equivalent to each other. In fact, the only unique direction through $P$ is along the perpendicular to the line charge. The electric field must be along this direction. If the charge is positive, the field will be outward.

Now, we construct a Gaussian surface. We draw a cylinder of length $l$ passing through $P$ and coaxial with the line charge. Let us close the cylinder with two plane surfaces perpendicular to the line charge. The curved surface of the cylinder together with the two plane parallel surfaces constitutes a closed surface as shown in figure (30.17). We use this surface as the Gaussian surface.

All the points on the curved part of this Gaussian surface are at the same perpendicular distance from the line charge. All these points are equivalent. The electric field at all these points will have the same magnitude $E$ as that at $P$. Also, the direction of the field at any point on the curved surface is normal to the line and hence normal to the cylindrical surface

element there. The flux through the curved part is, therefore,

$$\int \vec{E}\cdot d\vec{S} = \int E\, dS = E \int dS = E\, 2\pi rl.$$

Now, consider the flat parts of the Gaussian surface, that is, the lids of the cylinder. The electric field at any point is perpendicular to the line charge. The normal to any element on these plane surfaces is parallel to the line charge. Hence, the field and the area-vector make an angle of 90° with each other so that $\int \vec{E}\cdot d\vec{S} = 0$ on these parts. The total flux through the closed Gaussian surface is, therefore,

$$\oint \vec{E}\cdot d\vec{S} = E\, 2\pi rl. \qquad \ldots \text{(i)}$$

The charge enclosed in the Gaussian surface is $\lambda l$ as a length $l$ of the line charge is inside the closed surface. Using Gauss's law and (i),

$$E\, 2\pi rl = (\lambda l)/\varepsilon_0$$

or,
$$E = \frac{\lambda}{2\pi\varepsilon_0 r}. \qquad \ldots \text{(30.4)}$$

This is the field at a distance $r$ from the line. It is directed away from the line if the charge is positive and towards the line if the charge is negative.

**(D) Electric Field due to a Plane Sheet of Charge**

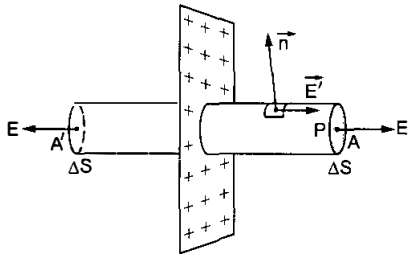



Figure 30.18

Consider a large plane sheet of charge with surface charge density (charge per unit area) $\sigma$. We have to find the electric field $E$ at a point $P$ in front of the sheet (Figure 30.18). What can be the direction of the electric field at $P$? The only unique direction we can identify is along the perpendicular to the plane. The field must be along this line. If the charge is positive, the field is away from the plane. Same is true for all points near the plane provided the sheet is large and the charge density is uniform. If these conditions are not fulfilled, the argument of symmetry may fail.

To calculate the field $E$ at $P$, we choose a Gaussian surface as follows. Draw a plane surface $A$ passing through $P$ and parallel to the charge sheet. Draw a cylinder with this surface as a cross section and extend it to the other side of the plane charge sheet. Close the cylinder on the other side by a cross section $A'$ such that $A$ and $A'$ are equidistant from the sheet. Also, $A$ and $A'$ have equal area $\Delta S$. The cylinder together with its cross sectional areas forms a closed surface and we apply Gauss's law to this surface.

The electric field at all the points of $A$ has the same magnitude $E$. The direction is along the positive normal to $A$. Thus, the flux of the electric field through $A$ is

$$\Phi = \vec{E}\cdot\Delta\vec{S} = E\,\Delta S.$$

Note that the two sides of the charge sheet are equivalent in all respect. As $A$ and $A'$ are equidistant from the sheet, the electric field at any point of $A'$ is also equal to $E$ and is along the positive normal (that is, the outward normal) to $A'$. Hence, the flux of the electric field through $A'$ is also $E\,\Delta S$. At the points on the curved surface, the field and the outward normal make an angle of 90° with each other and hence $\vec{E}\cdot\Delta\vec{S} = 0$. The total flux through the closed surface is

$$\Phi = \oint \vec{E}\cdot d\vec{S} = E\,\Delta S + E\,\Delta S + 0 = 2\,E\,\Delta S.$$

The area of the sheet enclosed in the cylinder is $\Delta S$. The charge contained in the cylinder is, therefore, $\sigma\,\Delta S$. Hence from Gauss's law,

$$2\,E\Delta S = \frac{\sigma\,\Delta S}{\varepsilon_0}$$

or,
$$E = \frac{\sigma}{2\varepsilon_0}. \qquad \ldots \text{(30.5)}$$

We see that the field is uniform and does not depend on the distance from the charge sheet. This is true as long as the sheet is large as compared to its distance from $P$.

**(E) Electric Field near a Charged Conducting Surface**

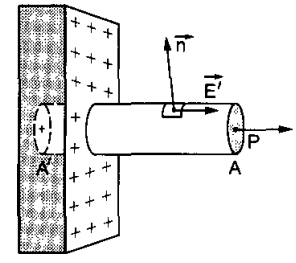



Figure 30.19

In figure (30.19), we show a large, plane conducting sheet. The surface on right has a uniform surface charge density $\sigma$. We have to find the electric field at a point $P$ near this surface and outside the conductor. As we know, the conducting surface is an equipotential surface and the electric field near the surface is perpendicular to the surface. For positive charge on the surface, the field is away from the surface. To find the electric field, we construct a Gaussian surface as follows. Take a small plane surface $A$ passing through

the point $P$ and parallel to the given conducting surface. Draw a cylinder with $A$ as a cross section and terminate it with another plane surface $A'$ parallel to $A$ and lying in the interior of the conducting sheet.

If the electric field at $P$ is $E$, the flux through the plane surface $A$ is

$$\Phi = E\,\Delta S,$$

where $\Delta S$ is the area of $A$. At the curved parts of the cylinder, the electric field is either zero (inside the conductor) or is parallel to the curved surface (outside the conductor). The field $\vec{E}$ and the area-vector $\Delta\vec{S}$ are perpendicular to each other making $\vec{E}\cdot\Delta\vec{S} = 0$ at these outside points. The flux on the curved part is, therefore, zero. Also, the flux on $A'$ is zero as the field inside the conductor is zero.

The total flux through the Gaussian surface constructed is, therefore, $E\,\Delta S$. The charge enclosed inside the closed surface is $\sigma\,\Delta S$ and hence from Gauss's law,

$$E\,\Delta S = \frac{\sigma\,\Delta S}{\varepsilon_0}$$

or,
$$E = \frac{\sigma}{\varepsilon_0}. \qquad \ldots (30.6)$$

*The electric field near a charged conducting surface is $\sigma/\varepsilon_0$ and it is normal to the surface.*

Compare this result with the field due to a plane sheet of charge of surface density $\sigma$ (equation 30.5). The field $E$ had a magnitude $\sigma/(2\varepsilon_0)$ in that case. Apparently, for a conductor also we have a plane sheet of charge of the same density but the field derived is $\sigma/\varepsilon_0$ and not $\sigma/(2\varepsilon_0)$. Consider the conducting sheet shown in figure (30.20). In fact, the field due to the charge on the right surface is indeed $\frac{\sigma}{2\varepsilon_0}$ at $P$. Where does the extra $\frac{\sigma}{2\varepsilon_0}$ field come from ?

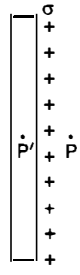



Figure 30.20

Consider a point $P'$ inside the sheet as shown in figure (30.20). The electric field at this point due to the charge sheet on the right surface of the conductor is $\sigma/(2\varepsilon_0)$ towards left. But $P'$ is a point inside the conductor and hence the field here must be zero. This means that the charge distribution shown in figure (30.20) is not complete as it does not ensure zero field inside the conductor. Apart from the surface charge of density $\sigma$ shown in the figure, there must be other charges nearby. These other nearby charges must create a field at $P'$ towards right so that the resultant field at $P'$ becomes zero. These other charges also create a field $\sigma/(2\varepsilon_0)$ towards right at $P$ which adds to the field due to the surface charge shown in figure (30.20). Thus, the field at $P$ becomes

$$\frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$$

towards right. As Gauss's law gives the net field, equation (30.6) gives $E = \sigma/\varepsilon_0$ which is the actual field due to all the charges and not the field due to the surface charge only. As examples, we give in figure (30.21) some of the possible complete charge distributions which ensure zero field inside the conductor.

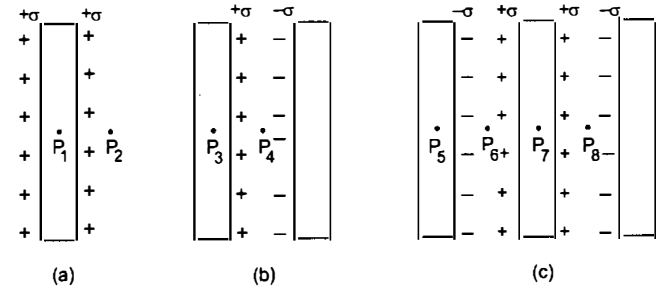



Figure 30.21

Calculate the electric field at the points indicated using the formula $E = \sigma/(2\varepsilon_0)$ for each charged surface. Verify that the field at each of the points $P_1$, $P_3$, $P_5$ and $P_7$ is zero and at each of the points $P_2$, $P_4$, $P_6$ and $P_8$ is $\sigma/\varepsilon_0$.

Note that electric field changes discontinuously at the surface of a conductor. Just outside the conductor it is $\sigma/\varepsilon_0$ and inside the conductor it is zero. In fact, the field gradually decreases from $\sigma/\varepsilon_0$ to zero in a small thickness of about 4–5 atomic layers at the surface. When we say 'the surface of a conductor' we actually mean this small thickness.

## 30.5 SPHERICAL CHARGE DISTRIBUTIONS

We have seen that the electric field due to a uniformly charged sphere at an external point is the same as that due to an equal point charge placed at the centre. Similar result was obtained for the gravitational field due to a uniform sphere. This similarity is expected because the Coulomb force

$$F = -\frac{q_1 q_2}{4\pi\varepsilon_0 r^2}$$

and the gravitational force

$$F = \frac{Gm_1m_2}{r^2}$$

have similar mathematical structure.

Many of the results derived for gravitational field, potential and potential energy may, therefore, be used for the corresponding electrical quantities. We state some of the useful results for a spherical charge distribution of radius $R$.

(a) The electric field due to a uniformly charged, thin spherical shell at an external point is the same as that due to an equal point charge placed at the centre of the shell, $E = Q/(4\pi\varepsilon_0 r^2)$.

(b) The electric field due to a uniformly charged thin spherical shell at an internal point is zero.

(c) The electric field due to a uniformly charged sphere at an external point is the same as that due to an equal point charge placed at the centre of the sphere.

(d) The electric field due to a uniformly charged sphere at an internal point is proportional to the distance of the point from the centre of the sphere. Thus, it is zero at the centre and increases linearly as one moves out towards the surface.

(e) The electric potential due to a uniformly charged, thin spherical shell at an external point is the same as that due to an equal point charge placed at the centre, $V = Q/(4\pi\varepsilon_0 r)$.

(f) The electric potential due to a uniformly charged, thin spherical shell at an internal point is the same everywhere and is equal to that at the surface, $V = Q/(4\pi\varepsilon_0 R)$.

(g) The electric potential due to a uniformly charged sphere at an external point is the same as that due to an equal point charge placed at the centre, $V = Q/(4\pi\varepsilon_0 r)$.

**Electric Potential Energy of a Uniformly Charged Sphere**

Consider a uniformly charged sphere of radius $R$ having a total charge $Q$. The electric potential energy of this sphere is equal to the work done in bringing the charges from infinity to assemble the sphere. Let us assume that at some instant, charge is assembled up to a radius $x$. In the next step, we bring some charge from infinity and put it on this sphere to increase the radius from $x$ to $x + dx$. The entire sphere is assembled as $x$ varies from 0 to $R$.

The charge density is

$$\rho = \frac{3Q}{4\pi R^3}.$$

When the radius of the sphere is $x$, the charge contained in it is,

$$q = \frac{4}{3}\pi x^3 \rho = \frac{Q}{R^3}x^3.$$

The potential at the surface is

$$V = \frac{q}{4\pi\varepsilon_0 x} = \frac{Q}{4\pi\varepsilon_0 R^3}x^2.$$

The charge needed to increase the radius from $x$ to $x + dx$ is

$$dq = (4\pi x^2 dx)\rho = \frac{3Q}{R^3}x^2 dx.$$

The work done in bringing the charge $dq$ from infinity to the surface of the sphere of radius $x$ is

$$dW = V(dq) = \frac{3Q^2}{4\pi\varepsilon_0 R^6}x^4 dx.$$

The total work done in assembling the charged sphere of radius $R$ is

$$W = \frac{3Q^2}{4\pi\varepsilon_0 R^6}\int_0^R x^4 dx = \frac{3Q^2}{20\pi\varepsilon_0 R}$$

This is the electric potential energy of the charged sphere.

**Electric Potential Energy of a Uniformly Charged, Thin Spherical Shell**

Consider a uniformly charged, thin spherical shell of radius $R$ having a total charge $Q$. The electric potential energy is equal to the work done in bringing charges from infinity and put them on the shell. Suppose at some instant, a charge $q$ is placed on the shell. The potential at the surface is

$$V = \frac{q}{4\pi\varepsilon_0 R}.$$

The work done in bringing a charge $dq$ from infinity to this shell is

$$dW = V(dq) = \frac{q\,dq}{4\pi\varepsilon_0 R}.$$

The total work done in assembling the charge on the shell is

$$W = \int_0^Q \frac{q\,dq}{4\pi\varepsilon_0 R} = \frac{Q^2}{8\pi\varepsilon_0 R}.$$

This is the electric potential energy of the charged spherical shell.

## 30.6 EARTHING A CONDUCTOR

The earth is a good conductor of electricity. If we assume that the earth is uncharged, its potential will be zero. In fact, the earth's surface has a negative charge of about 1 nC m$^{-2}$ and hence is at a constant potential $V$. All conductors which are not given any external charge, are also very nearly at the same potential. In turns out that for many practical

calculations, we can ignore the charge on the earth. The potential of the earth can then be taken as the same as that of a point far away from all charges, i.e., at infinity. So, the potential of the earth is often taken to be zero. Also, if a small quantity of charge is given to the earth or is taken away from it, the potential does not change by any appreciable extent. This is because of the large size of the earth.

If a conductor is connected to the earth, the potential of the conductor becomes equal to that of the earth, i.e., zero. If the conductor was at some other potential, charges will flow from it to the earth or from the earth to it to bring its potential to zero.

When a conductor is connected to the earth, the conductor is said to be *earthed* or *grounded*. Figure (30.22a) shows the symbol for earthing.

Suppose a spherical conductor of radius $R$ is given a charge $Q$. The charge will be distributed uniformly on the surface. So it is equivalent to a uniformly charged, thin spherical shell. Its potential will, therefore, become $Q/(4\pi\varepsilon_0 R)$. If this conductor is connected to the earth, the charge $Q$ will be transferred to the earth so that the potential will become zero.

Next suppose, a charge $+Q$ is placed at the centre of a spherical conducting shell. A charge $-Q$ will appear on its inner surface and $+Q$ on its outer surface (figure 30.22b). The potential of the sphere due to the charge at the centre and that due to the charge at the inner surface are $\frac{Q}{4\pi\varepsilon_0 R}$ and $\frac{-Q}{4\pi\varepsilon_0 R}$ respectively. The potential due to the charge on the outer surface is $\frac{Q}{4\pi\varepsilon_0 R}$. The net potential of the sphere is, therefore, $\frac{Q}{4\pi\varepsilon_0 R}$. If this sphere is now connected to the earth (figure 30.22c), the charge $Q$ on the outer surface flows to the earth and the potential of the sphere becomes zero.

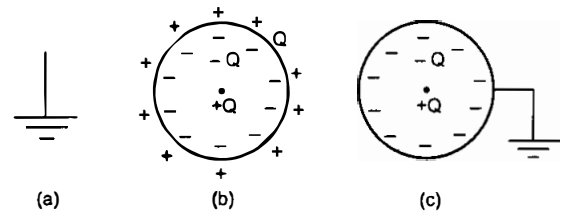



Figure 30.22

Earthing a conductor is a technical job. A thick metal plate is buried deep into the earth and wires are drawn from this plate. The electric wiring in our houses has three wires: live, neutral and earth. The live and neutral wires carry electric currents which come from the power station. The earth wire is connected to the metal plate buried in the earth. The metallic bodies of electric appliances such as electric iron, refrigerator, etc. are connected to the earth wire. This ensures that the metallic body remains at zero potential while an appliance is being used. If by any fault, the live wire touches the metallic body, charge flows to the earth and the potential of the metallic body remains zero. If it is not connected to the earth, the user may get an electric shock.

## *Worked Out Examples*

**1.** *A uniform electric field of magnitude* $E = 100\ \text{N C}^{-1}$ *exists in the space in x-direction. Calculate the flux of this field through a plane square area of edge* 10 cm *placed in the y–z plane. Take the normal along the positive x-axis to be positive.*

***Solution*** : The flux $\Phi = \int E\cos\theta\, dS$. As the normal to the area points along the electric field, $\theta = 0$. Also, $E$ is uniform, so

$$\Phi = E\,\Delta S$$

$$= (100\ \text{N C}^{-1})\,(0{\cdot}10\ \text{m})^2 = 1{\cdot}0\ \text{N m}^2\text{C}^{-1}$$

**2.** *A large plane charge sheet having surface charge density* $\sigma = 2{\cdot}0 \times 10^{-6}\ \text{C m}^{-2}$ *lies in the x-y plane. Find the flux of the electric field through a circular area of radius* 1 cm *lying completely in the region where x, y, z are all positive and with its normal making an angle of* 60° *with the z-axis.*

***Solution*** : The electric field near the plane charge sheet is $E = \sigma/2\varepsilon_0$ in the direction away from the sheet. At the given area, the field is along the $z$-axis.

The area $= \pi r^2 = 3{\cdot}14 \times 1\ \text{cm}^2 = 3{\cdot}14 \times 10^{-4}\ \text{m}^2$.

The angle between the normal to the area and the field is 60°.

Hence, the flux $= \vec{E}\cdot\Delta\vec{S} = E\,\Delta S\cos\theta = \frac{\sigma}{2\varepsilon_0}\pi r^2 \cos 60°$

$$= \frac{2{\cdot}0 \times 10^{-6}\ \text{C m}^{-2}}{2 \times 8{\cdot}85 \times 10^{-12}\ \text{C}^2\ \text{N m}^{-2}} \times (3{\cdot}14 \times 10^{-4}\ \text{m}^2)\,\frac{1}{2}$$

$$= 17{\cdot}5\ \text{N m}^2\ \text{C}^{-1}.$$

**3.** *A charge of* $4 \times 10^{-8}$ C *is distributed uniformly on the surface of a sphere of radius* 1 cm. *It is covered by a concentric, hollow conducting sphere of radius* 5 cm. *(a) Find the electric field at a point* 2 cm *away from the centre. (b) A charge of* $6 \times 10^{-8}$ C *is placed on the hollow sphere. Find the surface charge density on the outer surface of the hollow sphere.*

***Solution*** :

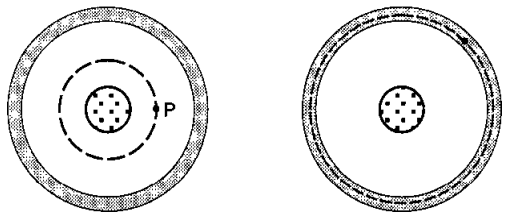



Figure 30-W1

(a) Let us consider figure (30-W1a). Suppose, we have to find the field at the point $P$. Draw a concentric spherical surface through $P$. All the points on this surface are equivalent and by symmetry, the field at all these points will be equal in magnitude and radial in direction.

The flux through this surface $= \oint \vec{E}\cdot d\vec{S}$

$$= \oint E\, dS = E \oint dS$$

$$= 4\pi x^2 E,$$

where $x = 2\text{ cm} = 2 \times 10^{-2}\text{ m}$.

From Gauss's law, this flux is equal to the charge $q$ contained inside the surface divided by $\varepsilon_0$. Thus,

$$4\pi x^2 E = q/\varepsilon_0$$

or,
$$E = \frac{q}{4\pi\varepsilon_0 x^2}$$

$$= (9 \times 10^9\text{ N m}^2\text{ C}^{-2}) \times \frac{4 \times 10^{-8}\text{ C}}{4 \times 10^{-4}\text{ m}^2}$$

$$= 9 \times 10^5\text{ N C}^{-1}.$$

(b) See figure (30-W1b). Take a Gaussian surface through the material of the hollow sphere. As the electric field in a conducting material is zero, the flux $\oint \vec{E}\cdot d\vec{S}$ through this Gaussian surface is zero. Using Gauss's law, the total charge enclosed must be zero. Hence, the charge on the inner surface of the hollow sphere is $-4 \times 10^{-8}$ C. But the total charge given to this hollow sphere is $6 \times 10^{-8}$ C. Hence, the charge on the outer surface will be $10 \times 10^{-8}$ C.

**4.** *Figure (30-W2a) shows three concentric thin spherical shells A, B and C of radii a, b and c respectively. The shells A and C are given charges q and –q respectively and the shell B is earthed. Find the charges appearing on the surfaces of B and C.*

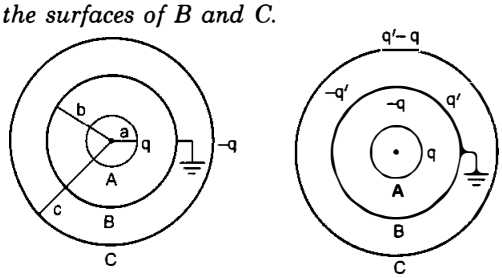



Figure 30-W2

***Solution*** :

As shown in the previous worked out example, the inner surface of $B$ must have a charge $-q$ from the Gauss's law. Suppose, the outer surface of $B$ has a charge $q'$. The inner surface of $C$ must have a charge $-q'$ from the Gauss's law. As the net charge on $C$ must be $-q$, its outer surface should have a charge $q' - q$. The charge distribution is shown in figure (30-W2b).

The potential at $B$ due to the charge $q$ on $A$

$$= \frac{q}{4\pi\varepsilon_0 b},$$

due to the charge $-q$ on the inner surface of $B$

$$= \frac{-q}{4\pi\varepsilon_0 b},$$

due to the charge $q'$ on the outer surface of $B$

$$= \frac{q'}{4\pi\varepsilon_0 b},$$

due to the charge $-q'$, on the inner surface of $C$

$$= \frac{-q'}{4\pi\varepsilon_0 c}$$

and due to the charge $q' - q$ on the outer surface of $C$

$$= \frac{q' - q}{4\pi\varepsilon_0 c}.$$

The net potential is

$$V_B = \frac{q'}{4\pi\varepsilon_0 b} - \frac{q}{4\pi\varepsilon_0 c}$$

This should be zero as the shell $B$ is earthed. Thus,

$$q' = \frac{b}{c} q.$$

The charges on various surfaces are as shown in figure (30-W3).

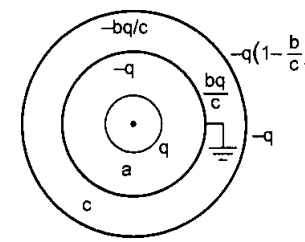



Figure 30-W3

**5.** *An electric dipole consists of charges* $\pm 2{\cdot}0 \times 10^{-8}$ C *separated by a distance of* $2{\cdot}0 \times 10^{-3}$ m. *It is placed near a long line charge of linear charge density* $4{\cdot}0 \times 10^{-4}\text{ C m}^{-1}$ *as shown in figure (30-W4), such that the negative charge is at a distance of* 2·0 cm *from the line charge. Find the force acting on the dipole.*

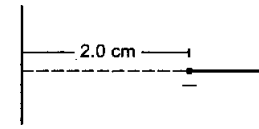



Figure 30-W4

***Solution*** : The electric field at a distance $r$ from the line charge of linear density $\lambda$ is given by

$$E = \frac{\lambda}{2\pi\varepsilon_0 r}.$$

Hence, the field at the negative charge is

$$E_1 = \frac{(4{\cdot}0 \times 10^{-4}\ \text{C m}^{-1})\,(2 \times 9 \times 10^{9}\ \text{N m}^2\ \text{C}^{-2})}{0{\cdot}02\ \text{m}}$$

$$= 3{\cdot}6 \times 10^{8}\ \text{N C}^{-1}.$$

The force on the negative charge is

$$F_1 = (3{\cdot}6 \times 10^{8}\ \text{N C}^{-1})\,(2{\cdot}0 \times 10^{-8}\ \text{C}) = 7{\cdot}2\ \text{N}$$

towards the line charge.

Similarly, the field at the positive charge, i.e., at $r = 0{\cdot}022$ m is

$$E_2 = 3{\cdot}3 \times 10^{8}\ \text{N C}^{-1}.$$

The force on the positive charge is

$$F_2 = (3{\cdot}3 \times 10^{8}\ \text{N C}^{-1}) \times (2{\cdot}0 \times 10^{-8}\ \text{C})$$

$$= 6{\cdot}6\ \text{N}$$ away from the line charge.

Hence, the net force on the dipole = ( 7·2 – 6·6 ) N

= 0·6 N towards the line charge.

**6.** *The electric field in a region is radially outward with magnitude $E = Ar$. Find the charge contained in a sphere of radius a centred at the origin. Take $A = 100\ \text{V m}^{-2}$ and $a = 20{\cdot}0$ cm.*

***Solution*** : The electric field at the surface of the sphere is $Aa$ and being radial it is along the outward normal. The flux of the electric field is, therefore,

$$\Phi = \oint E\,dS\cos\theta = Aa(4\pi a^2).$$

The charge contained in the sphere is, from Gauss's law,

$$Q_{inside} = \varepsilon_0 \Phi = 4\pi\varepsilon_0 Aa^3$$

$$= \left(\frac{1}{9 \times 10^{9}}\ \text{C}^2\ \text{N}^{-1}\ \text{m}^{-2}\right)(100\ \text{V m}^{-2})\,(0{\cdot}20\ \text{m})^3$$

$$= 8{\cdot}89 \times 10^{-11}\ \text{C}.$$

**7.** *A particle of mass $5 \times 10^{-6}$ g is kept over a large horizontal sheet of charge of density $4{\cdot}0 \times 10^{-6}\ \text{C m}^{-2}$ (figure 30-W5). What charge should be given to this particle so that if released, it does not fall down ? How many electrons are to be removed to give this charge ? How much mass is decreased due to the removal of these electrons ?*

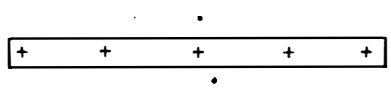


Figure 30-W5

***Solution*** : The electric field in front of the sheet is

$$E = \frac{\sigma}{2\varepsilon_0} = \frac{4{\cdot}0 \times 10^{-6}\ \text{C m}^{-2}}{2 \times 8{\cdot}85 \times 10^{-12}\ \text{C}^2\ \text{N}^{-1}\ \text{m}^{-2}}$$

$$= 2{\cdot}26 \times 10^{5}\ \text{N C}^{-1}.$$

If a charge $q$ is given to the particle, the electric force $qE$ acts in the upward direction. It will balance the weight of the particle if

$$q \times 2{\cdot}26 \times 10^{5}\ \text{N C}^{-1} = 5 \times 10^{-9}\ \text{kg} \times 9{\cdot}8\ \text{m s}^{-2}$$

or,
$$q = \frac{4{\cdot}9 \times 10^{-8}}{2{\cdot}26 \times 10^{5}}\ \text{C}$$

$$= 2{\cdot}21 \times 10^{-13}\ \text{C}.$$

The charge on one electron is $1{\cdot}6 \times 10^{-19}$ C. The number of electrons to be removed

$$= \frac{2{\cdot}21 \times 10^{-13}\ \text{C}}{1{\cdot}6 \times 10^{-19}\ \text{C}} = 1{\cdot}4 \times 10^{6}.$$

Mass decreased due to the removal of these electrons

$$= 1{\cdot}4 \times 10^{6} \times 9{\cdot}1 \times 10^{-31}\ \text{kg}$$

$$= 1{\cdot}3 \times 10^{-24}\ \text{kg}.$$

**8.** *Two conducting plates A and B are placed parallel to each other. A is given a charge $Q_1$ and B a charge $Q_2$. Find the distribution of charges on the four surfaces.*

***Solution*** : Consider a Gaussian surface as shown in figure (30-W6a). Two faces of this closed surface lie completely inside the conductor where the electric field is zero. The flux through these faces is, therefore, zero. The other parts of the closed surface which are outside the conductor are parallel to the electric field and hence the flux on these parts is also zero. The total flux of the electric field through the closed surface is, therefore, zero. From Gauss's law, the total charge inside this closed surface should be zero. The charge on the inner surface of *A* should be equal and opposite to that on the inner surface of *B*.

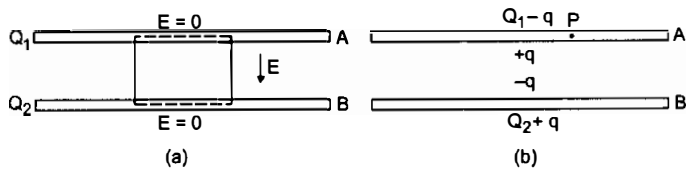



Figure 30-W6

The distribution should be like the one shown in figure (30-W6b). To find the value of $q$, consider the field at a point $P$ inside the plate *A*. Suppose, the surface area of the plate (one side) is *A*. Using the equation $E = \sigma/(2\varepsilon_0)$, the electric field at $P$

due to the charge $Q_1 - q = \frac{Q_1 - q}{2A\varepsilon_0}$ (downward),

due to the charge $+q = \frac{q}{2A\varepsilon_0}$ (upward),

due to the charge $-q = \frac{-q}{2A\varepsilon_0}$ (downward),

and due to the charge $Q_2 + q = \dfrac{Q_2 + q}{2A\varepsilon_0}$ (upward).

The net electric field at $P$ due to all the four charged surfaces is (in the downward direction)

$$\frac{Q_1 - q}{2A\varepsilon_0} - \frac{q}{2A\varepsilon_0} + \frac{q}{2A\varepsilon_0} - \frac{Q_2 + q}{2A\varepsilon_0}$$

As the point $P$ is inside the conductor, this field should be zero. Hence,

$$Q_1 - q - Q_2 - q = 0$$

or, $$q = \frac{Q_1 - Q_2}{2} \quad \ldots \text{(i)}$$

Thus, $$Q_1 - q = \frac{Q_1 + Q_2}{2} \quad \ldots \text{(ii)}$$

and $$Q_2 + q = \frac{Q_1 + Q_2}{2}$$

Using these equations, the distribution shown in the figure (30-W6) can be redrawn as in figure (30-W7).

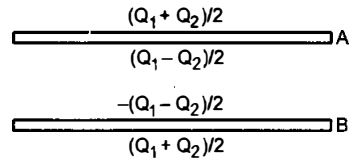



Figure 30-W7

This result is a special case of the following result. When charged conducting plates are placed parallel to each other, the two outermost surfaces get equal charges and the facing surfaces get equal and opposite charges.

□

## QUESTIONS FOR SHORT ANSWER

1. A small plane area is rotated in an electric field. In which orientation of the area is the flux of electric field through the area maximum ? In which orientation is it zero ?
2. A circular ring of radius $r$ made of a nonconducting material is placed with its axis parallel to a uniform electric field. The ring is rotated about a diameter through 180°. Does the flux of electric field change ? If yes, does it decrease or increase ?
3. A charge $Q$ is uniformly distributed on a thin spherical shell. What is the field at the centre of the shell ? If a point charge is brought close to the shell, will the field at the centre change ? Does your answer depend on whether the shell is conducting or nonconducting ?
4. A spherical shell made of plastic, contains a charge $Q$ distributed uniformly over its surface. What is the electric field inside the shell ? If the shell is hammered to deshape it without altering the charge, will the field inside be changed ? What happens if the shell is made of a metal ?
5. A point charge $q$ is placed in a cavity in a metal block. If a charge $Q$ is brought outside the metal, will the charge $q$ feel an electric force ?
6. A rubber balloon is given a charge $Q$ distributed uniformly over its surface. Is the field inside the balloon zero everywhere if the balloon does not have a spherical surface ?
7. It is said that any charge given to a conductor comes to its surface. Should all the protons come to the surface ? Should all the electrons come to the surface ? Should all the free electrons come to the surface ?

## OBJECTIVE I

1. A charge $Q$ is uniformly distributed over a large plastic plate. The electric field at a point $P$ close to the centre of the plate is 10 V m$^{-1}$. If the plastic plate is replaced by a copper plate of the same geometrical dimensions and carrying the same charge $Q$, the electric field at the point $P$ will become
   (a) zero (b) 5 V m$^{-1}$ (c) 10 V m$^{-1}$ (d) 20 V m$^{-1}$.
2. A metallic particle having no net charge is placed near a finite metal plate carrying a positive charge. The electric force on the particle will be
   (a) towards the plate (b) away from the plate
   (c) parallel to the plate (d) zero.
3. A thin, metallic spherical shell contains a charge $Q$ on it. A point charge $q$ is placed at the centre of the shell and another charge $q_1$ is placed outside it as shown in figure (30-Q1). All the three charges are positive. The force on the charge at the centre is

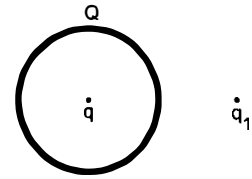



Figure 30-Q1

(a) towards left (b) towards right
(c) upward (d) zero.

4. Consider the situation of the previous problem. The force on the central charge due to the shell is
(a) towards left (b) towards right
(c) upward (d) zero.

5. Electric charges are distributed in a small volume. The flux of the electric field through a spherical surface of radius 10 cm surrounding the total charge is 25 V m. The flux over a concentric sphere of radius 20 cm will be
(a) 25 V m (b) 50 V m (c) 100 V m (d) 200 V m.

6. Figure (30-Q2a) shows an imaginary cube of edge $L/2$. A uniformly charged rod of length $L$ moves towards left at a small but constant speed $v$. At $t = 0$, the left end just touches the centre of the face of the cube opposite it. Which of the graphs shown in figure (30-Q2b) represents the flux of the electric field through the cube as the rod goes through it ?

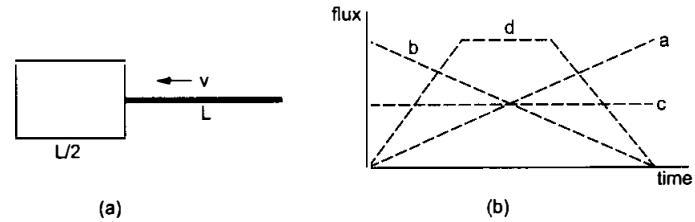



Figure 30-Q2

7. A charge $q$ is placed at the centre of the open end of a cylindrical vessel (figure 30-Q3). The flux of the electric field through the surface of the vessel is
(a) zero (b) $q/\varepsilon_0$ (c) $q/2\varepsilon_0$ (d) $2q/\varepsilon_0$.

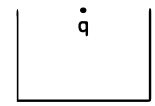



Figure 30-Q3

## OBJECTIVE II

1. Mark the correct options:
(a) Gauss's law is valid only for symmetrical charge distributions.
(b) Gauss's law is valid only for charges placed in vacuum.
(c) The electric field calculated by Gauss's law is the field due to the charges inside the Gaussian surface.
(d) The flux of the electric field through a closed surface due to all the charges is equal to the flux due to the charges enclosed by the surface.

2. A positive point charge $Q$ is brought near an isolated metal cube.
(a) The cube becomes negatively charged.
(b) The cube becomes positively charged.
(c) The interior becomes positively charged and the surface becomes negatively charged.
(d) The interior remains charge free and the surface gets nonuniform charge distribution.

3. A large nonconducting sheet $M$ is given a uniform charge density. Two uncharged small metal rods $A$ and $B$ are placed near the sheet as shown in figure (30-Q4).
(a) $M$ attracts $A$. (b) $M$ attracts $B$.
(c) $A$ attracts $B$. (d) $B$ attracts $A$.

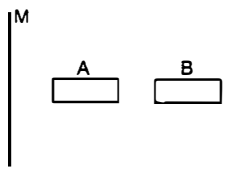



Figure 30-Q4

4. If the flux of the electric field through a closed surface is zero,
(a) the electric field must be zero everywhere on the surface
(b) the electric field may be zero everywhere on the surface
(c) the charge inside the surface must be zero
(d) the charge in the vicinity of the surface must be zero.

5. An electric dipole is placed at the centre of a sphere. Mark the correct options:
(a) The flux of the electric field through the sphere is zero.
(b) The electric field is zero at every point of the sphere.
(c) The electric field is not zero anywhere on the sphere.
(d) The electric field is zero on a circle on the sphere.

6. Figure (30-Q5) shows a charge $q$ placed at the centre of a hemisphere. A second charge $Q$ is placed at one of the positions $A$, $B$, $C$ and $D$. In which position(s) of this second charge, the flux of the electric field through the hemisphere remains unchanged ?
(a) $A$ (b) $B$ (c) $C$ (d) $D$.

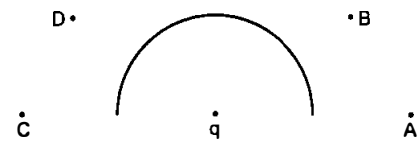



Figure 30-Q5

7. A closed surface $S$ is constructed around a conducting wire connected to a battery and a switch (figure 30-Q6). As the switch is closed, the free electrons in the wire start moving along the wire. In any time interval, the number of electrons entering the closed surface $S$ is equal to the number of electrons leaving it. On closing

the switch, the flux of the electric field through the closed surface
(a) is increased (b) is decreased
(c) remains unchanged (d) remains zero.

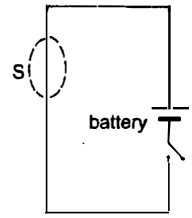



Figure 30-Q6

8. Figure (30-Q7) shows a closed surface which intersects a conducting sphere. If a positive charge is placed at the point P, the flux of the electric field through the closed surface
(a) will remain zero (b) will become positive
(c) will become negative (d) will become undefined.

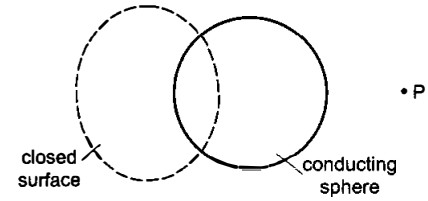



Figure 30-Q7

## EXERCISES

1. The electric field in a region is given by $\vec{E} = \frac{3}{5} E_0 \vec{i} + \frac{4}{5} E_0 \vec{j}$ with $E_0 = 2{\cdot}0 \times 10^3$ N C$^{-1}$. Find the flux of this field through a rectangular surface of area $0{\cdot}2$ m$^2$ parallel to the $y$–$z$ plane.

2. A charge $Q$ is uniformly distributed over a rod of length $l$. Consider a hypothetical cube of edge $l$ with the centre of the cube at one end of the rod. Find the minimum possible flux of the electric field through the entire surface of the cube.

3. Show that there can be no net charge in a region in which the electric field is uniform at all points.

4. The electric field in a region is given by $\vec{E} = \frac{E_0 x}{l} \vec{i}$. Find the charge contained inside a cubical volume bounded by the surfaces $x = 0$, $x = a$, $y = 0$, $y = a$, $z = 0$ and $z = a$. Take $E_0 = 5 \times 10^3$ N C$^{-1}$, $l = 2$ cm and $a = 1$ cm.

5. A charge $Q$ is placed at the centre of a cube. Find the flux of the electric field through the six surfaces of the cube.

6. A charge $Q$ is placed at a distance $a/2$ above the centre of a horizontal, square surface of edge $a$ as shown in figure (30-E1). Find the flux of the electric field through the square surface.

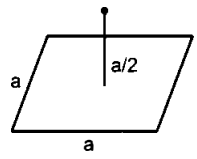



Figure 30-E1

7. Find the flux of the electric field through a spherical surface of radius $R$ due to a charge of $10^{-7}$ C at the centre and another equal charge at a point $2R$ away from the centre (figure 30-E2).

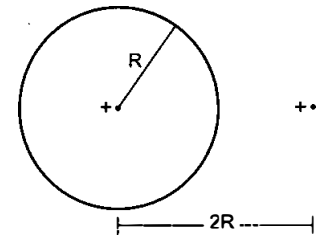



Figure 30-E2

8. A charge $Q$ is placed at the centre of an imaginary hemispherical surface. Using symmetry arguments and the Gauss's law, find the flux of the electric field due to this charge through the surface of the hemisphere (figure 30-E3).

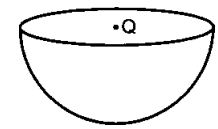



Figure 30-E3

9. A spherical volume contains a uniformly distributed charge of density $2{\cdot}0 \times 10^{-4}$ C m$^{-3}$. Find the electric field at a point inside the volume at a distance $4{\cdot}0$ cm from the centre.

10. The radius of a gold nucleus ($Z = 79$) is about $7{\cdot}0 \times 10^{-15}$ m. Assume that the positive charge is distributed uniformly throughout the nuclear volume. Find the strength of the electric field at (a) the surface of the nucleus and (b) at the middle point of a radius. Remembering that gold is a conductor, is it justified to assume that the positive charge is uniformly distributed over the entire volume of the nucleus and does not come to the outer surface ?

11. A charge $Q$ is distributed uniformly within the material of a hollow sphere of inner and outer radii $r_1$ and $r_2$ (figure 30-E4). Find the electric field at a point $P$ a

distance $x$ away from the centre for $r_1 < x < r_2$. Draw a rough graph showing the electric field as a function of $x$ for $0 < x < 2r_2$ (figure 30-E4).

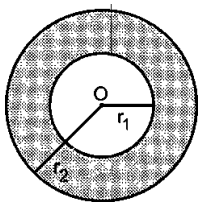



Figure 30-E4

**12.** A charge $Q$ is placed at the centre of an uncharged, hollow metallic sphere of radius $a$. (a) Find the surface charge density on the inner surface and on the outer surface. (b) If a charge $q$ is put on the sphere, what would be the surface charge densities on the inner and the outer surfaces? (c) Find the electric field inside the sphere at a distance $x$ from the centre in the situations (a) and (b).

**13.** Consider the following very rough model of a beryllium atom. The nucleus has four protons and four neutrons confined to a small volume of radius $10^{-15}$ m. The two $1s$ electrons make a spherical charge cloud at an average distance of $1{\cdot}3 \times 10^{-11}$ m from the nucleus, whereas the two $2s$ electrons make another spherical cloud at an average distance of $5{\cdot}2 \times 10^{-11}$ m from the nucleus. Find the electric field at (a) a point just inside the $1s$ cloud and (b) a point just inside the $2s$ cloud.

**14.** Find the magnitude of the electric field at a point 4 cm away from a line charge of density $2 \times 10^{-6}$ C m$^{-1}$.

**15.** A long cylindrical wire carries a positive charge of linear density $2{\cdot}0 \times 10^{-8}$ C m$^{-1}$. An electron revolves around it in a circular path under the influence of the attractive electrostatic force. Find the kinetic energy of the electron. Note that it is independent of the radius.

**16.** A long cylindrical volume contains a uniformly distributed charge of density $\rho$. Find the electric field at a point $P$ inside the cylindrical volume at a distance $x$ from its axis (figure 30-E5)

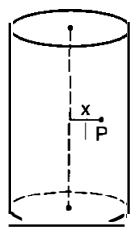



Figure 30-E5

**17.** A nonconducting sheet of large surface area and thickness $d$ contains uniform charge distribution of density $\rho$. Find the electric field at a point $P$ inside the plate, at a distance $x$ from the central plane. Draw a qualitative graph of $E$ against $x$ for $0 < x < d$.

**18.** A charged particle having a charge of $-2{\cdot}0 \times 10^{-6}$ C is placed close to a nonconducting plate having a surface charge density $4{\cdot}0 \times 10^{-6}$ C m$^{-2}$. Find the force of attraction between the particle and the plate.

**19.** One end of a 10 cm long silk thread is fixed to a large vertical surface of a charged nonconducting plate and the other end is fastened to a small ball having a mass of 10 g and a charge of $4{\cdot}0 \times 10^{-6}$ C. In equilibrium, the thread makes an angle of 60° with the vertical. Find the surface charge density on the plate.

**20.** Consider the situation of the previous problem. (a) Find the tension in the string in equilibrium. (b) Suppose the ball is slightly pushed aside and released. Find the time period of the small oscillations.

**21.** Two large conducting plates are placed parallel to each other with a separation of 2·00 cm between them. An electron starting from rest near one of the plates reaches the other plate in 2·00 microseconds. Find the surface charge density on the inner surfaces.

**22.** Two large conducting plates are placed parallel to each other and they carry equal and opposite charges with surface density $\sigma$ as shown in figure (30-E6). Find the electric field (a) at the left of the plates, (b) in between the plates and (c) at the right of the plates.

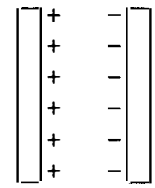


Figure 30-E6

**23.** Two conducting plates $X$ and $Y$, each having large surface area $A$ (on one side), are placed parallel to each other as shown in figure (30-E7). The plate $X$ is given a charge $Q$ whereas the other is neutral. Find (a) the surface charge density at the inner surface of the plate $X$, (b) the electric field at a point to the left of the plates, (c) the electric field at a point in between the plates and (d) the electric field at a point to the right of the plates.

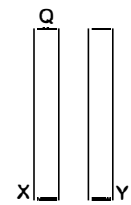



Figure 30-E7

**24.** Three identical metal plates with large surface areas are kept parallel to each other as shown in figure (30-E8). The leftmost plate is given a charge $Q$, the rightmost a charge $-2Q$ and the middle one remains neutral. Find the charge appearing on the outer surface of the rightmost plate.

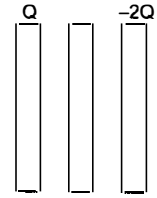



Figure 30-E8

## ANSWERS

### OBJECTIVE I

1. (c) 2. (a) 3. (d) 4. (b) 5. (a) 6. (d)
7. (c)

### OBJECTIVE II

1. (d) 2. (d) 3. all
4. (b), (c) 5. (a), (c) 6. (a), (c)
7. (c), (d) 8. (b)

### EXERCISES

1. $240\ \text{N m}^2\ \text{C}^{-1}$
2. $Q/(2\varepsilon_0)$
4. $2{\cdot}2 \times 10^{-12}\ \text{C}$
5. $Q/\varepsilon_0$
6. $Q/(6\varepsilon_0)$
7. $1{\cdot}1 \times 10^4\ \text{N m}^{-2}\ \text{C}^{-1}$
8. $Q/(2\varepsilon_0)$
9. $3{\cdot}0 \times 10^5\ \text{N C}^{-1}$
10. (a) $2{\cdot}32 \times 10^{21}\ \text{N C}^{-1}$ (b) $1{\cdot}16 \times 10^{21}\ \text{N C}^{-1}$
11. $\dfrac{Q(x^3 - r_1^3)}{4\pi\varepsilon_0 x^2 (r_2^3 - r_1^3)}$
12. (a) $-\dfrac{Q}{4\pi a^2}, \dfrac{Q}{4\pi a^2}$ (b) $-\dfrac{Q}{4\pi a^2}, \dfrac{Q+q}{4\pi a^2}$
    (c) $\dfrac{Q}{4\pi\varepsilon_0 x^2}$ in both situations
13. (a) $3{\cdot}4 \times 10^{13}\ \text{N C}^{-1}$ (b) $1{\cdot}1 \times 10^{12}\ \text{N C}^{-1}$
14. $9 \times 10^5\ \text{N C}^{-1}$
15. $2{\cdot}88 \times 10^{-17}\ \text{J}$
16. $\rho x/(2\varepsilon_0)$
17. $\rho x/\varepsilon_0$
18. 0·45 N
19. $7{\cdot}5 \times 10^{-7}\ \text{C m}^{-2}$
20. (a) 0·20 N (b) 0·45 s
21. $0{\cdot}505 \times 10^{-12}\ \text{C m}^{-2}$
22. (a) zero (b) $\sigma/\varepsilon_0$ (c) zero
23. (a) $\dfrac{Q}{2A}$ (b) $\dfrac{Q}{2A\varepsilon_0}$ towards left (c) $\dfrac{Q}{2A\varepsilon_0}$ towards right
    (d) $\dfrac{Q}{2A\varepsilon_0}$ towards right
24. $-Q/2$

□

## CHAPTER 31

# CAPACITORS

### 31.1 CAPACITOR AND CAPACITANCE

A combination of two conductors placed close to each other is called a *capacitor*. One of the conductors is given a positive charge and the other is given an equal negative charge. The conductor with the positive charge is called the *positive plate* and the other is called the *negative plate*. The charge on the positive plate is called *the charge on the capacitor* and the potential difference between the plates is called the *potential of the capacitor*. Figure (31.1a) shows two conductors. One of the conductors has a positive charge $+Q$ and the other has an equal, negative charge $-Q$. The first one is at a potential $V_+$ and the other is at a potential $V_-$. The charge on the capacitor is $Q$ and the potential of the capacitor is $V = V_+ - V_-$. Note that the term *charge on a capacitor* does not mean the total charge given to the capacitor. This total charge is $+Q - Q = 0$. Figure (31.1b) shows the symbol used to represent a capacitor.

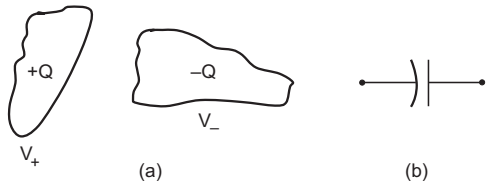



Figure 31.1

For a given capacitor, the charge $Q$ on the capacitor is proportional to the potential difference $V$ between the plates

Thus, $$Q \propto V$$

or, $$Q = CV. \quad \ldots \quad (31.1)$$

The proportionality constant $C$ is called the *capacitance* of the capacitor. It depends on the shape, size and geometrical placing of the conductors and the medium between them.

The SI unit of capacitance is coulomb per volt which is written as farad. The symbol F is used for it. This is a large unit on normal scales and microfarad (μF) is used more frequently.

To put equal and opposite charges on the two conductors, they may be connected to the terminals of a *battery*. We shall discuss in somewhat greater detail about the battery in the next chapter. Here we state the following properties of an ideal battery.

(a) A battery has two terminals.

(b) The potential difference $V$ between the terminals is constant for a given battery. The terminal with higher potential is called the *positive terminal* and that with lower potential is called the *negative terminal*.

(c) The value of this fixed potential difference is equal to the *electromotive force* or *emf* of the battery. If a conductor is connected to a terminal of a battery, the potential of the conductor becomes equal to the potential of the terminal. When the two plates of a capacitor are connected to the terminals of a battery, the potential difference between the plates of the capacitor becomes equal to the emf of the battery.

(d) The total charge in a battery always remains zero. If its positive terminal supplies a charge $Q$, its negative terminal supplies an equal, negative charge $-Q$.

(e) When a charge $Q$ passes through a battery of emf $\mathcal{E}$ from the negative terminal to the positive terminal, an amount $Q\mathcal{E}$ of work is done by the battery.

An ideal battery is represented by the symbol shown in figure (31.2). The potential difference between the facing parallel lines is equal to the emf $\mathcal{E}$ of the battery. The longer line is at the higher potential.

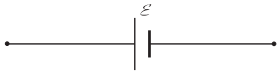



Figure 31.2

***Example 31.1***

*A capacitor gets a charge of* 60 μC *when it is connected to a battery of emf* 12 V. *Calculate the capacitance of the capacitor.*

***Solution*** : The potential difference between the plates is the same as the emf of the battery which is 12 V. Thus,

the capacitance is

$$C = \frac{Q}{V} = \frac{60\ \mu C}{12\ V} = 5\ \mu F.$$

## 31.2 CALCULATION OF CAPACITANCE

The procedure to calculate the capacitance of a given capacitor is simple. We assume that a charge $+Q$ is placed on the positive plate and a charge $-Q$ is placed on the negative plate of the capacitor. We calculate the electric field between the plates and from this the potential difference between the plates. The capacitance is then obtained using equation (31.1).

**Parallel-plate Capacitor**

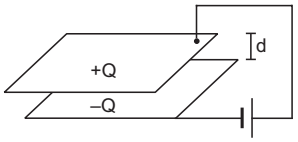



Figure 31.3

A parallel-plate capacitor consists of two large plane plates placed parallel to each other with a small separation between them (figure 31.3). Suppose, the area of each of the facing surfaces is $A$ and the separation between the two plates is $d$. Also, assume that the space between the plates contains vacuum.

Let us put a charge $Q$ on one plate and a charge $-Q$ on the other. The charges will appear on the facing surfaces. The charge density on each of these surfaces has a magnitude

$$\sigma = \frac{Q}{A}.$$

Suppose that the plates are large as compared to the separation between them. This means that any linear dimension of the plates is much larger than the separation $d$. For example, if the plates are square in shape, the length of a side should be much larger than $d$. If we use circular plates, the diameter should be much larger than $d$. The electric field between the plates is then uniform and perpendicular to the plates except for a small region near the edge. The magnitude of this uniform field $E$ may be calculated using Gauss's law.

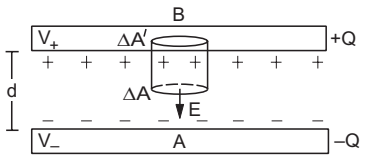



Figure 31.4

Let us draw a small area $\Delta A$ parallel to the plates and in between them (figure 31.4). Draw a cylinder with $\Delta A$ as a cross-section and terminate it by another symmetrically situated area $\Delta A'$ inside the positive plate. The cylinder and the two cross-sections $\Delta A$ and $\Delta A'$ form a Gaussian surface. The flux through $\Delta A'$ and through the curved part inside the plate is zero as the electric field is zero inside a conductor. The flux through the curved part outside the plates is also zero as the direction of the field $E$ is parallel to this surface. The flux through $\Delta A$ is

$$\Phi = \vec{E}\cdot\Delta\vec{A} = E\,\Delta A.$$

The only charge inside the Gaussian surface is

$$\Delta Q = \sigma\,\Delta A = \frac{Q}{A}\Delta A.$$

From Gauss's law,

$$\oint \vec{E}\cdot d\vec{S} = Q_{in}/\varepsilon_0$$

or, $$E\,\Delta A = \frac{Q}{\varepsilon_0 A}\Delta A$$

or, $$E = \frac{Q}{\varepsilon_0 A}.$$

The potential difference between the plates is

$$V = V_+ - V_- = -\int_A^B \vec{E}\cdot d\vec{r}.$$

As one goes from $A$ to $B$, the field $\vec{E}$ and the displacement $d\vec{r}$ are opposite in direction. Thus, $\vec{E}\cdot d\vec{r} = -E\,dr$ and

$$V = \int_A^B E\,dr$$

$$= Ed = \frac{Qd}{\varepsilon_0 A}.$$

The capacitance of the parallel-plate capacitor is

$$C = \frac{Q}{V} = \frac{Q\varepsilon_0 A}{Qd}$$

$$= \frac{\varepsilon_0 A}{d}. \qquad \ldots (31.2)$$

***Example 31.2***

*Show that the SI unit of* $\varepsilon_0$ *may be written as* farad metre$^{-1}$.

***Solution :***

We have $C = \frac{\varepsilon_0 A}{d}$

or, $$\varepsilon_0 = \frac{Cd}{A}.$$

As the SI units of $C$, $d$ and $A$ are farad, metre and metre$^2$ respectively, the SI unit of $\varepsilon_0$ is farad metre$^{-1}$.

***Example 31.3***

*Calculate the capacitance of a parallel-plate capacitor having* $20\ cm \times 20\ cm$ *square plates separated by a distance of* 1·0 mm.

***Solution*** : The capacitance is

$$C = \frac{\varepsilon_0 A}{d}$$

$$= \frac{8{\cdot}85 \times 10^{-12}\ \text{F m}^{-1} \times 400 \times 10^{-4}\ \text{m}^2}{1 \times 10^{-3}\ \text{m}}$$

$$= 3{\cdot}54 \times 10^{-10}\ \text{F} \approx 350\ \text{pF}.$$

**Spherical Capacitor**

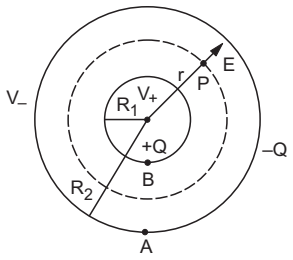



Figure 31.5

A spherical capacitor consists of a solid or a hollow spherical conductor surrounded by another concentric hollow spherical conductor. Suppose, the inner sphere has a radius $R_1$ and the outer sphere has a radius $R_2$. Suppose, the inner sphere is given a positive charge $Q$ and the outer is given a negative charge $-Q$.

The field at any point $P$ between the spheres is radially outward and its magnitude depends only on its distance $r$ from the centre. Let us draw a sphere through $P$ concentric with the given system. The flux of the electric field through this sphere is

$$\Phi = \oint \vec{E}\cdot d\vec{S} = \oint E\, dS$$

$$= E \oint dS = E\, 4\pi r^2.$$

The charge enclosed in this sphere is $Q$. Thus, from Gauss's law,

$$E\, 4\pi r^2 = \frac{Q}{\varepsilon_0}$$

or,
$$E = \frac{Q}{4\pi\varepsilon_0 r^2}.$$

The potential difference between the two conductors is

$$V = V_+ - V_- = -\int_A^B \vec{E}\cdot d\vec{r}$$

$$= -\int_{R_2}^{R_1} \frac{Q}{4\pi\varepsilon_0 r^2}\, dr$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{Q(R_2 - R_1)}{4\pi\varepsilon_0 R_1 R_2}.$$

The capacitance of the spherical capacitor is

$$C = \frac{Q}{V}$$

$$= \frac{4\pi\varepsilon_0 R_1 R_2}{R_2 - R_1}. \qquad \ldots\ (31.3)$$

**Isolated sphere**

If we assume that the outer sphere is at infinity, we get an isolated single sphere of radius $R_1$. The capacitance of such a single sphere can be obtained from equation (31.3) by taking the limit as $R_2 \to \infty$. Then

$$C = \frac{4\pi\varepsilon_0 R_1 R_2}{R_2 - R_1}$$

$$\approx \frac{4\pi\varepsilon_0 R_1 R_2}{R_2} = 4\pi\varepsilon_0 R_1.$$

If a charge $Q$ is placed on this sphere, its potential (with zero potential at infinity) becomes

$$V = \frac{Q}{C} = \frac{Q}{4\pi\varepsilon_0 R_1}.$$

**Parallel limit**

If both $R_1$ and $R_2$ are made large but $R_2 - R_1 = d$ is kept fixed, we can write

$$4\pi R_1 R_2 \approx 4\pi R^2 = A$$

where $R$ is approximately the radius of each sphere and $A$ is the area. Equation (31.3) then becomes

$$C = \frac{\varepsilon_0 A}{d}$$

which is the same as the equation for the capacitance of a parallel-plate capacitor.

**Cylindrical Capacitor**

A cylindrical capacitor consists of a solid or a hollow cylindrical conductor surrounded by another coaxial hollow cylindrical conductor. Let the length of the cylinders be $l$ and the radii of the inner and outer cylinders be $R_1$ and $R_2$ respectively. Suppose, a positive charge $Q$ is placed on the inner cylinder and a negative charge $-Q$ is placed on the outer cylinder. If the cylinders are long as compared to the separation between them, the electric field at a point between the cylinders will be radial and its magnitude will depend only on the distance of the point from the axis. Let $P$ be a point between the cylinders at a distance $r$ from the axis (figure 31.6).

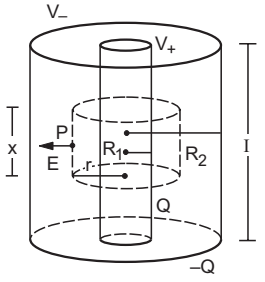



Figure 31.6

To calculate the electric field at the point *P,* let us draw a coaxial cylinder of length *x* through the point *P*. This cylinder together with its two cross sections forms a Gaussian surface. The flux through the cross sections is zero because the electric field is radial wherever it exists and hence is parallel to the cross sections. The flux through the curved part is

$$\Phi = \oint \vec{E}\cdot d\vec{S}$$

$$= \int E\, dS$$

$$= E \int dS = E\, 2\pi r x.$$

The charge enclosed by the Gaussian surface is

$$Q_{in} = \frac{Q}{l}\, x.$$

Thus, from Gauss's law,

$$E\, 2\pi r x = \left(\frac{Q}{l}\, x\right)/\varepsilon_0$$

or, $$E = \frac{Q}{2\pi\varepsilon_0\, rl}\,.$$

The potential difference between the cylinders is

$$V = V_+ - V_-$$

$$= -\int_A^B \vec{E}\cdot d\vec{r} = -\int_{R_2}^{R_1} E\, dr$$

$$= -\int_{R_2}^{R_1} \frac{Q}{2\pi\varepsilon_0\, rl}\, dr$$

$$= \frac{Q}{2\pi\varepsilon_0\, l} \ln \frac{R_2}{R_1}\,.$$

The capacitance is

$$C = \frac{Q}{V} = \frac{2\pi\varepsilon_0\, l}{\ln(R_2/R_1)}\,. \qquad \ldots \quad (31.4)$$

## 31.3 COMBINATION OF CAPACITORS

Two or more capacitors may be connected in a number of ways. The combination should have two points which may be connected to a battery to apply a potential difference. The battery supplies positive and negative charges to the system. If $V$ be the potential difference between the points and $Q$ be the magnitude of the charge supplied by either terminal of the battery, we define *equivalent capacitance* of the combination *between the two points* to be

$$C = \frac{Q}{V}\,.$$

If the combination is replaced by a single capacitor of this capacitance, the single capacitor will store the same amount of charge for a given potential difference as the combination does.

Two special methods of combination are frequently used, one known as *series* combination and the other as *parallel* combination.

### Series Combination

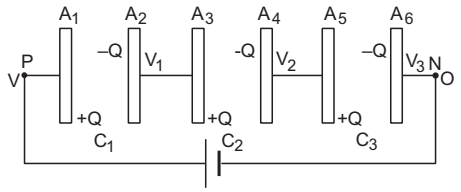



Figure 31.7

Figure (31.7) shows three capacitors connected in series. The capacitances are $C_1$, $C_2$ and $C_3$. The points $P$ and $N$ serve as the points through which a potential difference may be applied and a charge may be supplied to the combination. Let us connect the point $P$ to the positive terminal and the point $N$ to the negative terminal of a battery. The battery supplies a charge $+Q$ to the plate $A_1$ and a charge $-Q$ to the plate $A_6$. The charge $+Q$ given by the battery appears on the right surface of the plate $A_1$. The facing surface of $A_2$ must have a charge $-Q$ on it.

The plates $A_2$ and $A_3$ are connected and they together are isolated from everything else. The charge $-Q$ appearing on $A_2$ comes from the electrons drifted from the plate $A_3$ to $A_2$. This leaves a positive charge $+Q$ on the plate $A_3$. The facing surface of $A_4$ gets a charge $-Q$ from $A_5$ and a charge $+Q$ appears on the right surface of $A_5$. The facing surface of $A_6$ gets a charge $-Q$ from the battery. This completes the charge distribution. *In series combination, each capacitor has equal charge for any value of capacitances.*

Let us take the potential of the point $N$ to be zero. The potential of the plate $A_6$ is also zero as it is connected to $N$ by a conducting wire. The potential of the point $P$ as well as that of the plate $A_1$ is $V$. The plates $A_2$ and $A_3$ are at the same potential, say, $V_1$. Similarly, $A_4$ and $A_5$ are at the same potential, say, $V_2$.

The charge on the first capacitor is $Q$ and the potential difference is $V - V_1$. As the capacitance of this capacitor is $C_1$, we have

$$Q = C_1(V - V_1)$$

or, $$V - V_1 = \frac{Q}{C_1}\,. \qquad \ldots \quad \text{(i)}$$

Similarly, considering the other capacitors,

$$V_1 - V_2 = \frac{Q}{C_2} \qquad \ldots \quad \text{(ii)}$$

and $$V_2 - 0 = \frac{Q}{C_3}\,. \qquad \ldots \quad \text{(iii)}$$

Adding (i), (ii) and (iii);

$$V = Q\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right). \quad \ldots \text{ (iv)}$$

If the equivalent capacitance of the combination between the points $P$ and $N$ is $C$, we have

$$C = \frac{Q}{V}$$

and equation (iv) becomes

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}.$$

The above analysis may be extended to any number of capacitors, the equivalent capacitance $C$ is given by

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots \quad \ldots \quad (31.5)$$

***Example 31.4***

*Calculate the charge on each capacitor shown in figure (31.8).*

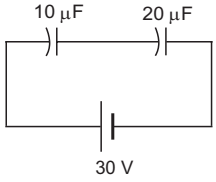



Figure 31.8

***Solution*** : The two capacitors are joined in series. Their equivalent capacitance is given by $\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$

or, $$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{(10\ \mu\text{F})\ (20\ \mu\text{F})}{30\ \mu\text{F}} = \frac{20}{3}\ \mu\text{F}.$$

The charge supplied by the battery is

$$Q = CV$$
$$= \left(\frac{20}{3}\ \mu\text{F}\right)(30\ \text{V}) = 200\ \mu\text{C}.$$

In series combination, each capacitor has equal charge and this charge equals the charge supplied by the battery. Thus, each capacitor has a charge of 200 μC.

**Parallel Combination**

Figure (31.9) shows three capacitors connected in parallel. The capacitances are $C_1$, $C_2$ and $C_3$. The points

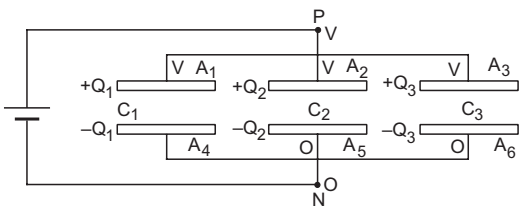



Figure 31.9

$P$ and $N$ are the two points through which a potential difference can be applied and charge can be supplied. Let us connect the point $P$ to the positive terminal of a battery and the point $N$ to its negative terminal. The battery supplies a charge $+Q$ which is distributed on the three positive plates $A_1$, $A_2$ and $A_3$ of the capacitors. Let the charges on the three plates $A_1$, $A_2$ and $A_3$ be $Q_1$, $Q_2$ and $Q_3$ respectively. The battery also supplies a charge $-Q$ which is distributed on the three plates $A_4$, $A_5$ and $A_6$. These plates must receive charges $-Q_1$, $-Q_2$ and $-Q_3$ respectively because the facing surfaces must have equal and opposite charges. We have

$$Q = Q_1 + Q_2 + Q_3. \quad \ldots \text{ (i)}$$

Let us take the potential of the point $N$ to be zero. The potentials of the plates $A_4$, $A_5$ and $A_6$ are also zero as they are all connected to $N$ by conducting wires. Let the potential of the point $P$ be $V$. This will also be the potential of the plates $A_1$, $A_2$ and $A_3$. Thus, *the potential differences of the capacitors connected in parallel are equal for any value of capacitances.* Using the equation $Q = CV$ for the three capacitors,

$$Q_1 = C_1 V \quad \ldots \text{ (ii)}$$

$$Q_2 = C_2 V \quad \ldots \text{ (iii)}$$

and $$Q_3 = C_3 V. \quad \ldots \text{ (iv)}$$

Adding (ii), (iii) and (iv) and using (i),

$$Q = (C_1 + C_2 + C_3)V$$

or, $$\frac{Q}{V} = C_1 + C_2 + C_3.$$

But $Q/V$ is the equivalent capacitance of the given combination. Thus,

$$C = C_1 + C_2 + C_3. \quad \ldots \quad (31.6)$$

*In parallel combination, all the positive plates are at the same potential and all the negative plates are at the same potential. The potential difference on each capacitor is the same in parallel combination but the charges on the capacitors may be different. In series combination, the charges on the capacitors are equal, the potential differences may be different.*

***Example 31.5***

*Find the equivalent capacitance of the combination shown in figure (31.10) between the points P and N.*

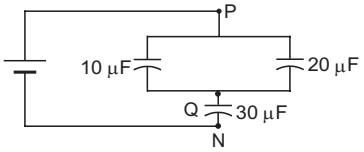



Figure 31.10

***Solution*** : The 10 μF and 20 μF capacitors are connected in parallel. Their equivalent capacitance is

10 μF + 20 μF = 30 μF. We can replace the 10 μF and the 20 μF capacitors by a single capacitor of capacitance 30 μF between $P$ and $Q$. This is connected in series with the given 30 μF capacitor. The equivalent capacitance $C$ of this combination is given by

$$\frac{1}{C} = \frac{1}{30\ \mu F} + \frac{1}{30\ \mu F} \text{ or, } C = 15\ \mu F.$$

We have used series–parallel combination to solve the above example. Sometimes it may not be easy to find the equivalent capacitance of a combination using the equations for series–parallel combinations. We may then use the general method which was applied to derive the equivalent capacitance in series and parallel combinations. For any given combination, one may proceed as follows:

**Step 1**

Identify the two points between which the equivalent capacitance is to be calculated. Call any one of them as $P$ and the other as $N$.

**Step 2**

Connect (mentally) a battery between $P$ and $N$ with the positive terminal connected to $P$ and the negative terminal to $N$. Send a charge $+Q$ from the positive terminal of the battery and $-Q$ from the negative terminal of the battery.

**Step 3**

Write the charges appearing on each of the plates of the capacitors. The charge conservation principle may be used. The facing surfaces of a capacitor will always have equal and opposite charges. Assume variables $Q_1$, $Q_2$ …, etc., for charges wherever needed.

**Step 4**

Take the potential of the negative terminal $N$ to be zero and that of the positive terminal $P$ to be $V$. Write the potential of each of the plates. If necessary, assume variables $V_1$, $V_2$ … .

**Step 5**

Write the capacitor equation $Q = CV$ for each capacitor. Eliminate $Q_1$, $Q_2$, … and $V_1$, $V_2$, …, etc., to obtain the equivalent capacitance $C = Q/V$.

***Example 31.6***

*Find the equivalent capacitance of the combination shown in figure (31.11a) between the points P and N.*

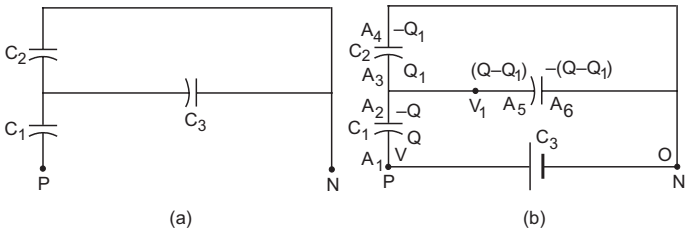



Figure 31.11

***Solution*** **:** Let us connect a battery between the points $P$ and $N$. The charges and the potentials are shown in figure (31.11b). The positive terminal of the battery supplies a charge $+Q$ which appears on the plate $A_1$. The facing plate $A_2$ gets a charge $-Q$. The plates $A_2$, $A_3$ and $A_5$ taken together form an isolated system. The total charge on these three plates should be zero. Let a charge $Q_1$ appear on $A_3$, then a charge $Q - Q_1$ will appear on $A_5$ to make the total charge zero on the three plates. The plate $A_4$ will get a charge $-Q_1$ (facing plate of $A_3$) and $A_6$ will get a charge $-(Q - Q_1)$ (facing plate of $A_5$). The total charge $-Q$ on $A_4$ and $A_6$ is supplied by the negative terminal of the battery. This completes the charge distribution.

Next, suppose the potential at the point $N$ is zero and at $P$ it is $V$. The potential of the plates $A_4$ and $A_6$ is also zero. The potential of the plate $A_1$ is $V$. The plates $A_2$, $A_3$ and $A_5$ are at the same potential. Let this common potential be $V_1$. This completes the potential distribution.

Applying the capacitor equation $Q = CV$ to the three capacitors,

$$Q = C_1(V - V_1) \quad \ldots \text{ (i)}$$

$$Q_1 = C_2V_1 \quad \ldots \text{ (ii)}$$

and $$Q - Q_1 = C_3V_1. \quad \ldots \text{ (iii)}$$

Adding (ii) and (iii),

$$Q = (C_2 + C_3)V_1$$

or, $$\frac{Q}{C_2 + C_3} = V_1. \quad \ldots \text{ (iv)}$$

From (i), $$\frac{Q}{C_1} = V - V_1. \quad \ldots \text{ (v)}$$

Adding (iv) and (v),

$$\frac{Q}{C_2 + C_3} + \frac{Q}{C_1} = V$$

or, $$\frac{(C_1 + C_2 + C_3)Q}{C_1(C_2 + C_3)} = V$$

or, $$C = \frac{Q}{V} = \frac{C_1(C_2 + C_3)}{C_1 + C_2 + C_3}.$$

It may be noted that the above example could be solved by using the equations for series–parallel combinations. However, the general method was used to demonstrate its application.

Symmetry arguments play important role in simplifying the algebra involved in the problem. The use of symmetry arguments in writing the charges on different plates will be demonstrated later in the section of worked out examples.

## 31.4 FORCE BETWEEN THE PLATES OF A CAPACITOR

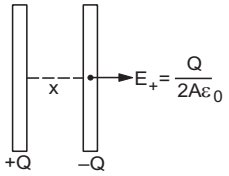



Figure 31.12

Consider a parallel-plate capacitor with plate area $A$. Suppose a positive charge $+Q$ is given to one plate and a negative charge $-Q$ to the other plate. The electric field due to only the positive plate is

$$E_+ = \frac{\sigma}{2\varepsilon_0} = \frac{Q}{2A\varepsilon_0}$$

at all points if the plate is large. The negative charge $-Q$ finds itself in the field of this positive charge. The force on $-Q$ is, therefore,

$$F = -QE_+$$
$$= (-Q)\frac{Q}{2A\varepsilon_0} = -\frac{Q^2}{2A\varepsilon_0}.$$

The magnitude of the force is

$$F = \frac{Q^2}{2A\varepsilon_0}.$$

This is the force with which the positive plate attracts the negative plate. This is also the force of attraction on the positive plate by the negative plate. Thus, the plates of a parallel-plate capacitor attract each other with a force

$$F = \frac{Q^2}{2A\varepsilon_0}. \qquad \ldots (31.7)$$

## 31.5 ENERGY STORED IN A CAPACITOR AND ENERGY DENSITY IN ELECTRIC FIELD

Let us consider a parallel-plate capacitor of plate area $A$ (figure 31.13). Suppose the plates of the capacitor are almost touching each other and a charge $Q$ is given to the capacitor. One of the plates, say $a$, is kept fixed and the other, say $b$, is slowly pulled away from $a$ to increase the separation from zero to $d$. The attractive force on the plate $b$ at any instant due to the first plate is, from equation (31.7),

$$F = \frac{Q^2}{2A\varepsilon_0}.$$

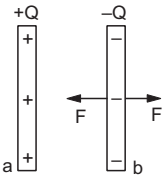



Figure 31.13

The person pulling the plate $b$ must apply an equal force $F$ in the opposite direction if the plate is only slowly moved.

The work done by the person during the displacement of the second plate is

$$W = Fd$$
$$= \frac{Q^2 d}{2A\varepsilon_0} = \frac{Q^2}{2C}$$

where $C$ is the capacitance of the capacitor in the final position. The work done by the person must be equal to the increase in the energy of the system. Thus, a capacitor of capacitance $C$ has a stored energy

$$U = \frac{Q^2}{2C} \qquad \ldots (31.8)$$

where $Q$ is the charge given to it. Using $Q = CV$, the above equation may also be written as

$$U = \frac{1}{2}CV^2 \qquad \ldots (31.9)$$

or,
$$U = \frac{1}{2}QV. \qquad \ldots (31.10)$$

***Example 31.7***

*Find the energy stored in a capacitor of capacitance* 100 μF *when it is charged to a potential difference of* 20 V.

***Solution*** : The energy stored in the capacitor is

$$U = \frac{1}{2}CV^2 = \frac{1}{2}(100\ \mu\text{F})(20\ \text{V})^2 = 0{\cdot}02\ \text{J}.$$

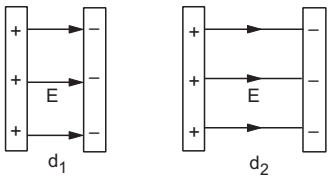



Figure 31.14

The energy stored in a capacitor is electrostatic potential energy. When we pull the plates of a capacitor apart, we have to do work against the electrostatic attraction between the plates. In which region of space is the energy stored ? When we increase the separation between the plates from $d_1$ to $d_2$, an amount $\frac{Q^2}{2A\varepsilon_0}(d_2 - d_1)$ of work is performed by us and

this much energy goes into the capacitor. On the other hand, new electric field is created in a volume $A(d_2 - d_1)$ (figure 31.14). We conclude that the energy $\frac{Q^2}{2A\varepsilon_0}(d_2 - d_1)$ is stored in the volume $A(d_2 - d_1)$ which is now filled with the electric field. Thus, an electric field has energy associated with it. The energy stored per unit volume in the electric field is

$$u = \frac{\frac{Q^2(d_2 - d_1)}{2A\varepsilon_0}}{A(d_2 - d_1)} = \frac{Q^2}{2A^2\varepsilon_0}$$
$$= \frac{1}{2}\varepsilon_0\left(\frac{Q}{A\varepsilon_0}\right)^2 = \frac{1}{2}\varepsilon_0 E^2$$

where $E$ is the intensity of the electric field.

Once it is established that a region containing electric field $E$ has energy $\frac{1}{2}\varepsilon_0 E^2$ per unit volume, the result can be used for any electric field whether it is due to a capacitor or otherwise.

### 31.6 DIELECTRICS

In dielectric materials, effectively there are no free electrons. The monatomic materials are made of atoms. Each atom consists of a positively charged nucleus surrounded by electrons. In general, the centre of the negative charge coincides with the centre of the positive charge. Polyatomic materials, on the other hand, are made of molecules. The centre of the negative charge distribution in a molecule may or may not coincide with the centre of the positive charge distribution. If it does not coincide, each molecule has a permanent dipole moment $\vec{p}$. Such materials are known as *polar materials*. However, different molecules have different directions of the dipole moment because of the random thermal agitation in the material. In any volume containing a large number of molecules (say more than a thousand), the net dipole moment is zero. If such a material is placed in an electric field, the individual dipoles experience torque due to the field and they try to align along the field. On the other hand, thermal agitation tries to randomise the orientation and hence, there is a partial alignment. As a result, we get a net dipole moment in any volume of the material.

In nonpolar materials, the centre of the positive charge distribution in an atom or a molecule coincides with the centre of the negative charge distribution. The atoms or the molecules do not have any permanent dipole moment. If such a material is placed in an electric field, the electron charge distribution is slightly shifted opposite to the electric field. This induces dipole moment in each atom or molecule and thus, we get a dipole moment in any volume of the material.

Thus, when a dielectric material is placed in an electric field, dipole moment appears in any volume in it. This fact is known as *polarization* of the material. The *polarization vector* $\vec{P}$ is defined as the dipole moment per unit volume. Its magnitude $P$ is often referred to as the polarization.

Consider a rectangular slab of a dielectric. The individual dipole moments are randomly oriented (figure 31.15a). In any volume containing a large number of molecules, the net charge is zero. When an electric field is applied, the dipoles get aligned along the field. Figure (31.15b) and (31.15c) show the effect of dipole alignment when a field is applied from left to right. We see that the interior is still charge free but the left surface of the slab gets negative charge and the right surface gets positive charge. The situation may be represented as in figure (31.15d). The charge appearing on the surface of a dielectric when placed in an electric field is called *induced charge*. As the induced charge appears due to a shift in the electrons bound to the nuclei, this charge is also called *bound charge*.

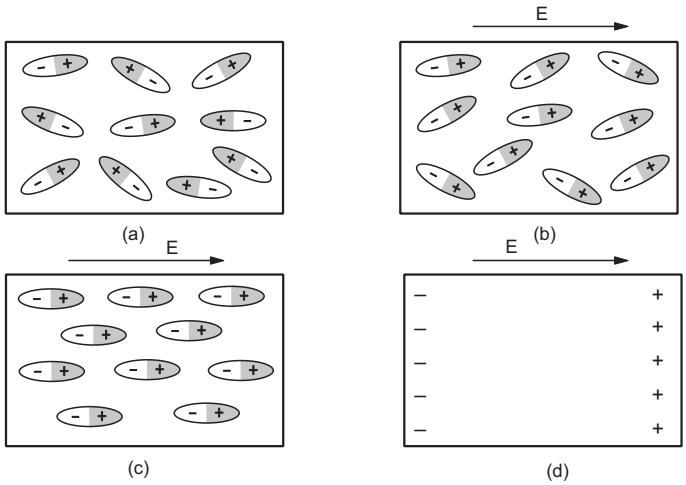



Figure 31.15

The surface charge density of the induced charge has a simple relationship with the polarization $P$. Suppose, the rectangular slab of figure (31.15) has a length $l$ and area of cross-section $A$. Let $\sigma_p$ be the magnitude of the induced charge per unit area on the faces. The dipole moment of the slab is then $(\sigma_p A)l = \sigma_p(Al)$. The polarization is dipole moment induced per unit volume. Thus,

$$P = \frac{\sigma_p (Al)}{Al} = \sigma_p. \qquad \ldots (31.11)$$

Although this result is deduced for a rectangular slab, it is true in general. The induced surface charge density is equal in magnitude to the polarization $P$.

**Dielectric Constant**

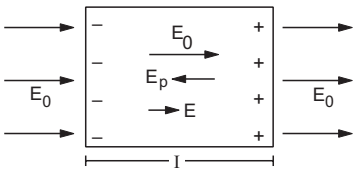



Figure 31.16

Because of the induced charges, an extra electric field is produced inside the material. Let $\vec{E}_0$ be the applied field due to external sources and $\vec{E}_p$ be the field due to polarization (figure 31.16). The resultant field is $\vec{E} = \vec{E}_0 + \vec{E}_p$. For homogeneous and isotropic dielectrics, the direction of $\vec{E}_p$ is opposite to the direction of $\vec{E}_0$. The resultant field $\vec{E}$ is in the same direction as the applied field $\vec{E}_0$ but its magnitude is reduced. We can write

$$\vec{E} = \frac{\vec{E}_0}{K}$$

where $K$ is a constant for the given dielectric which has a value greater than one. This constant $K$ is called the *dielectric constant* or *relative permittivity* of the dielectric. For vacuum, there is no polarization and hence $\vec{E} = \vec{E}_0$ and $K = 1$.

If a very high electric field is created in a dielectric, the outer electrons may get detached from their parent atoms. The dielectric then behaves like a conductor. This phenomenon is known as *dielectric breakdown*. The minimum field at which the breakdown occurs is called the *dielectric strength* of the material. Table (31.1) gives dielectric constants and dielectric strengths for some of the dielectrics.

## 31.7. PARALLEL-PLATE CAPACITOR WITH A DIELECTRIC

Consider a parallel-plate capacitor with plate area $A$ and separation $d$ between the plates (figure 31.17). A dielectric slab of dielectric constant $K$ is inserted in the space between the plates. Suppose, the slab almost completely fills the space between the plates. A charge $Q$ is given to the positive plate and $-Q$ to the negative plate of the capacitor. The electric field polarizes the dielectric so that induced charges $+Q_p$ and $-Q_p$ appear on the two faces of the slab.

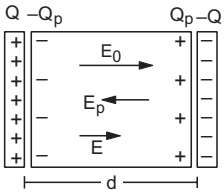



Figure 31.17

**Table 31.1 :** ***Dielectric constants and dielectric strengths***

| Material | Dielectric constant | Dielectric strength ($kVmm^{-1}$) |
|---|---|---|
| Vacuum | 1 | ∞ |
| Pyrex Glass | 5·6 | ≈ 14 |
| Mica | 3–6 | 12 |
| Neoprene rubber | 6·9 | 12 |
| Bakelite | 4·9 | 24 |
| Plexiglas | 3·40 | 40 |
| Fused quartz | 3·8 | 8 |
| Paper | 3·5 | 14 |
| Polystyrene | 2·6 | 25 |
| Teflon | 2·1 | 60 |
| Strontium titanate | 310 | 8 |
| Titanium dioxide | 100 | 6 |
| Water | 80 | – |
| Glycerin | 42·5 | – |
| Benzene | 2·3 | – |
| Air (1 atm) | 1·00059 | 3 |
| Air (100 atm) | 1·0548 | – |

The electric field at a point between the plates due to the charges $+Q, -Q$ on the capacitor plates is

$$E_0 = \frac{\sigma}{\varepsilon_0} = \frac{Q}{A\varepsilon_0} \qquad \ldots \text{ (i)}$$

in a direction left to right in the figure (31.17).

From the definition of dielectric constant, the resultant field is

$$E = \frac{E_0}{K} = \frac{Q}{\varepsilon_0 AK}. \qquad \ldots \text{ (ii)}$$

The potential difference between the plates is

$$V = Ed$$
$$= \frac{Qd}{\varepsilon_0 AK}.$$

The capacitance is

$$C = \frac{Q}{V} = \frac{K\varepsilon_0 A}{d} = KC_0 \qquad \ldots \text{ (31.12)}$$

where $C_0 = \frac{\varepsilon_0 A}{d}$ is the capacitance without the dielectric. Thus,

*The capacitance of a capacitor is increased by a factor of K when the space between the plates is filled with a dielectric of dielectric constant K.*

This result is often taken as the definition of the dielectric constant.

**Magnitude of the Induced Charge**

From (i), the electric field at a point between the plates due to the charges $+Q, -Q$ is

$$E_0 = \frac{Q}{A\varepsilon_0}.$$

The field due to the charges $Q_p$, $-Q_p$ is directed oppositely and has magnitude

$$E_p = \frac{\sigma_p}{\varepsilon_0} = \frac{Q_p}{A\varepsilon_0}.$$

The resultant field is

$$E = E_0 - E_p$$
$$= \frac{Q - Q_p}{A\varepsilon_0}. \quad \ldots \text{(iii)}$$

From equations (ii) and (iii),

$$\frac{Q - Q_p}{\varepsilon_0 A} = \frac{Q}{\varepsilon_0 AK}$$

or, $$Q - Q_p = \frac{Q}{K}$$

or, $$Q_p = Q\left(1 - \frac{1}{K}\right). \quad \ldots \text{(31.13)}$$

***Example 31.8***

*Two parallel-plate capacitors, each of capacitance* 40 μF, *are connected in series. The space between the plates of one capacitor is filled with a dielectric material of dielectric constant* $K = 4$. *Find the equivalent capacitance of the system.*

***Solution*** : The capacitance of the capacitor with the dielectric is

$$C_1 = KC_0 = 4 \times 40\ \mu\text{F} = 160\ \mu\text{F}.$$

The other capacitor has capacitance $C_2 = 40$ μF. As they are connected in series, the equivalent capacitance is

$$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{(160\ \mu\text{F})\ (40\ \mu\text{F})}{200\ \mu\text{F}} = 32\ \mu\text{F}.$$

***Example 31.9***

*A parallel-plate capacitor has plate area A and plate separation d. The space between the plates is filled up to a thickness x* (< *d*) *with a dielectric of dielectric constant K. Calculate the capacitance of the system.*

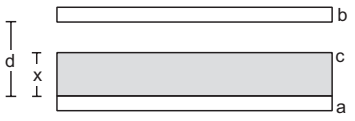



Figure 31.18

***Solution*** :

The situation is shown in figure (31.18). The given system is equivalent to the series combination of two capacitors, one between *a* and *c* and the other between *c* and *b*. Here *c* represents the upper surface of the dielectric. This is because the potential at the upper surface of the dielectric is constant and we can imagine a thin metal plate being placed there.

The capacitance of the capacitor between *a* and *c* is

$$C_1 = \frac{K\varepsilon_0 A}{x}$$

and that between *c* and *b* is

$$C_2 = \frac{\varepsilon_0 A}{d - x}.$$

The equivalent capacitance is

$$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{K\varepsilon_0 A}{Kd - x(K-1)}.$$

## 31.8 AN ALTERNATIVE FORM OF GAUSS'S LAW

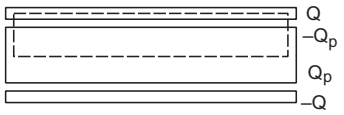



Figure 31.19

Let us again consider a parallel-plate capacitor with a charge $Q$. The space between the plates is filled with a dielectric slab of dielectric constant $K$. Let us consider a Gaussian surface as shown in figure (31.19). The charge enclosed by the surface is $Q - Q_p$. From Gauss's law,

$$\oint \vec{E}\cdot d\vec{S} = \frac{Q - Q_p}{\varepsilon_0} \quad \ldots \text{(i)}$$

$$= \frac{1}{\varepsilon_0}\left[Q - Q\left(1 - \frac{1}{K}\right)\right] = \frac{Q}{\varepsilon_0 K}$$

or, $$\oint K\vec{E}\cdot d\vec{S} = \frac{Q_{free}}{\varepsilon_0}. \quad \ldots \text{(31.14)}$$

$Q_{free}$ is used in place of $Q$ to emphasise that it is the free charge given to the plates and does not include the bound charge appearing due to polarization.

Equation (31.14) is taken as another form of Gauss's law. This form differs from the usual form of Gauss's law in two respects. Firstly, the charge $Q_{free}$ appearing on the right-hand side is not the total charge inside the Gaussian surface. It is the free charge or external charge inside the Gaussian surface. The bound charge $Q_p$ appearing due to polarization of the dielectric is left out. Secondly, an extra factor $K$ appears on the left-hand side. The two differences compensate the effects of each other and the two forms of Gauss's law are identical. Either of the two may be used in any case.

Though we derived this result for a special case of parallel-plate capacitor, it is true in any situation where the dielectric used is homogeneous and isotropic. Let us now write Gauss's law in yet another form valid for any case.

**Displacement Vector**

The field due to the polarization is

$$E_p = \frac{\sigma_p}{\varepsilon_0} = \frac{P}{\varepsilon_0}$$

where $P$ is the polarization (the dipole moment per unit volume). As the direction of $E_p$ is opposite to the polarization vector $\vec{P}$, we write

$$\vec{E}_p = -\frac{\vec{P}}{\varepsilon_0}.$$

Now, $$\vec{E} = \vec{E}_0 + \vec{E}_p$$

or, $$\vec{E} = \vec{E}_0 - \frac{\vec{P}}{\varepsilon_0}$$

or, $$\varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E}_0 \quad \ldots \text{(i)}$$

or, $$\oint (\varepsilon_0 \vec{E} + \vec{P}) \cdot d\vec{S} = \oint \varepsilon_0 \vec{E}_0 \cdot d\vec{S}$$

over any closed surface. As $\vec{E}_0$ is the field produced by the free charge $Q_{free}$, $\oint \varepsilon_0 \vec{E}_0 \cdot d\vec{S} = Q_{free}$ from Gauss's law. Thus,

$$\oint (\varepsilon_0 \vec{E} + \vec{P}) \cdot d\vec{S} = Q_{free}. \quad \ldots \text{(ii)}$$

The quantity $\varepsilon_0 \vec{E} + \vec{P}$ is known as the *electric displacement vector* $\vec{D}$. Equation (ii) above may be written in terms of $\vec{D}$ as

$$\oint \vec{D} \cdot d\vec{S} = Q_{free} \quad \ldots (31.15)$$

which is another form of Gauss's law.

If there is no polarization, $\vec{D} = \varepsilon_0 \vec{E}$ and $Q_{free}$ is equal to the total charge inside the Gaussian surface. Equation (31.15) then reduces to the usual form of Gauss's law.

In case of homogeneous and isotropic dielectrics, $\vec{E}_0 = K\vec{E}$ so that equation (i) above gives $\vec{D} = \varepsilon_0 K\vec{E}$ and equation (31.15) reduces to (31.14).

## 31.9 ELECTRIC FIELD DUE TO A POINT CHARGE *q* PLACED IN AN INFINITE DIELECTRIC

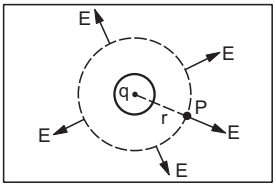



Figure 31.20

Suppose, a point charge $q$ is placed inside an infinite dielectric and we wish to calculate the electric field at a point $P$ at a distance $r$ from the charge $q$ (figure 31.20). We draw a spherical surface through $P$ with the centre at $q$. From Gauss's law,

$$\oint K \vec{E} \cdot d\vec{S} = \frac{q}{\varepsilon_0}$$

or, $$KE\, 4\pi r^2 = \frac{q}{\varepsilon_0}$$

or, $$E = \frac{q}{4\pi\varepsilon_0 K r^2}. \quad \ldots (31.16)$$

The field is radially away from the charge. Note that $q$ is the total *free charge* inside the Gaussian surface.

It should be clear that the field $\frac{q}{4\pi\varepsilon_0 K r^2}$ is due to the free charge $q$ and the polarization charges induced in the dielectric medium. Because of the radially outward field (assuming $q$ to be positive), negative charges shift inward. This produces an induced charge $-q\left(1 - \frac{1}{K}\right)$ on the surface of the cavity in the dielectric in which the charge $q$ is residing. The effective charge is, therefore, $q - q\left(1 - \frac{1}{K}\right) = q/K$ and hence the field is $\frac{q}{4\pi\varepsilon_0 K r^2}$.

## 31.10 ENERGY IN THE ELECTRIC FIELD IN A DIELECTRIC

Consider a parallel-plate capacitor filled with a dielectric of dielectric constant $K$. The energy stored in the capacitor is $U = \frac{1}{2} CV^2$. The energy density in the volume between the plates is

$$u = \frac{U}{Ad} = \frac{\frac{1}{2}\left(\frac{K\varepsilon_0 A}{d}\right)V^2}{Ad} = \frac{1}{2} K\varepsilon_0 \left(\frac{V}{d}\right)^2 = \frac{1}{2} K\varepsilon_0 E^2$$

where $E = V/d$ is the electric field between the plates.

We see that the energy density in dielectrics is greater than that in vacuum for the same electric field. The dipole moments interact with each other so as to give this additional energy.

## 31.11 CORONA DISCHARGE

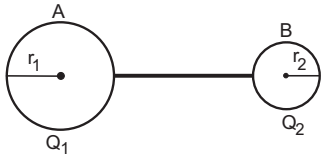



Figure 31.21

Let us consider two conducting spheres $A$ and $B$ connected to each other by a conducting wire. The radius of $A$ is $r_1$ which is larger than the radius $r_2$ of $B$. A charge $Q$ is given to this system. Suppose a part

$Q_1$ resides on the surface of $A$ and the rest $Q_2$ on the surface of $B$. The potential of the sphere $A$ is

$$V_1 = \frac{Q_1}{4\pi\varepsilon_0 r_1}$$

and that of the sphere $B$ is

$$V_2 = \frac{Q_2}{4\pi\varepsilon_0 r_2} .$$

As the two spheres are connected by a conducting wire, their potentials must be the same. Thus,

$$\frac{Q_1}{4\pi\varepsilon_0 r_1} = \frac{Q_2}{4\pi\varepsilon_0 r_2}$$

or, $$\sigma_1 r_1 = \sigma_2 r_2$$

or, $$\frac{\sigma_1}{\sigma_2} = \frac{r_2}{r_1} \quad \ldots (31.17)$$

where $\sigma_1$ and $\sigma_2$ are charge densities on the two spheres. We see that the sphere with smaller radius has larger charge density to maintain the same potential.

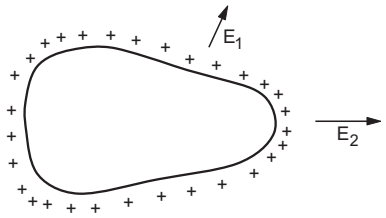



Figure 31.22

Now consider a single conductor with a nonspherical shape. If a charge is given to this conductor (figure 31.22), the charge density will not be uniform on the entire surface. A portion where the surface is more "flat" may be considered as part of a sphere of larger radius. The charge density at such a portion will be smaller from equation (31.17). At portions where the surface is more curved, the charge density will be larger. More precisely, the charge density will be larger where the radius of curvature is small.

The electric field just outside the surface of a conductor is $\sigma/\varepsilon_0$. Thus, the electric field near the portions of small radius of curvature (more curved part) is large as compared to the field near the portions of large radius of curvature (flatter part). If a conductor has a pointed shape like a needle and a charge is given to it, the charge density at the pointed end will be very high. Correspondingly, the electric field near these pointed ends will be very high which may cause dielectric breakdown in air. The charge may jump from the conductor to the air because of increased conductivity of the air. Often this discharge of air is accompanied by a visible glow surrounding the pointed end. This phenomenon is called *corona discharge*.

## 31.12 HIGH-VOLTAGE GENERATOR

In 1929, Robert J van de Graaff designed a machine which could produce large electrostatic potential difference, of the order of $10^7$ volts. This machine, known as *van de Graaff generator,* is now described.

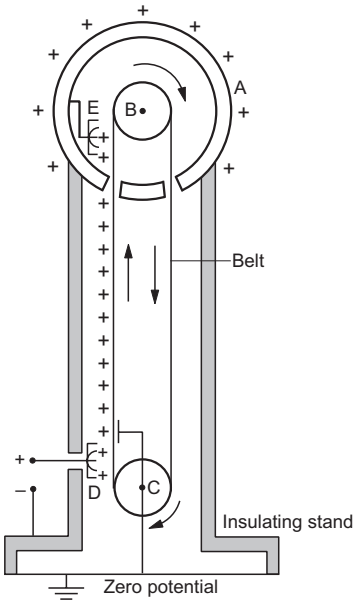



Figure 31.23

A hollow, metallic sphere $A$ is mounted on an insulating stand. A pulley $B$ is mounted at the centre of the sphere and another pulley $C$ is mounted near the bottom. A belt of insulating material (such as silk) goes over the pulleys. The pulley $C$ is continuously driven by an electric motor, or by hand for a smaller machine used for demonstration. The belt, therefore, continuously moves. Two comb-shaped conductors $D$ and $E$, having a number of metallic needles, are mounted near the pulleys. The needles point towards the belt. The lower comb $D$ is maintained at a positive potential of the order of $10^4$ volts by a power supply system. The upper comb $E$ is connected to the metallic sphere $A$.

Because of the high electric field near the needles of $D$, the air becomes conducting (corona discharge). The negative charges in the air move towards the needles and the positive charges towards the belt. This positive charge sticks to the belt. The negative charge neutralises some of the positive charge on the comb $D$. The power supply maintains the positive potential of the needles by supplying more positive charge to it. Effectively, positive charge is transferred from the power supply to the belt. As the belt moves, this positive charge is physically carried upwards. When it reaches near the upper comb $E$, corona discharge takes place and the air becomes conducting. The negative

charges of the air move towards the belt and the positive charges towards the needles of the comb. The negative charges neutralise the positive charge on the belt. The positive charges of the air which have moved to the comb are transferred to the sphere. Effectively, the positive charge on the belt is transferred to the sphere. This positive charge quickly goes to the outer surface of the sphere.

The machine, thus, continuously transfers positive charge to the sphere. The potential of the sphere keeps on increasing. The main limiting factor on the value of this high potential is the radius of the sphere. If the electric field just outside the sphere is sufficient for dielectric breakdown of air, no more charge can be transferred to it. The dielectric strength of air is $3 \times 10^6$ V m$^{-1}$. For a conducting sphere, the electric field just outside the sphere is $E = \frac{Q}{4\pi\varepsilon_0 R^2}$ and the potential of the sphere is $V = \frac{Q}{4\pi\varepsilon_0 R}$. Thus, $V = ER$. To have a field of $3 \times 10^6$ V m$^{-1}$ with a sphere of radius 1 m, its potential should be $3 \times 10^6$ V. Thus, the potential of a sphere of radius 1 m can be raised to $3 \times 10^6$ V by this method. The potential can be increased by enclosing the sphere in a highly evacuated chamber.

## *Worked Out Examples*

**1.** *A parallel-plate capacitor has plates of area* 200 cm$^2$ *and separation between the plates* 1·00 mm. *What potential difference will be developed if a charge of* 1·00 nC (*i.e.,* $1{\cdot}00 \times 10^{-9}$ C) *is given to the capacitor* ? *If the plate separation is now increased to* 2·00 mm, *what will be the new potential difference* ?

***Solution*** : The capacitance of the capacitor is $C = \frac{\varepsilon_0 A}{d}$

$$= 8{\cdot}85 \times 10^{-12}\ \text{Fm}^{-1} \times \frac{200 \times 10^{-4}\ \text{m}^2}{1 \times 10^{-3}\ \text{m}}$$

$$= 0{\cdot}177 \times 10^{-9}\ \text{F} = 0{\cdot}177\ \text{nF}.$$

The potential difference between the plates is

$$V = \frac{Q}{C} = \frac{1\ \text{nC}}{0{\cdot}177\ \text{nF}} = 5{\cdot}65\ \text{volts}.$$

If the separation is increased from 1·00 mm to 2·00 mm, the capacitance is decreased by a factor of 2. If the charge remains the same, the potential difference will increase by a factor of 2. Thus, the new potential difference will be

$$5{\cdot}65\ \text{volts} \times 2 = 11{\cdot}3\ \text{volts}.$$

**2.** *An isolated sphere has a capacitance of* 50 pF. *(a) Calculate its radius. (b) How much charge should be placed on it to raise its potential to* $10^4$ V ?

***Solution*** : (a) The capacitance of an isolated sphere is $C = 4\pi\varepsilon_0 R$. Thus,

$$50 \times 10^{-12}\ \text{F} = \frac{R}{9 \times 10^9\ \text{mF}^{-1}}$$

or, $$R = 50 \times 10^{-12} \times 9 \times 10^9\ \text{m} = 45\ \text{cm}.$$

(b) $$Q = CV$$

$$= 50 \times 10^{-12}\ \text{F} \times 10^4\ \text{V} = 0{\cdot}5\ \mu\text{C}.$$

**3.** *Consider the connections shown in figure (31-W1). (a) Find the capacitance between the points A and B. (b) Find the charges on the three capacitors. (c) Taking the potential at the point B to be zero, find the potential at the point D.*

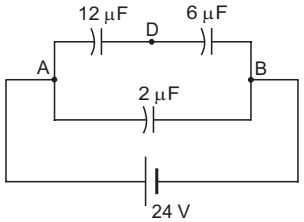



Figure 31-W1

***Solution*** : (a) The 12 μF and 6 μF capacitors are joined in series. The equivalent of these two will have a capacitance given by

$$\frac{1}{C} = \frac{1}{12\ \mu\text{F}} + \frac{1}{6\ \mu\text{F}},$$

or, $$C = 4\ \mu\text{F}.$$

The combination of these two capacitors is joined in parallel with the 2 μF capacitor. Thus, the equivalent capacitance between $A$ and $B$ is

$$4\ \mu\text{F} + 2\ \mu\text{F} = 6\ \mu\text{F}.$$

(b) The charge supplied by the battery is

$$Q = CV = 6\ \mu\text{F} \times 24\ \text{V} = 144\ \mu\text{C}.$$

The potential difference across the 2 μF capacitor is 24 V. The charge on this capacitor is, therefore,

$$2\ \mu\text{F} \times 24\ \text{V} = 48\ \mu\text{C}.$$

The charge on the 12 μF and 6 μF capacitor is, therefore,

$$144\ \mu\text{C} - 48\ \mu\text{C} = 96\ \mu\text{C}.$$

(c) The potential difference across the 6 μF capacitor is

$$\frac{96\ \mu\text{C}}{6\ \mu\text{F}} = 16\ \text{V}.$$

As the potential at the point $B$ is taken to be zero, the potential at the point $D$ is 16 V.

**4.** *If* 100 volts *of potential difference is applied between a and b in the circuit of figure (31-W2a), find the potential difference between c and d.*

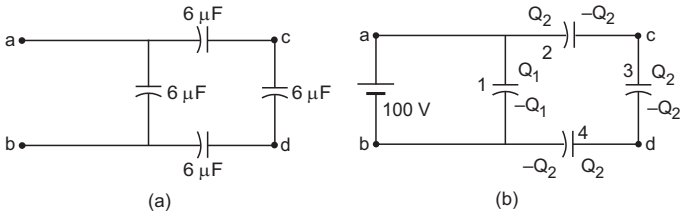



Figure 31-W2

***Solution*** **:** The charge distribution on different plates is shown in figure (31-W2b). Suppose charge $Q_1 + Q_2$ is given by the positive terminal of the battery, out of which $Q_1$ resides on the positive plate of capacitor (1) and $Q_2$ on that of (2). The remaining plates will have charges as shown in the figure.

Take the potential at the point $b$ to be zero. The potential at $a$ will be 100 V. Let the potentials at points $c$ and $d$ be $V_c$ and $V_d$ respectively. Writing the equation $Q = CV$ for the four capacitors, we get,

$$Q_1 = 6\ \mu\text{F} \times 100\ \text{V} = 600\ \mu\text{C} \quad \ldots\ (i)$$

$$Q_2 = 6\ \mu\text{F} \times (100\ \text{V} - V_c) \quad \ldots\ (ii)$$

$$Q_2 = 6\ \mu\text{F} \times (V_c - V_d) \quad \ldots\ (iii)$$

$$Q_2 = 6\ \mu\text{F} \times V_d. \quad \ldots\ (iv)$$

From (ii) and (iii),

$$100\ \text{V} - V_c = V_c - V_d$$

or, $$2\,V_c - V_d = 100\ \text{V} \quad \ldots\ (v)$$

and from (iii) and (iv),

$$V_c - V_d = V_d$$

or, $$V_c = 2\,V_d. \quad \ldots\ (vi)$$

From (v) and (vi),

$$V_d = \frac{100}{3}\ \text{V} \text{ and } V_c = \frac{200}{3}\ \text{V}$$

so that $V_c - V_d = \frac{100}{3}\ \text{V}$.

**5.** *Find the charges on the three capacitors shown in figure (31-W3a).*

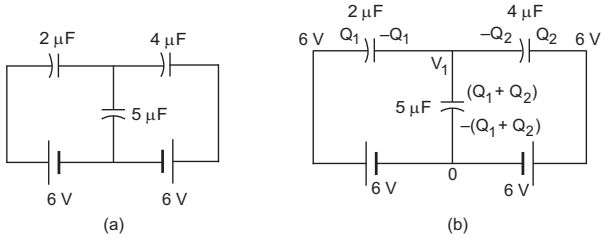



Figure 31-W3

***Solution*** **:** Take the potential at the junction of the batteries to be zero. Let the left battery supply a charge $Q_1$ and the right battery a charge $Q_2$. The charge on the 5 μF capacitor will be $Q_1 + Q_2$. Let the potential at the junction of the capacitors be $V_1$. The charges at different plates and potentials at different points are shown in figure (31-W3b).

Note that the charges on the three plates which are in contact add to zero. It should be so, because, these plates taken together form an isolated system which cannot receive charges from the batteries. Applying the equation $Q = CV$ to the three capacitors, we get,

$$Q_1 = 2\ \mu\text{F}(6\ \text{V} - V_1) \quad \ldots\ (i)$$

$$Q_2 = 4\ \mu\text{F}(6\ \text{V} - V_1) \quad \ldots\ (ii)$$

and $$Q_1 + Q_2 = 5\ \mu\text{F}(V_1 - 0). \quad \ldots\ (iii)$$

From (i) and (ii),

$$2\,Q_1 - Q_2 = 0 \text{ or, } Q_2 = 2\,Q_1.$$

From (ii) and (iii),

$$5\,Q_2 + 4(Q_1 + Q_2) = 20\ \mu\text{F} \times 6\ \text{V}$$

or, $$4\,Q_1 + 9\,Q_2 = 120\ \mu\text{C}$$

or, $$4\,Q_1 + 18\,Q_1 = 120\ \mu\text{C}$$

or, $$Q_1 = 5{\cdot}45\ \mu\text{C} \text{ and } Q_2 = 10{\cdot}9\ \mu\text{C}.$$

Thus, the charges on the 2 μF, 4 μF and 5 μF capacitors are 5·45 μC, 10·9 μC and 16·35 μC respectively.

**6.** *Find the equivalent capacitance of the system shown in figure (31-W4a) between the points a and b.*

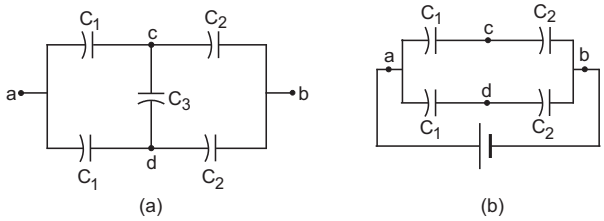



Figure 31-W4

***Solution*** **:** Suppose, the capacitor $C_3$ is removed from the given system and a battery is connected between $a$ and $b$. The remaining system is shown in figure (31-W4b).

From the symmetry of the figure, the potential at $c$ will be the same as the potential at $d$. Thus, if the capacitor $C_3$ is connected between $c$ and $d$, it will have no charge. The charges of all the remaining four capacitors will remain unchanged. Thus, the system of capacitors in figure (31-W4a) is equivalent to that in the figure (31-W4b). The equivalent capacitance of the system in figure (31-W4b) can be calculated by applying the formulae for series and parallel combinations. $C_1$ and $C_2$ are connected in series. Their equivalent capacitance is

$$C = \frac{C_1 C_2}{C_1 + C_2}.$$

Two such capacitors are joined in parallel. So the equivalent capacitance of the given system is

$$2C = \frac{2\,C_1C_2}{C_1 + C_2}\,.$$

**7.** *Find the equivalent capacitance between the point A and B in figure (31-W5a).*

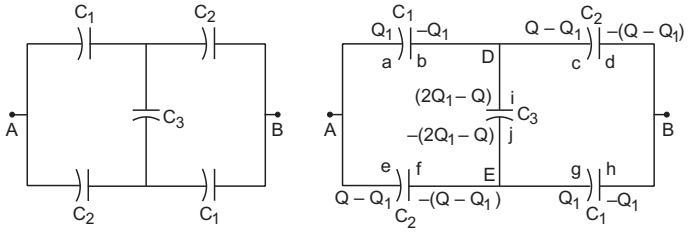



Figure 31-W5

***Solution*** **:** Let us connect a battery between the points *A* and *B*. The charge distribution is shown in figure (31-W5b). Suppose the positive terminal of the battery supplies a charge $+Q$ and the negative terminal a charge $-Q$. The charge $Q$ is divided between plates *a* and *e*. A charge $Q_1$ goes to the plate *a* and the rest $Q - Q_1$ goes to the plate *e*. The charge $-Q$ supplied by the negative terminal is divided between plates *d* and *h*. Using the symmetry of the figure, charge $-Q_1$ goes to the plate *h* and $-(Q - Q_1)$ to the plate *d*. This is because if you look into the circuit from *A* or from *B,* the circuit looks identical. The division of charge at *A* and at *B* should, therefore, be similar. The charges on the other plates may be written easily. The charge on the plate *i* is $2\,Q_1 - Q$ which ensures that the total charge on plates *b, c* and *i* remains zero as these three plates form an isolated system.

We have,

$$V_A - V_B = (V_A - V_D) + (V_D - V_B)$$
$$= \frac{Q_1}{C_1} + \frac{Q - Q_1}{C_2} \qquad \ldots \text{(i)}$$

Also, $$V_A - V_B = (V_A - V_D) + (V_D - V_E) + (V_E - V_B)$$
$$= \frac{Q_1}{C_1} + \frac{2\,Q_1 - Q}{C_3} + \frac{Q_1}{C_1}\,. \qquad \ldots \text{(ii)}$$

We have to eliminate $Q_1$ from these equations to get the equivalent capacitance $Q/(V_A - V_B)$.

The first equation may be written as

$$V_A - V_B = Q_1\left(\frac{1}{C_1} - \frac{1}{C_2}\right) + \frac{Q}{C_2}$$

or, $$\frac{C_1C_2}{C_2 - C_1}(V_A - V_B) = Q_1 + \frac{C_1}{C_2 - C_1}\,Q. \qquad \ldots \text{(iii)}$$

The second equation may be written as

$$V_A - V_B = 2\,Q_1\left(\frac{1}{C_1} + \frac{1}{C_3}\right) - \frac{Q}{C_3}$$

or, $$\frac{C_1C_3}{2(C_1 + C_3)}(V_A - V_B) = Q_1 - \frac{C_1}{2(C_1 + C_3)}\,Q. \qquad \ldots \text{(iv)}$$

Subtracting (iv) from (iii),

$$(V_A - V_B)\left[\frac{C_1C_2}{C_2 - C_1} - \frac{C_1C_3}{2(C_1 + C_3)}\right] = \left[\frac{C_1}{C_2 - C_1} + \frac{C_1}{2(C_1 + C_3)}\right]Q$$

or, $$(V_A - V_B)\,[2\,C_1C_2(C_1 + C_3) - C_1C_3(C_2 - C_1)]$$
$$= C_1[2\,(C_1 + C_3) + (C_2 - C_1)]\,Q$$

or, $$C = \frac{Q}{V_A - V_B} = \frac{2\,C_1C_2 + C_2C_3 + C_3C_1}{C_1 + C_2 + 2\,C_3}.$$

**8.** *Twelve capacitors, each having a capacitance C, are connected to form a cube (figure 31-W6a). Find the equivalent capacitance between the diagonally opposite corners such as A and B.*

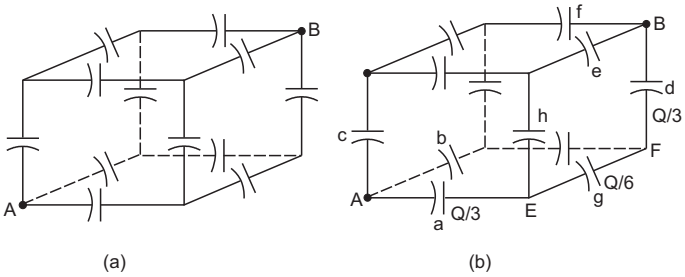



Figure 31-W6

***Solution*** **:** Suppose the points *A* and *B* are connected to a battery. The charges appearing on some of the capacitors are shown in figure (31-W6b). Suppose the positive terminal of the battery supplies a charge $+Q$ through the point *A*. This charge is divided on the three plates connected to *A*. Looking from *A,* the three sides of the cube have identical properties and hence, the charge will be equally distributed on the three plates. Each of the capacitors *a, b* and *c* will receive a charge $Q/3$.

The negative terminal of the battery supplies a charge $-Q$ through the point *B*. This is again divided equally on the three plates connected to *B*. Each of the capacitors *d, e* and *f* gets equal charge $Q/3$.

Now consider the capacitors *g* and *h*. As the three plates connected to the point *E* form an isolated system, their total charge must be zero. The negative plate of the capacitor *a* has a charge $-Q/3$. The two plates of *g* and *h* connected to *E* should have a total charge $Q/3$. By symmetry, these two plates should have equal charges and hence each of these has a charge $Q/6$.

The capacitors *a*, *g* and *d* have charges $Q/3$, $Q/6$ and $Q/3$ respectively.

We have,

$$V_A - V_B = (V_A - V_E) + (V_E - V_F) + (V_F - V_B)$$

$$= \frac{Q/3}{C} + \frac{Q/6}{C} + \frac{Q/3}{C} = \frac{5Q}{6C}$$

or, $$C_{eq} = \frac{Q}{V_A - V_B} = \frac{6}{5} C.$$

**9.** *The negative plate of a parallel plate capacitor is given a charge of* $-20 \times 10^{-8}$ C. *Find the charges appearing on the four surfaces of the capacitor plates.*

***Solution :***

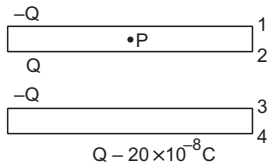



Figure 31-W7

Let the charge appearing on the inner surface of the negative plate be $-Q$. The charge on its outer surface will be $Q - 20 \times 10^{-8}$ C.

The charge on the inner surface of the positive plate will be $+Q$ from Gauss's law and that on the outer surface will be $-Q$ as the positive plate is electrically neutral. The distribution is shown in figure (31-W7).

To obtain the value of $Q$, consider the electric field at a point $P$ inside the upper plate.

Field due to surface (1) $= \frac{Q}{2\varepsilon_0 A}$ upward,

due to surface (2) $= \frac{Q}{2\varepsilon_0 A}$ upward,

due to surface (3) $= \frac{Q}{2\varepsilon_0 A}$ downward

and due to surface (4) $= \frac{Q - 20 \times 10^{-8}\text{ C}}{2\varepsilon_0 A}$ upward.

As $P$ is a point inside the conductor, the field here must be zero. Thus,

$$Q = -Q + 20 \times 10^{-8} \text{ C}$$

or, $$Q = 10 \times 10^{-8} \text{ C}.$$

The charges on the four surfaces may be written immediately from figure (31-W7).

**10.** *Three capacitors of capacitances* 2 μF, 3 μF *and* 6 μF *are connected in series with a* 12 V *battery. All the connecting wires are disconnected, the three positive plates are connected together and the three negative plates are connected together. Find the charges on the three capacitors after the reconnection.*

***Solution :*** The equivalent capacitance of the three capacitors joined in series is given by

$$\frac{1}{C} = \frac{1}{2\ \mu\text{F}} + \frac{1}{3\ \mu\text{F}} + \frac{1}{6\ \mu\text{F}}$$

or, $$C = 1\ \mu\text{F}.$$

The charge supplied by the battery $= 1\ \mu\text{F} \times 12\ \text{V}$
$= 12\ \mu\text{C}.$

As the capacitors are connected in series, 12 μC charge appears on each of the positive plates and –12 μC on each of the negative plates. The charged capacitors are now connected as shown in figure (31-W8).

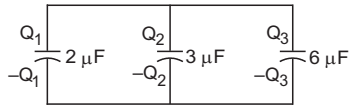



Figure 31-W8

The 36 μC charge on the three positive plates now redistribute as $Q_1$, $Q_2$ and $Q_3$ on the three connected positive plates. Similarly, –36 μC redistributes as $-Q_1$, $-Q_2$ and $-Q_3$. The three positive plates are now at a common potential and the three negative plates are also at a common potential. Let the potential difference across each capacitor be $V$. Then

$$Q_1 = (2\ \mu\text{F})\ V,$$

$$Q_2 = (3\ \mu\text{F})\ V,$$

and $$Q_3 = (6\ \mu\text{F})\ V.$$

Also, $$Q_1 + Q_2 + Q_3 = 36\ \mu\text{C}.$$

Solving these equations,

$$Q_1 = \frac{72}{11}\ \mu\text{C},\ Q_2 = \frac{108}{11}\ \mu\text{C} \text{ and } Q_3 = \frac{216}{11}\ \mu\text{C}.$$

**11.** *The connections shown in figure (31-W9a) are established with the switch S open. How much charge will flow through the switch if it is closed ?*

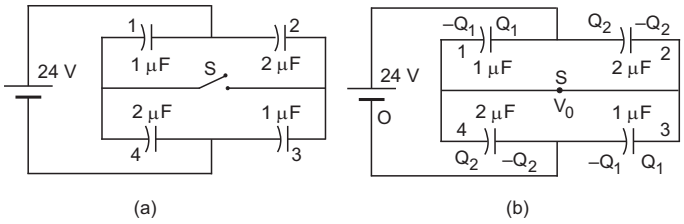



Figure 31-W9

***Solution :*** When the switch is open, capacitors (2) and (3) are in series. Their equivalent capacitance is $\frac{2}{3}$ μF. The charge appearing on each of these capacitors is, therefore, $24\ \text{V} \times \frac{2}{3}\ \mu\text{F} = 16\ \mu\text{C}$.

The equivalent capacitance of (1) and (4), which are also connected in series, is also $\frac{2}{3}$ μF and the charge on each of these capacitors is also 16 μC. The total charge on the two plates of (1) and (4) connected to the switch is, therefore, zero.

The situation when the switch $S$ is closed is shown in figure (31-W9b). Let the charges be distributed as shown in the figure. $Q_1$ and $Q_2$ are arbitrarily chosen for the positive plates of (1) and (2). The same magnitude of charges will appear at the negative plates of (3) and (4).

Take the potential at the negative terminal to be zero and at the switch to be $V_0$.

Writing equations for the capacitors (1), (2), (3) and (4),

$$Q_1 = (24\ \text{V} - V_0) \times 1\ \mu\text{F} \quad \ldots\ (\text{i})$$

$$Q_2 = (24\ \text{V} - V_0) \times 2\ \mu\text{F} \quad \ldots\ (\text{ii})$$

$$Q_1 = V_0 \times 1\ \mu\text{F} \quad \ldots\ (\text{iii})$$

$$Q_2 = V_0 \times 2\ \mu\text{F}. \quad \ldots\ (\text{iv})$$

From (i) and (iii), $V_0 = 12$ V.

Thus, from (iii) and (iv),

$$Q_1 = 12\ \mu\text{C} \text{ and } Q_2 = 24\ \mu\text{C}.$$

The charge on the two plates of (1) and (4) which are connected to the switch is, therefore, $Q_2 - Q_1 = 12\ \mu$C.

When the switch was open, this charge was zero. Thus, 12 μC of charge has passed through the switch after it was closed.

**12.** *Each of the three plates shown in figure (31-W10a) has an area of* 200 $\text{cm}^2$ *on one side and the gap between the adjacent plates is* 0·2 mm. *The emf of the battery is* 20 V. *Find the distribution of charge on various surfaces of the plates. What is the equivalent capacitance of the system between the terminal points* ?

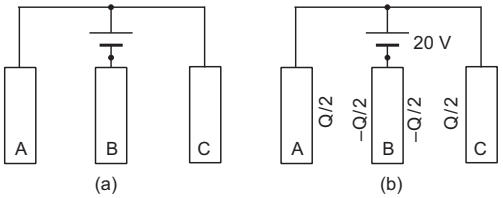



Figure 31-W10

***Solution*** **:** Suppose the negative terminal of the battery gives a charge $-Q$ to the plate $B$. As the situation is symmetric on the two sides of $B$, the two faces of the plate $B$ will share equal charges $-Q/2$ each. From Gauss's law, the facing surfaces will have charges $Q/2$ each. As the positive terminal of the battery has supplied just this much charge ($+Q$) to $A$ and $C$, the outer surfaces of $A$ and $C$ will have no charge. The distribution will be as shown in figure (31-W10b).

The capacitance between the plates $A$ and $B$ is

$$8{\cdot}85 \times 10^{-12}\ \text{F m}^{-1} \times \frac{200 \times 10^{-4}\ \text{m}^2}{2 \times 10^{-4}\ \text{m}}$$

$$= 8{\cdot}85 \times 10^{-10}\ \text{F} = 0{\cdot}885\ \text{nF}.$$

Thus, $$\frac{Q}{2} = 0{\cdot}885\ \text{nF} \times 20\ \text{V} = 17{\cdot}7\ \text{nC}.$$

The distribution of charge on various surfaces may be written from figure (31-W10b).

The equivalent capacitance is

$$\frac{Q}{20\ \text{V}} = 1{\cdot}77\ \text{nF}.$$

**13.** *Find the capacitance of the infinite ladder shown in figure (31-W11).*

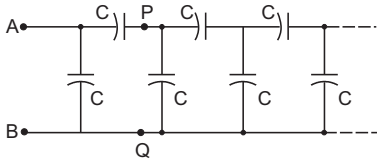



Figure 31-W11

***Solution*** **:** As the ladder is infinitely long, the capacitance of the ladder to the right of the points $P$, $Q$ is the same as that of the ladder to the right of the points $A$, $B$. If the equivalent capacitance of the ladder is $C_1$, the given ladder may be replaced by the connections shown in figure (31-W12).

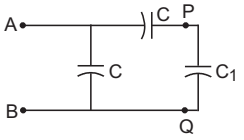



Figure 31-W12

The equivalent capacitance between $A$ and $B$ is easily found to be $C + \frac{CC_1}{C + C_1}$. But being equivalent to the original ladder, the equivalent capacitance is also $C_1$.

Thus, $$C_1 = C + \frac{CC_1}{C + C_1}$$

or, $$C_1 C + C_1^2 = C^2 + 2\,CC_1$$

or, $$C_1^2 - CC_1 - C^2 = 0$$

giving $$C_1 = \frac{C + \sqrt{C^2 + 4\,C^2}}{2} = \frac{1 + \sqrt{5}}{2}\,C.$$

Negative value of $C_1$ is rejected.

**14.** *Find the energy stored in the electric field produced by a metal sphere of radius R containing a charge Q.*

***Solution*** **:** Consider a thin spherical shell of radius $x$ ($> R$) and thickness $dx$ concentric with the given metal sphere. The energy density in the shell is

$$u = \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\varepsilon_0 \left(\frac{Q}{4\pi\varepsilon_0 x^2}\right)^2.$$

The volume of the shell is $4\pi x^2 dx$. The energy contained in the shell is, therefore,

$$dU = \frac{1}{2}\varepsilon_0 \left(\frac{Q}{4\pi\varepsilon_0 x^2}\right)^2 \times 4\pi x^2 dx = \frac{Q^2 dx}{8\pi\varepsilon_0 x^2}.$$

The energy contained in the whole space outside the sphere is

$$U = \int_R^\infty \frac{Q^2 dx}{8\pi\varepsilon_0 x^2} = \frac{Q^2}{8\pi\varepsilon_0 R}.$$

As the field inside the sphere is zero, this is also the total energy stored in the field.

**Alternative Method**

Considering a concentric spherical shell at infinity, we have a spherical capacitor. The capacitance is $C = 4\pi\varepsilon_0 R$. The energy stored in this capacitor is the energy stored in the entire electric field. This energy is

$$U = \frac{Q^2}{2C} = \frac{Q^2}{8\pi\varepsilon_0 R}.$$

**15.** *A capacitor of capacitance C is charged by connecting it to a battery of emf $\mathcal{E}$. The capacitor is now disconnected and reconnected to the battery with the polarity reversed. Calculate the heat developed in the connecting wires.*

***Solution*** **:** When the capacitor is connected to the battery, a charge $Q = C\mathcal{E}$ appears on one plate and $-Q$ on the other. When the polarity is reversed, a charge $-Q$ appears on the first plate and $+Q$ on the second. A charge $2Q$, therefore, passes through the battery from the negative to the positive terminal. The battery does a work

$$W = (2Q)\mathcal{E} = 2C\mathcal{E}^2$$

in the process. The energy stored in the capacitor is the same in the two cases. Thus, the work done by the battery appears as heat in the connecting wires. The heat produced is, therefore, $2C\mathcal{E}^2$.

**16.** *An uncharged capacitor is connected to a battery. Show that half the energy supplied by the battery is lost as heat while charging the capacitor.*

***Solution*** **:** Suppose the capacitance of the capacitor is $C$ and the emf of the battery is $V$. The charge given to the capacitor is $Q = CV$. The work done by the battery is

$$W = QV.$$

The battery supplies this energy. The energy stored in the capacitor is

$$U = \frac{1}{2}CV^2 = \frac{1}{2}QV.$$

The remaining energy $QV - \frac{1}{2}QV = \frac{1}{2}QV$ is lost as heat. Thus, half the energy supplied by the battery is lost as heat.

**17.** *A parallel-plate capacitor having plate area* 100 $\text{cm}^2$ *and separation* 1·0 mm *holds a charge of* 0·12 μC *when connected to a* 120 V *battery. Find the dielectric constant of the material filling the gap.*

***Solution*** **:**

The capacitance of the capacitor is

$$\frac{0{\cdot}12\ \mu\text{C}}{120\ \text{V}} = 1{\cdot}0 \times 10^{-9}\ \text{F}.$$

If $K$ be the dielectric constant, the capacitance is also given by $\frac{K\varepsilon_0 A}{d}$. Thus,

$$\frac{K \times 8{\cdot}85 \times 10^{-12}\ \text{F m}^{-1} \times 100 \times 10^{-4}\ \text{m}^2}{1{\cdot}0 \times 10^{-3}\ \text{m}} = 1{\cdot}0 \times 10^{-9}\ \text{F}$$

or, $K = 11{\cdot}3.$

**18.** *A parallel-plate capacitor is formed by two plates, each of area* 100 $\text{cm}^2$, *separated by a distance of* 1 mm. A *dielectric of dielectric constant* 5·0 *and dielectric strength* $1{\cdot}9 \times 10^7\ \text{V m}^{-1}$ *is filled between the plates. Find the maximum charge that can be stored on the capacitor without causing any dielectric breakdown.*

***Solution*** **:** If $Q$ be the charge on the capacitor, the surface charge density is $\sigma = Q/A$ and the electric field is $\frac{Q}{KA\varepsilon_0}$. This should not exceed the dielectric strength $1{\cdot}9 \times 10^7\ \text{V m}^{-1}$. Thus, the maximum charge is given by

$$\frac{Q}{KA\varepsilon_0} = 1{\cdot}9 \times 10^7\ \text{V m}^{-1}$$

or, $$Q = KA\varepsilon_0 \times 1{\cdot}9 \times 10^7\ \text{V m}^{-1}$$

$$= (5{\cdot}0)\ (10^{-2}\ \text{m}^2)\ (8{\cdot}85 \times 10^{-12}\ \text{F m}^{-1}) \times (1{\cdot}9 \times 10^7\ \text{V m}^{-1})$$

$$= 8{\cdot}4 \times 10^{-6}\ \text{C}.$$

**19.** *The space between the plates of a parallel-plate capacitor of capacitance C is filled with three dielectric slabs of identical size as shown in figure (31-W13). If the dielectric constants of the three slabs are $K_1$, $K_2$ and $K_3$, find the new capacitance.*

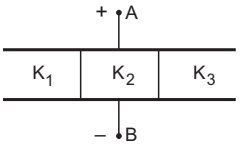



Figure 31-W13

***Solution*** **:** Consider each one third of the assembly as a separate capacitor. The three positive plates are connected together at point $A$ and the three negative plates are connected together at point $B$. Thus, the three capacitors are joined in parallel. As the plate area is one third of the original for each part, the capacitances of these parts will be $K_1C/3$, $K_2C/3$ and $K_3C/3$. The equivalent capacitance is, therefore,

$$C_{eq} = (K_1 + K_2 + K_3)\frac{C}{3}.$$

**20.** *Figure (31-W14a) shows a parallel-plate capacitor having square plates of edge a and plate-separation d. The gap between the plates is filled with a dielectric of dielectric constant K which varies parallel to an edge as*

$$K = K_0 + \alpha x$$

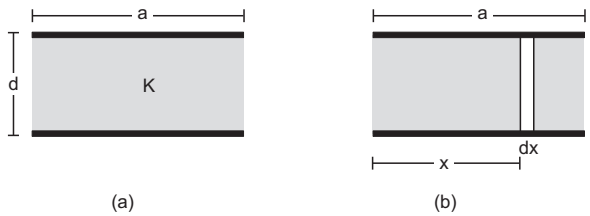



Figure 31-W14

*where K and α are constants and x is the distance from the left end. Calculate the capacitance.*

***Solution*** **:**

Consider a small strip of width $dx$ at a separation $x$ from the left end (figure 31-W14b). This strip forms a small capacitor of plate area $adx$. Its capacitance is

$$dC = \frac{(K_0 + \alpha x)\varepsilon_0 a\, dx}{d}.$$

The given capacitor may be divided into such strips with $x$ varying from 0 to $a$. All these strips are connected in parallel. The capacitance of the given capacitor is,

$$C = \int_0^a \frac{(K_0 + \alpha x)\varepsilon_0 a dx}{d}$$

$$= \frac{\varepsilon_0 a^2}{d}\left(K_0 + \frac{a\alpha}{2}\right).$$

**21.** *A parallel-plate capacitor of capacitance* 100 μF *is connected to a power supply of* 200 V. *A dielectric slab of dielectric constant* 5 *is now inserted into the gap between the plates. (a) Find the extra charge flown through the power supply and the work done by the supply. (b) Find the change in the electrostatic energy of the electric field in the capacitor.*

***Solution*** **:** (a) The original capacitance was 100 μF. The charge on the capacitor before the insertion of the dielectric was, therefore,

$$Q_1 = 100\ \mu\text{F} \times 200\ \text{V} = 20\ \text{mC}.$$

After the dielectric slab is introduced, the capacitance is increased to 500 μF. The new charge on the capacitor is, therefore, $500\ \mu\text{F} \times 200\ \text{V} = 100\ \text{mC}$. The charge flown through the power supply is, therefore, $100\ \text{mC} - 20\ \text{mC} = 80\ \text{mC}$. The work done by the power supply is $200\ \text{V} \times 80\ \text{mC} = 16\ \text{J}$.

(b) The electrostatic field energy of the capacitor without the dielectric slab is

$$U_1 = \frac{1}{2}CV^2$$

$$= \frac{1}{2} \times (100\ \mu\text{F}) \times (200\ \text{V})^2 = 2\ \text{J}$$

and that after the slab is inserted is

$$U_2 = \frac{1}{2} \times (500\ \mu\text{F}) \times (200\ \text{V})^2 = 10\ \text{J}.$$

Thus, the energy is increased by 8 J.

**22.** *Figure (31-W15) shows a parallel-plate capacitor with plates of width b and length l. The separation between the plates is d. The plates are rigidly clamped and connected to a battery of emf V. A dielectric slab of thickness d and dielectric constant K is slowly inserted between the plates. (a) Calculate the energy of the system when a length x of the slab is introduced into the capacitor. (b) What force should be applied on the slab to ensure that it goes slowly into the capacitor ? Neglect any effect of friction or gravity.*

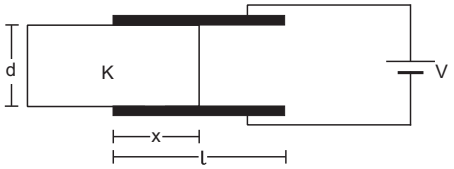



Figure 31-W15

***Solution*** **:** (a) The plate area of the part with the dielectric is $bx$. Its capacitance is

$$C_1 = \frac{K\varepsilon_0 bx}{d}.$$

Similarly, the capacitance of the part without the dielectric is

$$C_2 = \frac{\varepsilon_0 b(l - x)}{d}.$$

These two parts are connected in parallel. The capacitance of the system is, therefore,

$$C = C_1 + C_2$$

$$= \frac{\varepsilon_0 b}{d}[l + x(K - 1)]. \quad \ldots \text{(i)}$$

The energy of the capacitor is

$$U = \frac{1}{2}CV^2 = \frac{\varepsilon_0 bV^2}{2d}[l + x(K - 1)].$$

(b) Suppose, the electric field attracts the dielectric slab with a force $F$. An external force of equal magnitude $F$ should be applied in opposite direction so that the plate moves slowly (no acceleration).

Consider the part of motion in which the dielectric moves a distance $dx$ further inside the capacitor. The capacitance increases to $C + dC$. As the potential difference remains constant at $V$, the battery has to supply a further charge

$$dQ = (dC)V$$

to the capacitor. The work done by the battery is, therefore,

$$dW_b = V\, dQ = (dC)V^2.$$

The external force $F$ does a work

$$dW_e = (-F\, dx)$$

during the displacement. The total work done on the capacitor is

$$dW_b + dW_e = (dC)V^2 - Fdx.$$

This should be equal to the increase $dU$ in the stored energy. Thus,

$$\frac{1}{2}(dC)V^2 = (dC)V^2 - Fdx$$

or,

$$F = \frac{1}{2}V^2\frac{dC}{dx}.$$

Using equation (i),

$$F = \frac{\varepsilon_0 bV^2(K-1)}{2d}.$$

Thus, the electric field attracts the dielectric into the capacitor with a force $\frac{\varepsilon_0 bV^2(K-1)}{2d}$ and this much force should be applied in the opposite direction.

**23.** *A parallel-plate capacitor is placed in such a way that its plates are horizontal and the lower plate is dipped into a liquid of dielectric constant K and density* ρ. *Each plate has an area A. The plates are now connected to a battery which supplies a positive charge of magnitude Q to the upper plate. Find the rise in the level of the liquid in the space between the plates.*

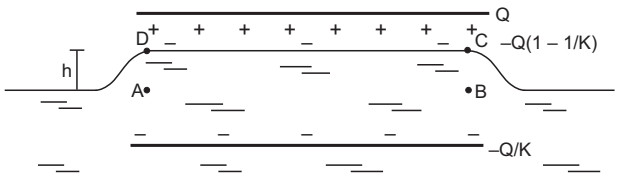



Figure 31-W16

***Solution :***

The situation is shown in figure (31-W16). A charge $-Q\left(1-\frac{1}{K}\right)$ is induced on the upper surface of the liquid and $Q\left(1-\frac{1}{K}\right)$ at the surface in contact with the lower plate. The net charge on the lower plate is $-Q+Q\left(1-\frac{1}{K}\right)=-\frac{Q}{K}$. Consider the equilibrium of the liquid in the volume *ABCD*. The forces on this liquid are

(a) the force due to the electric field at *CD*,
(b) the weight of the liquid,
(c) the force due to atmospheric pressure and
(d) the force due to the pressure of the liquid below *AB*.

As *AB* is in the same horizontal level as the outside surface, the pressure here is the same as the atmospheric pressure. The forces in (c) and (d), therefore, balance each other. Hence, for equilibrium, the forces in (a) and (b) should balance each other.

The electric field at *CD* due to the charge *Q* is

$$E_1 = \frac{Q}{2A\varepsilon_0}$$

in the downward direction. The field at *CD* due to the charge $-Q/K$ is

$$E_2 = \frac{Q}{2A\varepsilon_0 K}$$

also in the downward direction. The net field at *CD* is

$$E_1 + E_2 = \frac{(K+1)Q}{2A\varepsilon_0 K}.$$

The force on the charge $-Q\left(1-\frac{1}{K}\right)$ at *CD* is

$$F = Q\left(1-\frac{1}{K}\right)\frac{(K+1)Q}{2A\varepsilon_0 K}$$

$$= \frac{(K^2-1)Q^2}{2A\varepsilon_0 K^2}$$

in the upward direction. The weight of the liquid considered is $hA\rho g$. Thus,

$$hA\rho g = \frac{(K^2-1)Q^2}{2A\varepsilon_0 K^2}$$

or,
$$h = \frac{(K^2-1)Q^2}{2A^2K^2\varepsilon_0\rho g}.$$

□

## QUESTIONS FOR SHORT ANSWER

**1.** Suppose a charge $+Q_1$ is given to the positive plate and a charge $-Q_2$ to the negative plate of a capacitor. What is the "charge on the capacitor" ?

**2.** As $C=\left(\frac{1}{V}\right)Q$, can you say that the capacitance *C* is proportional to the charge *Q* ?

**3.** A hollow metal sphere and a solid metal sphere of equal radii are given equal charges. Which of the two will have higher potential ?

**4.** The plates of a parallel-plate capacitor are given equal positive charges. What will be the potential difference between the plates ? What will be the charges on the facing surfaces and on the outer surfaces ?

**5.** A capacitor has capacitance *C*. Is this information sufficient to know what maximum charge the capacitor can contain ? If yes, what is this charge ? If no, what other information is needed ?

**6.** The dielectric constant decreases if the temperature is increased. Explain this in terms of polarization of the material.

**7.** When a dielectric slab is gradually inserted between the plates of an isolated parallel-plate capacitor, the energy of the system decreases. What can you conclude about the force on the slab exerted by the electric field ?

## OBJECTIVE I

**1.** A capacitor of capacitance $C$ is charged to a potential $V$. The flux of the electric field through a closed surface enclosing the capacitor is
(a) $\frac{CV}{\varepsilon_0}$ (b) $\frac{2\,CV}{\varepsilon_0}$ (c) $\frac{CV}{2\varepsilon_0}$ (d) zero.

**2.** Two capacitors each having capacitance $C$ and breakdown voltage $V$ are joined in series. The capacitance and the breakdown voltage of the combination will be
(a) $2\,C$ and $2\,V$ (b) $C/2$ and $V/2$
(c) $2\,C$ and $V/2$ (d) $C/2$ and $2\,V$.

**3.** If the capacitors in the previous question are joined in parallel, the capacitance and the breakdown voltage of the combination will be
(a) $2\,C$ and $2\,V$ (b) $C$ and $2\,V$
(c) $2\,C$ and $V$ (d) $C$ and $V$.

**4.** The equivalent capacitance of the combination shown in figure (31-Q1) is
(a) $C$ (b) $2\,C$ (c) $C/2$ (d) none of these.

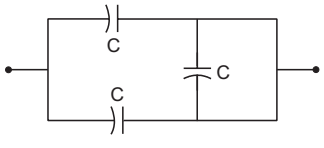



Figure 31-Q1

**5.** A dielectric slab is inserted between the plates of an isolated capacitor. The force between the plates will
(a) increase (b) decrease
(c) remain unchanged (d) become zero.

**6.** The energy density in the electric field created by a point charge falls off with the distance from the point charge as
(a) $\frac{1}{r}$ (b) $\frac{1}{r^2}$ (c) $\frac{1}{r^3}$ (d) $\frac{1}{r^4}$.

**7.** A parallel-plate capacitor has plates of unequal area. The larger plate is connected to the positive terminal of the battery and the smaller plate to its negative terminal. Let $Q_+$ and $Q_-$ be the charges appearing on the positive and negative plates respectively.
(a) $Q_+ > Q_-$ (b) $Q_+ = Q_-$ (c) $Q_+ < Q_-$
(d) The information is not sufficient to decide the relation between $Q_+$ and $Q_-$.

**8.** A thin metal plate $P$ is inserted between the plates of a parallel-plate capacitor of capacitance $C$ in such a way that its edges touch the two plates (figure 31-Q2). The capacitance now becomes
(a) $C/2$ (b) $2\,C$ (c) 0 (d) indeterminate.

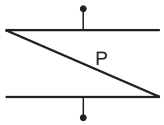



Figure 31-Q2

**9.** Figure (31-Q3) shows two capacitors connected in series and joined to a battery. The graph shows the variation in potential as one moves from left to right on the branch containing the capacitors.
(a) $C_1 > C_2$ (b) $C_1 = C_2$ (c) $C_1 < C_2$
(d) The information is not sufficient to decide the relation between $C_1$ and $C_2$.

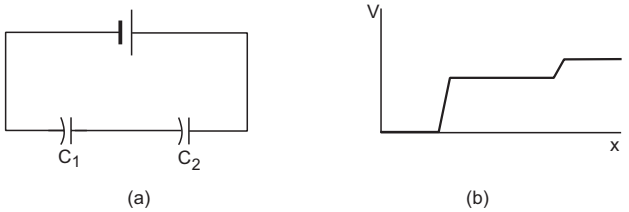



Figure 31-Q3

**10.** Two metal plates having charges $Q, -Q$ face each other at some separation and are dipped into an oil tank. If the oil is pumped out, the electric field between the plates will
(a) increase (b) decrease
(c) remain the same (d) become zero.

**11.** Two metal spheres of capacitances $C_1$ and $C_2$ carry some charges. They are put in contact and then separated. The final charges $Q_1$ and $Q_2$ on them will satisfy
(a) $\frac{Q_1}{Q_2} < \frac{C_1}{C_2}$ (b) $\frac{Q_1}{Q_2} = \frac{C_1}{C_2}$ (c) $\frac{Q_1}{Q_2} > \frac{C_1}{C_2}$ (d) $\frac{Q_1}{Q_2} = \frac{C_2}{C_1}$.

**12.** Three capacitors of capacitances 6 μF each are available. The minimum and maximum capacitances, which may be obtained are
(a) 6 μF, 18 μF (b) 3 μF, 12 μF
(c) 2 μF, 12 μF (d) 2 μF, 18 μF.

## OBJECTIVE II

**1.** The capacitance of a capacitor does not depend on
(a) the shape of the plates
(b) the size of the plates
(c) the charges on the plates
(d) the separation between the plates.

**2.** A dielectric slab is inserted between the plates of an isolated charged capacitor. Which of the following quantities will remain the same ?
(a) The electric field in the capacitor
(b) The charge on the capacitor

(c) The potential difference between the plates
(d) The stored energy in the capacitor

**3.** A dielectric slab is inserted between the plates of a capacitor. The charge on the capacitor is $Q$ and the magnitude of the induced charge on each surface of the dielectric is $Q'$.
(a) $Q'$ may be larger than $Q$.
(b) $Q'$ must be larger than $Q$.
(c) $Q'$ must be equal to $Q$.
(d) $Q'$ must be smaller than $Q$.

**4.** Each plate of a parallel plate capacitor has a charge $q$ on it. The capacitor is now connected to a battery. Now,
(a) the facing surfaces of the capacitor have equal and opposite charges
(b) the two plates of the capacitor have equal and opposite charges
(c) the battery supplies equal and opposite charges to the two plates
(d) the outer surfaces of the plates have equal charges

**5.** The separation between the plates of a charged parallel-plate capacitor is increased. Which of the following quantities will change ?
(a) Charge on the capacitor
(b) Potential difference across the capacitor
(c) Energy of the capacitor
(d) Energy density between the plates

**6.** A parallel-plate capacitor is connected to a battery. A metal sheet of negligible thickness is placed between the plates. The sheet remains parallel to the plates of the capacitor.
(a) The battery will supply more charge.
(b) The capacitance will increase.
(c) The potential difference between the plates will increase.
(d) Equal and opposite charges will appear on the two faces of the metal plate.

**7.** Following operations can be performed on a capacitor:
$X$ – connect the capacitor to a battery of emf $\mathcal{E}$.
$Y$ – disconnect the battery.
$Z$ – reconnect the battery with polarity reversed.
$W$ – insert a dielectric slab in the capacitor.
(a) In $XYZ$ (perform $X$, then $Y$, then $Z$) the stored electric energy remains unchanged and no thermal energy is developed.
(b) The charge appearing on the capacitor is greater after the action $XWY$ than after the action $XYW$.
(c) The electric energy stored in the capacitor is greater after the action $WXY$ than after the action $XYW$.
(d) The electric field in the capacitor after the action $XW$ is the same as that after $WX$.

## EXERCISES

**1.** When $1{\cdot}0 \times 10^{12}$ electrons are transferred from one conductor to another, a potential difference of 10 V appears between the conductors. Calculate the capacitance of the two-conductor system.

**2.** The plates of a parallel-plate capacitor are made of circular discs of radii 5·0 cm each. If the separation between the plates is 1·0 mm, what is the capacitance ?

**3.** Suppose, one wishes to construct a 1·0 farad capacitor using circular discs. If the separation between the discs be kept at 1·0 mm, what would be the radius of the discs ?

**4.** A parallel-plate capacitor having plate area 25 $cm^2$ and separation 1·00 mm is connected to a battery of 6·0 V. Calculate the charge flown through the battery. How much work has been done by the battery during the process ?

**5.** A parallel-plate capacitor has plate area 25·0 $cm^2$ and a separation of 2·00 mm between the plates. The capacitor is connected to a battery of 12·0 V. (a) Find the charge on the capacitor. (b) The plate separation is decreased to 1·00 mm. Find the extra charge given by the battery to the positive plate.

**6.** Find the charges on the three capacitors connected to a battery as shown in figure (31-E1). Take $C_1 = 2{\cdot}0\ \mu F$, $C_2 = 4{\cdot}0\ \mu F$, $C_3 = 6{\cdot}0\ \mu F$ and $V = 12$ volts.

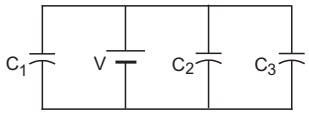



Figure 31-E1

**7.** Three capacitors having capacitances 20 μF, 30 μF and 40 μF are connected in series with a 12 V battery. Find the charge on each of the capacitors. How much work has been done by the battery in charging the capacitors ?

**8.** Find the charge appearing on each of the three capacitors shown in figure (31-E2).

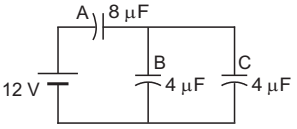



Figure 31-E2

**9.** Take $C_1 = 4{\cdot}0\ \mu F$ and $C_2 = 6{\cdot}0\ \mu F$ in figure (31-E3). Calculate the equivalent capacitance of the combination between the points indicated.

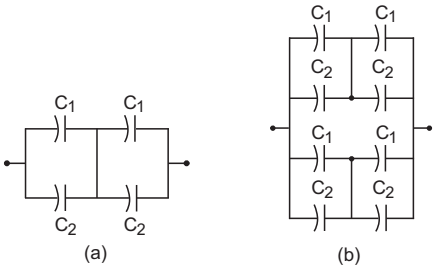



Figure 31-E3

**10.** Find the charge supplied by the battery in the arrangement shown in figure (31-E4).

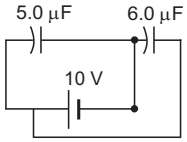



Figure 31-E4

**11.** The outer cylinders of two cylindrical capacitors of capacitance 2·2 μF each, are kept in contact and the inner cylinders are connected through a wire. A battery of emf 10 V is connected as shown in figure (31-E5). Find the total charge supplied by the battery to the inner cylinders.

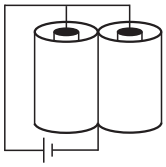


Figure 31-E5

**12.** Two conducting spheres of radii $R_1$ and $R_2$ are kept widely separated from each other. What are their individual capacitances ? If the spheres are connected by a metal wire, what will be the capacitance of the combination ? Think in terms of series–parallel connections.

**13.** Each of the capacitors shown in figure (31-E6) has a capacitance of 2 μF. Find the equivalent capacitance of the assembly between the points $A$ and $B$. Suppose, a battery of emf 60 volts is connected between $A$ and $B$. Find the potential difference appearing on the individual capacitors.

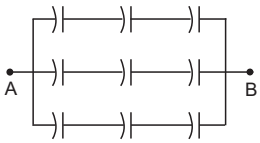



Figure 31-E6

**14.** It is required to construct a 10 μF capacitor which can be connected across a 200 V battery. Capacitors of capacitance 10 μF are available but they can withstand only 50 V. Design a combination which can yield the desired result.

**15.** Take the potential of the point $B$ in figure (31-E7) to be zero. (a) Find the potentials at the points $C$ and $D$. (b) If a capacitor is connected between $C$ and $D$, what charge will appear on this capacitor ?

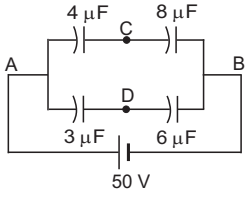



Figure 31-E7

**16.** Find the equivalent capacitance of the system shown in figure (31-E8) between the points $a$ and $b$.

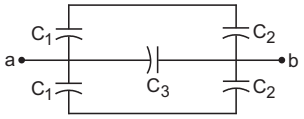



Figure 31-E8

**17.** A capacitor is made of a flat plate of area $A$ and a second plate having a stair-like structure as shown in figure (31-E9). The width of each stair is $a$ and the height is $b$. Find the capacitance of the assembly.

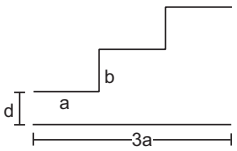



Figure 31-E9

**18.** A cylindrical capacitor is constructed using two coaxial cylinders of the same length 10 cm and of radii 2 mm and 4 mm. (a) Calculate the capacitance. (b) Another capacitor of the same length is constructed with cylinders of radii 4 mm and 8 mm. Calculate the capacitance.

**19.** A 100 pF capacitor is charged to a potential difference of 24 V. It is connected to an uncharged capacitor of capacitance 20 pF. What will be the new potential difference across the 100 pF capacitor ?

**20.** Each capacitor shown in figure (31-E10) has a capacitance of 5·0 μF. The emf of the battery is 50 V. How much charge will flow through $AB$ if the switch $S$ is closed ?

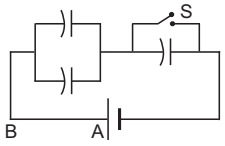



Figure 31-E10

**21.** The particle $P$ shown in figure (31-E11) has a mass of 10 mg and a charge of $-0{\cdot}01$ μC. Each plate has a surface area 100 $cm^2$ on one side. What potential difference $V$ should be applied to the combination to hold the particle $P$ in equilibrium ?

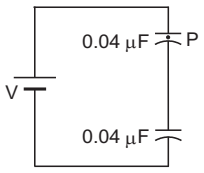



Figure 31-E11

**22.** Both the capacitors shown in figure (31-E12) are made of square plates of edge $a$. The separations between the plates of the capacitors are $d_1$ and $d_2$ as shown in the

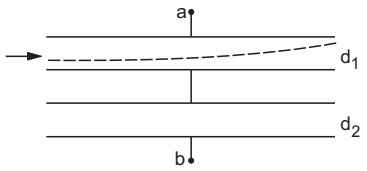



Figure 31-E12

figure. A potential difference $V$ is applied between the points $a$ and $b$. An electron is projected between the plates of the upper capacitor along the central line. With what minimum speed should the electron be projected so that it does not collide with any plate ? Consider only the electric forces.

**23.** The plates of a capacitor are 2·00 cm apart. An electron–proton pair is released somewhere in the gap between the plates and it is found that the proton reaches the negative plate at the same time as the electron reaches the positive plate. At what distance from the negative plate was the pair released ?

**24.** Convince yourself that parts (a), (b) and (c) of figure (31-E13) are identical. Find the capacitance between the points $A$ and $B$ of the assembly.

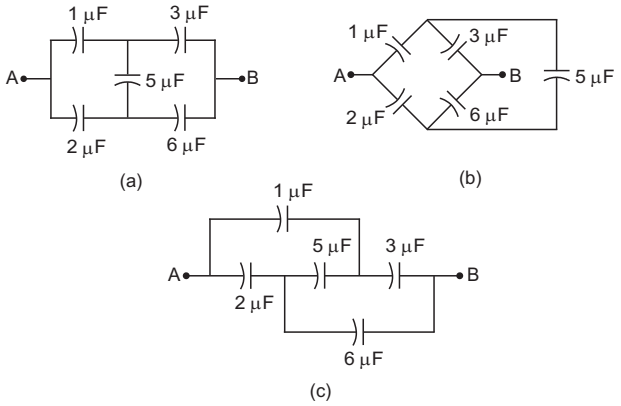



Figure 31-E13

**25.** Find the potential difference $V_a - V_b$ between the points $a$ and $b$ shown in each part of the figure (31-E14).

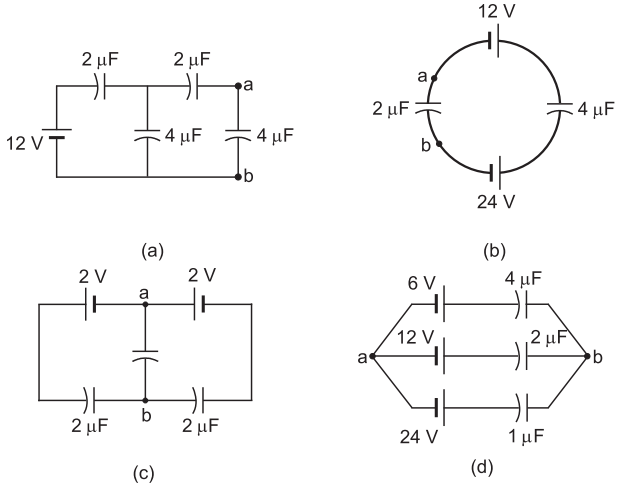



Figure 31-E14

**26.** Find the equivalent capacitances of the combinations shown in figure (31-E15) between the indicated points.

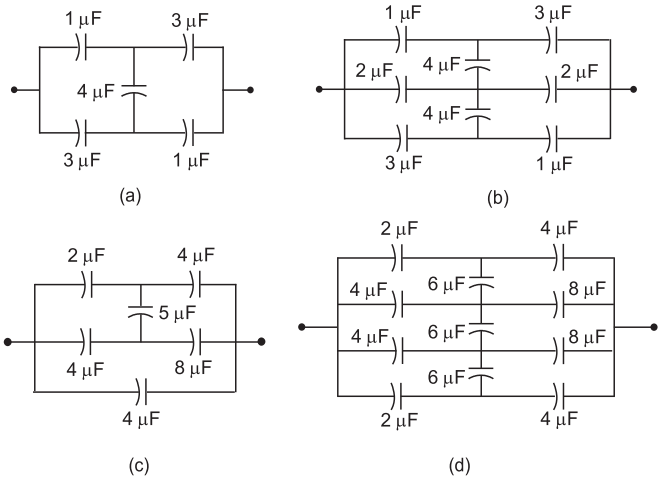



Figure 31-E15

**27.** Find the capacitance of the combination shown in figure (31-E16) between $A$ and $B$.

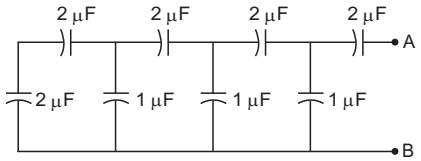



Figure 31-E16

**28.** Find the equivalent capacitance of the infinite ladder shown in figure (31-E17) between the points $A$ and $B$.

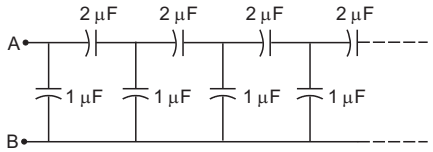



Figure 31-E17

**29.** A finite ladder is constructed by connecting several sections of 2 μF, 4 μF capacitor combinations as shown in figure (31-E18). It is terminated by a capacitor of capacitance $C$. What value should be chosen for $C$, such that the equivalent capacitance of the ladder between the points $A$ and $B$ becomes independent of the number of sections in between ?

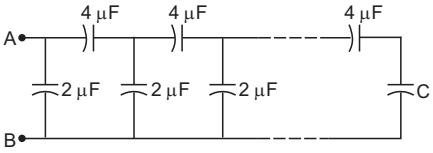



Figure 31-E18

**30.** A charge of + 2·0 × $10^{-8}$ C is placed on the positive plate and a charge of – 1·0 × $10^{-8}$ C on the negative plate of a parallel-plate capacitor of capacitance 1·2 × $10^{-3}$ μF.

Calculate the potential difference developed between the plates.

**31.** A charge of 20 μC is placed on the positive plate of an isolated parallel-plate capacitor of capacitance 10 μF. Calculate the potential difference developed between the plates.

**32.** A charge of 1 μC is given to one plate of a parallel-plate capacitor of capacitance 0·1 μF and a charge of 2 μC is given to the other plate. Find the potential difference developed between the plates.

**33.** Each of the plates shown in figure (31-E19) has surface area $(96/\varepsilon_0) \times 10^{-12}$ F–m on one side and the separation between the consecutive plates is 4·0 mm. The emf of the battery connected is 10 volts. Find the magnitude of the charge supplied by the battery to each of the plates connected to it.

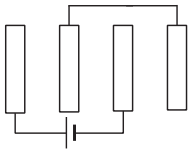


Figure 31-E19

**34.** The capacitance between the adjacent plates shown in figure (31-E20) is 50 nF. A charge of 1·0 μC is placed on the middle plate. (a) What will be the charge on the outer surface of the upper plate ? (b) Find the potential difference developed between the upper and the middle plates.

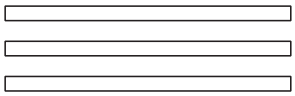


Figure 31-E20

**35.** Consider the situation of the previous problem. If 1·0 μC is placed on the upper plate instead of the middle, what will be the potential difference between (a) the upper and the middle plates and (b) the middle and the lower plates ?

**36.** Two capacitors of capacitances 20·0 pF and 50·0 pF are connected in series with a 6·00 V battery. Find (a) the potential difference across each capacitor and (b) the energy stored in each capacitor.

**37.** Two capacitors of capacitances 4·0 μF and 6·0 μF are connected in series with a battery of 20 V. Find the energy supplied by the battery.

**38.** Each capacitor in figure (31-E21) has a capacitance of 10 μF. The emf of the battery is 100 V. Find the energy stored in each of the four capacitors.

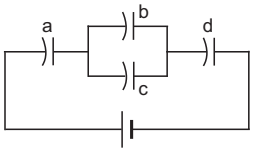



Figure 31-E21

**39.** A capacitor with stored energy 4·0 J is connected with an identical capacitor with no electric field in between. Find the total energy stored in the two capacitors.

**40.** A capacitor of capacitance 2·0 μF is charged to a potential difference of 12 V. It is then connected to an uncharged capacitor of capacitance 4·0 μF as shown in figure (31-E22). Find (a) the charge on each of the two capacitors after the connection, (b) the electrostatic energy stored in each of the two capacitors and (c) the heat produced during the charge transfer from one capacitor to the other.

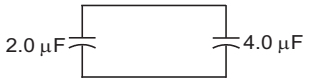



Figure 31-E22

**41.** A point charge $Q$ is placed at the origin. Find the electrostatic energy stored outside the sphere of radius $R$ centred at the origin.

**42.** A metal sphere of radius $R$ is charged to a potential $V$. (a) Find the electrostatic energy stored in the electric field within a concentric sphere of radius $2R$. (b) Show that the electrostatic field energy stored outside the sphere of radius $2R$ equals that stored within it.

**43.** A large conducting plane has a surface charge density $1·0 \times 10^{-4}$ C m$^{-2}$. Find the electrostatic energy stored in a cubical volume of edge 1·0 cm in front of the plane.

**44.** A parallel-plate capacitor having plate area 20 cm$^2$ and separation between the plates 1·00 mm is connected to a battery of 12·0 V. The plates are pulled apart to increase the separation to 2·0 mm. (a) Calculate the charge flown through the circuit during the process. (b) How much energy is absorbed by the battery during the process ? (c) Calculate the stored energy in the electric field before and after the process. (d) Using the expression for the force between the plates, find the work done by the person pulling the plates apart. (e) Show and justify that no heat is produced during this transfer of charge as the separation is increased.

**45.** A capacitor having a capacitance of 100 μF is charged to a potential difference of 24 V. The charging battery is disconnected and the capacitor is connected to another battery of emf 12 V with the positive plate of the capacitor joined with the positive terminal of the battery. (a) Find the charges on the capacitor before and after the reconnection. (b) Find the charge flown through the 12 V battery. (c) Is work done by the battery or is it done on the battery ? Find its magnitude. (d) Find the decrease in electrostatic field energy. (e) Find the heat developed during the flow of charge after reconnection.

**46.** Consider the situation shown in figure (31-E23). The switch $S$ is open for a long time and then closed. (a) Find the charge flown through the battery when the switch $S$ is closed. (b) Find the work done by the battery.

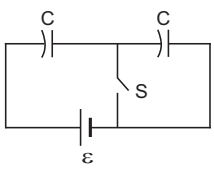



Figure 31-E23

(c) Find the change in energy stored in the capacitors. (d) Find the heat developed in the system.

**47.** A capacitor of capacitance 5·00 μF is charged to 24·0 V and another capacitor of capacitance 6·0 μF is charged to 12·0 V. (a) Find the energy stored in each capacitor. (b) The positive plate of the first capacitor is now connected to the negative plate of the second and vice versa. Find the new charges on the capacitors. (c) Find the loss of electrostatic energy during the process. (d) Where does this energy go ?

**48.** A 5·0 μF capacitor is charged to 12 V. The positive plate of this capacitor is now connected to the negative terminal of a 12 V battery and vice versa. Calculate the heat developed in the connecting wires.

**49.** The two square faces of a rectangular dielectric slab (dielectric constant 4·0) of dimensions 20 cm × 20 cm × 1·0 mm are metal-coated. Find the capacitance between the coated surfaces.

**50.** If the above capacitor is connected across a 6·0 V battery, find (a) the charge supplied by the battery, (b) the induced charge on the dielectric and (c) the net charge appearing on one of the coated surfaces.

**51.** The separation between the plates of a parallel-plate capacitor is 0·500 cm and its plate area is 100 $cm^2$. A 0·400 cm thick metal plate is inserted into the gap with its faces parallel to the plates. Show that the capacitance of the assembly is independent of the position of the metal plate within the gap and find its value.

**52.** A capacitor stores 50 μC charge when connected across a battery. When the gap between the plates is filled with a dielectric, a charge of 100 μC flows through the battery. Find the dielectric constant of the material inserted.

**53.** A parallel-plate capacitor of capacitance 5 μF is connected to a battery of emf 6 V. The separation between the plates is 2 mm. (a) Find the charge on the positive plate. (b) Find the electric field between the plates. (c) A dielectric slab of thickness 1 mm and dielectric constant 5 is inserted into the gap to occupy the lower half of it. Find the capacitance of the new combination. (d) How much charge has flown through the battery after the slab is inserted ?

**54.** A parallel-plate capacitor has plate area 100 $cm^2$ and plate separation 1·0 cm. A glass plate (dielectric constant 6·0) of thickness 6·0 mm and an ebonite plate (dielectric constant 4·0) are inserted one over the other to fill the space between the plates of the capacitor. Find the new capacitance.

**55.** A parallel-plate capacitor having plate area 400 $cm^2$ and separation between the plates 1·0 mm is connected to a power supply of 100 V. A dielectric slab of thickness 1·0 mm and dielectric constant 5·0 is inserted into the gap. (a) Find the increase in electrostatic energy. (b) If the power supply is now disconnected and the dielectric slab is taken out, find the further increase in energy. (c) Why does the energy increase in inserting the slab as well as in taking it out ?

**56.** Find the capacitances of the capacitors shown in figure (31-E24). The plate area is $A$ and the separation between the plates is $d$. Different dielectric slabs in a particular part of the figure are of the same thickness and the entire gap between the plates is filled with the dielectric slabs.

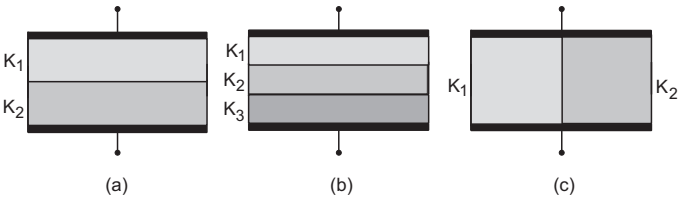



Figure 31-E24

**57.** A capacitor is formed by two square metal-plates of edge $a$, separated by a distance $d$. Dielectrics of dielectric constants $K_1$ and $K_2$ are filled in the gap as shown in figure (31-E25). Find the capacitance.

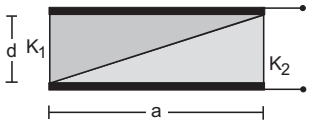



Figure 31-E25

**58.** Figure (31-E26) shows two identical parallel plate capacitors connected to a battery through a switch $S$. Initially, the switch is closed so that the capacitors are completely charged. The switch is now opened and the free space between the plates of the capacitors is filled with a dielectric of dielectric constant 3. Find the ratio of the initial total energy stored in the capacitors to the final total energy stored.

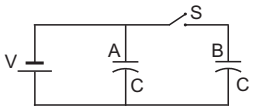



Figure 31-E26

**59.** A parallel-plate capacitor of plate area $A$ and plate separation $d$ is charged to a potential difference $V$ and then the battery is disconnected. A slab of dielectric constant $K$ is then inserted between the plates of the capacitor so as to fill the space between the plates. Find the work done on the system in the process of inserting the slab.

**60.** A capacitor having a capacitance of 100 μF is charged to a potential difference of 50 V. (a) What is the magnitude of the charge on each plate ? (b) The charging battery is disconnected and a dielectric of dielectric constant 2·5 is inserted. Calculate the new potential difference between the plates. (c) What charge would have produced this potential difference in absence of the dielectric slab. (d) Find the charge induced at a surface of the dielectric slab.

**61.** A sphercial capacitor is made of two conducting spherical shells of radii $a$ and $b$. The space between the shells is filled with a dielectric of dielectric constant $K$

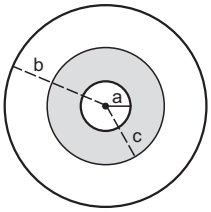



Figure 31-E27

up to a radius c as shown in figure (31-E27). Calculate the capacitance.

**62.** Consider an assembly of three conducting concentric spherical shells of radii *a, b* and *c* as shown in figure (31-E28). Find the capacitance of the assembly between the points *A* and *B*.

**63.** Suppose the space between the two inner shells of the

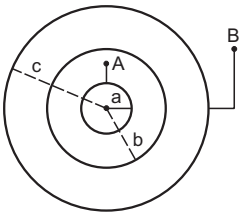



Figure 31-E28

previous problem is filled with a dielectric of dielectric constant *K*. Find the capacitance of the system between *A* and *B*.

**64.** An air-filled parallel-plate capacitor is to be constructed which can store 12 μC of charge when operated at 1200 V. What can be the minimum plate area of the capacitor? The dielectric strength of air is $3 \times 10^6$ V m$^{-1}$.

**65.** A parallel-plate capacitor with the plate area 100 cm$^2$ and the separation between the plates 1·0 cm is connected across a battery of emf 24 volts. Find the force of attraction between the plates.

**66.** Consider the situation shown in figure (31-E29). The width of each plate is b. The capacitor plates are rigidly clamped in the laboratory and connected to a battery of emf $\mathcal{E}$. All surfaces are frictionless. Calculate the value of *M* for which the dielectric slab will stay in equilibrium.

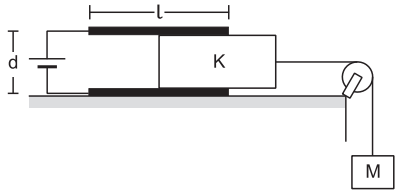



Figure 31-E29

**67.** Figure (31-E30) shows two parallel plate capacitors with fixed plates and connected to two batteries. The separation between the plates is the same for the two capacitors. The plates are rectangular in shape with width *b* and lengths $l_1$ and $l_2$. The left half of the dielectric slab has a dielectric constant $K_1$ and the right half $K_2$. Neglecting any friction, find the ratio of the emf of the left battery to that of the right battery for which the dielectric slab may remain in equilibrium.

**68.** Consider the situation shown in figure (31-E31). The

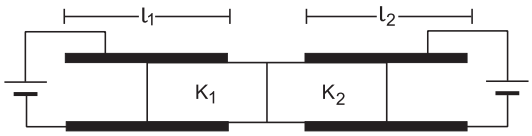



Figure 31-E30

plates of the capacitor have plate area *A* and are clamped in the laboratory. The dielectric slab is released from rest with a length *a* inside the capacitor. Neglecting any effect of friction or gravity, show that the slab will execute periodic motion and find its time period.

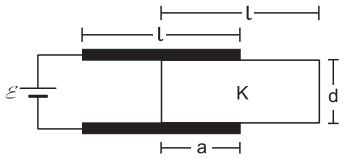



Figure 31-E31

□

## ANSWERS

### OBJECTIVE I

1. (d) 2. (d) 3. (c) 4. (b) 5. (c) 6. (d)
7. (b) 8. (d) 9. (c) 10. (a) 11. (b) 12. (d)

### OBJECTIVE II

1. (c) 2. (b) 3. (d) 4. (a), (c), (d) 5. (b), (c)
6. (d) 7. (b), (c), (d)

### EXERCISES

1. $1{\cdot}6 \times 10^{-8}$ F
2. $6{\cdot}95 \times 10^{-5}$ μF
3. 6 km
4. $1{\cdot}33 \times 10^{-10}$ C, $8{\cdot}0 \times 10^{-10}$ J
5. (a) $1{\cdot}33 \times 10^{-10}$ C (b) $1{\cdot}33 \times 10^{-10}$ C

6. 24 μC, 48 μC, 72 μC
7. 110 μC on each, $1{\cdot}33 \times 10^{-3}$ J
8. 48 μC on the 8 μF capacitor and 24 μC on each of the 4 μF capacitors
9. (a) 5 μF (b) 10 μF
10. 110 μC
11. 44 μC
12. $4\pi\varepsilon_0 R_1$, $4\pi\varepsilon_0 R_2$; $4\pi\varepsilon_0 (R_1 + R_2)$
13. 2 μF, 20 V
15. (a) 50/3 μV at each point (b) zero
16. $C_3 + \dfrac{2\,C_1C_2}{C_1 + C_2}$
17. $\dfrac{\varepsilon_0 A(3\,d^2 + 6\,bd + 2\,b^2)}{3\,d(d+b)\,(d+2\,b)}$
18. (a) 8 pF (b) same as in (a)
19. 20 V
20. $3{\cdot}3 \times 10^{-4}$ C
21. 43 mV
22. $\left(\dfrac{Vea^2}{md_1(d_1 + d_2)}\right)^{1/2}$
23. $1{\cdot}08 \times 10^{-3}$ cm
24. 2·25 μF
25. (a) $\dfrac{12}{11}$ V (b) – 8 V (c) zero (d) – 10·3 V
26. (a) $\dfrac{11}{6}$ μF (b) $\dfrac{11}{4}$ μF (c) 8 μF (d) 8 μF
27. 1 μF
28. 2 μF
29. 4 μF
30. 12·5 V
31. 1 V
32. 5 V
33. 0·16 μC
34. (a) 0·50 μC (b) 10 V
35. (a) 10 V (b) 10 V
36. (a) 1·71 V, 4·29 V (b) 184 pJ, 73·5 pJ
37. 960 μJ
38. 8 mJ in (a) and (d), 2 mJ in (b) and (c)
39. 2·0 J
40. (a) 8 μC, 16 μC (b) 16 μJ, 32 μJ, (c) 96 μJ
41. $\dfrac{Q^2}{8\pi\varepsilon_0 R}$
42. (a) $\pi\varepsilon_0 RV^2$
43. $5{\cdot}6 \times 10^{-4}$ J
44. (a) $1{\cdot}06 \times 10^{-10}$ C (b) $12{\cdot}7 \times 10^{-10}$ J
    (c) $12{\cdot}7 \times 10^{-10}$ J, $6{\cdot}35 \times 10^{-10}$ J (d) $6{\cdot}35 \times 10^{-10}$ J
45. (a) 2400 μC, 1200 μC (b) 1200 μC (c) 14·4 mJ
    (d) 21·6 mJ (e) 7·2 mJ
46. (a) $C\mathcal{E}/2$, (b) $C\mathcal{E}^2/2$ (c) $C\mathcal{E}^2/4$ (d) $C\mathcal{E}^2/4$
47. (a) 1·44 mJ, 0·432 mJ (b) 21·8 μC, 26·2 μC, (c) 1·77 mJ
48. 1·44 mJ
49. 1·42 nF
50. (a) 8·5 nC (b) 6·4 nC (c) 2·1 nC
51. 88 pF
52. 3
53. (a) 30 μC (b) $3 \times 10^3$ V m$^{-1}$ (c) 8·3 μF (d) 20 μC
54. 44 pF
55. (a) 1·18 μJ (b) 1·97 μJ
56. (a) $\dfrac{2\,K_1K_2\varepsilon_0 A}{d(K_1 + K_2)}$ (b) $\dfrac{3\varepsilon_0 A\,K_1K_2K_3}{d(K_1K_2 + K_2K_3 + K_3K_1)}$
    (c) $\dfrac{\varepsilon_0 A}{2\,d}(K_1 + K_2)$
57. $\dfrac{\varepsilon_0 K_1K_2a^2 \ln \dfrac{K_1}{K_2}}{(K_1 - K_2)d}$
58. 3 : 5
59. $\dfrac{\varepsilon_0 AV^2}{2\,d}\left(\dfrac{1}{K} - 1\right)$
60. (a) 5 mC (b) 20 V (c) 2 mC (d) 3 mC
61. $\dfrac{4\pi\varepsilon_0 Ka\,bc}{Ka(b-c) + b(c-a)}$
62. $\dfrac{4\pi\varepsilon_0 ac}{c-a}$
63. $\dfrac{4\pi\varepsilon_0 Kabc}{Ka(c-b) + c(b-a)}$
64. 0·45 m$^2$
65. $2{\cdot}5 \times 10^{-7}$ N
66. $\dfrac{\varepsilon_0\, b\,\mathcal{E}^2(K-1)}{2\,dg}$
67. $\sqrt{\dfrac{K_2 - 1}{K_1 - 1}}$
68. $8\sqrt{\dfrac{(l-a)\,lmd}{\varepsilon_0 A\mathcal{E}^2(K-1)}}$

□

## CHAPTER 32

# ELECTRIC CURRENT IN CONDUCTORS

### 32.1 ELECTRIC CURRENT AND CURRENT DENSITY

When there is a transfer of charge from one side of an area to the other, we say that there is an *electric current* through the area. If the moving charges are positive, the current is in the direction of motion. If they are negative, the current is opposite to the direction of motion. If a charge $\Delta Q$ crosses an area in time $\Delta t$, we define the average electric current through the area during this time as

$$\bar{i} = \frac{\Delta Q}{\Delta t}.$$

The current at time $t$ is

$$i = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt}. \qquad \ldots (32.1)$$

Thus, electric current through an area is the rate of transfer of charge from one side of the area to the other. The SI unit of current is ampere. If one coulomb of charge crosses an area in one second, the current is one ampere. It is one of the seven base units accepted in SI.

We shall now define a vector quantity known as *electric current density* at a point. To define the current density at a point $P$, we draw a small area $\Delta S$ through $P$ perpendicular to the flow of charges (figure 32.1a). If $\Delta i$ be the current through the area $\Delta S$, the average current density is

$$\bar{j} = \frac{\Delta i}{\Delta S}.$$

The current density at the point $P$ is

$$j = \lim_{\Delta S \to 0} \frac{\Delta i}{\Delta S} = \frac{di}{dS}.$$

The direction of the current density is the same as the direction of the current. Thus, it is along the motion of the moving charges if the charges are positive and opposite to the motion of the charges if the charges are negative. If a current $i$ is uniformly distributed over an area $S$ and is perpendicular to it,

$$j = \frac{i}{S}. \qquad \ldots (32.2)$$

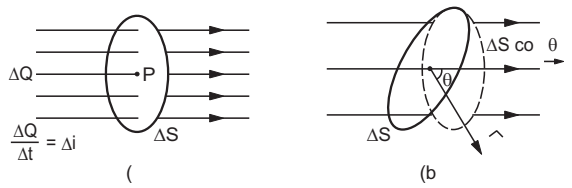



Figure 32.1

Now let us consider an area $\Delta S$ which is not necessarily perpendicular to the current (figure 32-1b). If the normal to the area makes an angle $\theta$ with the direction of the current, the current density is,

$$j = \frac{\Delta i}{\Delta S \cos\theta}$$

or, $$\Delta i = j\, \Delta S \cos\theta$$

where $\Delta i$ is the current through $\Delta S$. If $\Delta\vec{S}$ be the *area-vector* corresponding to the area $\Delta S$, we have

$$\Delta i = \vec{j} \cdot \Delta\vec{S}.$$

For a finite area,

$$i = \int \vec{j} \cdot d\vec{S}. \qquad \ldots (32.3)$$

Note carefully that an electric current has direction as well as magnitude but it is not a vector quantity. It does not add like vectors. The current density is a vector quantity.

***Example 32.1***

*An electron beam has an aperture* $1{\cdot}0\ \text{mm}^2$. *A total of* $6{\cdot}0 \times 10^{16}$ *electrons go through any perpendicular cross section per second. Find (a) the current and (b) the current density in the beam.*

***Solution :***

(a) The total charge crossing a perpendicular cross section in one second is

$$q = ne$$

$$= 6{\cdot}0 \times 10^{16} \times 1{\cdot}6 \times 10^{-19}\ \text{C}$$

$$= 9{\cdot}6 \times 10^{-3}\ \text{C}.$$

The current is

$$i = \frac{q}{t}$$

$$= \frac{9{\cdot}6 \times 10^{-3}\ \text{C}}{1\ \text{s}} = 9{\cdot}6 \times 10^{-3}\ \text{A}.$$

As the charge is negative, the current is opposite to the direction of motion of the beam.

(b) The current density is

$$j = \frac{i}{S} = \frac{9{\cdot}6 \times 10^{-3}\ \text{A}}{1{\cdot}0\ \text{mm}^2} = 9{\cdot}6 \times 10^3\ \text{A m}^{-2}.$$

Electric current can be obtained in a variety of ways. When a metal is heated to high temperatures, it emits electrons. These electrons travel in space. Considering any area perpendicular to their velocity, there is a current through it.

In many solutions, positive and negative ions wander. If the solution is placed in an electric field, the positive ions move (inside the solution) along the field and the negative ions move opposite to the field. Both movements contribute to a current in the direction of the field.

When a battery is connected across a capacitor, charges flow from the battery to the capacitor through the connecting wires. There is a current through any cross section of the wires as long as the charges keep going to the plates.

In this chapter, we shall study the electric current in a conductor when an electric field is established inside it.

## 32.2 DRIFT SPEED

A conductor contains a large number of loosely bound electrons which we call *free electrons* or *conduction electrons*. The remaining material is a collection of relatively heavy positive ions which we call *lattice*. These ions keep on vibrating about their mean positions. The average amplitude depends on the temperature. Occasionally, a free electron collides or interacts in some other fashion with the lattice. The speed and direction of the electron changes randomly at each such event. As a result, the electron moves in a zig-zag path. As there is a large number of free electrons moving in random directions, the number of electrons crossing an area $\Delta S$ from one side very nearly equals the number crossing from the other side in any given time interval. The electric current through the area is, therefore, zero.

When there is an electric field inside the conductor, a force acts on each electron in the direction opposite to the field. The electrons get biased in their random motion in favour of the force. As a result, the electrons drift slowly in this direction. At each collision, the electron starts afresh in a random direction with a random speed but gains an additional velocity $v'$ due to the electric field. This velocity $v'$ increases with time and suddenly becomes zero as the electron makes a collision with the lattice and starts afresh with a random velocity. As the time $t$ between successive collisions is small, the electron slowly and steadily drifts opposite to the applied field (figure 32.2). If the electron drifts a distance $l$ in a long time $t$, we define drift speed as

$$v_d = \frac{l}{t}.$$

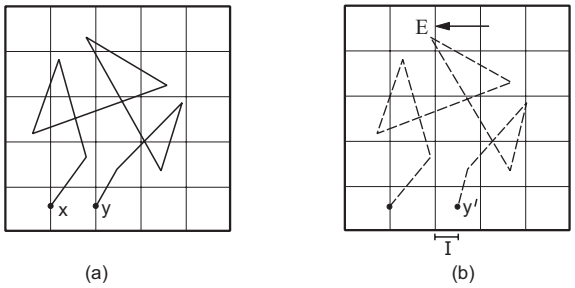



Figure 32.2

If $\tau$ be the average time between successive collisions, the distance drifted during this period is

$$l = \frac{1}{2}\, a(\tau)^2 = \frac{1}{2}\left(\frac{eE}{m}\right)(\tau)^2.$$

The drift speed is

$$v_d = \frac{l}{\tau} = \frac{1}{2}\left(\frac{eE}{m}\right)\tau.$$

It is proportional to the electric field $E$ and to the average collision-time $\tau$.

The random motion of free electrons does not contribute to the drift of these electrons. Also, the average collision-time is constant for a given material at a given temperature. We, therefore, make the following assumption for our present purpose of discussing electric current.

When no electric field exists in a conductor, the free electrons stay at rest ($v_d = 0$) and when a field $E$ exists, they move with a constant velocity

$$v_d = \frac{e\tau}{2\,m} E = kE \qquad \ldots \quad (32.4)$$

opposite to the field. The constant $k$ depends on the material of the conductor and its temperature.

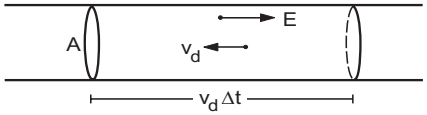



Figure 32.3

Let us now find the relation between the current density and the drift speed. Consider a cylindrical

conductor of cross-sectional area $A$ in which an electric field $E$ exists. Consider a length $v_d\,\Delta t$ of the conductor (figure 32.3). The volume of this portion is $Av_d\Delta t$. If there are $n$ free electrons per unit volume of the wire, the number of free electrons in this portion is $nAv_d\,\Delta t$. All these electrons cross the area $A$ in time $\Delta t$. Thus, the charge crossing this area in time $\Delta t$ is

$$\Delta Q = nAv_d\,\Delta t\, e$$

or, $$i = \frac{\Delta Q}{\Delta t} = nAv_d\, e$$

and $$j = \frac{i}{A} = nev_d. \quad \ldots\ (32.5)$$

***Example 32.2***

*Calculate the drift speed of the electrons when* 1 A *of current exists in a copper wire of cross section* 2 mm$^2$. *The number of free electrons in* 1 cm$^3$ *of copper is* $8{\cdot}5 \times 10^{22}$.

***Solution*** : We have

$$j = nev_d$$

or, $$v_d = \frac{j}{n\,e} = \frac{i}{A\,ne}$$

$$= \frac{1\text{ A}}{(2\times10^{-6}\text{ m}^2)\,(8{\cdot}5\times10^{22}\times10^{6}\text{ m}^{-3})\,(1{\cdot}6\times10^{-19}\text{ C})}$$

$$= 0{\cdot}036\text{ mm s}^{-1}.$$

We see that the drift speed is indeed small.

### 32.3 OHM'S LAW

Using equations (32.4) and (32.5),

$$j = nev_d = \frac{ne^2\tau}{2\,m}E$$

or, $$j = \sigma E \quad \ldots\ (32.6)$$

where $\sigma$ depends only on the material of the conductor and its temperature. This constant is called the *electrical conductivity* of the material. Equation (32.6) is known as *Ohm's law*.

The *resistivity* of a material is defined as

$$\rho = \frac{1}{\sigma}\cdot \quad \ldots\ (32.7)$$

Ohm's law tells us that the conductivity (or resistivity) of a material is independent of the electric field existing in the material. This is valid for conductors over a wide range of field.

Suppose we have a conductor of length $l$ and uniform cross sectional area $A$ (figure 32.4a). Let us apply a potential difference $V$ between the ends of the conductor. The electric field inside the conductor is $E = V/l$. If the current in the conductor is $i$, the current density is $j = \frac{i}{A}\cdot$ Ohm's law $j = \sigma E$ then becomes

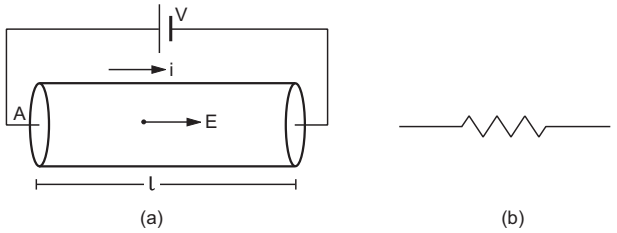



Figure 32.4

$$\frac{i}{A} = \sigma\frac{V}{l}$$

or, $$V = \frac{1}{\sigma}\frac{l}{A}i = \rho\frac{l}{A}i$$

or, $$V = R\,i \quad \ldots\ (32.8)$$

where $$R = \rho\frac{l}{A} \quad \ldots\ (32.9)$$

is called the *resistance* of the given conductor. The quantity $1/R$ is called *conductance*.

Equation (32.8) is another form of Ohm's law which is widely used in circuit analysis. The unit of resistance is called ohm and is denoted by the symbol Ω. An object of conducting material, having a resistance of desired value, is called a *resistor*. A resistor is represented by the symbol shown in figure (32.4b).

From equation (32.9), the unit of resistivity $\rho$ is ohm metre, also written as Ωm. The unit of conductivity ($\sigma$) is (ohm m)$^{-1}$ written as mho m$^{-1}$.

***Example 32.3***

*Calculate the resistance of an aluminium wire of length* 50 cm *and cross sectional area* 2·0 mm$^2$. *The resistivity of aluminium is* $\rho = 2{\cdot}6\times10^{-8}$ Ωm.

***Solution*** :

The resistance is $R = \rho\frac{l}{A}$

$$= \frac{(2{\cdot}6\times10^{-8}\ \Omega\text{m})\times(0{\cdot}50\text{ m})}{2\times10^{-6}\text{ m}^2} = 0{\cdot}0065\ \Omega.$$

We arrived at Ohm's law (equation 32.6 or 32.8) by making several assumptions about the existence and behaviour of the free electrons. These assumptions are not valid for semiconductors, insulators, solutions, etc. Ohm's law cannot be applied in such cases.

**Colour Code for Resistors**

Resistors of different values are commercially available. To make a resistor, carbon with a suitable binding agent is molded into a cylinder. Wire leads are

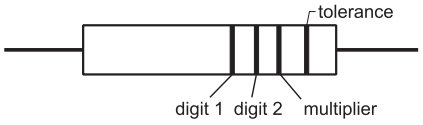



Figure 32.5

attached to this cylinder and the entire resistor is encased in a ceramic or plastic jacket. The two leads connect the resistor to a circuit. These resistors are widely used in electronic circuits such as those for radios, amplifiers, etc. The value of the resistance is indicated by four coloured-bands, marked on the surface of the cylinder (figure 32.5). The meanings of the four positions of the bands are shown in figure (32.5) and the meanings of different colours are given in table (32.1).

**Table 32.1 : *Resistance codes (resistance given in ohm)***

| Colour | Digit | Multiplier | Tolerance |
|---|---|---|---|
| Black | 0 | 1 | |
| Brown | 1 | 10 | |
| Red | 2 | $10^2$ | |
| Orange | 3 | $10^3$ | |
| Yellow | 4 | $10^4$ | |
| Green | 5 | $10^5$ | |
| Blue | 6 | $10^6$ | |
| Violet | 7 | $10^7$ | |
| Gray | 8 | $10^8$ | |
| White | 9 | $10^9$ | |
| Gold | | 0·1 | 5% |
| Silver | | 0·01 | 10% |

For example, suppose the colours on the resistor shown in figure (32.5) are brown, yellow, green and gold as read from left to right. Using table (32.1), the resistance is

$$(14 \times 10^5 \pm 5\%)\Omega = (1{\cdot}4 \pm 0{\cdot}07)\ \text{M}\Omega.$$

Sometimes, the tolerance band is missing from the code so that there are only three bands. This means the tolerance is 20%.

## 32.4 TEMPERATURE DEPENDENCE OF RESISTIVITY

As the temperature of a conductor is increased, the thermal agitation increases and the collisions become more frequent. The average time $\tau$ between the successive collisions decreases and hence the drift speed decreases. Thus, the conductivity decreases and the resistivity increases as the temperature increases. For small temperature variations, we can write for most of the materials,

$$\rho(T) = \rho(T_0)\ [1 + \alpha(T - T_0)]$$

where $\rho(T)$ and $\rho(T_0)$ are resistivities at temperatures $T$ and $T_0$ respectively and $\alpha$ is a constant for the given material. In fact, $\alpha$ depends to a small extent on the temperature. The constant $\alpha$ is called the *temperature coefficient of resistivity*. Table (32.2) lists the resistivity at room temperature and the average value of $\alpha$ for some materials.

**Table 32.2 : *Resistivities of different materials***

| Material | $\rho(\Omega\text{m})$ | $\alpha(\text{K}^{-1})$ |
|---|---|---|
| Silver | $1{\cdot}47 \times 10^{-8}$ | 0·0038 |
| Copper | $1{\cdot}72 \times 10^{-8}$ | 0·0039 |
| Gold | $2{\cdot}35 \times 10^{-8}$ | 0·0034 |
| Aluminium | $2{\cdot}63 \times 10^{-8}$ | 0·0039 |
| Tungsten | $5{\cdot}51 \times 10^{-8}$ | 0·0045 |
| Nickel | $86{\cdot}84 \times 10^{-8}$ | 0·0060 |
| Iron | $9{\cdot}71 \times 10^{-8}$ | 0·0050 |
| Magnesium | $44 \times 10^{-8}$ | 0·0000 |
| Mercury | $96 \times 10^{-8}$ | 0·0009 |
| Nichrome | $100 \times 10^{-8}$ | 0·0004 |
| Silicon | 640 | – 0·075 |
| Germanium | 0·46 | – 0·048 |
| Fused quartz | $7{\cdot}5 \times 10^{17}$ | |

The resistance of a given conductor depends on its length and area of cross section besides the resistivity (equation 32.9). As temperature changes, the length and the area also change. But these changes are quite small and the factor $l/A$ may be treated as constant. Then $R \propto \rho$ and hence

$$R(T) = R(T_0)\ [1 + \alpha(T - T_0)].$$

From table (32.2), we see that resistivity varies over a wide range. We have metals with resistivity of the order of $10^{-8}\ \Omega$m. They are good conductors of electricity. Fused quartz has resistivity as high as $7{\cdot}5 \times 10^{17}\ \Omega$m. This is an *insulator*. Then we have materials like silicon and germanium which have resistivity much smaller than that of insulators but much larger than that of the metals. They are called *semiconductors*.

### Thermistor

The temperature coefficient of resistivity is negative for semiconductors. This means that the resistivity decreases as we raise the temperature of such a material. The picture of a large number of free electrons colliding with each other and with the lattice is not adequate for semiconductors. The magnitude of the temperature coefficient of resistivity is often quite large for a semiconducting material. This fact is used to construct thermometers to detect small changes in temperatures. Such a device is called a *thermistor*. Thermistors are usually prepared from oxides of various metals such as nickel, iron, cobalt, copper, etc. These compounds are also semiconductors. A thermistor is usually enclosed in a capsule with epoxy

surface. The thermistor is dipped in the bath whose temperature is to be measured. The circuit is completed by connecting a battery. The current through the thermistor is measured. If the temperature increases, the current also increases because of the decrease in resistivity. Thus, by noting the change in the current, one can find the change in temperature. A typical thermistor can easily measure a change in temperature of the order of $10^{-3}$ °C.

**Superconductors**

There are certain materials for which the resistivity suddenly becomes zero below a certain temperature. This temperature is called the *critical temperature* for this transition. The material in this state is called a *superconductor*. Above the critical temperature, the resistivity follows the trend of a normal metal (figure 32.6). This phenomenon was observed for mercury in 1911 by H Kamerlingh Onnes. The critical temperature for mercury is 4·2 K.

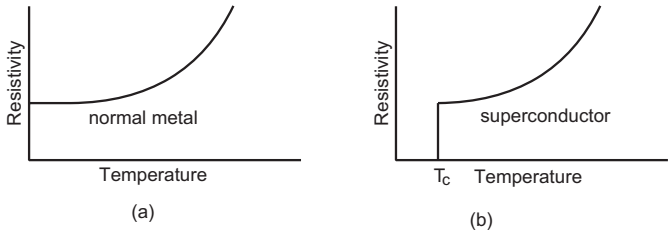



Figure 32.6

If an electric current is set up in a superconducting material, it can persist for long time without any applied emf. Steady currents have been observed for several years in superconducting loops without any observable decrease. Superconductors are used to construct very strong magnets. This has useful applications in material science research and high-energy particle physics. Possible applications of superconductors are ultrafast computer switches and transmission of electric power through superconducting power lines. However, the requirement of low temperature is posing difficulty. Scientists are putting great effort to construct compounds and alloys which would be superconducting at room temperature (300 K). Superconductivity at around 125 K has already been achieved and efforts are on to improve upon this.

## 32.5 BATTERY AND EMF

A battery is a device which maintains a potential difference between its two terminals $A$ and $B$. Figure (32.7) shows a schematic diagram of a battery. Some internal mechanism exerts forces on the charges of the battery material. This force drives the positive charges of the battery material towards $A$ and the negative charges of the battery material towards $B$. We show the force on a positive charge $q$ as $\vec{F}_b$. As positive charge accumulates on $A$ and negative charge on $B$, a potential difference develops and grows between $A$ and $B$. An electric field $\vec{E}$ is developed in the battery material from $A$ to $B$ and exerts a force $\vec{F}_e = q\vec{E}$ on a charge $q$. The direction of this force is opposite to that of $\vec{F}_b$. In steady state, the charge accumulation on $A$ and $B$ is such that $F_b = F_e$. No further accumulation takes place.

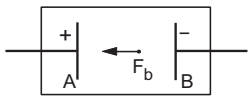



Figure 32.7

If a charge $q$ is taken from the terminal $B$ to the terminal $A$, the work done by the battery force $F_b$ is $W = F_b\, d$ where $d$ is the distance between $A$ and $B$. The work done by the battery force per unit charge is

$$\mathcal{E} = \frac{W}{q} = \frac{F_b d}{q}. \qquad \ldots \quad (32.10)$$

This quantity is called the *emf* of the battery. The full form of emf is *electromotive force.* The name is misleading in the sense that emf is not a force, it is work done/charge. We shall continue to denote this quantity by the short name emf. If nothing is connected externally between $A$ and $B$,

$$F_b = F_e = qE$$

or, $$F_b d = qEd = qV,$$

where $V = Ed$ is the potential difference between the terminals. Thus,

$$\mathcal{E} = \frac{F_b d}{q} = V.$$

Thus, *the emf of a battery equals the potential difference between its terminals when the terminals are not connected externally.*

Potential difference and emf are two different quantities whose magnitudes may be equal in certain conditions. The emf is the work done per unit charge by the battery force $F_b$ which is nonelectrostatic in nature. The potential difference originates from the electrostatic field created by the charges accumulated on the terminals of the battery.

A battery is often prepared by putting two rods or plates of different metals in a chemical solution. Such a battery, using chemical reactions to generate emf, is often called a *cell.*

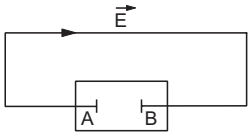



Figure 32.8

Now suppose the terminals of a battery are connected by a conducting wire as shown in figure (32.8). As the terminal $A$ is at a higher potential than $B$, there is an electric field in the wire in the direction shown in the figure. The free electrons in the wire move in the opposite direction and enter the battery at the terminal $A$. Some electrons are withdrawn from the terminal $B$ which enter the wire through the right end. Thus, the potential difference between $A$ and $B$ tends to decrease. If this potential difference decreases, the electrostatic force $F_e$ inside the battery also decreases. The force $F_b$ due to the battery mechanism remains the same. Thus, there is a net force on the positive charges of the battery material from $B$ to $A$. The positive charges rush towards $A$ and neutralise the effect of the electrons coming at $A$ from the wire. Similarly, the negative charges rush towards $B$. Thus, the potential difference between $A$ and $B$ is maintained.

For calculation of current, motion of a positive charge in one direction is equivalent to the motion of a negative charge in opposite direction. Using this fact, we can describe the above situation by a simpler model. The positive terminal of the battery supplies positive charges to the wire. These charges are pushed through the wire by the electric field and they reach the negative terminal of the battery. The battery mechanism drives these charges back to the positive terminal against the electric field existing in the battery and the process continues. This maintains a steady current in the circuit.

Current can also be driven into a battery in the reverse direction. In such a case, positive charge enters the battery at the positive terminal, moves inside the battery to the negative terminal and leaves the battery from the negative terminal. Such a process is called *charging* of the battery. The more common process in which the positive charge comes out of the battery from the positive terminal is called *discharging* of the battery.

## 32.6 ENERGY TRANSFER IN AN ELECTRIC CIRCUIT

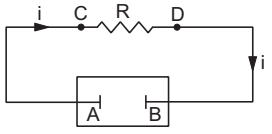



Figure 32.9

Figure (32.9) shows a simple circuit in which a resistor $CD$ having a resistance $R$ is connected to a battery of emf $\mathcal{E}$ through two connecting wires $CA$ and $DB$. The connecting wires are assumed to have negligible resistance. This ensures that potential differences across $AC$ and across $BD$ are zero even when there is a current through them. The potential difference across the resistor is the same as that across the battery. If the current in the circuit is $i$, this potential difference is

$$V = V_A - V_B = V_C - V_D = iR.$$

### Thermal Energy Produced in the Resistor

In time $t$, a charge $q = it$ goes through the circuit. As this charge moves from $C$ to $D$, the electric potential energy decreases by

$$U = qV = (it)\,(iR) = i^2Rt. \quad \ldots\ (32.11)$$

This loss in electric potential energy appears as increased thermal energy of the resistor. Thus, a current $i$ for a time $t$ through a resistance $R$ increases the thermal energy by $i^2Rt$. The power developed is

$$P = \frac{U}{t} = i^2R. \quad \ldots\ (32.12)$$

Using Ohm's law, this can also be written as

$$P = \frac{V^2}{R} = Vi.$$

***Example 32.4***

*A resistor develops* 400 J *of thermal energy in* 10 s *when a current of* 2 A *is passed through it. (a) Find its resistance. (b) If the current is increased to* 4 A, *what will be the energy developed in* 10 s.

***Solution*** :

(a) Using $U = i^2Rt$,

$$400\ \text{J} = (2\ \text{A})^2\, R(10\ \text{s})$$

or, $R = 10\ \Omega$.

(b) The thermal energy developed, when the current is 4 A, is

$$U = i^2Rt$$

$$= (4\ \text{A})^2 \times (10\ \Omega) \times (10\ \text{s}) = 1600\ \text{J}.$$

### Internal Resistance of a Battery

As the charge $q = it$ goes through the battery from the negative terminal $B$ to the positive terminal $A$, work is done by the nonelectrostatic battery force $F_b$. This work is $U_1 = q\mathcal{E} = \mathcal{E}it$. As the potential of $A$ is higher than the potential of $B$ by an amount $V$, the electric potential energy increases by an amount

$$U_2 = V(it).$$

The remaining energy $U_1 - U_2$ appears as thermal energy of the battery material. The fraction appearing as thermal energy depends on the battery material and the battery mechanism. If no thermal energy is developed as the charge goes through the battery,

$$\mathcal{E} it = Vit$$

or, $$\mathcal{E} = V.$$

Such a battery is called an *ideal battery*. The potential difference between the terminals of an ideal battery remains equal to its emf even if there is a current through it. As discussed earlier, an ideal battery is denoted by the symbol shown in figure (32.10a). The potential difference between the facing parallel lines is $V = \mathcal{E}$, the longer line being at the higher potential.

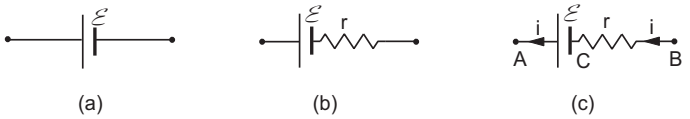



Figure 32.10

A nonideal battery develops thermal energy as a current passes through it and the potential difference between the terminals is smaller than the emf. Such a battery may be represented by the symbol shown in figure (32.10b). This is a combination of an ideal battery of emf $\mathcal{E}$ and a resistance $r$. If there is a current $i$ through the battery in the direction indicated in figure (32.10c), the potential difference between the terminals is

$$V_A - V_B = (V_A - V_C) - (V_B - V_C)$$
$$= \mathcal{E} - ir.$$

The thermal energy developed in time $t$ is $i^2rt$. The addition of a resistance $r$ accounts for the difference between $\mathcal{E}$ and $V$ as well as for the thermal energy developed in the battery. This resistance is called the *internal resistance* of the battery.

***Example 32.5***

*A battery of emf* 2·0 V *and internal resistance* 0·50 Ω *supplies a current of* 100 mA. *Find (a) the potential difference across the terminals of the battery and (b) the thermal energy developed in the battery in* 10 s.

***Solution*** **:** The situation is the same as that shown in figure (32.10c).

(a) The potential difference across the terminals is

$$V_A - V_B = (V_A - V_C) - (V_B - V_C)$$
$$= \mathcal{E} - ir$$
$$= 2{\cdot}0 \text{ V} - (0{\cdot}100 \text{ A})\ (0{\cdot}50\ \Omega) = 1{\cdot}95 \text{ V}.$$

(b) The thermal energy developed in the battery is

$$U = i^2rt = (0{\cdot}100 \text{ A})^2\ (0{\cdot}50\ \Omega)\ (10 \text{ s}) = 0{\cdot}05 \text{ J}.$$

## 32.7 KIRCHHOFF'S LAWS

### The Junction Law

*The sum of all the currents directed towards a point in a circuit is equal to the sum of all the currents directed away from the point.*

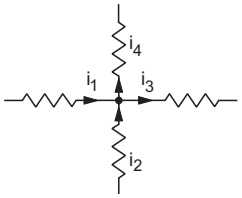



Figure 32.11

Thus, in figure (32.11),

$$i_1 + i_2 = i_3 + i_4. \quad \ldots \text{ (i)}$$

If we take the current directed towards a point as positive and that directed away from the point as negative, we can restate the junction law as, *the algebraic sum of all the currents directed towards a point is zero.*

In figure (32.11), the currents directed towards the junction point are $i_1$, $i_2$, $-i_3$ and $-i_4$.

Thus,

$$i_1 + i_2 + (-i_3) + (-i_4) = 0$$

which is the same as (i).

The junction law follows from the fact that no point in a circuit keeps on accumulating charge or keeps on supplying charge. Charges pass through the point. So, the net charge coming towards the point should be equal to that going away from it in the same time.

### The Loop Law

*The algebraic sum of all the potential differences along a closed loop in a circuit is zero.*

While using this rule, one starts from a point on the loop and goes along the loop, either clockwise or anticlockwise, to reach the same point again. Any potential *drop* encountered is taken to be positive and any potential *rise* is taken to be negative. The net

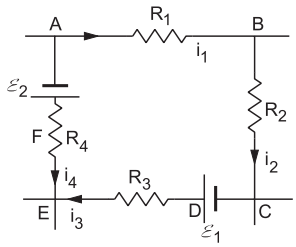



Figure 32.12

sum of all these potential differences should be zero. In figure (32.12), we show a loop *ABCDEFA* of a circuit. As we start from *A* and go along the loop clockwise to reach the same point *A*, we get the following potential differences:

$$V_A - V_B = i_1 R_1$$
$$V_B - V_C = i_2 R_2$$
$$V_C - V_D = -\mathcal{E}_1$$
$$V_D - V_E = i_3 R_3$$
$$V_E - V_F = -i_4 R_4$$
$$V_F - V_A = \mathcal{E}_2 .$$

Adding all these,

$$0 = i_1 R_1 + i_2 R_2 - \mathcal{E}_1 + i_3 R_3 - i_4 R_4 + \mathcal{E}_2 .$$

The loop law follows directly from the fact that electrostatic force is a conservative force and the work done by it in any closed path is zero.

## 32.8 COMBINATION OF RESISTORS IN SERIES AND PARALLEL

Several resistors may be combined to form a network. The combination should have two end points to connect it with a battery or other circuit elements. If a potential difference $V$ is applied to the combination, it draws some current $i$. We define *equivalent resistance* of the combination as

$$R_{eq} = V/i.$$

This single resistance draws the same current as the given combination when the same potential difference is applied across the end points.

### Series Combination

*Two or more resistors are said to be connected in series if the same current passes through all the resistors.*

Figure (32.13) shows three resistors having resistances $R_1$, $R_2$ and $R_3$ connected in series. The combination has two points $P$ and $N$ through which it can be connected to a battery or other circuit elements. Any current going through $R_1$ also goes through $R_2$ and $R_3$.

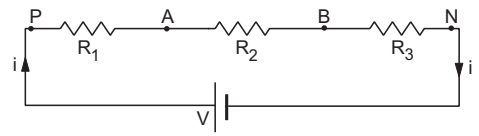



Figure 32.13

Suppose we apply a potential difference $V$ between the points $P$ and $N$. A current $i$ passes through all the resistors. Using Kirchhoff's loop law for the loop *PABNP,*

$$iR_1 + iR_2 + iR_3 - V = 0$$

or,
$$i = \frac{V}{R_1 + R_2 + R_3}$$

or,
$$\frac{V}{i} = R_1 + R_2 + R_3.$$

Thus, the equivalent resistance is

$$R_{eq} = R_1 + R_2 + R_3.$$

This argument may be extended for any number of resistors connected in series.

$$R_{eq} = R_1 + R_2 + R_3 + \ldots \qquad \ldots \quad (32.13)$$

### Parallel Combination

*Two or more resistors are said to be connected in parallel if the same potential difference exists across all the resistors.*

Figure (32.14) shows three resistors having resistances $R_1$, $R_2$ and $R_3$ connected in parallel. The combination has two end points $P$ and $N$. One end of each resistor is joined to $P$ and other end to $N$. Thus, the potential difference across any resistor is the same.

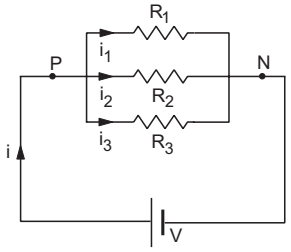



Figure 32.14

To find the equivalent resistance of the combination, let us apply a potential difference $V$ between the points $P$ and $N$. If the current through $P$ is $i$, the equivalent resistance is

$$R_{eq} = V/i. \qquad \ldots \quad (i)$$

The current $i$ is divided at the junction $P$. Suppose a current $i_1$ goes through $R_1$, $i_2$ through $R_2$ and $i_3$ through $R_3$. These combine at $N$ to give a total current $i$. Using Kirchhoff's junction law at $P$,

$$i = i_1 + i_2 + i_3. \qquad \ldots \quad (ii)$$

The potential difference across each resistor is $V_P - V_N = V$. Using Ohm's law for the resistances $R_1$, $R_2$ and $R_3$ separately;

$$i_1 = \frac{V}{R_1}, \; i_2 = \frac{V}{R_2} \text{ and } i_3 = \frac{V}{R_3}.$$

Adding the above three equations and using (ii),

$$i = V\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right).$$

Using (i),

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.$$

The process may be generalised for any number of resistors connected in parallel, so that

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots \quad \ldots \quad (32.14)$$

For two resistors in parallel,

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \text{ or, } R_{eq} = \frac{R_1 R_2}{R_1 + R_2}.$$

Note that the equivalent resistance is smaller than the smallest individual resistance.

***Example 32.6***

*Find the equivalent resistance of the network shown in figure (32.15) between the points A and B.*

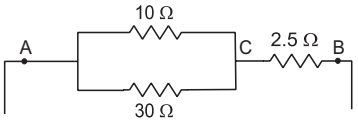



Figure 32.15

***Solution*** : The 10 Ω resistor and the 30 Ω resistor are connected in parallel. The equivalent resistance between $A$ and $C$ is

$$R_1 = \frac{(10\ \Omega)\ (30\ \Omega)}{10\ \Omega + 30\ \Omega} = 7{\cdot}5\ \Omega.$$

This is connected with 2·5 Ω in series. The equivalent resistance between $A$ and $B$ is $7{\cdot}5\ \Omega + 2{\cdot}5\ \Omega = 10\ \Omega$.

**Division of Current in Resistors Joined in Parallel**

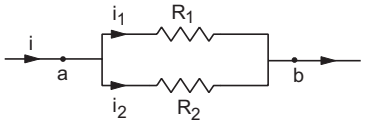



Figure 32.16

Consider the situation shown in figure (32.16). Using Ohm's law on resistors $R_1$ and $R_2$,

$$V_a - V_b = i_1 R_1 \text{ and } V_a - V_b = i_2 R_2.$$

Thus, $i_1 R_1 = i_2 R_2$

or, $$\frac{i_1}{i_2} = \frac{R_2}{R_1}. \quad \ldots \quad \text{(i)}$$

*We see that the current is divided in resistors, connected in parallel, in inverse ratio of the resistances.*

From (i), $$\frac{i_1}{i_1 + i_2} = \frac{R_2}{R_1 + R_2}$$

or, $$\frac{i_1}{i} = \frac{R_2}{R_1 + R_2}$$

or, $$i_1 = \frac{R_2}{R_1 + R_2}\, i.$$

Similarly,

$$i_2 = \frac{R_1}{R_1 + R_2}\, i.$$

## 32.9 GROUPING OF BATTERIES

**Series Connection**

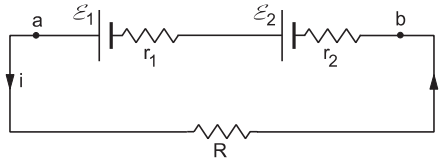



Figure 32.17

Suppose two batteries having emfs $\mathcal{E}_1$ and $\mathcal{E}_2$ and internal resistances $r_1$ and $r_2$ are connected in series as shown in figure (32.17). The points $a$ and $b$ act as the terminals of the combination. Suppose an external resistance $R$ is connected across the combination. From Kirchhoff's loop law,

$$Ri + r_2 i - \mathcal{E}_2 + r_1 i - \mathcal{E}_1 = 0$$

or, $$i = \frac{\mathcal{E}_1 + \mathcal{E}_2}{R + (r_1 + r_2)} = \frac{\mathcal{E}_0}{R + r_0}.$$

where $i$ is the current through the resistance $R$.

We see that the combination acts as a battery of emf $\mathcal{E}_0 = \mathcal{E}_1 + \mathcal{E}_2$ having an internal resistance $r_0 = r_1 + r_2$.

If the polarity of one of the batteries is reversed, the equivalent emf will be $|\mathcal{E}_1 - \mathcal{E}_2|$.

**Parallel Connection**

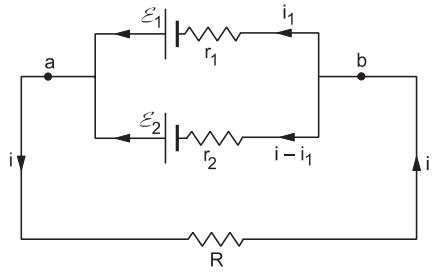



Figure 32.18

Now suppose the batteries are connected in parallel as shown in figure (32.18). The currents are also shown in the figure. Applying Kirchhoff's loop law in the loop containing $\mathcal{E}_1$, $r_1$ and $R$,

$$Ri + r_1 i_1 - \mathcal{E}_1 = 0. \quad \ldots \quad \text{(i)}$$

Similarly, applying Kirchhoff's law in the loop containing $\mathcal{E}_2, r_2$ and $R$,

$$Ri + r_2(i - i_1) - \mathcal{E}_2 = 0 \quad \ldots \quad \text{(ii)}$$

Multiply (i) by $r_2$, (ii) by $r_1$ and add. This gives,

$$iR(r_1 + r_2) + r_1 r_2 i - \mathcal{E}_1 r_2 - \mathcal{E}_2 r_1 = 0$$

or, $$i = \frac{\mathcal{E}_1 r_2 + \mathcal{E}_2 r_1}{R(r_1 + r_2) + r_1 r_2} = \frac{\dfrac{\mathcal{E}_1 r_2 + \mathcal{E}_2 r_1}{r_1 + r_2}}{R + \dfrac{r_1 r_2}{r_1 + r_2}} = \frac{\mathcal{E}_0}{R + r_0}.$$

We see that the combination acts as a battery of emf

$$\mathcal{E}_0 = \frac{\mathcal{E}_1 r_2 + \mathcal{E}_2 r_1}{r_1 + r_2}$$

and internal resistance

$$r_0 = \frac{r_1 r_2}{r_1 + r_2}.$$

If $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$ and $r_1 = r_2 = r$, $\mathcal{E}_0 = \mathcal{E}$ and $r_0 = r/2$.

## 32.10 WHEATSTONE BRIDGE

Wheatstone bridge is an arrangement of four resistances which can be used to measure one of them in terms of the rest.

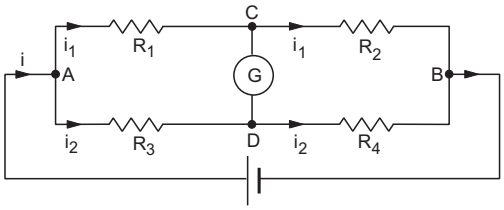



Figure 32.19

The arrangement is shown in figure (32.19). Four resistors with resistances $R_1, R_2, R_3$ and $R_4$ are connected to form a loop. There are four joints *A, B, C* and *D*. A battery is connected between two opposite joints $A$ and $B$ and a galvanometer is connected between the other two opposite joints $C$ and $D$.

We shall discuss the construction and working of a galvanometer later. Here, we only state that a galvanometer has a needle which deflects when an electric current passes through the galvanometer. The needle deflects towards left if the current passes in one direction and towards right if the current is reversed.

The current $i$ from the battery is divided at $A$ in two parts. A part $i_1$ goes through $R_1$ and the rest $i_2$ goes through $R_3$. For a particular relation between the resistances, there is no current through the galvanometer. The Wheatstone bridge is then said to be *balanced*. In this case, the current in $R_2$ is the same as the current in $R_1$ and the current in $R_4$ is the same as that in $R_3$. As there is no current through the galvanometer, the potential difference across its terminals is zero. Thus,

$$V_C = V_D.$$

Applying Ohm's law to $R_1$ and $R_2$,

$$V_A - V_C = i_1 R_1$$

and $$V_C - V_B = i_1 R_2.$$

Thus, $$\frac{V_A - V_C}{V_C - V_B} = \frac{R_1}{R_2}. \quad \ldots \text{(ii)}$$

Applying Ohm's law to $R_3$ and $R_4$,

$$V_A - V_D = i_2 R_3$$

and $$V_D - V_B = i_2 R_4.$$

Thus, $$\frac{V_A - V_D}{V_D - V_B} = \frac{R_3}{R_4}. \quad \ldots \text{(iii)}$$

As $V_C = V_D$, left sides of (ii) and (iii) are equal. Thus,

$$\frac{R_1}{R_2} = \frac{R_3}{R_4}. \quad \ldots (32.15)$$

This is the condition for which a Wheatstone bridge is balanced.

To measure the resistance of a resistor, it is connected as one of the four resistors in the bridge. One of the other three should be a variable resistor. Let us suppose $R_4$ is the resistance to be measured and $R_3$ is the variable resistance. When the Wheatstone bridge is connected, in general, there will be a deflection in the galvanometer. The value of the variable resistance $R_3$ is adjusted so that the deflection in the galvanometer becomes zero. In this case, the bridge is balanced and from equation (32.15),

$$R_4 = \frac{R_2}{R_1} R_3.$$

Knowing $R_1, R_2$ and $R_3$, the value of $R_4$ is calculated.

***Example 32.7***

*Find the value of R in figure (32.20) so that there is no current in the* 50 Ω *resistor.*

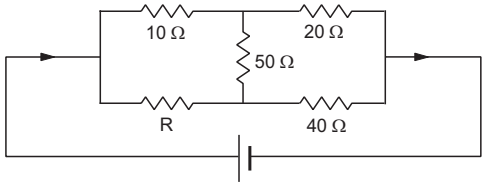



Figure 32.20

***Solution*** :

This is a Wheatstone bridge with the galvanometer replaced by the 50 Ω resistor. There will be no current in the 50 Ω resistor if the bridge is balanced.

In this case,

$$\frac{10\ \Omega}{20\ \Omega} = \frac{R}{40\ \Omega}$$

or, $R = 20\ \Omega.$

## 32.11 AMMETER AND VOLTMETER

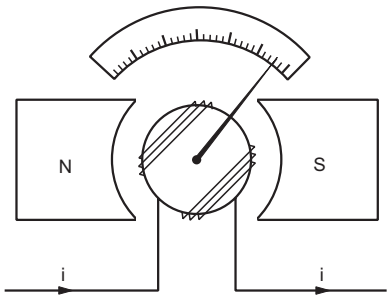



Figure 32.21

*Ammeter* is a device to measure an electric current and *voltmeter* is a device to measure a potential difference. In both the instruments there is a coil, suspended between the poles of a magnet. When a current is passed through the coil, it deflects. The angle of deflection is proportional to the current going through the coil. A needle is fixed to the coil. When the coil deflects, the needle moves on a graduated scale.

### Ammeter

In an ammeter, a resistor having a small resistance is connected in parallel with the coil. This resistor is called the *shunt*. The current to be measured is passed through the ammeter by connecting it in series with the segment which carries the current. Plus and minus signs are marked near the terminals of the ammeter. The current should enter the ammeter through the terminal marked "plus". When no current passes through the ammeter, the needle stays at zero which is marked at the left extreme of the scale.

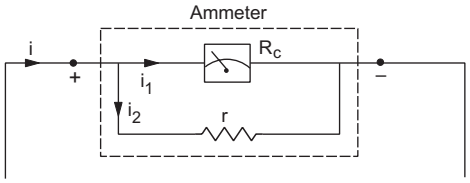



Figure 32.22

Suppose the coil has a resistance $R_c$ and the small resistance connected in parallel (shunt) has a value $r$. When a current $i$ is sent through the ammeter, the current gets divided in two parts. A part $i_1$ goes through the coil and the rest, $i - i_1$, through the shunt. As the potential difference across $R_c$ is the same as that across $r$,

$$i_1 R_c = (i - i_1) r$$

or, $$i_1 = \frac{r}{R_c + r}\, i. \qquad \ldots \text{(i)}$$

The deflection is proportional to $i_1$ and hence to $i$. The scale is graduated to read the value of $i$ directly.

The equivalent resistance of an ammeter is given by

$$R_{eq} = \frac{R_c r}{R_c + r}\,.$$

When the ammeter is connected in a segment of a circuit, the resistance of the segment increases by this amount $R_{eq}$. This reduces the main current which we wish to measure. To minimise this error, the equivalent resistant $R_{eq}$ should be small. This is one reason why the shunt having a small resistance $r$ is connected in parallel to the coil. This makes $R_{eq}$ small.

Galvanometer is very similar to an ammeter in construction. When no current passes through it, the needle stays in the middle of the graduated scale. This point is marked zero. Current can be passed through the galvanometer in either direction. The needle deflects accordingly towards left or towards right.

***Example 32.8***

*The ammeter shown in figure (32.23) consists of a* 480 Ω *coil connected in parallel to a* 20 Ω *shunt. Find the reading of the ammeter.*

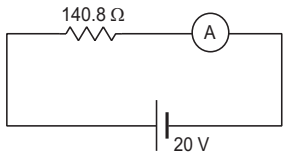



Figure 32.23

***Solution*** : The equivalent resistance of the ammeter is

$$\frac{(480\ \Omega)\ (20\ \Omega)}{480\ \Omega + 20\ \Omega} = 19{\cdot}2\ \Omega.$$

The equivalent resistance of the circuit is

$$140{\cdot}8\ \Omega + 19{\cdot}2\ \Omega = 160\ \Omega\ .$$

The current is $i = \dfrac{20\ \text{V}}{160\ \Omega} = 0{\cdot}125\ \text{A}.$

This current goes through the ammeter and hence the reading of the ammeter is 0·125 A.

### Voltmeter

In a voltmeter, a resistor having a high resistance $R$ is connected in series with the coil. The end points (terminals) are connected to the points $A$ and $B$ between which the potential difference is to be measured. Plus and minus signs are marked on the terminals. The terminal marked "plus" should be connected to the point at higher potential. When no potential difference is applied between the terminals, the needle stays at zero which is marked at the left

extreme of the scale. When the potential difference is applied, a current passes through the coil and the high resistance. If $R_c$ be the resistance of the coil and $V$ be the potential difference applied to the voltmeter, the current in the coil is

$$i = \frac{V}{R_c + R}.$$

The deflection is proportional to the current $i$ and hence to $V$. The scale is graduated to read the potential difference directly.

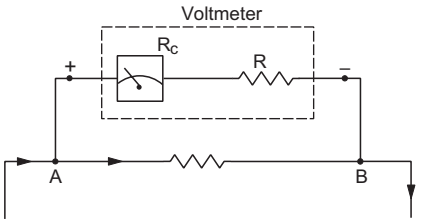



Figure 32.24

When the voltmeter is used in a circuit, its resistance $R_{eq} = R_c + R$ is connected in parallel to some element of the circuit. This changes the overall current in the circuit and hence, the potential difference to be measured is also changed. To minimise the error due to this, the equivalent resistance $R_{eq}$ of the voltmeter should be large. (When a large resistance is connected in parallel to a small resistance, the equivalent resistance is only slightly less than the smaller one.) That is why, a large resistance $R$ is added in series with the coil of a voltmeter.

### 32.12 STRETCHED-WIRE POTENTIOMETER

An ideal voltmeter which does not change the original potential difference, should have infinite resistance. But in the design described above, the resistance cannot be made infinite. *Potentiometer* is a device which does not draw any current from the given circuit and still measures the potential difference. Thus, it is equivalent to an ideal voltmeter.

The stretched-wire potentiometer consists of a long wire $AB$, usually 5 to 10 metres long, fixed on a wooden platform (figure 32.25). The wire has a uniform cross section. Usually, separate pieces of wire, each 1 m long, are fixed parallel to each other on the platform. The wires are joined to each other by thick copper strips so that the combination acts as a single wire of desired length (5 to 10 metres). The ends $A$ and $B$ are connected to a driving circuit consisting of a strong battery, a plug key and a rheostat. The driving circuit sends a constant current $i$ through the wire $AB$. Thus, the potential gradually decreases from $A$ to $B$. One end of a galvanometer is connected to a metal rod fixed on

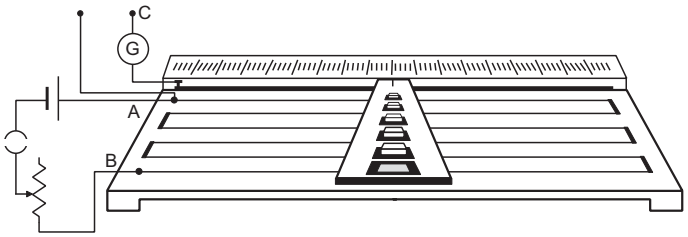



Figure 32.25

the wooden platform. A "jockey" may be slid on this metal rod and may touch the wire $AB$ at any desired point. In this way the galvanometer gets connected to the point of $AB$ which is touched by the jockey. The length of the wire between the end $A$ and this point can be measured with the help of a metre scale fixed on the platform. The other end $C$ of the galvanometer and the high-potential end $A$ of the wire, form the two end points (terminals) of the potentiometer. These points are connected to the points between which the potential difference is to be measured.

Suppose, we have to measure the potential difference between the points $a$ and $b$. Also let $a$ be at a higher potential and $b$ at a lower potential. The end $A$ of the wire $AB$ is connected to the point $a$ and the end $C$ of the galvanometer is connected to the point $b$. The circuit is represented schematically in figure (35.26).

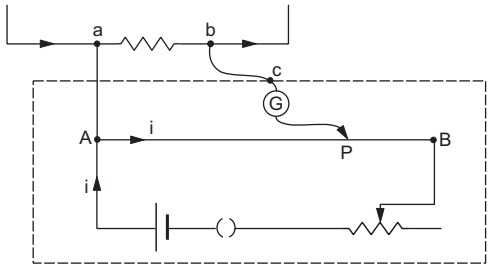



Figure 32.26

The connecting wire $Aa$ has a negligible resistance and hence potentials of $A$ and $a$ are equal. Suppose, the potential drop across $ab$ is smaller than the potential drop across $AB$. Then there will be a point $P$ on $AB$ which will have the same potential as $b$. If the jockey is slid to touch the wire at this point $P$, the potential difference across the galvanometer is zero and there will be no current through it. The process of measurement is to search for a point $P$ so that there is no deflection in the galvanometer.

Suppose, the driving circuit sets up a potential difference $V_0$ between the ends $A$ and $B$ of the potentiometer wire. As the wire is uniform, the resistance of a piece of the wire is proportional to its length. Hence, the potential difference across a piece of wire is also proportional to its length. If $AB = L$ and

$AP = l$, the potential difference between the points $A$ and $P$ is

$$V = V_0 \frac{l}{L} . \quad \ldots \text{(i)}$$

This is equal to the potential difference between $a$ and $b$ which we had to measure.

In order to get the value of the potential difference $V$, the total potential drop $V_0$ on $AB$ must be known. One way to do this is to use a standard cell having known and constant emf in place of $ab$. If the emf of the standard cell is $\mathcal{E}$ and the potentiometer is balanced (no deflection in the galvanometer) when $AP = l_0$, we have, from (i),

$$\mathcal{E} = V_0 \frac{l_0}{L}$$

or,
$$V_0 = \frac{L}{l_0} \mathcal{E} .$$

The potential difference $V$ between $a$ and $b$ is, from (i)

$$V = \frac{l}{l_0} \mathcal{E} .$$

This process of finding $V_0$ is called *calibration of the potentiometer*. Note that there is no current through the standard cell when the potentiometer is balanced during its calibration. Thus, the emf $\mathcal{E}$ equals the potential difference between its terminals.

**Comparison of Emf's of Two Batteries**

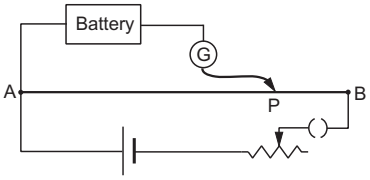



Figure 32.27

The driving circuit of the potentiometer is set up with a strong battery so that the potential difference $V_0$ across $AB$ is larger than the emf of either battery. One of the batteries is connected between the positive end $A$ and the galvanometer. The jockey is adjusted to touch the wire at a point $P_1$ so that there is no deflection in the galvanometer. The length $AP_1 = l_1$ is noted. Now, the first battery is replaced by the second and the length $AP_2 = l_2$ for the balance is noted. If the length $AB = L$, the emf of the first battery is, from (i) above,

$$\mathcal{E}_1 = \frac{l_1}{L} V_0$$

and that of the second battery is

$$\mathcal{E}_2 = \frac{l_2}{L} V_0.$$

Thus,
$$\frac{\mathcal{E}_1}{\mathcal{E}_2} = \frac{l_1}{l_2} .$$

Note that no calibration is needed in this case.

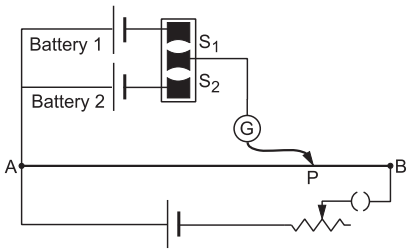



Figure 32.28

One can use a two-way key to connect both the batteries together as shown in figure (32.28). When the key is pressed in the plug $S_1$, the first battery is brought into the circuit. When the key is taken out from $S_1$ and pressed in the plug $S_2$, the second battery is brought into the circuit.

The value of emf of a battery can also be obtained by this same method by taking the other battery to be a standard cell. The emf of the standard cell is known and hence the emf of the given battery can be obtained.

**Measurement of Internal Resistance of a Battery**

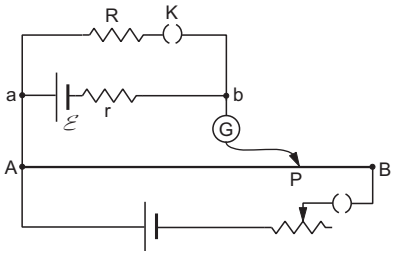



Figure 32.29

Figure (32.29) shows the arrangement for measuring the internal resistance of a battery. The emf of the battery is $\mathcal{E}$ and its internal resistance is $r$. A known resistance $R$ is connected across the battery together with a plug key $K$. The potentiometer circuit is set up as usual. The plug key $K$ is opened and the balance point $P$ is searched on the wire $AB$ so that there is no deflection in the galvanometer. As the key is open, there is no current through the resistance $R$. Hence, there is no current through the battery and the potential difference across the terminals $a$, $b$ is the same as the emf $\mathcal{E}$ of the battery. If $AP = l$, we have

$$\mathcal{E} = \frac{l}{L} V_0 \quad \ldots \text{(i)}$$

with the symbols having their usual meanings.

Now the key $K$ is closed and the new balance point $P'$ is searched. There is a current

$$i = \frac{\mathcal{E}}{R + r} \text{ through the battery.}$$

The potential difference between $a$ and $b$ is

$$V_a - V_b = Ri = \frac{\mathcal{E}R}{R + r}.$$

If $AP' = l'$, we have

$$\frac{\mathcal{E}R}{R + r} = \frac{l'}{L} V_0. \qquad \ldots \text{ (ii)}$$

Dividing (ii) by (i),

$$\frac{R}{R + r} = \frac{l'}{l}$$

or, $$r = \frac{R(l - l')}{l'}.$$

## 32.13 CHARGING AND DISCHARGING OF CAPACITORS

### Charging

When a capacitor is connected to a battery, positive charge appears on one plate and negative charge on the other. The potential difference between the plates ultimately becomes equal to the emf of the battery. The whole process takes some time and during this time there is an electric current through the connecting wires and the battery. Figure (32.30a) shows a typical connection. The resistance of the connecting wires and the internal resistance of the battery taken together is shown as the resistance $R$. The capacitor has capacitance $C$.

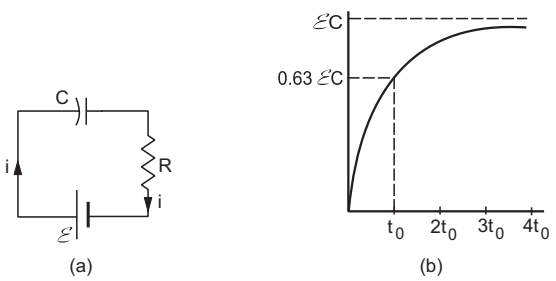



Figure 32.30

Suppose, the battery is connected at $t = 0$. Suppose the charge on the capacitor and the current in the circuit are $q$ and $i$ respectively at time $t$. The potential drop on the capacitor is $q/C$ and on the resistor it is $Ri$. Also, the charge deposited on the positive plate in time $dt$ is $dq = idt$

so that $$i = \frac{dq}{dt}.$$

Using Kirchhoff's loop law,

$$\frac{q}{C} + Ri - \mathcal{E} = 0$$

or, $$Ri = \mathcal{E} - \frac{q}{C}$$

or, $$R\frac{dq}{dt} = \frac{\mathcal{E}C - q}{C}$$

or, $$\int_0^q \frac{dq}{\mathcal{E}C - q} = \int_0^t \frac{1}{CR}\, dt$$

or, $$-\ln \frac{\mathcal{E}C - q}{\mathcal{E}C} = \frac{t}{CR}$$

or, $$1 - \frac{q}{\mathcal{E}C} = e^{-t/CR}$$

or, $$q = \mathcal{E}C(1 - e^{-t/CR}). \qquad \ldots \text{ (32.16)}$$

This gives the charge on the capacitor at time $t$. As $t$ increases, $q$ also increases. The maximum charge is obtained, in principle, at $t = \infty$ and its value is $\mathcal{E}C$. The constant $CR$ has dimensions of time and is called *time constant* of the circuit. In one time constant $\tau (= CR)$, the charge accumulated on the capacitor is

$$q = \mathcal{E}C\left(1 - \frac{1}{e}\right) = 0{\cdot}63\, \mathcal{E}C.$$

Thus, 63% of the maximum charge is deposited in one time constant. Figure (32.30b) shows a plot of $q$ versus $t$.

### Discharging

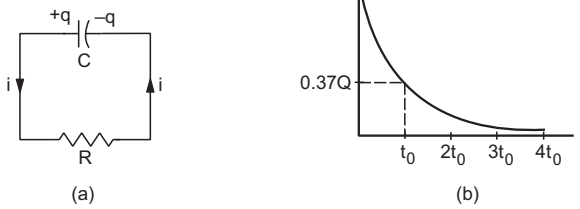



Figure 32.31

If the plates of a charged capacitor are connected through a conducting wire, the capacitor gets discharged. Again there is a flow of charge through the wires and hence there is a current. Suppose a capacitor of capacitance $C$ has a charge $Q$. At $t = 0$, the plates are connected through a resistance $R$ (figure 32.31a). Let the charge on the capacitor be $q$ and the current in the circuit be $i$ at time $t$.

Using Kirchhoff's loop law,

$$\frac{q}{C} - Ri = 0.$$

Here $i = -\frac{dq}{dt}$ because the charge $q$ decreases as time passes.

Thus, $$R\frac{dq}{dt} = -\frac{q}{C}.$$

or, $$\frac{dq}{q} = -\frac{1}{CR}dt$$

or, $$\int_{Q}^{q}\frac{dq}{q} = \int_{0}^{t} -\frac{1}{CR}dt$$

or, $$\ln\frac{q}{Q} = -\frac{t}{CR}$$

or, $$q = Q\,e^{-t/CR}. \quad \ldots (32.17)$$

In principle, discharging is complete only at $t = \infty$. The constant $CR$ is the time constant. At $t = CR$, the remaining charge is $q = \frac{Q}{e} = 0{\cdot}37\,Q$. Thus, 63% of the discharging is complete in one time constant. Figure (32.31b) shows the charge as a function of time.

***Example 32.9***

*A capacitor of capacitance* 100 μF *is charged by connecting it to a battery of emf* 12 V *and internal resistance* 2 Ω. *(a) Find the time constant of the circuit. (b) Find the time taken before* 99% *of the maximum charge is stored on the capacitor.*

***Solution*** : The time constant is

$$\tau = CR = (100\ \mu\text{F})\ (2\ \Omega) = 200\ \mu\text{s}.$$

The charge at time $t$ is

$$q = \mathscr{E}C\,(1 - e^{-t/CR}).$$

Putting $$q = 0{\cdot}99\,\mathscr{E}C,$$

$$0{\cdot}99 = 1 - e^{-t/(200\ \mu\text{s})}$$

or, $$-\frac{t}{200\ \mu\text{s}} = \ln(0{\cdot}01)$$

or, $$t = 920\ \mu\text{s} = 0{\cdot}92\ \text{ms}.$$

***Example 32.10***

*The plates of a* 50 μF *capacitor charged to* 400 μC *are connected through a resistance of* 1·0 kΩ. *Find the charge remaining on the capacitor* 1 s *after the connection is made.*

***Solution*** : The time constant is

$$CR = (50\ \mu\text{F})\ (1{\cdot}0\ \text{k}\Omega) = 50\ \text{ms}.$$

At $t = 1$ s, $t/CR = 1\ \text{s}/50\ \text{ms} = 20$.

The charge remaining on the capacitor is

$$q = Q\,e^{-t/CR}$$

$$= (400\ \mu\text{C})\ e^{-20} = 8{\cdot}2 \times 10^{-7}\ \mu\text{C}.$$

We see that in a typical charging or discharging circuit, the time constant is of the order of a millisecond. Also, four to five time constants are sufficient for 99% of the charging or discharging. Thus, for practical purposes, we can assume that charging or discharging is complete in a fraction of a second.

## 32.14 ATMOSPHERIC ELECTRICITY

The earth and the atmosphere surrounding it show very interesting electric phenomena. The earth has a negative charge spread with approximately uniform density over its surface. The average surface charge density on the earth is little less than one nanocoulomb per square metre. There is a corresponding electric field of about 100 V m$^{-1}$ in the atmosphere above the earth. This field is in the vertically downward direction. This means, if you look at a flat desert, the electric potential increases by about 100 V as you move up by 1 m. The potential keeps on increasing as one goes higher in atmosphere but the magnitude of the electric field gradually decreases. At about 50 km from the earth's surface, the field is negligible. The total potential difference between the earth's surface and the top of the atmosphere is about 400 kV.

The atmosphere contains a number of ions, both positively charged and negatively charged. The main source of these ions is *cosmic rays* which come from outside the earth, even from outside the solar system. These rays come down to the earth and ionize molecules in the air. Air contains dust particles which become charged by friction as they move through the air. This is another source of the presence of charged ions in air. Because of the electric field in the atmosphere, positive ions come down and negative ions go up. Thus, there is an electric current in the atmosphere. This current is about $3{\cdot}5 \times 10^{-12}$ A over a square metre area parallel to the earth's surface. When the total surface area of the earth is considered, 1800 A of current reaches the earth.

The density of ions increases with height over the earth's surface. Also, the density of air decreases and the ions can travel larger distances between collisions. Both these factors contribute to the fact that "conductivity of air" increases with altitude. At about 50 km above the earth's surface, the air becomes highly conducting. We can draw an equivalent picture by assuming that at about this height there is a perfectly conducting surface having a potential of 400 kV and current comes down from this surface to the earth.

If 1800 A of current flows towards the earth, the entire negative charge of the earth should get neutralised in about half an hour and the electric field in the atmosphere should reduce to zero. But it is not so. So, there must be some mechanism which brings negative charge back to the earth, so that the 400 kV potential difference is maintained. This situation is like that of a battery. The current provided by a battery discharges it. There is a source of emf which maintains the potential difference across the battery's terminals. So, what is the source that charges our atmospheric battery. The answer is *thunderstorms* and *lightning*.

Because of the difference in temperature and pressure between different parts of the atmosphere, air packets keep on moving in a rather systematic fashion. As the upper atmosphere is cool (temperature is around –10°C at a height of 3–4 km and –20°C at a height of 6–7 km), water vapour condenses to form small water droplets and tiny ice particles. A parcel of air with these droplets and ice particles forms a thunderstorm. A typical thunderstorm may have an average horizontal extension of about 7–8 km and a vertical extension of about 3 km. A matured thunderstorm is formed with its lower end at a height of about 3–4 km above the earth's surface and the upper end at about 6–7 km above the earth's surface.

The upper part of a thunderstorm contains excess positive charge and the lower part contains excess negative charge. The density of negative charge in the clouds in the lower part of the storm is very high. This negative charge creates a potential difference of 20 to 100 MV between these clouds and the earth. Note that

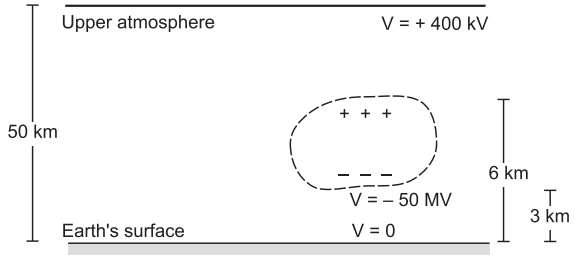



Figure 32.32

this potential difference is much larger than the 400 kV between the earth and the top of atmosphere and is opposite in sign. Figure (32.32) represents a typical situation.

The high electric field between the lower part of the storm and the earth is often sufficient for the dielectric breakdown of air and the air becomes conducting. Negative charge thus jumps from the cloud to the earth's surface. This phenomenon is called *lightning*. The positive charge in the upper part of the storm gradually moves up to enter the high-altitude ($\approx 50$ km) layer of high conductivity. In one lightning stroke, about 20 C of negative charge is deposited to the earth. It takes about 5 s for the clouds to regain the charge for the next lightning stroke. There are a number of thunderstorms everyday throughout the earth. They charge the atmospheric battery by supplying negative charge to the earth and positive charge to the upper atmosphere. In the area of clear weather, the battery is discharged by the movement of positive ions towards the earth and negative ions away from it (the 1800 A current discussed earlier).

### *Worked Out Examples*

**1.** *An electron moves in a circle of radius* 10 cm *with a constant speed of* $4{\cdot}0 \times 10^{6}\ \text{m s}^{-1}$. *Find the electric current at a point on the circle.*

***Solution*** **:** Consider a point $A$ on the circle. The electron crosses this point once in every revolution. In one revolution, the electron travels $2\pi \times (10\ \text{cm})$ distance. Hence, the number of revolutions made by the electron in one second is

$$\frac{4{\cdot}0 \times 10^{6}\ \text{m}}{20\pi \times 10^{-2}\ \text{m}} = \frac{2}{\pi} \times 10^{7}.$$

The charge crossing the point $A$ per second is

$$\frac{2}{\pi} \times 10^{7} \times 1{\cdot}6 \times 10^{-19}\ \text{C} = \frac{3{\cdot}2}{\pi} \times 10^{-12}\ \text{C}.$$

Thus, the electric current at this point is

$$\frac{3{\cdot}2}{\pi} \times 10^{-12}\ \text{A} \approx 1{\cdot}0 \times 10^{-12}\ \text{A}.$$

**2.** *A current of* 2·0 A *exists in a wire of cross sectional area* $1{\cdot}0\ \text{mm}^{2}$. *If each cubic metre of the wire contains* $6{\cdot}0 \times 10^{28}$ *free electrons, find the drift speed.*

***Solution*** **:** The current density in the wire is

$$j = \frac{i}{A} = \frac{2{\cdot}0\ \text{A}}{1\ \text{mm}^{2}} = 2{\cdot}0 \times 10^{6}\ \text{A m}^{-2}.$$

The drift speed is

$$v = \frac{j}{ne} = \frac{2{\cdot}0 \times 10^{6}\ \text{A m}^{-2}}{6{\cdot}0 \times 10^{28}\ \text{m}^{-3} \times 1{\cdot}6 \times 10^{-19}\ \text{C}}$$

$$= 2{\cdot}1 \times 10^{-4}\ \text{m s}^{-1}.$$

**3.** *Find the resistance of a copper coil of total wire-length* 10 m *and area of cross section* 1·0 $mm^2$. *What would be the resistance of a similar coil of aluminium ? The resistivity of copper* $= 1{\cdot}7 \times 10^{-8}\ \Omega$ m *and that of aluminium* $= 2{\cdot}6 \times 10^{-8}\ \Omega$ m.

***Solution*** **:** The resistance of the copper coil is

$$\rho \frac{l}{A} = \frac{(1{\cdot}7 \times 10^{-8}\ \Omega\ \text{m}) \times 10\ \text{m}}{1{\cdot}0 \times 10^{-6}\ \text{m}^2} = 0{\cdot}17\ \Omega.$$

The resistance of the similar aluminium coil will be

$$\frac{(2{\cdot}6 \times 10^{-8}\ \Omega\ \text{m}) \times 10\ \text{m}}{1{\cdot}0 \times 10^{-6}\ \text{m}^2} = 0{\cdot}26\ \Omega.$$

**4.** *A parallel-plate capacitor has plates of area* 10 $cm^2$ *separated by a distance of* 1 mm. *It is filled with the dielectric mica and connected to a battery of emf* 6 volts. *Find the leakage current through the capacitor. Resistivity of mica* $= 1 \times 10^{13}\ \Omega$ m.

***Solution*** **:** The resistance of the mica between the two faces is

$$\rho \frac{l}{A} = \frac{(1 \times 10^{13}\ \Omega\ \text{m}) \times 10^{-3}\ \text{m}}{10{\cdot}0 \times 10^{-4}\ \text{m}^2} = 1 \times 10^{13}\ \Omega.$$

The leakage current $= \dfrac{6\ \text{V}}{1 \times 10^{13}\ \Omega} = 6 \times 10^{-13}$ A.

**5.** *Find the resistance of a hollow cylindrical conductor of length* 1·0 m *and inner and outer radii* 1·0 mm *and* 2·0 mm *respectively. The resistivity of the material is* $2{\cdot}0 \times 10^{-8}\ \Omega$ m.

***Solution*** **:** The area of cross section of the conductor through which the charges will flow is

$$A = \pi(2{\cdot}0\ \text{mm})^2 - \pi(1{\cdot}0\ \text{mm})^2 = 3{\cdot}0 \times \pi\ \text{mm}^2.$$

The resistance of the wire is, therefore,

$$R = \rho \frac{l}{A} = \frac{(2{\cdot}0 \times 10^{-8}\ \Omega\ \text{m}) \times 1{\cdot}0\ \text{m}}{3{\cdot}0 \times \pi \times 10^{-6}\ \text{m}^2} = 2{\cdot}1 \times 10^{-3}\ \Omega.$$

**6.** *A battery of emf* 2 V *and internal resistance* 0·5 Ω *is connected across a resistance of* 9·5 Ω. *How many electrons cross through a cross section of the resistance in* 1 *second ?*

***Solution*** **:** The current in the circuit is

$$i = \frac{2\ \text{V}}{9{\cdot}5\ \Omega + 0{\cdot}5\ \Omega} = 0{\cdot}2\ \text{A}.$$

Thus, a net transfer of 0·2 C per second takes place across any cross section in the circuit. The number of electrons crossing the section in 1 second is, therefore,

$$\frac{0{\cdot}2\ \text{C}}{1{\cdot}6 \times 10^{-19}\ \text{C}} = 0{\cdot}125 \times 10^{19} = 1{\cdot}25 \times 10^{18}.$$

**7.** *A battery of emf* 2·0 volts *and internal resistance* 0·10 Ω *is being charged with a current of* 5·0 A. *What is the potential difference between the terminals of the battery ?*

***Solution*** **:**

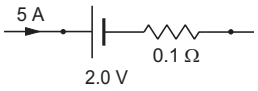



Figure 32-W1

As the battery is being charged, the current goes into the positive terminal as shown in figure (32-W1).

The potential drop across the internal resistance is

$$5{\cdot}0\ \text{A} \times 0{\cdot}10\ \Omega = 0{\cdot}50\ \text{V}.$$

Hence, the potential drop across the terminals will be

$$2{\cdot}0\ \text{V} + 0{\cdot}50\ \text{V} = 2{\cdot}5\ \text{V}.$$

**8.** *Figure (32-W2) shows n batteries connected to form a circuit. The resistances denote the internal resistances of the batteries which are related to the emf's as* $r_i = k\mathcal{E}_i$ *where k is a constant. The solid dots represent the terminals of the batteries. Find (a) the current through the circuit and (b) the potential difference between the terminals of the ith battery.*

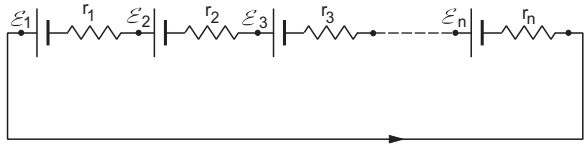



Figure 32-W2

***Solution*** **:** (a) Suppose the current is $i$ in the indicated direction. Applying Kirchoff's loop law,

$$\mathcal{E}_1 - ir_1 + \mathcal{E}_2 - ir_2 + \mathcal{E}_3 - ir_3 + \ldots + \mathcal{E}_n - ir_n = 0$$

or, $$i = \frac{\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \ldots + \mathcal{E}_n}{r_1 + r_2 + r_3 + \ldots + r_n}$$

$$= \frac{\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \ldots + \mathcal{E}_n}{k(\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \ldots + \mathcal{E}_n)} = \frac{1}{k}.$$

(b) The potential difference between the terminals of the $i$th battery is

$$\mathcal{E}_i - ir_i = \mathcal{E}_i - \left(\frac{1}{k}\right)(k\mathcal{E}_i) = 0.$$

**9**. *A copper rod of length* 20 cm *and cross-sectional area* 2 $mm^2$ *is joined with a similar aluminium rod as shown in figure (32-W3). Find the resistance of the combination between the ends. Resistivity of copper* $= 1{\cdot}7 \times 10^{-8}\ \Omega$ m *and that of aluminium* $= 2{\cdot}6 \times 10^{-8}\ \Omega$ m.

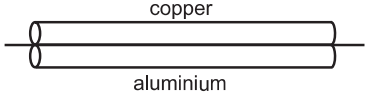



Figure 32-W3

***Solution*** : The resistance of the copper rod

$$= \rho \frac{l}{A} = \frac{(1{\cdot}7 \times 10^{-8}\ \Omega\ \text{m}) \times (20 \times 10^{-2}\ \text{m})}{2{\cdot}0 \times 10^{-6}\ \text{m}^2}$$

$$= 1{\cdot}7 \times 10^{-3}\ \Omega.$$

Similarly, the resistance of the aluminium rod

$$= 2{\cdot}6 \times 10^{-3}\ \Omega.$$

These rods are joined in parallel so that the equivalent resistance $R$ between the ends is given by

$$\frac{1}{R} = \frac{1}{1{\cdot}7 \times 10^{-3}\ \Omega} + \frac{1}{2{\cdot}6 \times 10^{-3}\ \Omega}$$

or, $$R = \frac{1{\cdot}7 \times 2{\cdot}6}{4{\cdot}3} \times 10^{-3}\ \Omega \approx 1{\cdot}0\ \text{m}\Omega.$$

**10.** *A wire of resistance* 10 Ω *is bent to form a complete circle. Find its resistance between two diametrically opposite points.*

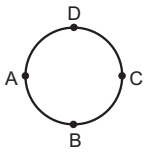



Figure 32-W4

***Solution*** :

Let *ABCDA* be the wire of resistance 10 Ω. We have to calculate the resistance of this loop between the diametrically opposite points *A* and *C*. The wires *ADC* and *ABC* will have resistances 5 Ω each. These two are joined in parallel between *A* and *C*. The equivalent resistance $R$ between $A$ and $C$ is, therefore, given by

$$R = \frac{5\ \Omega \times 5\ \Omega}{5\ \Omega + 5\ \Omega} = 2{\cdot}5\ \Omega.$$

**11.** *Find the currents in the different resistors shown in figure (32-W5).*

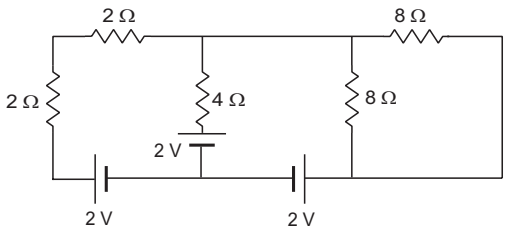



Figure 32-W5

***Solution*** : The two 2 Ω resistors are in series so that their equivalent resistance is 4 Ω. The two 8 Ω resistors are in parallel and their equivalent resistance is also 4 Ω. The circuit may be redrawn as in figure (32-W6a).

Suppose the middle 4 Ω resistor is removed. The remaining circuit is redrawn in figure (32-W6b). It is easy to see that no current will go through any resistor. If we take the potential at $b$ to be zero, the potential at

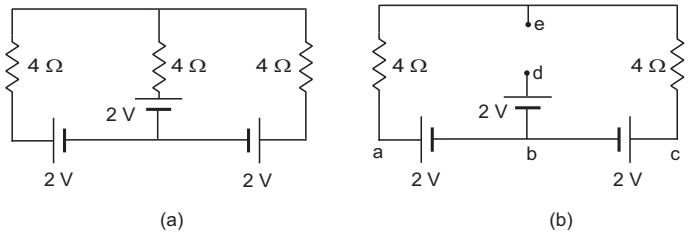



Figure 32-W6

$d$ will be 2 V. The potential at $a$ and $c$ will also be 2 V. As there is no current in the 4 Ω resistors, the potential at $e$ will also be 2 V. Thus, there is no potential difference between $d$ and $e$. When a 4 Ω resistor is added between $d$ and $e$, no current will be drawn into it and hence no change will occur in the remaining part of the circuit. This circuit is then the same as the given circuit. Thus, the current in all the resistors in the given circuit is zero.

**12.** *Find the current supplied by the battery in the circuit shown in figure (32-W7).*

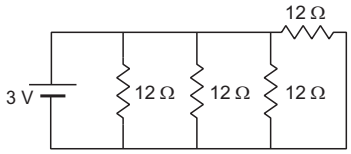



Figure 32-W7

***Solution*** : All the resistors shown in the figure are connected in parallel between the terminals of the battery. The equivalent resistance $R$ between the terminals is, therefore, given by

$$\frac{1}{R} = \frac{1}{12\ \Omega} + \frac{1}{12\ \Omega} + \frac{1}{12\ \Omega} + \frac{1}{12\ \Omega}$$

or, $$R = 3\ \Omega.$$

The current supplied by the battery is

$$i = \frac{V}{R} = \frac{3\ \text{V}}{3\ \Omega} = 1\ \text{A}.$$

**13.** *Find the equivalent resistance between the points a and b of the network shown in figure (32-W8).*

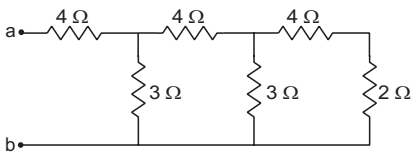



Figure 32-W8

***Solution*** : The two resistors 4 Ω and 2 Ω at the right end are joined in series and may be replaced by a single resistor of 6 Ω. This 6 Ω is connected with the adjacent 3 Ω resistor in parallel. The equivalent resistance of these two is

$$\frac{6\ \Omega \times 3\ \Omega}{6\ \Omega + 3\ \Omega} = 2\ \Omega.$$

This is connected in series with the adjacent 4 Ω resistor giving an equivalent resistance of 6 Ω which is connected in parallel with the 3 Ω resistor. Their equivalent resistance is 2 Ω which is connected in series with the first 4 Ω resistor from left. Thus, the equivalent resistance between $a$ and $b$ is 6 Ω.

**14.** *Find the effective resistance between the points A and B in figure (32-W9).*

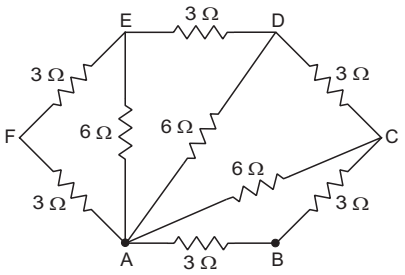



Figure 32-W9

***Solution*** : The resistors $AF$ and $FE$ are in series. Their equivalent resistance is $3\ \Omega + 3\ \Omega = 6\ \Omega$. This is connected in parallel with $AE$. Their equivalent between $A$ and $E$ is, therefore,

$$\frac{6\ \Omega \times 6\ \Omega}{6\ \Omega + 6\ \Omega} = 3\ \Omega.$$

This 3 Ω resistance between $A$ and $E$ is in series with $ED$ and the combination is in parallel with $AD$. Their equivalent between $A$ and $D$ is again 3 Ω.

Similarly, the equivalent of this 3 Ω, $DC$ and $AC$ is 3 Ω. This 3 Ω is in series with $CB$ and the combination is in parallel with $AB$. The equivalent resistance between $A$ and $B$ is, therefore,

$$\frac{6\ \Omega \times 3\ \Omega}{6\ \Omega + 3\ \Omega} = 2\ \Omega.$$

**15.** *Find the equivalent resistance of the network shown in figure (32-W10) between the points a and b when (a) the switch S is open and (b) the switch S is closed.*

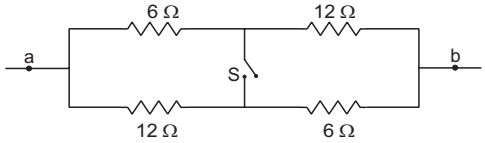



Figure 32-W10

***Solution*** : (a) When the switch is open, 6 Ω and 12 Ω resistors on the upper line are in series giving an equivalent of 18 Ω. Similarly, the resistors on the lower line have equivalent resistance 18 Ω. These two 18 Ω resistances are connected in parallel between $a$ and $b$ so that the equivalent resistance is 9 Ω.

(b) When the switch is closed, the 6 Ω and 12 Ω resistors on the left are in parallel giving an equivalent resistance of 4 Ω. Similarly, the two resistors on the right half are equivalent to 4 Ω. These two are connected in series between $a$ and $b$ so that the equivalent resistance is 8 Ω.

**16.** *Each resistor shown in figure (32-W11) has a resistance of* 10 Ω *and the battery has an emf of* 6 V. *Find the current supplied by the battery.*

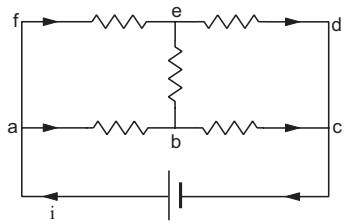



Figure 32-W11

***Solution*** : Suppose a current $i$ starts from the positive terminal of the battery. By symmetry, it divides equally in the resistors $ab$ and $fe$, so that each of these carries a current $i/2$. The current going into the negative terminal is also $i$ and by symmetry, equal currents should come from $ed$ and $bc$. Thus, the current in $ed$ is also $i/2$ and hence there will be no current in $eb$.

We have,

$$V_a - V_c = (V_a - V_b) + (V_b - V_c)$$

or, $$6\ \text{V} = \frac{i}{2} \times 10\ \Omega + \frac{i}{2} \times 10\ \Omega$$

giving $$i = 0{\cdot}6\ \text{A}.$$

This is a balanced Wheatstone bridge.

**17.** *Find the equivalent resistance of the network shown in figure (32-W12) between the points A and B.*

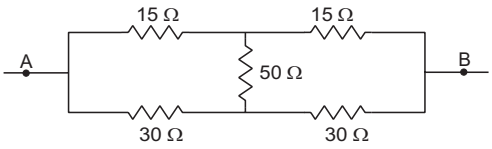



Figure 32-W12

***Solution*** : Suppose an ideal battery of emf $\mathcal{E}$ is connected across the points $A$ and $B$. The circuit is a Wheatstone bridge with the galvanometer replaced by a 50 Ω resistance. As the bridge is balanced ($R_1/R_2 = R_3/R_4$), there will be no current through the 50 Ω resistance. We can just remove the 50 Ω resistance without changing any other current. The circuit is then equivalent to two resistances 30 Ω and 60 Ω connected in parallel. The equivalent resistance is

$$R = \frac{(30\ \Omega) \times (60\ \Omega)}{(30\ \Omega) + (60\ \Omega)} = 20\ \Omega.$$

**18.** *In the circuit shown in figure (32-W13a) E, F, G and H are cells of emf* 2, 1, 3 *and* 1 V *respectively. The resistances* 2, 1, 3 *and* 1 Ω *are their respective internal resistances. Calculate (a) the potential difference between B and D and (b) the potential differences across the terminals of each of the cells G and H.*

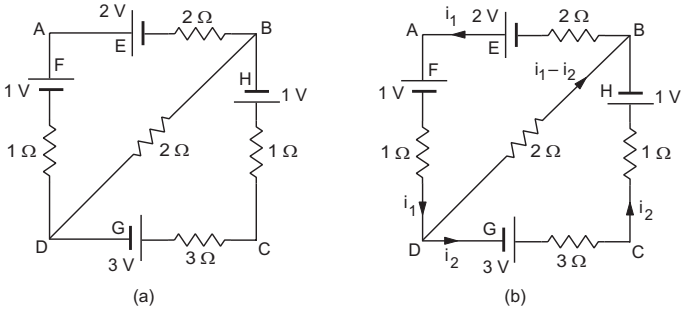



Figure 32-W13

***Solution*** **:** Suppose a current $i_1$ goes in the branch *BAD* and a current $i_2$ in the branch *DCB*. The current in *DB* will be $i_1 - i_2$ from the junction law. The circuit with the currents shown is redrawn in figure (32-W13b). Applying the loop law to *BADB* we get,

$$(2\ \Omega)i_1 - 2\ \text{V} + 1\ \text{V} + (1\ \Omega)i_1 + (2\ \Omega)(i_1 - i_2) = 0$$

or, $(5\ \Omega)i_1 - (2\ \Omega)i_2 = 1\ \text{V}.$ … (i)

Applying the same law to the loop *DCBD,* we get

$$-3\ \text{V} + (3\ \Omega)i_2 + (1\ \Omega)i_2 + 1\ \text{V} - (2\ \Omega)\ (i_1 - i_2) = 0$$

or, $-(2\ \Omega)i_1 + (6\ \Omega)i_2 = 2\ \text{V}.$ … (ii)

From (i) and (ii),

$$i_1 = \frac{5}{13}\ \text{A},\ i_2 = \frac{6}{13}\ \text{A}$$

so that $i_1 - i_2 = -\frac{1}{13}\ \text{A}.$

The current in *BD* is from *B* to *D*.

(a) $V_B - V_D = (2\ \Omega)\left(\frac{1}{13}\ \text{A}\right) = \frac{2}{13}\ \text{V}.$

(b) The potential difference across the cell *G* is

$$V_C - V_D = -(3\ \Omega)i_2 + 3\ \text{V}$$

$$= \left(3\ \text{V} - \frac{18}{13}\ \text{V}\right) = \frac{21}{13}\ \text{V}.$$

The potential difference across the cell *H* is

$$V_C - V_B = (1\ \Omega)i_2 + 1\ \text{V} = (1\ \Omega)\left(\frac{6}{13}\ \text{A}\right) + 1\ \text{V} = \frac{19}{13}\ \text{V}.$$

**19.** *Find the equivalent resistance between the points a and b of the circuit shown in figure (32-W14a).*

***Solution*** **:** Suppose a current $i$ enters the circuit at the point $a$, a part $i_1$ goes through the 10 Ω resistor and the rest $i - i_1$ through the 5 Ω resistor. By symmetry, the current $i$ coming out from the point $b$ will be composed of a part $i_1$ from the 10 Ω resistor and $i - i_1$ from the 5 Ω resistor. Applying Kirchhoff's junction law, we can find the current through the middle 5 Ω resistor. The current distribution is shown in figure (32-W14b).

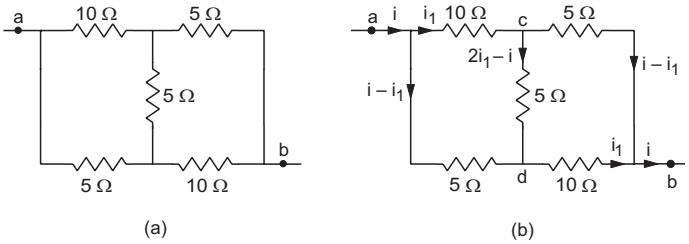



Figure 32-W14

We have

$$V_a - V_b = (V_a - V_c) + (V_c - V_b)$$

$$= (10\ \Omega)i_1 + (5\ \Omega)\ (i - i_1)$$

$$= (5\ \Omega)i + (5\ \Omega)\ i_1. \quad \ldots \text{(i)}$$

Also, $V_a - V_b = (V_a - V_c) + (V_c - V_d) + (V_d - V_b)$

$$= (10\ \Omega)i_1 + (5\ \Omega)\ (2i_1 - i) + (10\ \Omega)i_1$$

$$= -(5\ \Omega)i + (30\ \Omega)i_1 \quad \ldots \text{(ii)}$$

Multiplying (i) by 6 and subtracting (ii) from it, we eliminate $i_1$ and get,

$$5(V_a - V_b) = (35\ \Omega)i$$

or, $$\frac{V_a - V_b}{i} = 7\ \Omega.$$

Thus, the equivalent resistance between the points $a$ and $b$ is 7 Ω.

**20.** *Find the currents going through the three resistors $R_1$, $R_2$ and $R_3$ in the circuit of figure (32-W15a).*

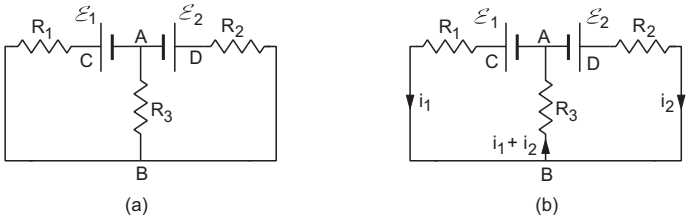



Figure 32-W15

***Solution*** **:** Let us take the potential of the point *A* to be zero. The potential at *C* will be $\mathcal{E}_1$ and that at *D* will be $\mathcal{E}_2$. Let the potential at *B* be *V*. The currents through the three resistors are $i_1$, $i_2$ and $i_1 + i_2$ as shown in figure (32.15b). Note that the current directed towards *B* equals the current directed away from *B*.

Applying Ohm's law to the three resistors $R_1$, $R_2$ and $R_3$, we get

$$\mathcal{E}_1 - V = R_1 i_1 \quad \ldots \text{(i)}$$

$$\mathcal{E}_2 - V = R_2 i_2 \quad \ldots \text{(ii)}$$

and $$V - 0 = R_3(i_1 + i_2). \quad \ldots \text{(iii)}$$

Adding (i) and (iii),

$$\mathcal{E}_1 = R_1 i_1 + R_3(i_1 + i_2)$$
$$= (R_1 + R_3) i_1 + R_3 i_2 \quad \ldots \text{(iv)}$$

and adding (ii) and (iii),

$$\mathcal{E}_2 = R_2 i_2 + R_3(i_1 + i_2)$$
$$= (R_2 + R_3) i_2 + R_3 i_1. \quad \ldots \text{(v)}$$

Equations (iv) and (v) may be directly written from Kirchhoff's loop law applied to the left half and the right half of the circuit.

Multiply (iv) by $(R_2 + R_3)$, (v) by $R_3$ and subtract to eliminate $i_2$. This gives

$$i_1 = \frac{\mathcal{E}_1(R_2 + R_3) - \mathcal{E}_2 R_3}{(R_1 + R_3)(R_2 + R_3) - R_3^2}$$
$$= \frac{\mathcal{E}_1(R_2 + R_3) - \mathcal{E}_2 R_3}{R_1 R_2 + R_2 R_3 + R_3 R_1}.$$

Similarly, eliminating $i_1$ from (iv) and (v) we obtain,

$$i_2 = \frac{\mathcal{E}_2(R_1 + R_3) - \mathcal{E}_1 R_3}{R_1 R_2 + R_2 R_3 + R_3 R_1}.$$

And so,

$$i_1 + i_2 = \frac{\mathcal{E}_1 R_2 + \mathcal{E}_2 R_1}{R_1 R_2 + R_2 R_3 + R_3 R_1}.$$

**21.** *Find the equivalent resistance between the points a and c of the network shown in figure (32-W16a). Each resistance is equal to r.*

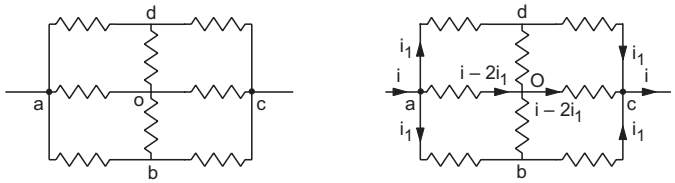



Figure 32-W16

***Solution*** **:** Suppose a potential difference $V$ is applied between $a$ and $c$ so that a current $i$ enters at $a$ and the same current leaves at $c$ (figure 32-W16b). The current $i$ divides in three parts at $a$. By symmetry, the part in $ad$ and in $ab$ will be equal. Let each of these currents be $i_1$. The current through $ao$ is $i - 2i_1$. Similarly, currents from $dc$, $bc$ and $oc$ combine at $c$ to give the total current $i$. Since the situation at $c$ is equivalent to that at $a$, by symmetry, the currents in $dc$ and $bc$ will be $i_1$ and that in $oc$ will be $i - 2i_1$.

Applying Kirchhoff's junction law at $d$, we see that the current in $do$ is zero. Similarly, the current in $ob$ is zero. We can remove $do$ and $ob$ for further analysis. It is then equivalent to three resistances, each of value $2r$, in parallel. The equivalent resistance is, therefore, $2r/3$.

**22.** *Twelve wires, each having resistance r, are joined to form a cube as shown in figure (32-W17). Find the equivalent resistance between the ends of a face diagonal such as a and c.*

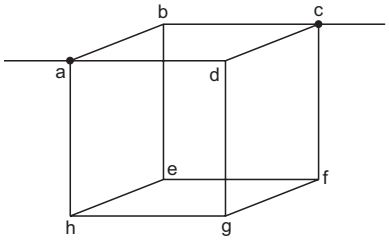



Figure 32-W17

***Solution*** **:** Suppose a potential difference $V$ is applied between the points $a$ and $c$ so that a current $i$ enters at $a$ and the same current leaves at $c$. The current distribution is shown in figure (32-W18a).

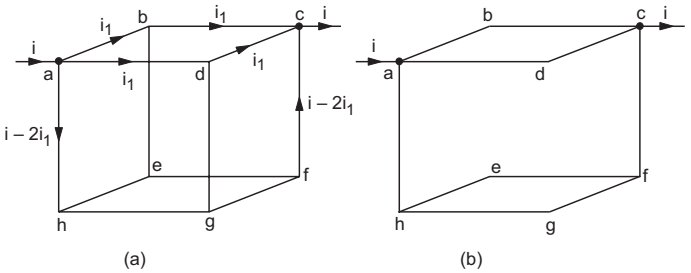



Figure 32-W18

By symmetry, the paths $ad$ and $ab$ are equivalent and hence will carry the same current $i_1$. The path $ah$ will carry the remaining current $i - 2i_1$ (using Kirchhoff's junction law). Similarly at junction $c$, currents coming from $dc$ and $bc$ will be $i_1$ each and from $fc$ will be $i - 2i_1$. Kirchhoff's junction law at $b$ and $d$ shows that currents through $be$ and $dg$ will be zero and hence may be ignored for further analysis. Omitting these two wires, the circuit is redrawn in figure (32-W18b).

The wire $hef$ and $hgf$ are joined in parallel and have equivalent resistance $\frac{(2r)(2r)}{(2r) + (2r)} = r$ between $h$ and $f$. This is joined in series with $ah$ and $fc$ giving equivalent resistance $r + r + r = 3r$. This $3r$ is joined in parallel with $adc$ $(2r)$ and $abc$ $(2r)$ between $a$ and $c$.

The equivalent resistance $R$ between $a$ and $c$ is, therefore, given by

$$\frac{1}{R} = \frac{1}{3r} + \frac{1}{2r} + \frac{1}{2r},$$

giving $$R = \frac{3}{4} r.$$

**23.** *Find the equivalent resistance of the circuit of the previous problem between the ends of an edge such as a and b in figure (32-W19a).*

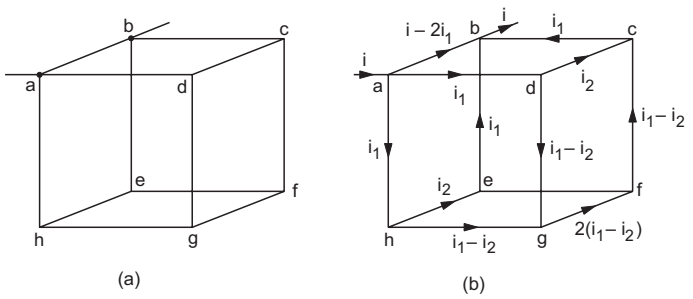



Figure 32-W19

***Solution*** **:** Suppose a current $i$ enters the circuit at the point $a$ and the same current leaves the circuit at the point $b$. The current distribution is shown in figure (32-W19b). The paths through $ad$ and $ah$ are equivalent and carry equal current $i_1$. The current through $ab$ is then $i - 2i_1$.

The same distribution holds at the junction $b$. Currents in $eb$ and $cb$ are $i_1$ each. The current $i_1$ in $ah$ is divided into a part $i_2$ in $he$ and $i_1 - i_2$ in $hg$. Similar is the division of current $i_1$ in $ad$ into $dc$ and $dg$. The rest of the currents may be written easily using Kirchhoff's junction law.

The potential difference $V$ between $a$ and $b$ may be written from the paths $ab$, $aheb$ and $ahgfcb$ as

$$V = (i - 2i_1)r$$

$$V = (i_1 + i_2 + i_1)r$$

and $$V = [i_1 + (i_1 - i_2) + 2(i_1 - i_2) + (i_1 - i_2) + i_1]r$$

which may be written as

$$V = (i - 2i_1)r$$

$$V = (2i_1 + i_2)r$$

and $$V = (6i_1 - 4i_2)r.$$

Eliminating $i_1$ and $i_2$ from these equations,

$$\frac{V}{i} = \frac{7}{12}r$$

which is the equivalent resistance.

**24.** *Find the equivalent resistance between the points a and b of the infinite ladder shown in figure (32-W20a).*

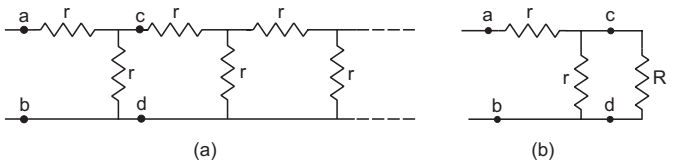



Figure 32-W20

***Solution*** **:** Let the equivalent resistance between $a$ and $b$ be $R$. As the ladder is infinite, $R$ is also the equivalent resistance of the ladder to the right of the points $c$ and $d$. Thus, we can replace the part to the right of $cd$ by a resistance $R$ and redraw the circuit as in figure (32-W20b).

This gives

$$R = r + \frac{rR}{r + R}$$

or, $$rR + R^2 = r^2 + 2rR$$

or, $$R^2 - rR - r^2 = 0$$

or, $$R = \frac{r + \sqrt{r^2 + 4r^2}}{2} = \frac{1 + \sqrt{5}}{2}r.$$

**25.** *Find the equivalent resistance of the network shown in figure (32-W21) between the points a and b.*

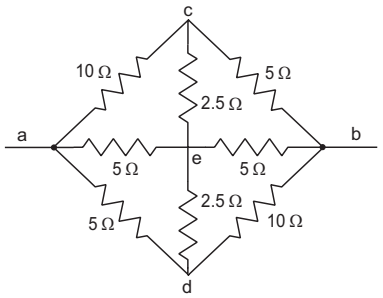



Figure 32-W21

***Solution*** **:**

Suppose a current $i$ enters the network at point $a$ and the same current leaves it at point $b$. Suppose, the currents in $ac$, $ad$ and $ae$ are $i_1$, $i_2$ and $i_3$ respectively. Similar will be the distribution of current at $b$. The current $i$ leaving at $b$ is composed of $i_1$ from $db$, $i_2$ from $cb$ and $i_3$ from $eb$. The situation is shown in figure (32-W22a).

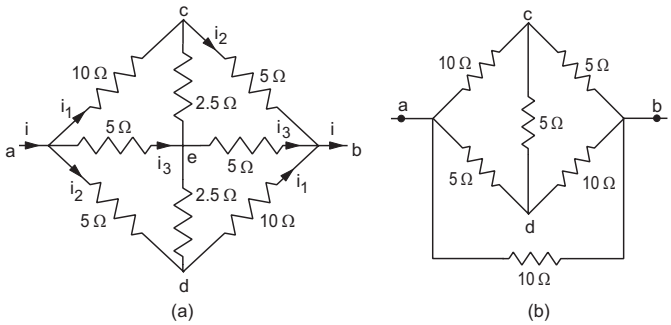



Figure 32-W22

As the current in $ae$ is equal to that in $eb$, the current in $ce$ will be equal to the current in $ed$ from the junction law. If we assume that the branches $ced$ and $aeb$ do not physically touch at $e$, nothing will be changed in the current distribution. We can then represent the branch $aeb$ by a single resistance of 10 Ω connected between $a$ and $b$. Similarly, the branch $ced$ may be replaced a single 5 Ω resistor between $c$ and $d$. The circuit is redrawn in figure (32-W22b). This is same as the circuit in figure (32-W14a) connected in parallel with a resistance of 10 Ω. So the network is equivalent to a parallel combination of 7 Ω and 10 Ω resistor. The equivalent resistance of the whole network is, therefore,

$$R = \frac{(7\ \Omega) \times (10\ \Omega)}{7\ \Omega + 10\ \Omega} \approx 4{\cdot}1\ \Omega.$$

**26.** *(a) Find the current i supplied by the battery in the network shown in figure (32-W23) in steady state. (b) Find the charge on the capacitor.*

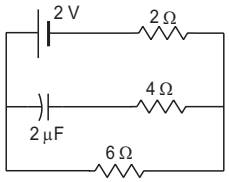



Figure 32-W23

***Solution*** **:** (a) Once the capacitor is charged, no current will go through it and hence the current through the middle branch of the circuit is zero in steady state. The 4 Ω resistor will have no current in it and may be omitted for current analysis. The 2 Ω and 6 Ω resistors are, therefore, connected in series and hence

$$i = \frac{2\ \text{V}}{2\ \Omega + 6\ \Omega} = 0{\cdot}25\ \text{A}.$$

(b) The potential drop across the 6 Ω resistor is 6 Ω × 0·25 A = 1·5 V. As there is no current in the 4 Ω resistor, there is no potential drop across it. The potential difference across the capacitor is, therefore, 1·5 V. The charge on this capacitor is

$$Q = CV = 2\ \mu\text{F} \times 1{\cdot}5\ \text{V} = 3\ \mu\text{C}.$$

**27.** *A part of a circuit in steady state along with the currents flowing in the branches, the values of resistances, etc., is shown in figure (32-W24). Calculate the energy stored in the capacitor.*

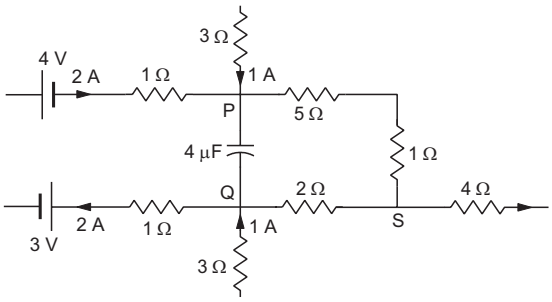



Figure 32-W24

***Solution*** **:** To get the energy stored in the capacitor, we shall calculate the potential difference between the points $P$ and $Q$. In steady state, there is no current in the capacitor branch. Applying Kirchhoff's junction law at $P$, the current in the 5 Ω – 1 Ω branch will be 3 A and hence

$$V_P - V_S = 6\ \Omega \times 3\ \text{A} = 18\ \text{V}.$$

Applying the same theorem at $Q$, the current in the 2 Ω resistor will be 1 A towards $Q$ so that

$$V_S - V_Q = 2\ \Omega \times 1\ \text{A} = 2\ \text{V}.$$

Thus, $V_P - V_Q = (V_P - V_S) + (V_S - V_Q) = 20\ \text{V}$.

The energy stored in the capacitor

$$= \frac{1}{2} CV^2 = \frac{1}{2} \times 4\ \mu\text{F} \times 400\ \text{V}^2$$

$$= 800\ \mu\text{J}.$$

**28.** *(a) Find the potential drops across the two resistors shown in figure (32-W25a). (b) A voltmeter of resistance* 600 Ω *is used to measure the potential drop across the* 300 Ω *resistor. What will be the measured potential drop ?*

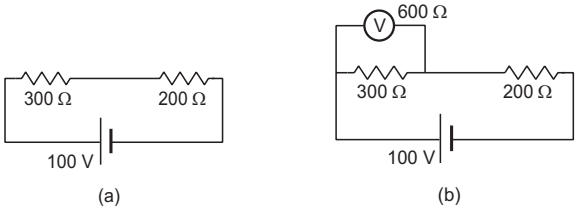



Figure 32-W25

***Solution*** **:**

(a) The current in the circuit is $\frac{100\ \text{V}}{300\ \Omega + 200\ \Omega} = 0{\cdot}2\ \text{A}$.

The potential drop across the 300 Ω resistor is

$$300\ \Omega \times 0{\cdot}2\ \text{A} = 60\ \text{V}.$$

Similarly, the drop across the 200 Ω resistor is 40 V.

(b) The equivalent resistance, when the voltmeter is connected across 300 Ω, is (figure 32-W25b)

$$R = 200\ \Omega + \frac{600\ \Omega \times 300\ \Omega}{600\ \Omega + 300\ \Omega} = 400\ \Omega.$$

Thus, the main current from the battery is

$$i = \frac{100\ \text{V}}{400\ \Omega} = 0{\cdot}25\ \text{A}.$$

The potential drop across the 200 Ω resistor is, therefore, 200 Ω × 0·25 A = 50 V and that across 300 Ω is also 50 V. This is also the potential drop across the voltmeter and hence the reading of the voltmeter is 50 V.

**29.** *A galvanometer has a coil of resistance* 100 Ω *showing a full-scale deflection at* 50 μA. *What resistance should be added to use it as (a) a voltmeter of range* 50 V *(b) an ammeter of range* 10 mA ?

***Solution*** **:** (a) When a potential difference of 50 V is applied across the voltmeter, full-scale deflection should take place. Thus, 50 μA should go through the coil. We

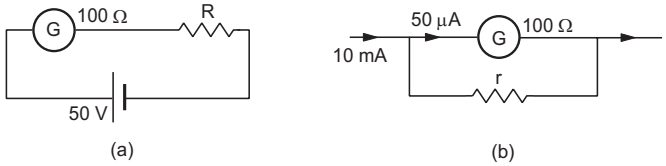



Figure 32-W26

add a resistance $R$ in series with the given coil to achieve this (figure 32-W26a).

We have,

$$50\ \mu A = \frac{50\ V}{100\ \Omega + R}$$

or, $$R = 10^6\ \Omega - 100\ \Omega \approx 10^6\ \Omega.$$

(b) When a current of 10 mA is passed through the ammeter, 50 μA should go through the coil. We add a resistance $r$ in parallel to the coil to achieve this (figure 32-W26b).

The current through the coil is

$$50\ \mu A = (10\ mA)\frac{r}{r + 100\ \Omega}$$

or, $$r \approx 0{\cdot}5\ \Omega.$$

**30.** *The electric field between the plates of a parallel-plate capacitor of capacitance* 2·0 μF *drops to one third of its initial value in* 4·4 μs *when the plates are connected by a thin wire. Find the resistance of the wire.*

***Solution*** **:** The electric field between the plates is

$$E = \frac{Q}{A\varepsilon_0} = \frac{Q_0}{A\varepsilon_0} e^{-t/RC}$$

or, $$E = E_0 e^{-t/RC}.$$

In the given problem, $E = \frac{1}{3}E_0$ at $t = 4{\cdot}4\ \mu s$.

Thus, $$\frac{1}{3} = e^{-\frac{4{\cdot}4\ \mu s}{RC}}$$

or, $$\frac{4{\cdot}4\ \mu s}{RC} = \ln 3 = 1{\cdot}1$$

or, $$R = \frac{4{\cdot}4\ \mu s}{1{\cdot}1 \times 2{\cdot}0\ \mu F} = 2{\cdot}0\ \Omega.$$

**31.** *A capacitor is connected to a* 12 V *battery through a resistance of* 10 Ω. *It is found that the potential difference across the capacitor rises to* 4·0 V *in* 1 μs. *Find the capacitance of the capacitor.*

***Solution*** **:** The charge on the capacitor during charging is given by

$$Q = Q_0(1 - e^{-t/RC}).$$

Hence, the potential difference across the capacitor is

$$V = Q/C = Q_0/C(1 - e^{-t/RC}).$$

Here at $t = 1\ \mu s$, the potential difference is 4 V whereas the steady state potential difference is $Q_0/C = 12$ V. So,

$$4\ V = 12\ V(1 - e^{-t/RC})$$

or, $$1 - e^{-t/RC} = \frac{1}{3}$$

or, $$e^{-t/RC} = \frac{2}{3}$$

or, $$\frac{t}{RC} = \ln\left(\frac{3}{2}\right) = 0{\cdot}405$$

or, $$RC = \frac{t}{0{\cdot}405} = \frac{1\ \mu s}{0{\cdot}405} = 2{\cdot}469\ \mu s$$

or, $$C = \frac{2{\cdot}469\ \mu s}{10\ \Omega} \approx 0{\cdot}25\ \mu F.$$

**32.** *A capacitor charged to* 50 V *is discharged by connecting the two plates at* $t = 0$. *If the potential difference across the plates drops to* 1·0 V *at* $t = 10$ ms, *what will be the potential difference at* $t = 20$ ms ?

***Solution*** **:** The potential difference at time $t$ is given by

$$V = Q/C = (Q_0/C)\, e^{-t/RC}$$

or, $$V = V_0 e^{-t/RC}.$$

According to the given data,

$$1\ V = (50\ V)\, e^{-10\ ms/RC}$$

or, $$e^{-10\ ms/RC} = \frac{1}{50}.$$

The potential difference at $t$ = 20 ms is

$$V = V_0 e^{-t/RC}$$

$$= (50\ V)\, e^{-20\ ms/RC} = (50\ V)\left(e^{-10\ ms/RC}\right)^2$$

$$= 0{\cdot}02\ V.$$

**33.** *A* 5·0 μF *capacitor having a charge of* 20 μC *is discharged through a wire of resistance* 5·0 Ω. *Find the heat dissipated in the wire between* 25 *to* 50 μs *after the connections are made.*

***Solution*** **:** The charge on the capacitor at time $t$ after the connections are made is

$$Q = Q_0 e^{-t/RC}$$

or, $$i = \frac{dQ}{dt} = -(Q_0/RC)\, e^{-t/RC}.$$

Heat dissipated during the time $t_1$ to $t_2$ is

$$U = \int_{t_1}^{t_2} i^2 R\, dt$$

$$= \int_{t_1}^{t_2} \frac{{Q_0}^2}{RC^2} e^{-2t/RC}\, dt$$

$$= \frac{Q_0^2}{2C}\left(e^{-\frac{2t_1}{RC}} - e^{-\frac{2t_2}{RC}}\right). \quad \ldots (i)$$

The time constant $RC$ is $5\ \Omega \times 5{\cdot}0\ \mu F = 25\ \mu s$.

Putting $t_1 = 25\ \mu s$, $t_2 = 50\ \mu s$ and other values in (i),

$$U = \frac{(20\ \mu C)^2}{2 \times 5{\cdot}0\ \mu F}(e^{-2} - e^{-4}) = 4{\cdot}7\ \mu J.$$

□

## QUESTIONS FOR SHORT ANSWER

**1**. Suppose you have three resistors each of value 30 Ω. List all the different resistances you can obtain using them.

**2**. A proton beam is going from east to west. Is there an electric current ? If yes, in what direction ?

**3**. In an electrolyte, the positive ions move from left to right and the negative ions from right to left. Is there a net current ? If yes, in what direction ?

**4**. In a TV tube, the electrons are accelerated from the rear to the front. What is the direction of the current ?

**5**. The drift speed is defined as $v_d = \Delta l / \Delta t$ where $\Delta l$ is the distance drifted in a long time $\Delta t$. Why don't we define the drift speed as the limit of $\Delta l / \Delta t$ as $\Delta t \to 0$ ?

**6**. One of your friends argues that he has read in previous chapters that there can be no electric field inside a conductor. And hence there can be no current through it. What is the fallacy in this argument ?

**7.** When a current is established in a wire, the free electrons drift in the direction opposite to the current. Does the number of free electrons in the wire continuously decrease ?

**8.** A fan with copper winding in its motor consumes less power as compared to an otherwise similar fan having aluminium winding. Explain.

**9.** The thermal energy developed in a current-carrying resistor is given by $U = i^2Rt$ and also by $U = Vit$. Should we say that $U$ is proportional to $i^2$ or to $i$ ?

**10.** Consider a circuit containing an ideal battery connected to a resistor. Do "work done by the battery" and "the thermal energy developed" represent two names of the same physical quantity ?

**11.** Is work done by a battery always equal to the thermal energy developed in electrical circuits ? What happens if a capacitor is connected in the circuit ?

**12.** A nonideal battery is connected to a resistor. Is work done by the battery equal to the thermal energy developed in the resistor ? Does your answer change if the battery is ideal ?

**13.** Sometimes it is said that "heat is developed" in a resistance when there is an electric current in it. Recall that heat is defined as the energy being transferred due to the temperature difference. Is the statement under quotes technically correct ?

**14.** We often say "a current is going through the wire". What goes through the wire, the charge or the current ?

**15.** Would you prefer a voltmeter or a potentiometer to measure the emf of a battery ?

**16.** Does a conductor become charged when a current is passed through it ?

**17.** Can the potential difference across a battery be greater than its emf ?

## OBJECTIVE I

**1.** A metallic resistor is connected across a battery. If the number of collisions of the free electrons with the lattice is somehow decreased in the resistor (for example, by cooling it), the current will
(a) increase (b) decrease
(c) remain constant (d) become zero.

**2.** Two resistors $A$ and $B$ have resistances $R_A$ and $R_B$ respectively with $R_A < R_B$. The resistivities of their materials are $\rho_A$ and $\rho_B$.
(a) $\rho_A > \rho_B$ (b) $\rho_A = \rho_B$ (c) $\rho_A < \rho_B$
(d) The information is not sufficient to find the relation between $\rho_A$ and $\rho_B$.

**3.** The product of resistivity and conductivity of a cylindrical conductor depends on
(a) temperature (b) material
(c) area of cross section (d) none of these.

**4.** As the temperature of a metallic resistor is increased, the product of its resistivity and conductivity
(a) increases (b) decreases
(c) remains constant (d) may increase or decrease.

**5.** In an electric circuit containing a battery, the charge (assumed positive) inside the battery
(a) always goes from the positive terminal to the negative terminal
(b) may go from the positive terminal to the negative terminal
(c) always goes from the negative terminal to the positive terminal
(d) does not move.

**6.** A resistor of resistance $R$ is connected to an ideal battery. If the value of $R$ is decreased, the power dissipated in the resistor will
(a) increase (b) decrease (c) remain unchanged.

**7.** A current passes through a resistor. Let $K_1$ and $K_2$ represent the average kinetic energy of the conduction electrons and the metal ions respectively.
(a) $K_1 < K_2$ (b) $K_1 = K_2$
(c) $K_1 > K_2$ (d) Any of these three may occur.

**8.** Two resistors $R$ and $2R$ are connected in series in an electric circuit. The thermal energy developed in $R$ and $2R$ are in the ratio
(a) 1 : 2 (b) 2 : 1 (c) 1 : 4 (d) 4 : 1.

**9.** Two resistances $R$ and $2R$ are connected in parallel in an electric circuit. The thermal energy developed in $R$ and $2R$ are in the ratio
(a) 1 : 2 (b) 2 : 1 (c) 1 : 4 (d) 4 : 1.

**10.** A uniform wire of resistance 50 Ω is cut into 5 equal parts. These parts are now connected in parallel. The

equivalent resistance of the combination is
(a) 2 Ω (b) 10 Ω (c) 250 Ω (d) 6250 Ω.

**11.** Consider the following two statements:
(A) Kirchhoff's junction law follows from conservation of charge.
(B) Kirchhoff's loop law follows from conservative nature of electric field.
(a) Both *A* and *B* are correct.
(b) *A* is correct but *B* is wrong.
(c) *B* is correct but *A* is wrong.
(d) Both *A* and *B* are wrong.

**12.** Two nonideal batteries are connected in series. Consider the following statements:
(A) The equivalent emf is larger than either of the two emfs.
(B) The equivalent internal resistance is smaller than either of the two internal resistances.
(a) Each of *A* and *B* is correct.
(b) *A* is correct but *B* is wrong.
(c) *B* is correct but *A* is wrong.
(d) Each of *A* and *B* is wrong.

**13.** Two nonideal batteries are connected in parallel. Consider the following statements:
(A) The equivalent emf is smaller than either of the two emfs.
(B) The equivalent internal resistance is smaller than either of the two internal resistances.
(a) Both *A* and *B* are correct.
(b) *A* is correct but *B* is wrong.
(c) *B* is correct but *A* is wrong.
(d) Both *A* and *B* are wrong.

**14.** The net resistance of an ammeter should be small to ensure that
(a) it does not get overheated
(b) it does not draw excessive current
(c) it can measure large currents
(d) it does not appreciably change the current to be measured.

**15.** The net resistance of a voltmeter should be large to ensure that
(a) it does not get overheated
(b) it does not draw excessive current
(c) it can measure large potential differences
(d) it does not appreciably change the potential difference to be measured.

**16.** Consider a capacitor-charging circuit. Let $Q_1$ be the charge given to the capacitor in a time interval of 10 ms and $Q_2$ be the charge given in the next time interval of 10 ms. Let 10 μC charge be deposited in a time interval $t_1$ and the next 10 μC charge is deposited in the next time interval $t_2$.
(a) $Q_1 > Q_2, t_1 > t_2$ (b) $Q_1 > Q_2, t_1 < t_2$
(c) $Q_1 < Q_2, t_1 > t_2$ (d) $Q_1 < Q_2, t_1 < t_2$

## OBJECTIVE II

**1.** Electrons are emitted by a hot filament and are accelerated by an electric field as shown in figure (32-Q1). The two stops at the left ensure that the electron beam has a uniform cross-section.
(a) The speed of the electron is more at *B* than at *A*.
(b) The electric current is from left to right.
(c) The magnitude of the current is larger at *B* than at *A*.
(d) The current density is more at *B* than at *A*.

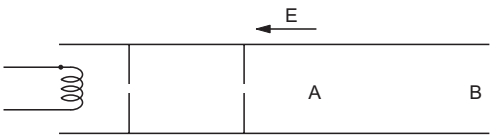



Figure 32-Q1

**2.** A capacitor with no dielectric is connected to a battery at $t = 0$. Consider a point *A* in the connecting wires and a point *B* in between the plates.
(a) There is no current through *A*.
(b) There is no current through *B*.
(c) There is a current through *A* as long as the charging is not complete.
(d) There is a current through *B* as long as the charging is not complete.

**3.** When no current is passed through a conductor,
(a) the free electrons do not move
(b) the average speed of a free electron over a large period of time is zero
(c) the average velocity of a free electron over a large period of time is zero
(d) the average of the velocities of all the free electrons at an instant is zero.

**4.** Which of the following quantities do not change when a resistor connected to a battery is heated due to the current ?
(a) Drift speed (b) Resistivity
(c) Resistance (d) Number of free electrons

**5.** As the temperature of a conductor increases, its resistivity and conductivity change. The ratio of resistivity to conductivity
(a) increases (b) decreases (c) remains constant
(d) may increase or decrease depending on the actual temperature.

**6.** A current passes through a wire of nonuniform cross-section. Which of the following quantities are independent of the cross section ?
(a) The charge crossing in a given time interval
(b) Drift speed
(c) Current density
(d) Free-electron density

**7.** Mark out the correct options.
(a) An ammeter should have small resistance.
(b) An ammeter should have large resistance.

(c) A voltmeter should have small resistance.
(d) A voltmeter should have large resistance.

**8.** A capacitor of capacitance 500 μF is connected to a battery through a 10 kΩ resistor. The charge stored on the capacitor in the first 5 s is larger than the charge stored in the next
(a) 5 s (b) 50 s (c) 500 s (d) 5000 s

**9.** A capacitor $C_1$ of capacitance 1 μF and a capacitor $C_2$ of capacitance 2 μF are separately charged by a common battery for a long time. The two capacitors are then separately discharged through equal resistors. Both the discharge circuits are connected at $t = 0$.
(a) The current in each of the two discharging circuits is zero at $t = 0$.
(b) The currents in the two discharging circuits at $t = 0$ are equal but not zero.
(c) The currents in the two discharging circuits at $t = 0$ are unequal.
(d) $C_1$ loses 50% of its initial charge sooner than $C_2$ loses 50% of its initial charge.

## EXERCISES

**1.** The amount of charge passed in time $t$ through a cross-section of a wire is

$$Q(t) = At^2 + Bt + C.$$

(a) Write the dimensional formulae for $A$, $B$ and $C$.
(b) If the numerical values of $A$, $B$ and $C$ are 5, 3 and 1 respectively in SI units, find the value of the current at $t = 5$ s.

**2.** An electron gun emits $2{\cdot}0 \times 10^{16}$ electrons per second. What electric current does this correspond to?

**3.** The electric current existing in a discharge tube is 2·0 μA. How much charge is transferred across a cross-section of the tube in 5 minutes ?

**4.** The current through a wire depends on time as

$$i = i_0 + \alpha t,$$

where $i_0 = 10$ A and $\alpha = 4$ A s$^{-1}$. Find the charge crossed through a section of the wire in 10 seconds.

**5.** A current of 1·0 A exists in a copper wire of cross-section 1·0 mm$^2$. Assuming one free electron per atom calculate the drift speed of the free electrons in the wire. The density of copper is 9000 kg m$^{-3}$.

**6.** A wire of length 1 m and radius 0·1 mm has a resistance of 100 Ω. Find the resistivity of the material.

**7.** A uniform wire of resistance 100 Ω is melted and recast in a wire of length double that of the original. What would be the resistance of the wire ?

**8.** Consider a wire of length 4 m and cross-sectional area 1 mm$^2$ carrying a current of 2 A. If each cubic metre of the material contains $10^{29}$ free electrons, find the average time taken by an electron to cross the length of the wire.

**9.** What length of a copper wire of cross-sectional area 0·01 mm$^2$ will be needed to prepare a resistance of 1 kΩ ? Resistivity of copper = $1{\cdot}7 \times 10^{-8}$ Ω m.

**10.** Figure (32-E1) shows a conductor of length $l$ having a circular cross section. The radius of cross section varies linearly from $a$ to $b$. The resistivity of the material is ρ. Assuming that $b - a << l$, find the resistance of the conductor.

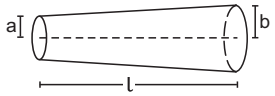


Figure 32-E1

**11.** A copper wire of radius 0·1 mm and resistance 1 kΩ is connected across a power supply of 20 V. (a) How many electrons are transferred per second between the supply and the wire at one end ? (b) Write down the current density in the wire.

**12.** Calculate the electric field in a copper wire of cross-sectional area 2·0 mm$^2$ carrying a current of 1 A. The resistivity of copper = $1{\cdot}7 \times 10^{-8}$ Ω m.

**13.** A wire has a length of 2·0 m and a resistance of 5·0 Ω. Find the elecric field existing inside the wire if it carries a current of 10 A.

**14.** The resistances of an iron wire and a copper wire at 20°C are 3·9 Ω and 4·1 Ω respectively. At what temperature will the resistances be equal ? Temperature coefficient of resistivity for iron is $5{\cdot}0 \times 10^{-3}$ K$^{-1}$ and for copper it is $4{\cdot}0 \times 10^{-3}$ K$^{-1}$. Neglect any thermal expansion.

**15.** The current in a conductor and the potential difference across its ends are measured by an ammeter and a voltmeter. The meters draw negligible currents. The ammeter is accurate but the voltmeter has a zero error (that is, it does not read zero when no potential difference is applied). Calculate the zero error if the readings for two different conditions are 1·75 A, 14·4 V and 2·75 A, 22·4 V.

**16.** Figure (32-E2) shows an arrangement to measure the emf $\mathcal{E}$ and internal resistance $r$ of a battery. The voltmeter has a very high resistance and the ammeter also has some resistance. The voltmeter reads 1·52 V when the switch $S$ is open. When the switch is closed the voltmeter reading drops to 1·45 V and the ammeter reads 1·0 A. Find the emf and the internal resistance of the battery.

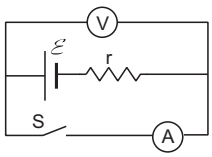


Figure 32-E2

**17.** The potential difference between the terminals of a battery of emf 6·0 V and internal resistance 1 Ω drops to 5·8 V when connected across an external resistor. Find the resistance of the external resistor.

**18.** The potential difference between the terminals of a 6·0 V battery is 7·2 V when it is being charged by a current of 2·0 A. What is the internal resistance of the battery ?

**19.** The internal resistance of an accumulator battery of emf 6 V is 10 Ω when it is fully discharged. As the battery gets charged up, its internal resistance decreases to 1 Ω. The battery in its completely discharged state is connected to a charger which maintains a constant potential difference of 9 V. Find the current through the battery (a) just after the connections are made and (b) after a long time when it is completely charged.

**20.** Find the value of $i_1 / i_2$ in figure (32-E3) if (a) $R = 0{\cdot}1$ Ω, (b) $R = 1$ Ω (c) $R = 10$ Ω. Note from your answers that in order to get more current from a combination of two batteries they should be joined in parallel if the external resistance is small and in series if the external resistance is large as compared to the internal resistances.

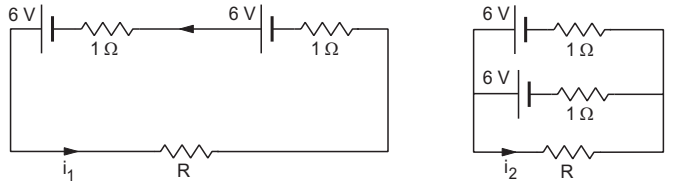



Figure 32-E3

**21.** Consider $N = n_1 n_2$ identical cells, each of emf $\mathcal{E}$ and internal resistance $r$. Suppose $n_1$ cells are joined in series to form a line and $n_2$ such lines are connected in parallel. The combination drives a current in an external resistance $R$. (a) Find the current in the external resistance. (b) Assuming that $n_1$ and $n_2$ can be continuously varied, find the relation between $n_1$, $n_2$, $R$ and $r$ for which the current in $R$ is maximum.

**22.** A battery of emf 100 V and a resistor of resistance 10 kΩ are joined in series. This system is used as a source to supply current to an external resistance $R$. If $R$ is not greater than 100 Ω, the current through it is constant up to two significant digits. Find its value. This is the basic principle of a *constant-current source.*

**23.** If the reading of ammeter $A_1$ in figure (32-E4) is 2·4 A, what will the ammeters $A_2$ and $A_3$ read ? Neglect the resistances of the ammeters.

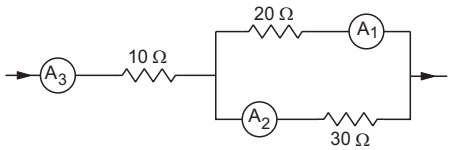



Figure 32-E4

**24.** The resistance of the rheostat shown in figure (32-E5) is 30 Ω. Neglecting the meter resistance, find the minimum and maximum currents through the ammeter as the rheostat is varied.

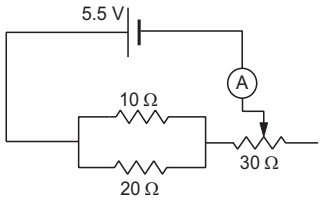



Figure 32-E5

**25.** Three bulbs, each having a resistance of 180 Ω, are connected in parallel to an ideal battery of emf 60 V. Find the current delivered by the battery when (a) all the bulbs are switched on, (b) two of the bulbs are switched on and (c) only one bulb is switched on.

**26.** Suppose you have three resistors of 20 Ω, 50 Ω and 100 Ω. What minimum and maximum resistances can you obtain from these resistors ?

**27.** A bulb is made using two filaments. A switch selects whether the filaments are used individually or in parallel. When used with a 15 V battery, the bulb can be operated at 5 W, 10 W or 15 W. What should be the resistances of the filaments ?

**28.** Figure (32-E6) shows a part of a circuit. If a current of 12 mA exists in the 5 kΩ resistor, find the currents in the other three resistors. What is the potential difference between the points $A$ and $B$ ?

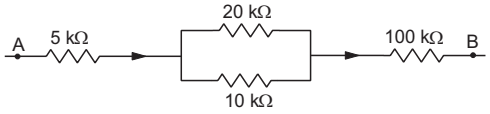



Figure 32-E6

**29.** An ideal battery sends a current of 5 A in a resistor. When another resistor of value 10 Ω is connected in parallel, the current through the battery is increased to 6 A. Find the resistance of the first resistor.

**30.** Find the equivalent resistance of the network shown in figure (32-E7) between the points $a$ and $b$.

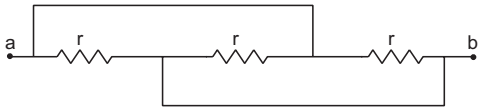



Figure 32-E7

**31.** A wire of resistance 15·0 Ω is bent to form a regular hexagon *ABCDEFA*. Find the equivalent resistance of the loop between the points (a) $A$ and $B$, (b) $A$ and $C$ and (c) $A$ and $D$.

**32.** Consider the circuit shown in figure (32-E8). Find the current through the 10 Ω resistor when the switch $S$ is (a) open (b) closed.

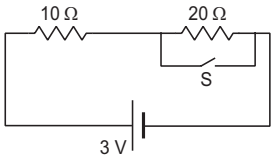



Figure 32-E8

**33.** Find the currents through the three resistors shown in figure (32-E9).

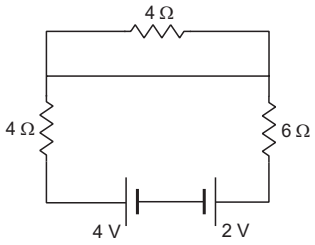



Figure 32-E9

**34.** Figure (32-E10) shows a part of an electric circuit. The potentials at the points *a, b* and *c* are 30 V, 12 V and 2 V respectively. Find the currents through the three resistors.

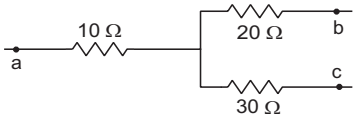



Figure 32-E10

**35.** Each of the resistors shown in figure (32-E11) has a resistance of 10 Ω and each of the batteries has an emf of 10 V. Find the currents through the resistors *a* and *b* in the two circuits.

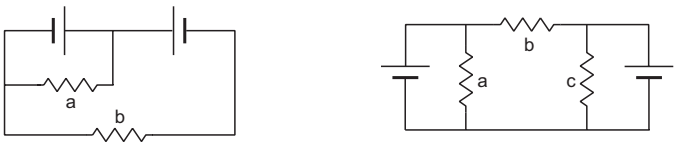



Figure 32-E11

**36.** Find the potential difference $V_a - V_b$ in the circuits shown in figure (32-E12).

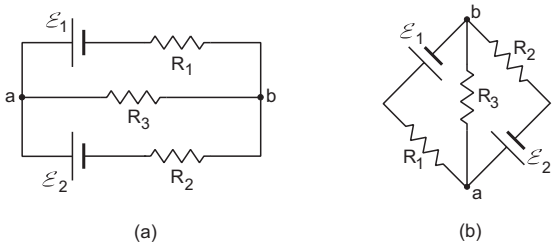



Figure 32-E12

**37.** In the circuit shown in figure (32-E13), $\mathcal{E}_1 = 3$ V, $\mathcal{E}_2 = 2$ V, $\mathcal{E}_3 = 1$ V and $r_1 = r_2 = r_3 = 1\ \Omega$. Find the potential difference between the points *A* and *B* and the current through each branch.

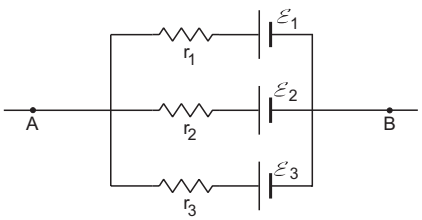



Figure 32-E13

**38.** Find the current through the 10 Ω resistor shown in figure (32-E14).

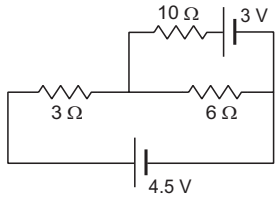



Figure 32-E14

**39.** Find the current in the three resistors shown in figure (32-E15).

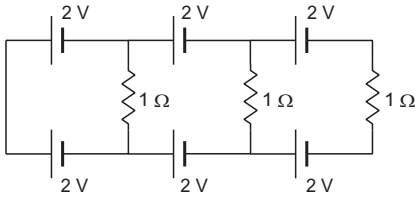



Figure 32-E15

**40.** What should be the value of *R* in figure (32-E16) for which the current in it is zero ?

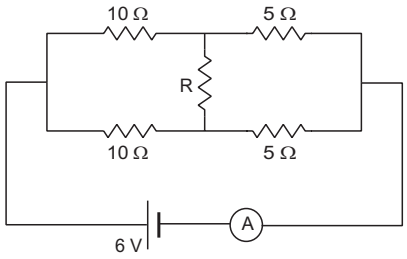



Figure 32-E16

**41.** Find the equivalent resistance of the circuits shown in figure (32-E17) between the points *a* and *b*. Each resistor has a resistance *r*.

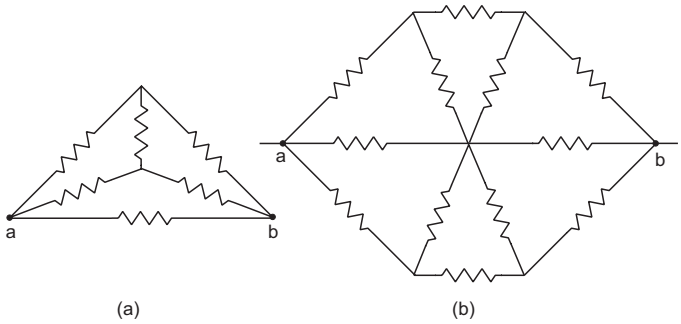



Figure 32-E17

**42.** Find the current measured by the ammeter in the circuit shown in figure (32-E18).

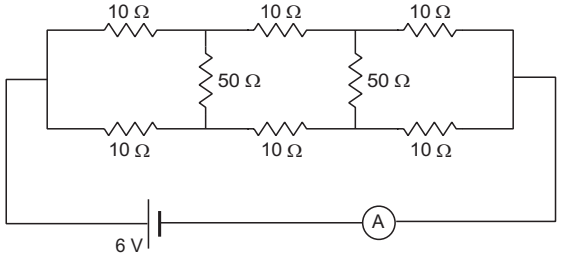



Figure 32-E18

**43.** Consider the circuit shown in figure (32-E19a). Find (a) the current in the circuit, (b) the potential drop across the 5 Ω resistor, (c) the potential drop across the 10 Ω resistor. (d) Answer the parts (a), (b) and (c) with reference to figure (32-E19b).

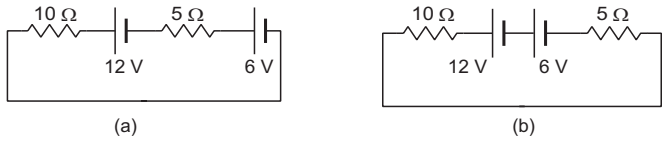



Figure 32-E19

**44.** Twelve wires, each having equal resistance $r$, are joined to form a cube as shown in figure (32-E20). Find the equivalent resistance between the diagonally opposite points $a$ and $f$.

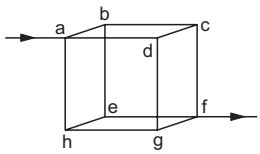



Figure 32-E20

**45.** Find the equivalent resistances of the networks shown in figure (32-E21) between the points $a$ and $b$.

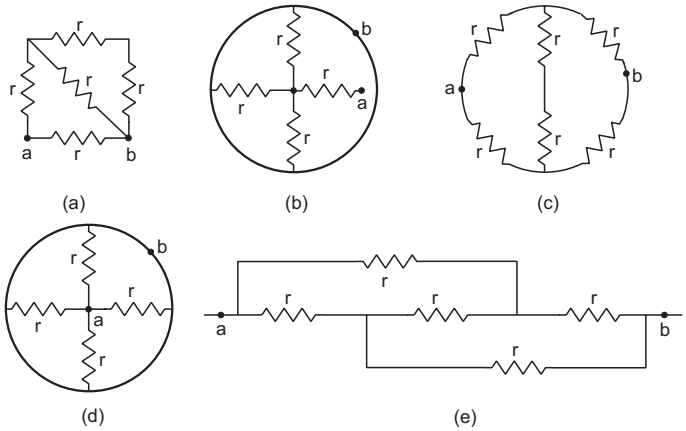



Figure 32-E21

**46.** An infinite ladder is constructed with 1 Ω and 2 Ω resistors as shown in figure (32-E22). (a) Find the effective resistance between the points $A$ and $B$. (b) Find the current that passes through the 2 Ω resistor nearest to the battery.

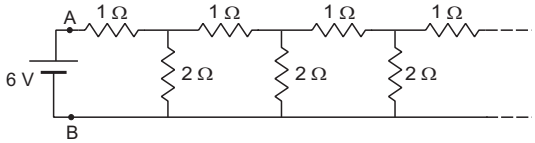



Figure 32-E22

**47.** The emf $\mathcal{E}$ and the internal resistance $r$ of the battery shown in figure (32-E23) are 4·3 V and 1·0 Ω respectively. The external resistance $R$ is 50 Ω. The resistances of the ammeter and voltmeter are 2·0 Ω and 200 Ω respectively. (a) Find the readings of the two meters. (b) The switch is thrown to the other side. What will be the readings of the two meters now ?

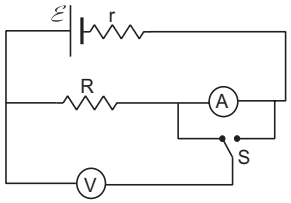



Figure 32-E23

**48.** A voltmeter of resistance 400 Ω is used to measure the potential difference across the 100 Ω resistor in the circuit shown in figure (32-E24). (a) What will be the reading of the voltmeter ? (b) What was the potential difference across 100 Ω before the voltmeter was connected ?

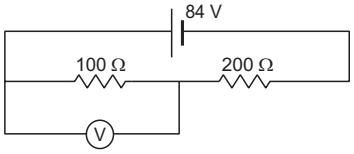



Figure 32-E24

**49.** The voltmeter shown in figure (32-E25) reads 18 V across the 50 Ω resistor. Find the resistance of the voltmeter.

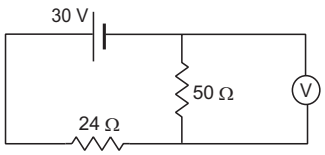



Figure 32-E25

**50.** A voltmeter consists of a 25 Ω coil connected in series with a 575 Ω resistor. The coil takes 10 mA for full scale deflection. What maximum potential difference can be measured on this voltmeter ?

**51.** An ammeter is to be constructed which can read currents up to 2·0 A. If the coil has a resistance of 25 Ω and takes 1 mA for full-scale deflection, what should be the resistance of the shunt used ?

**52.** A voltmeter coil has resistance 50·0 Ω and a resistor of 1·15 kΩ is connected in series. It can read potential differences upto 12 volts. If this same coil is used to construct an ammeter which can measure currents up to 2·0 A, what should be the resistance of the shunt used ?

**53.** The potentiometer wire $AB$ shown in figure (32-E26) is 40 cm long. Where should the free end of the galvanometer be connected on $AB$ so that the galvanometer may show zero deflection ?

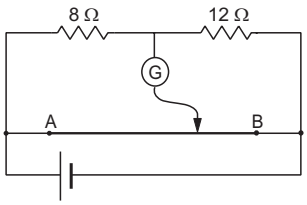



Figure 32-E26

**54.** The potentiometer wire $AB$ shown in figure (32-E27) is 50 cm long. When $AD = 30$ cm, no deflection occurs in the galvanometer. Find $R$.

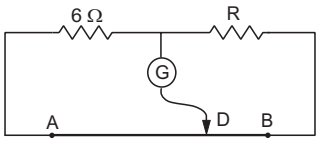



Figure 32-E27

**55.** A 6-volt battery of negligible internal resistance is connected across a uniform wire $AB$ of length 100 cm. The positive terminal of another battery of emf 4 V and internal resistance 1 Ω is joined to the point $A$ as shown in figure (32-E28). Take the potential at $B$ to be zero. (a) What are the potentials at the points $A$ and $C$ ? (b) At which point $D$ of the wire $AB$, the potential is equal to the potenial at $C$ ? (c) If the points $C$ and $D$ are connected by a wire, what will be the current through it ? (d) If the 4 V battery is replaced by 7·5 V battery, what would be the answers of parts (a) and (b) ?

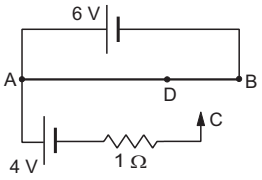



Figure 32-E28

**56.** Consider the potentiometer circuit arranged as in figure (32-E29). The potentiometer wire is 600 cm long. (a) At what distance from the point $A$ should the jockey touch the wire to get zero deflection in the galvanometer? (b) If the jockey touches the wire at a distance of 560 cm from $A$, what will be the current in the galvanometer ?

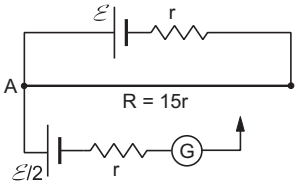



Figure 32-E29

**57.** Find the charge on the capacitor shown in figure (32-E30).

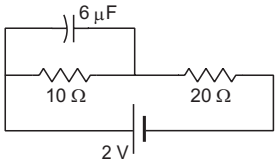



Figure 32-E30

**58.** (a) Find the current in the 20 Ω resistor shown in figure (32-E31). (b) If a capacitor of capacitance 4 μF is joined between the points $A$ and $B$, what would be the electrostatic energy stored in it in steady state ?

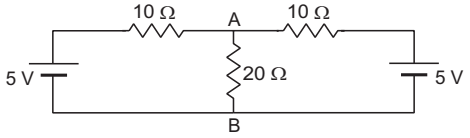



Figure 32-E31

**59.** Find the charges on the four capacitors of capacitances 1 μF, 2 μF, 3 μF and 4 μF shown in figure (32-E32).

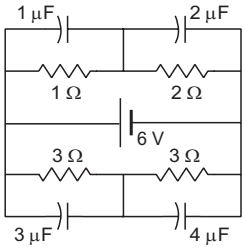



Figure 32-E32

**60.** Find the potential difference between the points $A$ and $B$ and between the points $B$ and $C$ of figure (32-E33) in steady state.

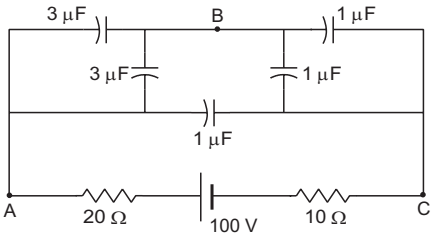



Figure 32-E33

**61.** A capacitance $C$, a resistance $R$ and an emf $\mathcal{E}$ are connected in series at $t = 0$. What is the maximum value of (a) the potential difference across the resistor, (b) the current in the circuit, (c) the potential difference across the capacitor, (d) the energy stored in the capacitor, (e) the power delivered by the battery and (f) the power converted into heat.

**62.** A parallel-plate capacitor with plate area 20 $cm^2$ and plate separation 1·0 mm is connected to a battery. The resistance of the circuit is 10 kΩ. Find the time constant of the circuit.

**63.** A capacitor of capacitance 10 μF is connected to a battery of emf 2 V. It is found that it takes 50 ms for the charge on the capacitor to become 12·6 μC. Find the resistance of the circuit.

**64.** A 20 μF capacitor is joined to a battery of emf 6·0 V through a resistance of 100 Ω. Find the charge on the capacitor 2·0 ms after the connections are made.

**65.** The plates of a capacitor of capacitance 10 μF, charged to 60 μC, are joined together by a wire of resistance 10 Ω at $t = 0$. Find the charge on the capacitor in the circuit at (a) $t = 0$, (b) $t = 30$ μs, (c) $t = 120$ μs and (d) $t = 1·0$ ms.

**66.** A capacitor of capacitance 8·0 μF is connected to a battery of emf 6·0 V through a resistance of 24 Ω. Find

the current in the circuit (a) just after the connections are made and (b) one time constant after the connections are made.

**67.** A parallel-plate capacitor of plate area 40 $cm^2$ and separation between the plates 0·10 mm is connected to a battery of emf 2·0 V through a 16 Ω resistor. Find the electric field in the capacitor 10 ns after the connections are made.

**68.** A parallel-plate capacitor has plate area 20 $cm^2$, plate separation 1·0 mm and a dielectric slab of dielectric constant 5·0 filling up the space between the plates. This capacitor is joined to a battery of emf 6·0 V through a 100 kΩ resistor. Find the energy of the capacitor 8·9 μs after the connections are made.

**69.** A 100 μF capacitor is joined to a 24 V battery through a 1·0 MΩ resistor. Plot qualitative graphs (a) between current and time for the first 10 minutes and (b) between charge and time for the same period.

**70.** How many time constants will elapse before the current in a charging *RC* circuit drops to half of its initial value ? Answer the same question for a discharging *RC* circuit.

**71.** How many time constants will elapse before the charge on a capacitor falls to 0·1% of its maximum value in a discharging *RC* circuit ?

**72.** How many time constants will elapse before the energy stored in the capacitor reaches half of its equilibrium value in a charging *RC* circuit ?

**73.** How many time constants will elapse before the power delivered by the battery drops to half of its maximum value in an *RC* circuit ?

**74.** A capacitor of capacitance $C$ is connected to a battery of emf $\mathcal{E}$ at $t = 0$ through a resistance $R$. Find the maximum rate at which energy is stored in the capacitor. When does the rate has this maximum value?

**75.** A capacitor of capacitance 12·0 μF is connected to a battery of emf 6·00 V and internal resistance 1·00 Ω through resistanceless leads. 12·0 μs after the connections are made, what will be (a) the current in the circuit, (b) the power delivered by the battery, (c) the power dissipated in heat and (d) the rate at which the energy stored in the capacitor is increasing.

**76.** A capacitance $C$ charged to a potential difference $V$ is discharged by connecting its plates through a resistance $R$. Find the heat dissipated in one time constant after the connections are made. Do this by calculating $\int i^2 R\, dt$ and also by finding the decrease in the energy stored in the capacitor.

**77.** By evaluating $\int i^2 R dt$, show that when a capacitor is charged by connecting it to a battery through a resistor, the energy dissipated as heat equals the energy stored in the capacitor.

**78.** A parallel-plate capacitor is filled with a dielectric material having resistivity ρ and dielectric constant $K$. The capacitor is charged and disconnected from the charging source. The capacitor is slowly discharged through the dielectric. Show that the time constant of the discharge is independent of all geometrical parameters like the plate area or separation between the plates. Find this time constant.

**79.** Find the charge on each of the capacitors 0·20 ms after the switch $S$ is closed in figure (32-E34).

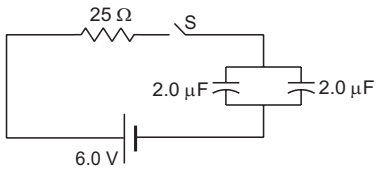



Figure 32-E34

**80.** The switch $S$ shown in figure (32-E35) is kept closed for a long time and is then opened at $t$ = 0. Find the current in the middle 10 Ω resistor at $t$ = 1·0 ms.

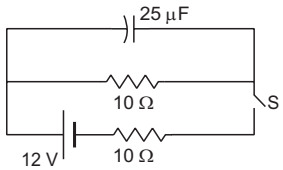



Figure 32-E35

**81.** A capacitor of capacitance 100 μF is connected across a battery of emf 6·0 V through a resistance of 20 kΩ for 4·0 s. The battery is then replaced by a thick wire. What will be the charge on the capacitor 4·0 s after the battery is disconnected ?

**82.** Consider the situation shown in figure (32-E36). The switch is closed at $t$ = 0 when the capacitors are uncharged. Find the charge on the capacitor $C_1$ as a function of time $t$.

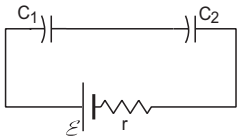



Figure 32-E36

**83.** A capacitor of capacitance $C$ is given a charge $Q$. At $t = 0$, it is connected to an uncharged capacitor of equal capacitance through a resistance $R$. Find the charge on the second capacitor as a function of time.

**84.** A capacitor of capacitance $C$ is given a charge $Q$. At $t = 0$, it is connected to an ideal battery of emf $\mathcal{E}$ through a resistance $R$. Find the charge on the capacitor at time $t$.

□

## ANSWERS

### OBJECTIVE I

| | | | | | |
|---|---|---|---|---|---|
| 1. (a) | 2. (d) | 3. (d) | 4. (c) | 5. (b) | 6. (a) |
| 7. (c) | 8. (a) | 9. (b) | 10. (a) | 11. (a) | 12. (b) |
| 13. (c) | 14. (d) | 15. (d) | 16. (b) | | |

### OBJECTIVE II

| | | |
|---|---|---|
| 1. (a) | 2. (b), (c) | 3. (c), (d) |
| 4. (d) | 5. (a) | 6. (a), (d) |
| 7. (a), (d) | 8. all | 9. (b), (d) |

### EXERCISES

1. (a) $IT^{-1}, I, IT$ (b) 53 A
2. $3{\cdot}2 \times 10^{-3}$ A
3. $6{\cdot}0 \times 10^{-4}$ C
4. 300 C
5. 0·074 mm $s^{-1}$
6. $\pi \times 10^{-6}$ Ω m
7. 400 Ω
8. $3{\cdot}2 \times 10^{4}$ s ≈ 8·9 hours
9. 0·6 km
10. $\dfrac{\rho l}{\pi a b}$
11. (a) $1{\cdot}25 \times 10^{17}$ (b) $6{\cdot}37 \times 10^{5}$ A/$m^2$
12. 8·5 mV $m^{-1}$
13. 25 V $m^{-1}$
14. 84·5°C
15. 0·4 V
16. 1·52 V, 0·07 Ω
17. 29 Ω
18. 0·6 Ω
19. (a) 0·3 A (b) 3 A
20. (a) 0·57 (b) 1 (c) 1·75
21. (a) $\dfrac{n_1 \mathcal{E}}{R + \dfrac{n_1 r}{n_2}}$ (b) $r n_1 = R n_2$
22. 10 mA
23. 1·6 A, 4·0 A
24. 0·15 A, 0·83 A
25. (a) 1·0 A (b) 0·67 A (c) 0·33 A
26. 12·5 Ω, 170 Ω
27. 45 Ω, 22·5 Ω
28. 4 mA in 20 kΩ resistor, 8 mA in 10 kΩ resistor and 12 mA in 100 kΩ resistor, 1340 V
29. 2 Ω
30. $r/3$
31. (a) 2·08 Ω (b) 3·33 Ω (c) 3·75 Ω
32. (a) 0·1 A (b) 0·3 A
33. zero in the upper 4 Ω resistor and 0·2 A in the rest two
34. 1 A through 10 Ω, 0·4 Ω through 20 Ω and 0·6 A through 30 Ω
35. 1 A in $a$ and zero in $b$ in both the circuits
36. (a) $\dfrac{\dfrac{\mathcal{E}_1}{R_1} + \dfrac{\mathcal{E}_2}{R_2}}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}}$ (b) same as (a)
37. 2 V, $i_1 = 1$ A, $i_2 = 0$, $i_3 = -1$ A
38. zero
39. zero
40. any value of $R$ will do
41. (a) $r/2$ (b) $4r/5$
42. 0·4 A
43. (a) 1·2 A (b) 6 V (c) 12 V (d) same as the parts (a), (b) and (c)
44. $\dfrac{5}{6} r$
45. (a) $\dfrac{5}{8} r$ (b) $\dfrac{4}{3} r$ (c) $r$ (d) $\dfrac{r}{4}$ (e) $r$
46. (a) 2 Ω (b) 1·5 A
47. (a) 0·1 A, 4·0 V (b) 0·08 A, 4·2 V
48. (a) 24 V (b) 28 V
49. 130 Ω
50. 6 V
51. $1{\cdot}25 \times 10^{-2}$ Ω
52. 0·251 Ω
53. 16 cm from $A$
54. 4 Ω
55. (a) 6 V, 2 V (b) $AD$ = 66·7 cm (c) zero (d) 6 V, − 1·5 V, no such point $D$ exists.
56. (a) 320 cm (b) $\dfrac{3\mathcal{E}}{22 r}$
57. 4 μC
58. (a) 0·2 A (b) 32 μJ
59. 2 μC, 8 μC, 9 μC and 12 μC
60. 25 V, 75 V
61. (a) $\mathcal{E}$ (b) $\dfrac{\mathcal{E}}{R}$ (c) $\mathcal{E}$ (d) $\dfrac{1}{2} C \mathcal{E}^2$ (e) $\dfrac{\mathcal{E}^2}{R}$ (f) $\dfrac{\mathcal{E}^2}{R}$
62. 0·18 μs
63. 5 kΩ
64. 76 μC
65. (a) 60 μC (b) 44 μC (c) 18 μC (d) 0·003 μC
66. (a) 0·25 A (b) 0·09 A
67. $1{\cdot}7 \times 10^{4}$ V $m^{-1}$
68. $6{\cdot}3 \times 10^{-10}$ J
70. 0·69 in both cases
71. 6·9

72. 1·23
73. 0·69
74. $\frac{\mathcal{E}^2}{4R}$, $CR\ln 2$
75. (a) 2·21 A (b) 13·2 W (c) 4·87 W (d) 8·37 W
76. $\frac{1}{2}(1-1/e^2)CV^2$
78. $\varepsilon_0 \rho K$
79. 10·37 μC
80. 11 mA
81. 70 μC
82. $q = \mathcal{E}C(1-e^{-t/rc})$, where $C = \frac{C_1C_2}{C_1+C_2}$.
83. $\frac{Q}{2}(1-e^{-2t/RC})$
84. $C\mathcal{E}(1-e^{-t/CR}) + Qe^{-t/CR}$

□

CHAPTER 33

# THERMAL AND CHEMICAL EFFECTS OF ELECTRIC CURRENT

We have seen that an electric current through a resistor increases its thermal energy. Also, there are other situations in which an electric current can produce or absorb thermal energy. All these are termed as thermal effects of electric current. We shall study them in sections (33.1) to (33.6). The chemical effects, such as electrolysis, are discussed in sections (33.7) onwards.

## 33.1 JOULE'S LAWS OF HEATING

When there is an electric current in a resistor, the thermal energy of the resistor increases. If the potential difference between the ends of a resistor is $V$ and a current $i$ passes through it, the work done by the electric field on the free electrons in time $t$ is

$$\begin{aligned} W &= (\text{potential difference}) \times (\text{charge}) \\ &= V(it) \\ &= (iR)\,(it) = i^2Rt. \qquad \ldots \quad (33.1) \end{aligned}$$

The work by the field is converted into thermal energy of the resistor through the collisions with the lattice. This thermal energy is generally referred to as the *heat produced in the resistor* and is denoted by $H$. (Strictly speaking, this energy is not heat as it does not correspond to any temperature difference. Because of the increased thermal energy, the temperature of the resistor may rise. It may then transfer "heat" to the surrounding.) It follows from equation (33.1) that

*(a) the heat produced in a given resistor in a given time is proportional to the square of the current in it, i.e.,*

$$H \propto i^2$$

*(b) the heat produced in a given resistor by a given current is proportional to the time for which the current exists in it, i.e.,*

$$H \propto t$$

*(c) the heat produced in a resistor by a given current in a given time is proportional to its resistance, i.e.,*

$$H \propto R.$$

The heating effects of an electric current were studied by James Prescott Joule and he arrived at the three laws stated above. These are thus known as *Joule's laws.*

***Example 33.1***

*Find the heat developed in each of the three resistors shown in figure (33.1) in* 1 *minute.*

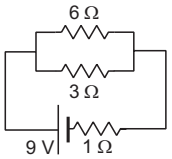



Figure 33.1

***Solution*** : The equivalent resistance of 6 Ω and 3 Ω resistors is

$$\frac{(6\ \Omega) \times (3\ \Omega)}{6\ \Omega + 3\ \Omega} = 2\ \Omega.$$

This is connected in series with the 1 Ω resistor. The equivalent resistance of the circuit is

$$R = 2\ \Omega + 1\ \Omega = 3\ \Omega.$$

The current through the battery is

$$i = \frac{9\ \text{V}}{3\ \Omega} = 3\ \text{A}.$$

The current through the 1 Ω resistor is, therefore, 3 A. The heat developed in this resistor is

$$\begin{aligned} H &= i^2Rt \\ &= (3\ \text{A})^2 \times (1\ \Omega) \times (60\ \text{s}) = 540\ \text{J}. \end{aligned}$$

The current through the 6 Ω resistor is

$$(3\ \text{A}) \times \frac{3\ \Omega}{6\ \Omega + 3\ \Omega} = 1\ \text{A}.$$

The heat developed in it

$$= (1\ \text{A})^2 \times (6\ \Omega) \times (60\ \text{s}) = 360\ \text{J}.$$

The current through the 3 Ω resistor is 3 A – 1 A = 2 A.
The heat developed in it

$$= (2\text{ A})^2 \times (3\ \Omega) \times (60\text{ s}) = 720\text{ J}.$$

## 33.2 VERIFICATION OF JOULE'S LAWS

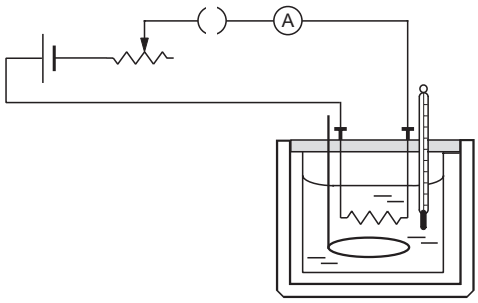



Figure 33.2

Figure (33.2) shows the apparatus and the connections. K-oil is taken in a copper calorimeter provided with a nonconducting lid and a stirrer. A resistor is dipped in the oil. It is joined to the external circuit through the leads coming out of the lid of the calorimeter. A thermometer is provided to measure the temperature of the oil. The calorimeter is called *Joule's calorimeter*.

### $H \propto i^2$

The external circuit consists of a battery, a rheostat and a plug key in series with the resistor. An ammeter is also connected in the circuit to measure the current. The temperature $\theta_1$ of the K-oil is noted. The plug key is closed to pass a constant current through the circuit for a known time $t$. The value of the current $i_1$ is measured by the ammeter. The liquid is stirred continuously and the final temperature $\theta_2$ is noted. Thus, the rise in temperature $\Delta\theta_1 = \theta_2 - \theta_1$ is calculated.

The system is allowed to cool down to room temperature. The resistance of the rheostat is changed and the key is closed. The current $i_2$ is measured by the ammeter and is passed for the same time $t$. The rise in temperature $\Delta\theta_2$ is found as above.

The heat produced in the resistor is used to increase the temperature of the K-oil. Thus, the heat produced is proportional to the rise in temperature. It is found that

$$\frac{\Delta\theta_1}{\Delta\theta_2} = \frac{i_1^2}{i_2^2}.$$

This shows that $\Delta\theta \propto i^2$

or, $H \propto i^2$.

### $H \propto t$

The arrangement described above can also be used to verify the second law, $H \propto t$. A current $i$ is passed through the resistor and the temperature is noted at regular time intervals. It is found that the temperature rises uniformly, i.e., it increases by equal amounts in equal times. This shows that equal amounts of heat are produced in equal time intervals. Thus, the heat produced is proportional to the time.

### $H \propto R$

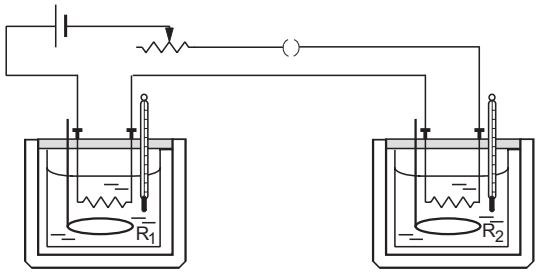



Figure 33.3

To verify this law, two Joule's calorimeters are taken (figure 33.3). Equal amounts of K-oil are taken in the two calorimeters. Different resistances $R_1$ and $R_2$ are dipped in the K-oil and the two are joined in series. The system is connected to a battery, a rheostat and a plug key. The initial temperatures of the two calorimeters are noted and a current is passed for some time. The temperatures of the two calorimeters are noted at the end. Let $\Delta\theta_1$ be the rise in the temperature of the first calorimeter and $\Delta\theta_2$ be the rise in the temperature of the second calorimeter. The heat produced in each resistor is proportional to the rise in the temperature of the corresponding calorimeter. It is found that

$$\frac{\Delta\theta_1}{\Delta\theta_2} = \frac{R_1}{R_2}$$

or, $\Delta\theta \propto R$

or, $H \propto R$.

## 33.3 SEEBECK EFFECT

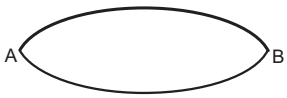



Figure 33.4

Figure (33.4) shows two metallic strips, made of different metals and joined at the ends to form a loop. If the junctions are kept at different temperatures, there is an electric current in the loop. This effect is called the *Seebeck effect* and the emf developed is called the *Seebeck emf* or *thermo-emf*.

The magnitude and the direction of the emf depend on the metals and the temperatures of the hot and

cold junctions. Such a combination of two metals is called a *thermocouple*.

**Thermoelectric Series**

For given temperatures of hot and cold junctions, the direction of the current in a thermocouple depends on the metals chosen. Metals are arranged in a particular sequence which may be used to predict the direction of the current in the temperature range 0°C to 100°C. This sequence known as the *thermoelectric series*, is as follows:

antimony, nichrome, iron, zinc, copper, gold, silver, lead, aluminium, mercury, platinum–rhodium, platinum, nickel, costantan, bismuth.

At the cold junction, the current is *from* the metal coming earlier in the series *to* the metal coming latter in the series. For example, in a copper–iron thermocouple, the current will be from iron to copper at the cold junction. Also, the series gives an idea of the relative magnitude of emf for different thermocouples. Farther apart two metals lie in the series, larger is the emf produced.

**Neutral and Inversion Temperature**

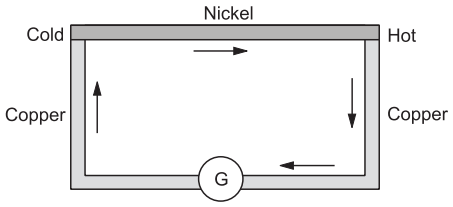



Figure 33.5

Figure (33.5) shows a copper–nickel thermocouple. A sensitive galvanometer is connected in series to measure the current. Suppose one of the junctions is kept at a fixed low temperature 0°C and the other is gradually heated. There will be a current in the circuit as shown in the figure. At the cold junction it is from copper to nickel and at the hot junction it is from nickel to copper. As the temperature of the hot junction is gradually increased, the magnitude of the current increases till the temperature becomes nearly 390°C. After this, the current decreases till the temperature of the hot junction becomes 780°C. At this temperature the current becomes zero. If the hot junction is heated further, the direction of the current is reversed and the magnitude increases.

If the cold junction of the copper–nickel thermocouple is not at 0°C but say at 10°C, the current will be again maximum when the hot junction is at 390°C. But the inversion of the direction of the current will take place at 770°C instead of 780°C. As the current is proportional to the emf developed, the above observations also indicate the behaviour of the thermo-emf.

Copper and nickel were taken above only as an example. The behaviour of any other thermocouple will be, in general, similar. The numeric values will, of course, be different. Also even for a copper–nickel thermocouple, the neutral temperature depends on the purity, heat treatment, etc., of the metals.

The temperature of the hot junction at which the thermo-emf is maximum is called the *neutral temperature* and the temperature at which the thermo-emf changes its sign (current reverses) is called the *inversion temperature*. If $\theta_c$, $\theta_n$ and $\theta_i$ denote the temperature of the cold junction, the neutral temperature and the inversion temperature respectively, we have

$$\theta_n - \theta_c = \theta_i - \theta_n. \qquad \ldots \quad (33.2)$$

At this stage, let us explain the sign convention. The thermo-emf developed in a thermocouple of metals $A$ and $B$ is denoted by $\mathcal{E}_{AB}$ and is taken to be positive if the direction of the current is from $A$ to $B$ at the hot junction.

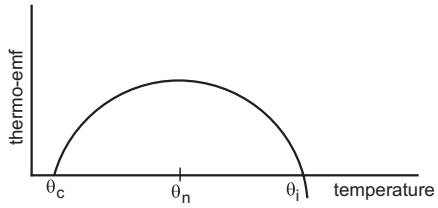



Figure 33.6

Figure (33.6) shows graphically the variation in thermo-emf as the temperature of the hot junction changes. If the cold junction is at 0°C and the hot junction at θ (in Celsius), the thermo-emf depends on the temperature as

$$\mathcal{E}_{AB} = a_{AB}\theta + \frac{1}{2} b_{AB}\theta^2 \qquad \ldots \quad (33.3)$$

where $a_{AB}$ and $b_{AB}$ are constants for the pair of metals $A$ and $B$.

This gives

$$\frac{d\mathcal{E}_{AB}}{d\theta} = a_{AB} + b_{AB}\theta. \qquad \ldots \quad (33.4)$$

The quantity $\frac{d\mathcal{E}_{AB}}{d\theta}$ is called *thermoelectric power* at temperature θ.

The emf is maximum when $\frac{d\mathcal{E}_{AB}}{d\theta} = 0$ or, $\theta = -\frac{a_{AB}}{b_{AB}}$. This is the neutral temperature. The emf becomes zero at $\theta = -2a_{AB}/b_{AB}$. This is the inversion temperature.

***Example 33.2***

*The cold junction of a thermocouple is maintained at* 10°C. *No thermo-emf is developed when the hot junction is maintained at* 530°C. *Find the neutral temperature.*

***Solution*** : Clearly, 530°C is the inversion temperature $\theta_i$ of the couple. If $\theta_n$ be the neutral temperature and $\theta_c$ be the temperature of the cold junction,

$$\theta_i - \theta_n = \theta_n - \theta_c$$

or, $$\theta_n = \frac{\theta_i + \theta_c}{2} = \frac{530°\text{C} + 10°\text{C}}{2} = 270°\text{C}.$$

Table (33.1) gives the values of $a$ and $b$ for some of the metals with lead. Note that there is no neutral temperature or inversion temperature above 0°C for the thermocouple if $a$ and $b$ have the same sign.

**Table 33.1 :** ***Coefficients a and b for thermocouples***

| Metal with lead (Pb) | $a$ $\mu V°C^{-1}$ | $b$ $\mu V°C^{-2}$ |
|---|---|---|
| Aluminium | – 0·47 | 0·003 |
| Bismuth | – 43·7 | – 0·47 |
| Copper | 2·76 | 0·012 |
| Gold | 2·90 | 0·0093 |
| Iron | 16·6 | – 0·030 |
| Nickel | 19·1 | – 0·030 |
| Platinum | – 1·79 | – 0·035 |
| Silver | 2·50 | 0·012 |
| Steel | 10·8 | – 0·016 |

**Law of Intermediate Metal**

Suppose $\mathcal{E}_{AB}$ = thermo-emf between metals $A$ and $B$

$\mathcal{E}_{AC}$ = thermo-emf between metals $A$ and $C$

and $\mathcal{E}_{BC}$ = thermo-emf between metals $B$ and $C$.

Also, suppose the temperatures of the cold junctions are the same in the three cases and the temperatures of the hot junctions are also the same in the three cases.

Then, $$\mathcal{E}_{AB} = \mathcal{E}_{AC} - \mathcal{E}_{BC}. \quad \ldots \quad (33.5)$$

This law is known as the *law of intermediate metal.*

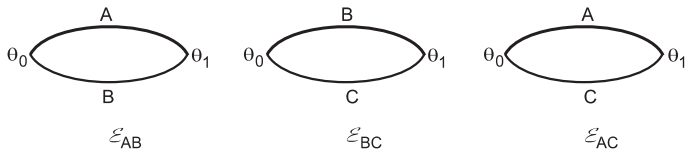



Figure 33.7

We have,

$$\mathcal{E}_{AB} = a_{AB}\theta + \frac{1}{2} b_{AB}\theta^2$$

$$\mathcal{E}_{BC} = a_{BC}\theta + \frac{1}{2} b_{BC}\theta^2$$

$$\mathcal{E}_{AC} = a_{AC}\theta + \frac{1}{2} b_{AC}\theta^2.$$

Equation (33.5) gives

$$a_{AB} = a_{AC} - a_{BC} \text{ and } b_{AB} = b_{AC} - b_{BC.}$$

Table (33.1) may, therefore, be used to find the values of $a$ and $b$ for any pair of metals listed.

***Example 33.3***

*Using table (33.1), find a and b coefficients for a copper–iron thermocouple.*

***Solution*** :

$$a_{Cu,\,Fe} = a_{Cu,\,Pb} - a_{Fe,\,Pb}$$

$$= 2{\cdot}76\ \mu V°C^{-1} - 16{\cdot}6\ \mu V°C^{-1} = -13{\cdot}8\ \mu V°C^{-1}.$$

$$b_{Cu,\,Fe} = b_{Cu,\,Pb} - b_{Fe,\,Pb}$$

$$= 0{\cdot}012\ \mu V°C^{-2} + 0{\cdot}030\ \mu V°C^{-2} = 0{\cdot}042\ \mu V°C^{-2}.$$

**Law of Intermediate Temperatures**

Let $\mathcal{E}_{\theta 1,\,\theta 2}$ denote the thermo-emf of a given thermocouple when the temperatures of the junctions are maintained at $\theta_1$ and $\theta_2$. Then,

$$\mathcal{E}_{\theta 1,\,\theta 2} = \mathcal{E}_{\theta 1,\,\theta 3} + \mathcal{E}_{\theta 3,\,\theta 2}. \quad \ldots \quad (33.6)$$

This is known as the *law of intermediate temperature.*

The two laws given by equations (33.5) and (33.6) show that we can include any metal wire or a galvanometer, etc. in the thermocouple circuit without changing the emf in the circuit.

## 33.4 PELTIER EFFECT

Suppose the two junctions of a thermocouple are initially at the same temperature and an electric current is passed through the circuit by using an external battery. It is observed that heat is produced at one junction and is absorbed at the other. Thus, one junction is warmed up and the other is cooled down due to the currents through the junctions. It is reverse of the Seebeck effect and is called the *Peltier effect*. If the direction of the current is reversed, the cooling and warming are also reversed. This means, the junction which was originally warmed up, now cools down and vice-versa. The heat absorbed or liberated at the junction is proportional to the charge passed through the junction. If an amount $\Delta H$ of heat is produced or absorbed when a charge $\Delta Q$ is passed through the junction, we define *Peltier emf* as

$$\Pi_{AB} = \frac{\Delta H}{\Delta Q} = \frac{\text{Peltier heat}}{\text{charge transferred}}.$$

The Peltier emf $\Pi_{AB}$ across a junction of two metals $A$ and $B$ is taken as positive if heat is absorbed by the junction when there is an electric current from $A$ to $B$ through the junction. When two different metals are joined at a point, this much emf is developed across the junction.

The heat developed in Peltier effect should not be confused with the Joule heat when a current is passed through a resistor. Joule heat always warms up the resistor whatever be the direction of the current. But one has both Peltier heating and Peltier cooling of the junction depending on the direction of the current.

Another difference is that the Peltier heat in a given time is proportional to the current through the junction whereas the Joule heat is proportional to the square of the current,

$$H_{Peltier} \propto i, \qquad H_{Joule} \propto i^2.$$

Also, the Peltier heating or cooling is observed only at a junction, whereas the Joule heating is throughout the resistor.

### 33.5 THOMSON EFFECT

If a metallic wire has a nonuniform temperature and a current is passed through it, heat may be absorbed or produced in different sections of the wire. This heat is over and above the Joule heat $i^2Rt$ and is called the *Thomson heat*. The effect itself is called the *Thomson effect*.

Thomson heat, produced or absorbed in a small section of a given wire, is proportional to the charge passed through the section and the temperature difference between the ends of the section.

If a charge $\Delta Q$ is passed through a small section of the wire having a temperature difference $\Delta T$ between the ends, the Thomson heat is

$$\Delta H = \sigma(\Delta Q)\,(\Delta T)$$

where $\sigma$ is a constant for a given metal at a given temperature. The quantity

$$\sigma \Delta T = \frac{\Delta H}{\Delta Q} = \frac{\text{Thomson heat}}{\text{charge transferred}}$$

is called the *Thomson emf*.

In fact, this much amount of emf is produced when the ends of the section of the wire are maintained at different temperatures.

The constant $\sigma$ is called the *Thomson coefficient*. It is taken to be positive if heat is absorbed when a current is passed from the low-temperature end to the high-temperature end. Copper, silver, zinc, antimony, cadmium, etc., have positive $\sigma$. Iron, cobalt, nickel, bismuth, platinum, etc., have negative $\sigma$. In these metals heat is absorbed when current is passed from hotter end to the colder end. In lead, $\sigma$ is almost zero.

### 33.6 EXPLANATION OF SEEBECK, PELTIER AND THOMSON EFFECTS

The density of free electrons is different in different metals. When two different metals are joined to form a junction, the electrons tend to diffuse from the side with higher concentration to the side with lower concentration. This produces an emf across the junction. This emf is the Peltier emf. If a current is forced through the junction, positive or negative work is done on the charge carriers depending on the direction of the current. Accordingly, thermal energy is either produced or absorbed.

If the temperature of a metal piece is not uniform everywhere, density of free electrons varies inside the metal. The electrons tend to diffuse from the higher-concentration regions to the lower-concentration regions. This gives rise to an emf between the hot and the cold parts of the metal. This emf is the Thomson emf. If a current is forced through a wire having nonuniform temperature, positive or negative work is done on the charge carriers depending on the direction of the current. Accordingly, thermal energy is either produced or absorbed.

The density of free electrons in a metal depends on the temperature. Hence, the Peltier emf developed across a junction depends on the temperature of the junction. In a thermocouple, there are two junctions. If these junctions are maintained at the same temperature, the Peltier emf's developed across the two junctions balance each other and there is no net Peltier emf in the loop. If the junctions are at different temperatures, the emf's developed across the junctions are different and there is a net Peltier emf in the loop. Also, when the junctions of a thermocouple are kept at different temperatures, each of the two metal pieces has nonuniform temperature. Thus, a Thomson emf is developed across its ends. The emf's are different for the two metals and hence there is a net Thomson emf in the loop because of this effect. The actual emf developed in a thermocouple loop is the algebraic sum of the net Peltier emf and the net Thomson emf developed in the loop.

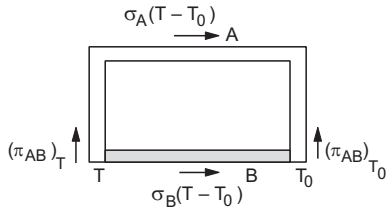



Figure 33.8

Referring to figure (33.8), we can write the thermo-emf $\mathcal{E}_{AB}$ in the loop as

$$\mathcal{E}_{AB} = (\Pi_{AB})_T - (\Pi_{AB})_{T_0} + (T - T_0)(\sigma_A - \sigma_B). \quad \ldots \quad (33.6)$$

Thus, Seebeck emf is a combination of two Peltier emf's and two Thomson emf's.

## 33.7 ELECTROLYSIS

Several liquids are found to be good conductors of electricity. Solutions of inorganic salts in water, dilute acids and bases are examples of conducting liquids. Such a liquid is called an *electrolyte* and the vessel with the electrolyte is called an *electrolytic cell.* The mechanism of electric conduction in electrolytes is quite different from that in metals. Let us understand the process with a specific example.

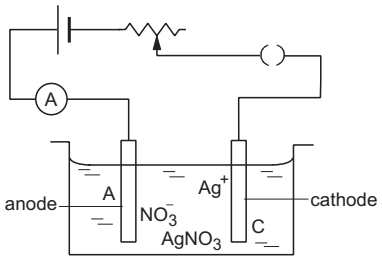



Figure 33.9

Let us take a solution of silver nitrate ($AgNO_3$) in an electrolytic cell and immerse two silver rods in it (figure 33.9). Electric current is passed through the solution by connecting the rods to an external battery. The current enters the solution through the rod *A*, called the *anode* and it leaves the solution through the rod *C*, called the *cathode.* These rods are collectively called *electrodes.* We find that as time passes, silver gets deposited on the cathode *C* and an equal amount of silver is removed from the anode *A.*

A fraction of the molecules of the dissolved $AgNO_3$ are separated in two parts, $Ag^+$ and $NO_3^-$, each of which has electric charge. These are called *ions.* An ion with positive charge is called *cation* and an ion with negative charge is called *anion.* Here $Ag^+$ is the cation and $NO_3^-$ is the anion. These ions move freely in the solution. When a battery is connected to the electrodes, electric field is produced in the solution from the anode towards the cathode. Thus, the cations move towards the cathode and the anions move towards the anode. The ions give up their charges at the electrodes and the substance making up the ions is *liberated.* The liberated substance may get deposited on the electrodes or may take part in some secondary chemical reaction. In case of $AgNO_3$, the $Ag^+$ ions (cation) move to the cathode, the charge is given up there and the silver atoms are deposited on the cathode. In fact an $Ag^+$ ion receives an electron from the cathode to become neutral Ag atom,

$$Ag^+ + e = Ag.$$

The $NO_3^-$ ion (anion) moves to the anode and gives its extra electron to it. The $NO_3^-$ ion is converted to $NO_3$,

$$NO_3^- = NO_3 + e.$$

The $NO_3$ so formed reacts with a silver atom of the anode to form $AgNO_3$ which gets dissolved in the solution. This way, silver is continuously removed from the anode and deposited on the cathode with the concentration of the electrolyte remaining unchanged. The movement of cations (positive charge) towards the cathode and anions (negative charge) towards the anode make the current in the electrolyte.

The electron given up by an anion to the anode is passed through the battery back to the cathode which has supplied one electron to the cation. This way, the potential difference between the electrodes is maintained. Also, the number of electrons passed through the battery is equal to the number of electrons absorbed or released at each electrode. In other words, the charge passed through the circuit is equal to the charge released at each electrode.

When a current passes through an electrolyte, chemical changes occur in the electrolyte and substances are liberated at the electrodes. This process is called *electrolysis.*

## 33.8 FARADAY'S LAWS OF ELECTROLYSIS

After systematically studying electrolysis, Faraday discovered two laws:

(a) The mass of a substance liberated at an electrode is proportional to the charge passing through the electrolyte.

(b) The mass of a substance liberated at an electrode by a given amount of charge is proportional to the chemical equivalent of the substance.

If an electric current $i$ is passed through an electrolyte for a time $t$, the amount of charge passed is $Q = it$. According to the first law, the amount of the substance liberated at an electrode is

$$m \propto Q \quad \ldots \quad \text{(i)}$$

or, $$m \propto it$$

or, $$m = Zit \quad \ldots \quad (33.7)$$

where $Z$ is a constant for the substance being liberated. The constant $Z$ is called the *electrochemical equivalent* (ECE) of the substance. The SI unit of ECE is kilogram coulomb$^{-1}$ written as kg C$^{-1}$.

The chemical equivalent of a substance is equal to its relative atomic mass divided by its valency. Relative atomic mass of a substance is the ratio of the

mass of its atom to the 1/12 of the mass of a $^{12}C$ atom. The relative atomic mass of silver is 108 and its valency is 1. Thus, the chemical equivalent of silver is 108. The relative atomic mass and valency of copper are 63·5 and 2 respectively. The chemical equivalent is, therefore, 31·75. For oxygen, the chemical equivalent is 16/2 = 8.

If $E$ denotes the chemical equivalent of a substance being liberated at an electrode, from the second law,

$$m \propto E.$$

Combining with (i) it gives

$$m \propto EQ.$$

or, $$m = \frac{1}{K} EQ \quad \ldots \quad (33.8)$$

where $K$ is a universal constant having the value $9{\cdot}6485 \times 10^{7}$ C kg$^{-1}$.

***Example 33.4***

*Calculate the electric current required to deposit* 0·972 g *of chromium in three hours. ECE of chromium is* 0·00018 g C$^{-1}$.

***Solution*** : We have

$$m = Zit$$

or, $$0{\cdot}972 \text{ g} = (0{\cdot}00018 \text{ g C}^{-1})\, i\, (3 \times 3600 \text{ s})$$

or, $$i = \frac{0{\cdot}972}{0{\cdot}00018 \times 3 \times 3600} \text{ A} = 0{\cdot}50 \text{ A}.$$

**Verification of Faraday's Laws**

**1st law**

An electrolytic cell is connected to a battery, a rheostat, a key and an ammeter in series as in figure (33.9). The cathode is cleaned, dried, weighed and then inserted in the electrolytic cell. A constant current $i_1$ is passed for a measured time $t$. The current is measured by the ammeter. Slight adjustment of the rheostat may be necessary to keep the current constant. The cathode is taken out, washed without rubbing in gently-flowing water and is dried. It is weighed again and the mass $m_1$ of the deposit is obtained. The cathode is reinserted in the cell and the rheostat position is changed. This allows a different current $i_2$ when the circuit is completed. This current is passed for the same time $t$ and the mass $m_2$ of the deposit is obtained. It is found that

$$\frac{m_1}{m_2} = \frac{i_1}{i_2}.$$

Thus, $$m \propto i. \quad \ldots \quad (i)$$

Similarly, two experiments are done with the same current but for different times $t_1$ and $t_2$. If the masses of the deposits are $m_1'$ and $m_2'$ respectively, it is found that

$$\frac{m_1'}{m_2'} = \frac{t_1}{t_2}$$

or, $$m \propto t. \quad \ldots \quad (ii)$$

From (i) and (ii),

$$m \propto it$$

or, $$m \propto Q.$$

So, the first law is verified.

**2nd law**

To verify the second law, one can take two electrolytic cells containing different electroytes. To be specific, one may contain $CuSO_4$ solution with copper electrodes and the other $AgNO_3$ solution with silver electrodes. The electrodes are connected in series with an external circuit as shown in figure (33.10). This arrangement ensures that the currents in the two cells are the same.

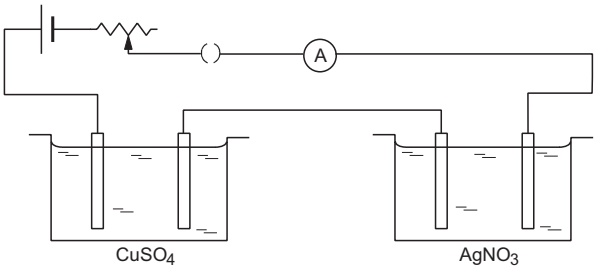



Figure 33.10

The cathodes are cleaned, dried, weighed and then inserted in the respective cells. The current is passed for some time and then the cathodes are taken out. They are washed without rubbing in gently-flowing water, dried and weighed. Thus, the masses of the deposits are obtained. If $m_1$ and $m_2$ are the masses of copper and silver deposited, it is found that

$$\frac{m_1}{m_2} = \frac{E_1}{E_2}$$

where $E_1$ and $E_2$ are the chemical equivalents of copper and silver respectively. So, the second law is verified.

**Faraday's Laws and Ionic Theory**

Faraday's laws of electrolysis can be easily understood in terms of the ionic character for the conduction in an electrolyte. Suppose a substance of valency $v$ is liberated at an electrode. Its ion will carry a positive or a negative charge of magnitude $ve$. To neutralise this ion, the battery has to supply or take up a charge $ve$. If the total charge passing through the battery is $Q$, the number of ions neutralised is

$$N = \frac{Q}{ve}.$$

Suppose, the relative atomic mass of the substance is $A$. Now $\frac{1}{12}$ of the mass of a $^{12}C$ atom is one atomic mass unit (amu) and has the value $1{\cdot}6605 \times 10^{-27}$ kg. Let us denote this quantity as $m_u$. The mass of an atom of the substance is then

$$m_a = A\, m_u.$$

This can also be taken as the mass of an ion because the difference in ionic mass and atomic mass is negligible. The mass of the $N$ ions neutralised at the electrode is

$$m = N A\, m_u$$
$$= \frac{Q}{ve} A\, m_u.$$

But $\frac{A}{v}$ is the chemical equivalent $E$, so that

$$m = \frac{1}{e/m_u} EQ.$$

Thus, $m \propto Q$ and $m \propto E$ which are Faraday's laws.

The constant

$$\frac{e}{m_u} = K = \frac{1{\cdot}6022 \times 10^{-19}\ \text{C}}{1{\cdot}6605 \times 10^{-27}\ \text{kg}}$$
$$= 9{\cdot}6485 \times 10^{7}\ \text{C kg}^{-1}$$

as mentioned earlier.

**The Unit "faraday" and Faraday Constant**

The charge of 1 mole of electrons is called 1 faraday. So faraday is a unit of charge and its relation with coulomb is

$$1\ \text{faraday} = (1{\cdot}6022 \times 10^{-19}\ \text{C}) \times (6{\cdot}022 \times 10^{23})$$
$$= 96485\ \text{C}.$$

The quantity *charge per mole of electrons* is called Faraday constant and is denoted by the symbol $F$. Thus,

$$F = 96485\ \text{C mol}^{-1}$$
$$= 1\ \text{faraday mol}^{-1}.$$

Suppose, 1 faraday of charge is passed through an electrolyte. The amount of electrons taken up or supplied at an electrode is 1 mole. Since each ion takes up or gives up $v$ electrons, the amount of ions liberated is $\frac{1}{v}$ moles. The mass of 1 mole of the substance is $A$ gram ($A$ = relative atomic mass), so that the mass of these ions is $\frac{A}{v}$ gram = $E$ gram. The amount $E$ gram of a substance is called 1 gram-equivalent of this substance. So, 1 *faraday of charge liberates one gram-equivalent of any substance in electrolysis.*

## 33.9 VOLTAMETER OR COULOMBMETER

An electrolytic cell can be used to measure electric currents or amounts of charge. The current to be measured is passed through the electrolytic cell for a known time $t$. If the mass deposited in time $t$ is $m$ and the electrochemical equivalent of the material is $Z$,

$$m = Zit$$

or,
$$i = \frac{m}{Zt}.$$

The charge passed is $it = m/Z$.

When an electrolytic cell is used to measure electric current it is called a *voltameter* or a *coulombmeter*. Quite often, the word voltameter is used to mean just the electrolytic cell whatever be its use.

A copper voltameter may be used for routine laboratory measurements. The electrolyte is $CuSO_4$ solution in water in the ratio of about 1 : 4 by weight. A copper plate placed in the middle serves as the cathode and two connected copper plates on the two sides of the cathode are used as the anode. In this arrangement, shown schematically in figure (33.11), both sides of the cathode plate receive the deposit.

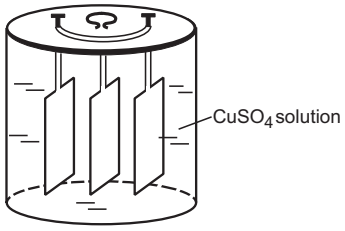



Figure 33.11

A silver voltameter (figure 33.12) is used when high accuracy is needed. The electrolyte is silver nitrate solution in water in the ratio of 1 : 5 to 1 : 6. The solution is taken in a platinum cup. The cup itself acts as the cathode. The anode is made of a silver rod placed in the middle. A porcelain cup which is porous to the electrolyte surrounds the anode. This prevents the impurities in the anode to reach the cathode.

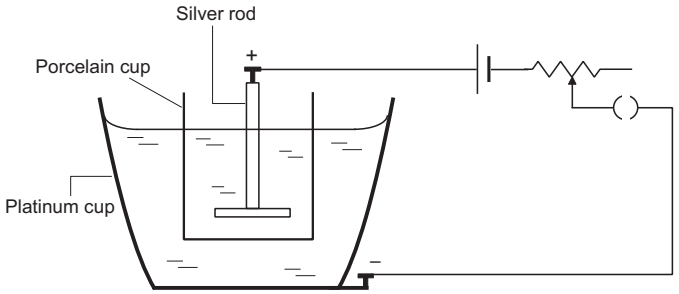



Figure 33.12

Because of the high accuracy attainable, silver voltameters are used in laboratories for standardization purposes.

## 33.10 PRIMARY AND SECONDARY CELLS

When a metal electrode is dipped in an electrolyte, negative or positive ions tend to go from the electrolyte to the electrode. Thus, an emf is produced between the electrolyte and the electrode. If two electrodes of different metals are dipped in an electrolyte, the emf's produced at the two electrodes are different and there is a net emf between the electrodes. This forms a *voltaic cell*, also called simply a *cell*. The electrode at higher potential is called the *positive electrode* and that at lower potential is called the *negative electrode*. Terminals connected to these electrodes are called positive and negative terminals respectively. This is the basic theory of a voltaic cell in which electrolysis is used to produce an emf between two terminals.

A cell is called *primary* if it is used only for discharge. The current leaves the cell at the positive terminal, goes through the external circuit and enters the cell at the negative terminal. A *secondary* cell on the other hand can be discharged as well as charged; the current can go both ways in the cell.

## 33.11 PRIMARY CELLS

### Daniell Cell

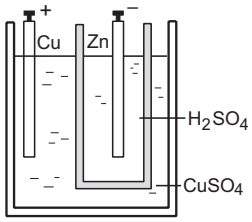



Figure 33.13

In the simplest form, a Daniell cell consists of a zinc electrode in a dilute $H_2SO_4$ solution and a copper electrode in $CuSO_4$ solution. The two solutions are separated by a porous cup which allows any gas to pass through, but generally prevents the liquids to mix. The zinc surface is amalgamated to avoid the local effects of impurities.

The $H_2SO_4$ solution consists of $H^+$ ions and $SO_4^{--}$ ions. The $SO_4^{--}$ ions move to the zinc electrode to form $ZnSO_4$. Negative charge is given up to the zinc electrode in the process. The $H^+$ ions move out through the porous cup. These $H^+$ ions combine with the $SO_4^{--}$ ions in the $CuSO_4$ solution to form $H_2SO_4$. The $Cu^{++}$ ions in the outer vessel move towards the copper electrode. Copper is deposited on the copper electrode and positive charge is given up to this electrode in the process. This arrangement prevents hydrogen from collecting near the anode. Any hydrogen collected near the anode may stop the function of the cell. This problem is called the *polarization* of the cell.

Because of the chemical actions described, the positive charge is accumulated on the copper electrode and negative charge on the zinc electrode. Thus an emf is produced. The zinc electrode works as the negative electrode and the copper electrode works as the positive electrode.

Daniel cell is used when a continuous current is needed. The emf of a Daniell cell is around 1·09 V and its internal resistance is around 1 Ω.

### Leclanche Cell

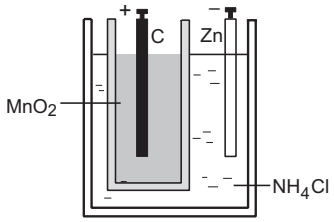



Figure 33.14

A Leclanche cell has a carbon and a zinc electrode in a solution of $NH_4Cl$. The carbon electrode is packed in a porous cup containing $MnO_2$. The zinc electrode forms the negative terminal and the carbon forms the positive terminal. When a current passes through the cell, the $Cl^-$ ions combine with zinc and the $NH_4^+$ ions move towards the carbon electrode. The $NH_4^+$ separates into ammonia ($NH_3$) and hydrogen ($H^4$). The $H^+$ ion enters the porous cup. The positive charge is given to the carbon electrode. The hydrogen reacts with $MnO_2$ present there to form $Mn_2O_3$ and water. Thus, $MnO_2$ prevents hydrogen from collecting on the anode which could otherwise stop the cell's function.

The depolarizing action (absorbing hydrogen in some chemical reaction) is quite slow in Leclanche cell. Thus, if the cell is used continuously, hydrogen starts collecting at the carbon electrode and the cell is not able to supply enough current. Thus, the cell is used when intermittent currents are needed. Its emf is about 1·5 V.

### Dry Cell

This is a special kind of Leclanche cell in which both $NH_4Cl$ and $MnO_2$ are prepared in the form of a paste. The paste is contained in a zinc container which itself works as the negative electrode. The whole system is sealed so that the paste does not dry up. The internal resistance of a dry cell is very small, of the order of 0·1 Ω.

### 33.12 SECONDARY CELL : LEAD ACCUMULATOR

In a secondary cell, one can pass current in both directions. When current leaves the cell at the positive terminal and enters the cell at the negative terminal, the cell is discharged. This is the normal working of the cell. Chemical energy is converted into electrical energy in this case. If the cell is connected to some other source of larger emf, current may enter the cell at the positive terminal and leave it at the negative terminal. The electric energy is then converted into chemical energy and the cell gets charged. The most commonly used secondary cell is a lead accumulator.

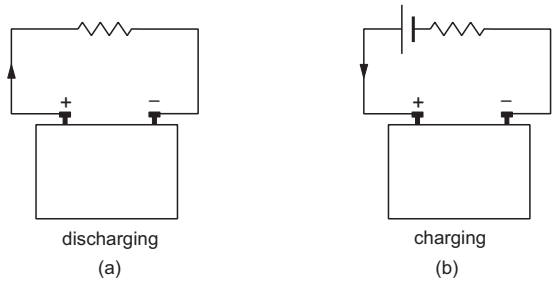



Figure 33.15

**Lead Accumulator**

In principle, a lead accumulator consists of electrodes made of $PbO_2$ and of Pb immersed in dilute sulphuric acid ($H_2SO_4$). The specific gravity of the solution should be between 1·20 and 1·28. $PbO_2$ acts as the positive electrode and Pb as the negative electrode. While discharging, the $SO_4^{--}$ ions move towards the Pb electrode, give up the negative charge and form $PbSO_4$ there (figure 33.16a). The $H^+$ ions move to the $PbO_2$ electrode, give up the positive charge and reduce $PbO_2$ to PbO,

$$PbO_2 + 2\ H = PbO + H_2O.$$

The PbO so formed reacts with the $H_2SO_4$ to form $PbSO_4$ and water.

$$PbO + H_2SO_4 = PbSO_4 + H_2O$$

Thus, $PbSO_4$ is formed at both the electrodes. As the sulphuric acid is used up in discharging, the specific gravity of the acid decreases. When the specific gravity falls to 1·15, the cell is considered to be fully discharged and any further current drawn from it may permanently damage the electrodes.

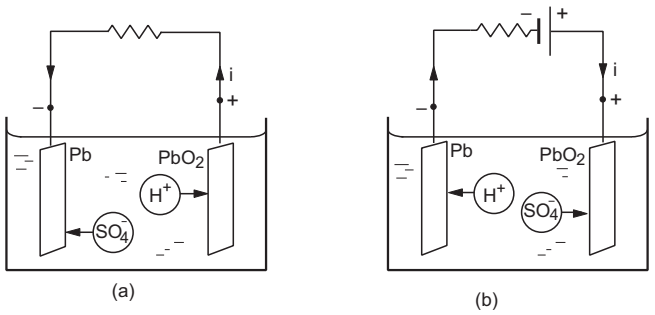



Figure 33.16

The charging process is reverse of discharging. A current is forced from the positive to the negative electrode inside the cell. The $H^+$ ions move towards the negative electrode and react with the $PbSO_4$ present there (which was formed during discharging).

$$PbSO_4 + 2\ H = Pb + H_2SO_4.$$

At the positive electrode, the reaction is

$$PbSO_4 + SO_4 + 2\ H_2O = PbO_2 + 2\ H_2SO_4.$$

Thus, the $PbSO_4$ deposited at the two electrodes is dissolved, Pb is deposited at the negative electrode and $PbO_2$ at the positive electrode. This restores the capacity of the cell to provide current. The emf of a lead accumulator is about 2·05 V when fully charged. In the discharged condition, the emf may fall to 1·8 V.

A practical lead accumulator contains several plates of lead connected together to form the negative electrode and several plates of lead peroxide connected together to form the positive electrode. This increases the capacity of the accumulator. These plates are separated from each other by insulating separators.

For commercial use, several such cells are connected in series and assembled together in one case. A six volt battery is obtained by connecting three such cells and a twelve volt battery is obtained by connecting six such cells.

The capacity of an accumulator is measured in ampere–hour. Thus, a 100 ampere–hour accumulator can supply 20 A current for 5 hours or 10 A current for 10 hours.

### *Worked Out Examples*

**1.** *A current of* 30 A *is registered when the terminals of a dry cell of emf* 1·5 *volts are connected through an ammeter. Neglecting the meter resistance, find the amount of heat produced in the battery in* 10 *seconds.*

***Solution*** : The current in the circuit will be

$$i = \frac{\mathscr{E}}{r}$$

or, $$30\ \text{A} = \frac{1{\cdot}5\ \text{V}}{r} \text{ giving } r = 0{\cdot}05\ \Omega.$$

The amount of heat produced in the battery

$$= i^2 rt = (30\ \text{A})^2 \times (0{\cdot}5\ \Omega) \times 10\ \text{s} = 450\ \text{J}.$$

**2.** *A room heater is rated* 500 W, 220 V. *(a) Find the resistance of its coil. (b) If the supply voltage drops to* 200 V, *what will be the power consumed ? (c) If an electric bulb rated* 100 W, 220 V *is connected in series with this heater, what will be the power consumed by the heater and by the bulb when the supply is at* 220 V ?

***Solution*** :

(a) The power consumed by a coil of resistance $R$ when connected across a supply $V$ is

$$P = \frac{V^2}{R}.$$

The resistance of the heater coil is, therefore,

$$R = \frac{(220\ \text{V})^2}{500\ \text{W}} = 96{\cdot}8\ \Omega.$$

(b) If the supply voltage drops to 200 V, the power consumed will be

$$P = \frac{V^2}{R} = \frac{(200\ \text{V})^2}{96{\cdot}8\ \Omega} = 413\ \text{W}.$$

(c) The resistance of the 100 W, 220 V bulb is

$$R = \frac{(220\ \text{V})^2}{100\ \text{W}} = 484\ \Omega.$$

If this is connected in series with the heater of 96·8 Ω, the current $i$ will be

$$i = \frac{220\ \text{V}}{484\ \Omega + 96{\cdot}8\ \Omega} = 0{\cdot}379\ \text{A}.$$

Thus, the power consumed by the heater

$$= i^2 \times 96{\cdot}8\ \Omega = 0{\cdot}144 \times 96{\cdot}8\ \text{W} = 13{\cdot}9\ \text{W}$$

and that by the bulb

$$= i^2 \times 484\ \Omega = 69{\cdot}7\ \text{W}.$$

**3.** *A battery of emf* $\mathcal{E}$ *and internal resistance r is used in a circuit with a variable external resistance R. Find the value of R for which the power consumed in R is maximum.*

***Solution*** : The current in the resistance $R$ is

$$i = \frac{\mathcal{E}}{r + R}.$$

The power consumed in $R$ is

$$P = i^2 R = \frac{\mathcal{E}^2 R}{(r + R)^2}.$$

It is maximum when $\frac{dP}{dR} = 0$. We have

$$\frac{dP}{dR} = \mathcal{E}^2 \left[\frac{(r + R)^2 - 2R(r + R)}{(r + R)^4}\right].$$

It is zero when

$$(r + R)^2 = 2R(r + R)$$

or, $$R = r.$$

**4.** *The junctions of a* Ni–Cu *thermocouple are maintained at* 0°C *and* 100°C. *Calculate the Seebeck emf produced in the loop.* $a_{Ni,\,Cu} = 16{\cdot}3 \times 10^{-6}$ V°C$^{-1}$ *and* $b_{Ni,\,Cu} = -\,0{\cdot}042 \times 10^{-6}$ V°C$^{-2}$.

***Solution*** :

$$\mathcal{E}_{Ni,\,Cu} = a_{Ni,\,Cu}\theta + \frac{1}{2} b_{Ni,\,Cu}\theta^2$$

$$= (16{\cdot}3 \times 10^{-6} \times 100)\text{V} + \frac{1}{2}(-\,0{\cdot}042 \times 10^{-6} \times 10^4)\text{V}$$

$$= 1{\cdot}42 \times 10^{-3}\ \text{V}.$$

**5.** *Find the neutral and inversion temperatures for* Ni–Cu *thermocouple with the cold junction at* 0°C. *Use data from previous example.*

***Solution*** : The neutral temperature is

$$\theta_n = -\frac{a}{b}$$

$$= \frac{16{\cdot}3 \times 10^{-6}}{0{\cdot}042 \times 10^{-6}}\ °\text{C} = 388°\text{C}.$$

The inversion temperature is double the neutral temperature, i.e., 776°C.

**6.** *An electric current of* 0·4 A *is passed through a silver voltameter for half an hour. Find the amount of silver deposited on the cathode. ECE of silver* $= 1{\cdot}12 \times 10^{-6}$ kg C$^{-1}$.

***Solution*** : Using the formula $m = Zit$, the mass of silver deposited

$$= (1{\cdot}12 \times 10^{-6}\ \text{kg C}^{-1})\ (0{\cdot}4\ \text{A})\ (30 \times 60\ \text{s})$$

$$= 8{\cdot}06 \times 10^{-4}\ \text{kg} = 0{\cdot}806\ \text{g}.$$

**7.** *A silver and a copper voltameter are connected in series with a* 12·0 V *battery of negligible resistance. It is found that* 0·806 g *of silver is deposited in half an hour. Find (a) the mass of the copper deposited and (b) the energy supplied by the battery. ECE of silver* $= 1{\cdot}12 \times 10^{-6}$ kg C$^{-1}$ *and that of copper* $= 6{\cdot}6 \times 10^{-7}$ kg C$^{-1}$.

***Solution*** :

(a) For silver voltameter, the formula $m = Zit$ gives

$$0{\cdot}806\ \text{g} = (1{\cdot}12 \times 10^{-6}\ \text{kg C}^{-1})\ i(30 \times 60\ \text{s})$$

or, $$i = 0{\cdot}4\ \text{A}.$$

As the two voltameters are connected in series, the same current passes through the copper voltameter. The mass of copper deposited is

$$m = Zit$$

$$= (6{\cdot}6 \times 10^{-7}\ \text{kg}^{-1}\text{C})\ (0{\cdot}4\ \text{A})\ (30 \times 60\ \text{s})$$

$$= 4{\cdot}75 \times 10^{-4}\ \text{kg} = 0{\cdot}475\ \text{g}.$$

This could also be obtained by using $\frac{m_1}{m_2} = \frac{Z_1}{Z_2}$ for series circuit.

(b) Energy supplied by the battery = $Vit$

$= (12\ \text{V})\ (0{\cdot}4\ \text{A})\ (30 \times 60\ \text{s}) = 8{\cdot}64\ \text{kJ}.$

**8.** *A current of* 1 A *is passed through a dilute solution of sulphuric acid for some time to liberate* 1 g *of oxygen. How much hydrogen is liberated during this period? How long was the current passed? Faraday constant* $= 96500\ \text{C mol}^{-1}$.

***Solution*** **:** The relative atomic mass of oxygen = 16 and its valency = 2 so that the chemical equivalent $E = \frac{16}{2} = 8$.

Chemical equivalent of hydrogen = 1.

$$\frac{m_{oxygen}}{m_{hydrogen}} = \frac{E_{oxygen}}{E_{hydrogen}} = \frac{8}{1}$$

or, $$m_{hydrogen} = \frac{m_{oxygen}}{8} = \frac{1\ \text{g}}{8} = 0{\cdot}125\ \text{g}.$$

We have, 1 g of oxygen $= \frac{1}{8}$ gram-equivalent.

The charge needed to liberate $\frac{1}{8}$ gram-equivalent

$$= \frac{1}{8}\ \text{faraday}$$

$$= \frac{96500}{8}\ \text{C} \cong 1{\cdot}12 \times 10^4\ \text{C}.$$

As the current is 1 A, the time taken is

$$t = \frac{Q}{i} = \frac{1{\cdot}2 \times 10^4\ \text{C}}{1\ \text{A}}$$

$$= 1{\cdot}2 \times 10^4\ \text{s}$$

$$= 3\ \text{hours}\ \ 20\ \ \text{minutes}.$$

□

## QUESTIONS FOR SHORT ANSWER

**1.** If a constant potential difference is applied across a bulb, the current slightly decreases as time passes and then becomes constant. Explain.

**2.** Two unequal resistances $R_1$ and $R_2$ are connected across two identical batteries of emf $\mathcal{E}$ and internal resistance $r$ (figure 33-Q1). Can the thermal energies developed in $R_1$ and $R_2$ be equal in a given time. If yes, what will be the condition ?

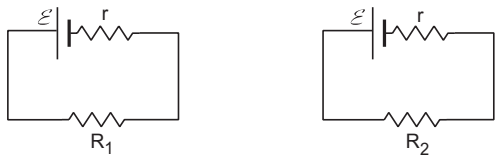



Figure 33-Q1

**3.** When a current passes through a resistor, its temperature increases. Is it an adiabatic process ?

**4.** Apply the first law of thermodynamics to a resistor carrying a current $i$. Identify which of the quantities $\Delta Q$, $\Delta U$ and $\Delta W$ are zero, which are positive and which are negative.

**5.** Do all the thermocouples have a neutral temperature ?

**6.** Is inversion temperature always double of the neutral temperature ? Does the unit of temperature have an effect in deciding this question ?

**7.** Is neutral temperature always the arithmetic mean of the inversion temperature and the temperature of the cold junction ? Does the unit of temperature have an effect in deciding this question ?

**8.** Do the electrodes in an electrolytic cell have fixed polarity like a battery ?

**9.** As temperature increases, the viscosity of liquids decreases considerably. Will this decrease the resistance of an electrolyte as the temperature increases ?

## OBJECTIVE I

**1.** Which of the following plots may represent the thermal energy produced in a resistor in a given time as a function of the electric current ?

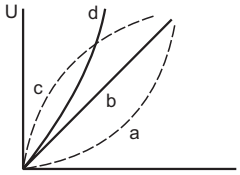



Figure 33-Q2

**2.** A constant current $i$ is passed through a resistor. Taking the temperature coefficient of resistance into account, indicate which of the plots shown in figure (33-Q3) best

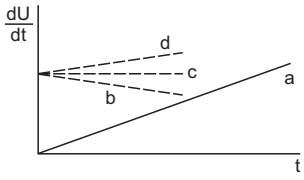



Figure 33-Q3

represents the rate of production of thermal energy in the resistor.

**3.** Consider the following statements regarding a thermocouple.
(A) The neutral temperature does not depend on the temperature of the cold junction.
(B) The inversion temperature does not depend on the temperature of the cold junction.
(a) Both A and B are correct.
(b) *A* is correct but B is wrong.
(c) *B* is correct but A is wrong.
(d) Both A and B are wrong.

**4.** The heat developed in a system is proportional to the current through it.
(a) It cannot be Thomson heat.
(b) It cannot be Peltier heat.
(c) It cannot be Joule heat.
(d) It can be any of the three heats mentioned above.

**5.** Consider the following two statements.
(A) Free-electron density is different in different metals.
(B) Free-electron density in a metal depends on temperature.
Seebeck effect is caused
(a) due to both A and B
(b) due to A but not due to B
(c) due to B but not due to A
(d) neither due to A nor due to B.

**6.** Consider the statements A and B in the previous question. Peltier effect is caused
(a) due to both A and B
(b) due to A but not due to B
(c) due to B but not due to A
(d) neither due to A nor due to B.

**7.** Consider the statements A and B in question 5. Thomson effect is caused
(a) due to both A and B
(b) due to A but not due to B
(c) due to B but not due to A
(d) neither due to A nor due to B.

**8.** Faraday constant
(a) depends on the amount of the electrolyte
(b) depends on the current in the electrolyte
(c) is a universal constant
(d) depends on the amount of charge passed through the electrolyte.

## OBJECTIVE II

**1.** Two resistors having equal resistances are joined in series and a current is passed through the combination. Neglect any variation in resistance as the temperature changes. In a given time interval,
(a) equal amounts of thermal energy must be produced in the resistors
(b) unequal amounts of thermal energy may be produced
(c) the temperature must rise equally in the resistors
(d) the temperature may rise equally in the resistors.

**2.** A copper strip *AB* and an iron strip *AC* are joined at *A*. The junction *A* is maintained at 0°C and the free ends *B* and *C* are maintained at 100°C. There is a potential difference between
(a) the two ends of the copper strip
(b) the copper end and the iron end at the junction
(c) the two ends of the iron strip
(d) the free ends *B* and *C*.

**3.** The constants $a$ and $b$ for the pair silver–lead are $2{\cdot}50\ \mu V°C^{-1}$ and $0{\cdot}012\ \mu V°C^{-2}$ respectively. For a silver–lead thermocouple with colder junction at 0°C,
(a) there will be no neutral temperature
(b) there will be no inversion temperature
(c) there will not be any thermo-emf even if the junctions are kept at different temperatures
(d) there will be no current in the thermocouple even if the junctions are kept at different temperatures.

**4.** An electrolysis experiment is stopped and the battery terminals are reversed.
(a) The electrolysis will stop.
(b) The rate of liberation of material at the electrodes will increase.
(c) The rate of liberation of material will remain the same.
(d) Heat will be produced at a greater rate.

**5.** The electrochemical equivalent of a material depends on
(a) the nature of the material
(b) the current through the electrolyte containing the material
(c) the amount of charge passed through the electrolyte
(d) the amount of this material present in the electrolyte.

## EXERCISES

**1.** An electric current of 2·0 A passes through a wire of resistance 25 Ω. How much heat will be developed in 1 minute ?

**2.** A coil of resistance 100 Ω is connected across a battery of emf 6·0 V. Assume that the heat developed in the coil is used to raise its temperature. If the heat capacity of the coil is $4{\cdot}0\ J\ K^{-1}$, how long will it take to raise the temperature of the coil by 15°C ?

**3.** The specification on a heater coil is 250 V, 500 W. Calculate the resistance of the coil. What will be the resistance of a coil of 1000 W to operate at the same voltage ?

**4.** A heater coil is to be constructed with a nichrome wire ($\rho = 1{\cdot}0 \times 10^{-6}$ Ωm) which can operate at 500 W when connected to a 250 V supply. (a) What would be the resistance of the coil ? (b) If the cross-sectional area of the wire is 0·5 $mm^2$, what length of the wire will be needed ? (c) If the radius of each turn is 4·0 mm, how many turns will be there in the coil ?

**5.** A bulb with rating 250 V, 100 W is connected to a power supply of 220 V situated 10 m away using a copper wire of area of cross section 5 $mm^2$. How much power will be consumed by the connecting wires ? Resistivity of copper = $1{\cdot}7 \times 10^{-8}$ Ωm.

**6.** An electric bulb, when connected across a power supply of 220 V, consumes a power of 60 W. If the supply drops to 180 V, what will be the power consumed ? If the supply is suddenly increased to 240 V, what will be the power consumed ?

**7.** A servo voltage stabiliser restricts the voltage output to 220 V ± 1%. If an electric bulb rated at 220 V, 100 W is connected to it, what will be the minimum and maximum power consumed by it ?

**8.** An electric bulb marked 220 V, 100 W will get fused if it is made to consume 150 W or more. What voltage fluctuation will the bulb withstand ?

**9.** An immersion heater rated 1000 W, 220 V is used to heat 0·01 $m^3$ of water. Assuming that the power is supplied at 220 V and 60% of the power supplied is used to heat the water, how long will it take to increase the temperature of the water from 15°C to 40°C ?

**10.** An electric kettle used to prepare tea, takes 2 minutes to boil 4 cups of water (1 cup contains 200 cc of water) if the room temperature is 25°C. (a) If the cost of power consumption is Re 1·00 per unit (1 unit = 1000 watt–hour), calculate the cost of boiling 4 cups of water. (b) What will be the corresponding cost if the room temperature drops to 5°C ?

**11.** The coil of an electric bulb takes 40 watts to start glowing. If more than 40 W is supplied, 60% of the extra power is converted into light and the remaining into heat. The bulb consumes 100 W at 220 V. Find the percentage drop in the light intensity at a point if the supply voltage changes from 220 V to 200 V.

**12.** The 2·0 Ω resistor shown in figure (33-E1) is dipped into a calorimeter containing water. The heat capacity of the calorimeter together with water is 2000 J $K^{-1}$. (a) If the circuit is active for 15 minutes, what would be the rise in the temperature of the water ? (b) Suppose the 6·0 Ω resistor gets burnt. What would be the rise in the temperature of the water in the next 15 minutes ?

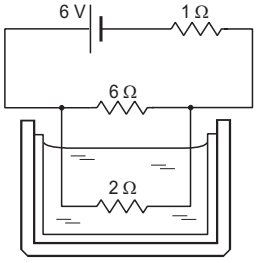



Figure 33-E1

**13.** The temperatures of the junctions of a bismuth–silver thermocouple are maintained at 0°C and 0·001°C. Find the thermo-emf (Seebeck emf) developed. For bismuth–silver, $a = -46 \times 10^{-6}$ V°$C^{-1}$ and $b = -0{\cdot}48 \times 10^{-6}$ V°$C^{-2}$.

**14.** Find the thermo-emf developed in a copper–silver thermocouple when the junctions are kept at 0°C and 40°C. Use the data in table (33.1).

**15.** Find the neutral temperature and inversion temperature of copper–iron thermocouple if the reference junction is kept at 0°C. Use the data in table (33.1).

**16.** Find the charge required to flow through an electrolyte to liberate one atom of (a) a monovalent material and (b) a divalent material.

**17.** Find the amount of silver liberated at cathode if 0·500 A of current is passed through $AgNO_3$ electrolyte for 1 hour. Atomic weight of silver is 107·9 g $mol^{-1}$.

**18.** An electroplating unit plates 3·0 g of silver on a brass plate in 3·0 minutes. Find the current used by the unit. The electrochemical equivalent of silver is $1{\cdot}12 \times 10^{-6}$ kg $C^{-1}$.

**19.** Find the time required to liberate 1·0 litre of hydrogen at STP in an electrolytic cell by a current of 5·0 A.

**20.** Two voltameters, one having a solution of silver salt and the other of a trivalent-metal salt, are connected in series and a current of 2 A is maintained for 1·50 hours. It is found that 1·00 g of the trivalent metal is deposited. (a) What is the atomic weight of the trivalent metal ? (b) How much silver is deposited during this period ? Atomic weight of silver is 107·9 g $mol^{-1}$.

**21.** A brass plate having surface area 200 $cm^2$ on one side is electroplated with 0·10 mm thick silver layers on both sides using a 15 A current. Find the time taken to do the job. The specific gravity of silver is 10·5 and its atomic weight is 107·9 g $mol^{-1}$.

**22.** Figure (33-E2) shows an electrolyte of AgCl through which a current is passed. It is observed that 2·68 g of silver is deposited in 10 minutes on the cathode. Find the heat developed in the 20 Ω resistor during this period. Atomic weight of silver is 107·9 g $mol^{-1}$.

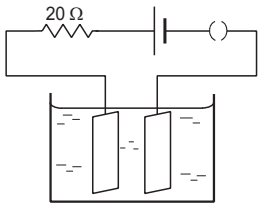



Figure 33-E2

**23.** The potential difference across the terminals of a battery of emf 12 V and internal resistance 2 Ω drops to 10 V when it is connected to a silver voltameter. Find the silver deposited at the cathode in half an hour. Atomic weight of silver is 107·9 g $mol^{-1}$.

**24.** A plate of area 10 $cm^2$ is to be electroplated with copper (density 9000 kg $m^{-3}$) to a thickness of 10 micrometres on both sides, using a cell of 12 V. Calculate the energy spent by the cell in the process of deposition. If this

energy is used to heat 100 g of water, calculate the rise in the temperature of the water. ECE of copper $= 3 \times 10^{-7}$ kg C$^{-1}$ and specific heat capacity of water $= 4200$ J kg$^{-1}$ K$^{-1}$.

□

## ANSWERS

### OBJECTIVE I

1. (a) 2. (d) 3. (b) 4. (c) 5. (a) 6. (b)
7. (c) 8. (c)

### OBJECTIVE II

1. (a), (d) 2. all 3. (a), (b)
4. (c) 5. (a)

### EXERCISES

1. $6{\cdot}0 \times 10^{3}$ J
2. 2·8 min
3. 125 Ω, 62·5 Ω
4. (a) 125 Ω (b) 62·5 m (c) ≈ 2500 turns
5. 8·4 mW
6. 40 W, 71 W
7. 98 W, 102 W
8. up to 270 V
9. 29 minutes
10. (a) 7 paise (b) 9 paise
11. 29%
12. (a) 2·9°C (b) 3·6°C
13. $-4{\cdot}6 \times 10^{-8}$ V
14. $1{\cdot}04 \times 10^{-5}$ V
15. 330°C, 659°C
16. (a) $1{\cdot}6 \times 10^{-19}$ C (b) $3{\cdot}2 \times 10^{-19}$ C
17. 2·01 g
18. 15 A
19. 29 minutes
20. (a) 26·8 g mol$^{-1}$ (b) 12·1 g
21. 42 minutes
22. 190 kJ
23. 2 g
24. 7·2 kJ, 17 K

□

## CHAPTER 34

# MAGNETIC FIELD

### 34.1 INTRODUCTION

If a charge $q$ is placed at rest at a point $P$ near a metallic wire carrying a current $i$, it experiences almost no force. We conclude that there is no appreciable electric field at the point $P$. This is expected because in any volume of wire (which contains several thousand atoms) there are equal amounts of positive and negative charges. The wire is electrically neutral and does not produce an electric field.*

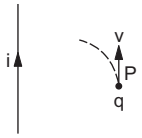



Figure 34.1

However, if the charge $q$ is projected from the point $P$ in the direction of the current (figure 34.1), it is deflected towards the wire ($q$ is assumed positive). There must be a field at $P$ which exerts a force on the charge when it is projected, but not when it is kept at rest. This field is different from the electric field which always exerts a force on a charged particle whether it is at rest or in motion. This new field is called *magnetic field* and is denoted by the symbol $\vec{B}$. The force exerted by a magnetic field is called *magnetic force*.

### 34.2 DEFINITION OF MAGNETIC FIELD $\vec{B}$

If a charged particle is projected in a magnetic field, in general, it experiences a magnetic force. By projecting the particle in different directions from the same point $P$ with different speeds, we can observe the following facts about the magnetic force:

(a) There is one line through the point $P$, such that, if the velocity of the particle is along this line, there is no magnetic force. We define the direction of the magnetic field to be along this line (the direction is not uniquely defined yet, because there are two opposite directions along any line).

(b) If the speed of the particle is $v$ and it makes an angle $\theta$ with the line identified in (a), i.e., with the direction of the magnetic field, the magnitude of the magnetic force is proportional to $| v \sin\theta |$.

(c) The direction of the magnetic force is perpendicular to the direction of the magnetic field as well as to the direction of the velocity.

(d) The force is proportional to the magnitude of the charge $q$ and its direction is opposite for positive and negative charges.

All the above facts may be explained if we define the magnetic field by the equation

$$\vec{F} = q\vec{v} \times \vec{B}. \qquad \ldots \quad (34.1)$$

By measuring the magnetic force $\vec{F}$ acting on a charge $q$ moving at velocity $\vec{v}$, we can obtain $\vec{B}$. If $\vec{v} \parallel \vec{B}$, the force is zero. By taking magnitudes in equation (34.1), we see that the force is proportional to $| v \sin\theta |$. By the rules of vector product, the force is perpendicular to both $\vec{B}$ and $\vec{v}$. Also, the observation (d) follows from equation (34.1).

Equation (34.1) uniquely determines the direction of $\vec{B}$ from the rules of vector product. The SI unit of magnetic field is newton/ampere meter. This is written as tesla and abbreviated as T. Another unit in common use is gauss (G). The relation between gauss and tesla is $1\ \text{T} = 10^4\ \text{G}$.

The unit weber/meter$^2$ is also used for magnetic field and is the same as tesla. Tesla is quite a large unit for many practical applications. We have a magnetic field of the order of $10^{-5}$ T near the earth's surface. Large superconducting magnets are needed to produce a field of the order of 10 T in laboratories.

---

* In fact, there is a small charge density on the surface of the wire which does produce an electric field near the wire. This field is very small and we shall neglect it.

In the older days, the term *magnetic induction* was used for this field $\vec{B}$.

***Example 34.1***

*A proton is projected with a speed of* $3 \times 10^6$ m s$^{-1}$ *horizontally from east to west. A uniform magnetic field* $\vec{B}$ *of strength* $2{\cdot}0 \times 10^{-3}$ T *exists in the vertically upward direction. (a) Find the force on the proton just after it is projected. (b) What is the acceleration produced?*

***Solution :***

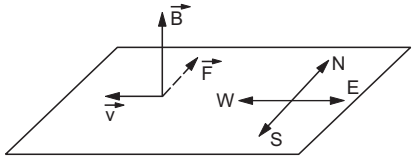



Figure 34.2

(a) The situation is shown in figure (34.2). The force is perpendicular to $\vec{B}$ hence it is in the horizontal plane through the proton. In this plane, it is perpendicular to the velocity $\vec{v}$. Thus, it is along the north-south line. The rule for vector product shows that $\vec{v} \times \vec{B}$ is towards north. As the charge on the proton is positive, the force $\vec{F} = q\vec{v} \times \vec{B}$ is also towards north. The magnitude of the force is

$$F = qvB \sin\theta$$
$$= (1{\cdot}6 \times 10^{-19}\ \text{C})\ (3{\cdot}0 \times 10^6\ \text{m s}^{-1})\ (2{\cdot}0 \times 10^{-3}\ \text{T})$$
$$= 9{\cdot}6 \times 10^{-16}\ \text{N}.$$

(b) The acceleration of the proton is

$$a = \frac{F}{m} = \frac{9{\cdot}6 \times 10^{-16}\ \text{N}}{1{\cdot}67 \times 10^{-27}\ \text{kg}}$$
$$= 5{\cdot}8 \times 10^{11}\ \text{m s}^{-2}.$$

## 34.3 RELATION BETWEEN ELECTRIC AND MAGNETIC FIELDS

Figure (34.3) shows a long wire carrying a current $i$ and a charge $q$ having a velocity $v$ parallel to the current as seen by an observer $S$. There is no electric field, but there is a magnetic field which exerts a force on the charge and the charge is attracted towards the wire.

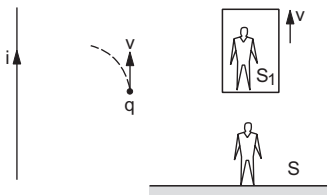



Figure 34.3

Now consider another observer $S_1$ who is moving at a uniform velocity $v$ parallel to the wire. In this frame the charge is at rest and hence, the magnetic field (if any) cannot exert any force on the charge. However, the observer $S_1$ also sees that the charge is attracted by the wire. In fact, the acceleration of the charge is the same for both $S$ and $S_1$ as they are unaccelerated with respect to each other. Hence, there must be an electric field in the frame of $S_1$. What was a pure magnetic field in the frame of $S$ turns out to be a combination of electric field and magnetic field in the frame of $S_1$. We conclude that the electric field and the magnetic field are not basically independent. They are two aspects of the same entity which we call *electromagnetic field*. Whether the electromagnetic field will show up as an electric field or a magnetic field or a combination, depends on the frame from which we are looking at the field. If we are confined to a particular frame, we can treat the electric field and the magnetic field as separate entities.

## 34.4 MOTION OF A CHARGED PARTICLE IN A UNIFORM MAGNETIC FIELD

As the magnetic force on a particle is perpendicular to the velocity, it does not do any work on the particle. Hence, the kinetic energy or the speed of the particle does not change due to the magnetic force.

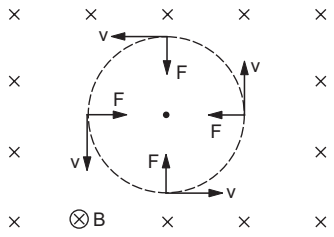



Figure 34.4

In figure (34.4), the magnetic field $B$ is perpendicular to the paper and going into it. (This direction is, by convention, shown as ⊗. The direction coming out of the paper is, by convention, shown as ⊙. The circle around the cross or around the dot is quite often omitted.) A charge $q$ is projected with a speed $v$ in the plane of the paper. The velocity is perpendicular to the magnetic field. The force is $F = qvB$ in the direction perpendicular to both $v$ and $B$. This force will deflect the particle without changing the speed and the particle will move along a circle perpendicular to the field. The magnetic force provides the centripetal force. If $r$ be the radius of the circle,

$$qvB = m\frac{v^2}{r}$$

or,
$$r = \frac{mv}{qB}. \qquad \ldots (34.2)$$

The time taken to complete the circle (time period) is

$$T = \frac{2\pi r}{v} = \frac{2\pi m}{qB}. \qquad \ldots (34.3)$$

We see that the time period is independent of the speed $v$. If the particle moves faster, the radius is larger, it has to move a longer distance to complete the circle so that the time taken is the same.

The frequency of revolution is

$$\nu = \frac{1}{T} = \frac{qB}{2\pi m} \cdot \quad \ldots \ (34.4)$$

This frequency is called the *cyclotron frequency.*

***Example 34.2***

*A particle having a charge of* 100 μC *and a mass of* 10 mg *is projected in a uniform magnetic field of* 25 mT *with a speed of* 10 m $s^{-1}$. *If the velocity is perpendicular to the magnetic field, how long will it take for the particle to come back to its original position for the first time after being projected.*

***Solution*** : The particle moves along a circle and returns to its original position after completing the circle, that is after one time period. The time period is

$$T = \frac{2\pi m}{qB}$$

$$= \frac{2\pi \times (10 \times 10^{-6}\ \text{kg})}{(100 \times 10^{-6}\ \text{C}) \times (25 \times 10^{-3}\ \text{T})} = 25\ \text{s}.$$

If the velocity of the charge is not perpendicular to the magnetic field, we can break the velocity in two components— $v_{||}$, parallel to the field and $v_{\perp}$, perpendicular to the field. The component $v_{||}$ remains unchanged as the force $q\vec{v} \times \vec{B}$ is perpendicular to it. In the plane perpendicular to the field, the particle traces a circle of radius $r = \frac{mv_{\perp}}{qB}$ as given by equation (34.2). The resultant path is a helix (figure 34.5).

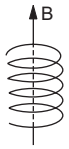



Figure 34.5

## 34.5 MAGNETIC FORCE ON A CURRENT-CARRYING WIRE

Suppose a conducting wire, carrying a current $i$, is placed in a magnetic field $\vec{B}$. Consider a small element $dl$ of the wire (figure 34.6). The free electrons drift with a speed $v_d$ opposite to the direction of the current. The relation between the current $i$ and the drift speed $v_d$ is

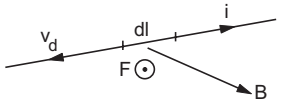



Figure 34.6

$$i = jA = nev_dA. \quad \ldots \ (i)$$

Here $A$ is the area of cross-section of the wire and $n$ is the number of free electrons per unit volume. Each electron experiences an average magnetic force

$$\vec{f} = -e\vec{v_d} \times \vec{B}.$$

The number of free electrons in the small element considered is $nAdl$. Thus, the magnetic force on the wire of length $dl$ is

$$d\vec{F} = (nAdl)\ (-e\vec{v_d} \times \vec{B}).$$

If we denote the length $dl$ along the direction of the current by $d\vec{l}$, the above equation becomes

$$d\vec{F} = nAev_d\ d\vec{l} \times \vec{B}.$$

Using (i),

$$d\vec{F} = i\ d\vec{l} \times \vec{B}. \quad \ldots \ (34.5)$$

The quantity $i\ d\vec{l}$ is called a *current element.*

If a straight wire of length $l$ carrying a current $i$ is placed in a uniform magnetic field $\vec{B}$, the force on it is

$$\vec{F} = i\ \vec{l} \times \vec{B}. \quad \ldots \ (34.6)$$

***Example 34.3***

*Figure (34.7) shows a triangular loop PQR carrying a current i. The triangle is equilateral with edge-length l. A uniform magnetic field B exists in a direction parallel to PQ. Find the forces acting on the three wires PQ, QR and RP separately.*

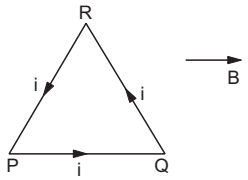



Figure 34.7

***Solution*** : The force on the wire $PQ$ is

$$\vec{F_1} = i\ \vec{PQ} \times \vec{B} = 0$$

as the field $\vec{B}$ is parallel to $\vec{PQ}$.

The force on $QR$ is

$$\vec{F_2} = i\ \vec{QR} \times \vec{B}$$

or, $$F_2 = ilB \sin 120°$$

$$= \frac{\sqrt{3}}{2} ilB.$$

From the rule of vector product, this force is perpendicular to the plane of the diagram and is going into it.

The force on $RP$ is

$$\vec{F_3} = i\ \vec{RP} \times \vec{B}$$

or, $$F_3 = i\ lB \sin 120° = \frac{\sqrt{3}}{2} ilB.$$

From the rule of vector product, this force is perpendicular to the plane of the diagram and is coming out of it.

### 34.6 TORQUE ON A CURRENT LOOP

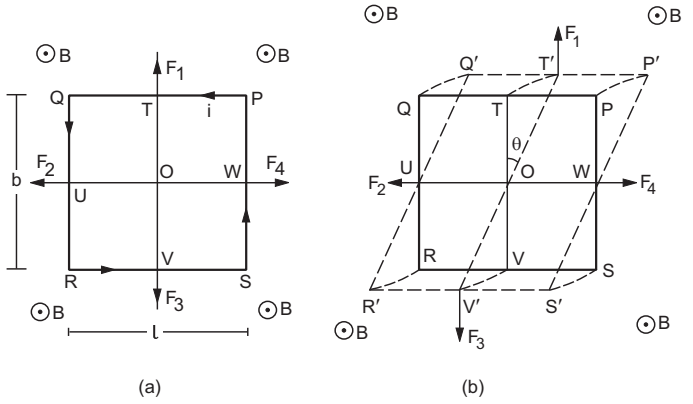



Figure 34.8

Figure (34.8a) shows a rectangular loop *PQRS* carrying a current $i$ and placed in a uniform magnetic field $B$. The magnetic forces $F_1$, $F_2$, $F_3$ and $F_4$ on the wires *PQ, QR, RS* and *SP* are obtained by using the equation $\vec{F} = i\vec{l} \times \vec{B}$. These forces act from the middle points *T, U, V, W* of the respective sides. Clearly, $F_1 = F_3 = ilB$ and $F_2 = F_4 = ibB$ in figure (34.8a). The resultant force is, therefore, zero. Also, $F_1$ and $F_3$ have the same line of action so they together produce no torque. Similarly, $F_2$ and $F_4$ together produce no torque. Hence, the resultant torque on the loop is zero.

Now suppose the loop is rotated through an angle θ about the line *WU* (figure 34.8b). The wire *PQ* shifts parallel to itself so that the force $\vec{F}_1 = i\vec{l} \times \vec{B}$ on it remains unchanged in magnitude and direction. Its point of application *T* shifts to *T′*. Similarly, the force on *RS* remains $\vec{F}_3$ but the point of application shifts to *V′*. The line *TV* gets rotated by an angle θ to take the position *T′V′*. This line makes an angle θ with the force $F_1$ and $F_3$. The torque of $F_1$ about *O* has magnitude

$$\left|\vec{OT'} \times \vec{F}_1\right| = \left(\frac{b}{2}\right) \times F_1 \times \sin\theta = \frac{b}{2}(ilB)\sin\theta.$$

This torque acts along the line *UW*. The torque of $F_3$ about *O* is also $\frac{b}{2}(ilB)\sin\theta$ along the same direction.

As the wire *QR* rotates about *WU*, the plane containing the wire and the magnetic field does not change. The force on the wire is perpendicular to this plane and hence its direction remains unchanged. Also, the point of application *U* remains the same. Similar is the case for the wire *SP*. The forces on *QR* and *SP* are, therefore, equal and opposite and act along the same line. They together produce no torque.

The net torque acting on the loop is, therefore,

$$\Gamma = \frac{b}{2}(ilB)\sin\theta + \frac{b}{2}(ilB)\sin\theta = b(ilB)\sin\theta$$
$$= iAB\sin\theta.$$

Let us define the *area-vector* $\vec{A}$ of the loop in the following way. The magnitude of $\vec{A}$ is equal to the area enclosed by the loop and the direction of $\vec{A}$ is perpendicular to the plane of the loop and is towards the side from which the current looks anticlockwise. Thus, in figure (34.8a), the area-vector $\vec{A}$ points towards the viewer. It is drawn from the centre *O* of the loop. Another way to get the direction of $\vec{A}$ is to use the right-hand thumb rule. If you curl your fingers of the right hand along the current, the stretched thumb gives the direction of $\vec{A}$.

The definition of area-vector is valid for a closed loop of any shape, not necessarily rectangular.

In figure (34.8a), the angle between the area-vector $\vec{A}$ and the magnetic field $\vec{B}$ is zero. As the loop rotates, the area-vector also rotates by an angle θ and hence the angle between $\vec{A}$ and $\vec{B}$ becomes θ. Taking the direction of the torque (along *UW*) into consideration,

$$\vec{\Gamma} = i\vec{A} \times \vec{B} = \vec{\mu} \times \vec{B} \qquad \ldots (34.7)$$

where $\vec{\mu} = i\vec{A}$ is called the *magnetic dipole moment* or simply *magnetic moment* of the current loop.

We have already discussed an electric dipole. A pair of charges $-q$, $+q$ separated by a distance $l$ forms an electric dipole of dipole moment $p = ql$. The direction is from $-q$ to $+q$. If such a dipole is placed in a uniform electric field, a torque

$$\vec{\Gamma} = \vec{p} \times \vec{E}$$

acts on the dipole. Equation (34.7) is similar to this equation in structure and hence $\vec{\mu}$ is called magnetic dipole moment.

If there are $n$ turns in the loop, each turn experiences a torque. The net torque is

$$\vec{\Gamma} = ni\vec{A} \times \vec{B}.$$

We still write it as $\vec{\Gamma} = \vec{\mu} \times \vec{B}$ with the magnetic dipole moment defined as

$$\vec{\mu} = ni\vec{A}. \qquad \ldots (34.8)$$

Equations (34.7) and (34.8) are obtained by considering a rectangular loop. However, these equations are valid for plane loops of any shape.

***Example 34.4***

*A current of* 10·0 nA *is established in a circular loop of radius* 5·0 cm. *Find the magnetic dipole moment of the current loop.*

***Solution*** : The magnetic dipole moment is

$$\vec{\mu} = i\vec{A}.$$

Thus, $\mu = i\pi r^2 = (10 \times 10^{-9}\ \text{A})\ (3{\cdot}14) \times (5 \times 10^{-2}\ \text{m})^2$

$$= 7{\cdot}85 \times 10^{-11}\ \text{A m}^2.$$

## *Worked Out Examples*

**1.** *A charge of* 2·0 μC *moves with a speed of* $2{\cdot}0 \times 10^6$ m $s^{-1}$ *along the positive x-axis. A magnetic field* $\vec{B}$ *of strength* $(0{\cdot}20\,\vec{j} + 0{\cdot}40\,\vec{k})$T *exists in space. Find the magnetic force acting on the charge.*

***Solution*** : The force on the charge

$$= q\vec{v} \times \vec{B}$$

$$= (2{\cdot}0 \times 10^{-6}\ \text{C})\ (2{\cdot}0 \times 10^6\ \text{m s}^{-1}\,\vec{i}) \times (0{\cdot}20\,\vec{j} + 0{\cdot}40\,\vec{k})\ \text{T}$$

$$= 4{\cdot}0(0{\cdot}20\,\vec{i} \times \vec{j} + 0{\cdot}40\,\vec{i} \times \vec{k})\ \text{N}$$

$$= (0{\cdot}8\,\vec{k} - 1{\cdot}6\,\vec{j})\ \text{N}.$$

**2.** *A wire is bent in the form of an equilateral triangle PQR of side* 10 cm *and carries a current of* 5·0 A. *It is placed in a magnetic field B of magnitude* 2·0 T *directed perpendicularly to the plane of the loop. Find the forces on the three sides of the triangle.*

***Solution*** :

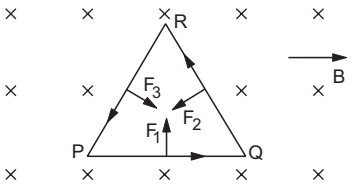



Figure 34-W1

Suppose the field and the current have directions as shown in figure (34-W1). The force on *PQ* is

$$\vec{F}_1 = i\,\vec{l} \times \vec{B}$$

or, $F_1 = 5{\cdot}0\ \text{A} \times 10\ \text{cm} \times 2{\cdot}0\ \text{T} = 1{\cdot}0\ \text{N}.$

The rule of vector product shows that the force $F_1$ is perpendicular to *PQ* and is directed towards the inside of the triangle.

The forces $\vec{F}_2$ and $\vec{F}_3$ on *QR* and *RP* can also be obtained similarly. Both the forces are 1·0 N directed perpendicularly to the respective sides and towards the inside of the triangle.

The three forces $\vec{F}_1, \vec{F}_2,$ and $\vec{F}_3$ will have zero resultant, so that there is no net magnetic force on the triangle. This result can be generalised. Any closed current loop, placed in a homogeneous magnetic field, does not experience a net magnetic force.

**3.** *Figure (34-W2) shows two long metal rails placed horizontally and parallel to each other at a separation l. A uniform magnetic field B exists in the vertically downward direction. A wire of mass m can slide on the rails. The rails are connected to a constant current source which drives a current i in the circuit. The friction coefficient between the rails and the wire is* μ. *(a) What should be the minimum value of* μ *which can prevent the wire from sliding on the rails ? (b) Describe the motion of the wire if the value of* μ *is half the value found in the previous part.*

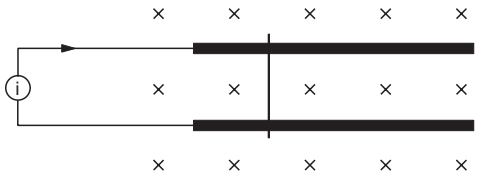



Figure 34-W2

***Solution*** :

(a) The force on the wire due to the magnetic field is

$$\vec{F} = i\,\vec{l} \times \vec{B}$$

or, $F = ilB.$

It acts towards right in the given figure. If the wire does not slide on the rails, the force of friction by the rails should be equal to *F*. If $\mu_0$ be the minimum coefficient of friction which can prevent sliding, this force is also equal to $\mu_0\ mg$. Thus,

$$\mu_0\ mg = ilB$$

or, $$\mu_0 = \frac{ilB}{mg}.$$

(b) If the friction coefficient is $\mu = \frac{\mu_0}{2} = \frac{ilB}{2\ mg}$, the wire will slide towards right. The frictional force by the rails is

$$f = \mu mg = \frac{ilB}{2} \text{ towards left.}$$

The resultant force is $ilB - \frac{ilB}{2} = \frac{ilB}{2}$ towards right. The acceleration will be $a = \frac{ilB}{2\ m}$. The wire will slide towards right with this acceleration.

**4.** *A proton, a deuteron and an alpha particle moving with equal kinetic energies enter perpendicularly into a magnetic field. If* $r_p, r_d$ *and* $r_a$ *are the respective radii of the circular paths, find the ratios* $r_p/r_d$ and $r_p/r_a$.

***Solution*** : We have $r = \frac{mv}{qB} = \frac{\sqrt{2\ mK}}{qB}$

where $K = \frac{1}{2}\ mv^2 =$ kinetic energy.

Thus, $r_p = \dfrac{\sqrt{2\, m_p K}}{q_p B}$, $r_d = \dfrac{\sqrt{2\, m_d K}}{q_d B}$

and $r_a = \dfrac{\sqrt{2\, m_a K}}{q_a B}$.

We get $\dfrac{r_p}{r_d} = \dfrac{q_d}{q_p}\sqrt{\dfrac{m_p}{m_d}} = \dfrac{q_p}{q_p}\sqrt{\dfrac{m_p}{2\, m_p}} = \dfrac{1}{\sqrt{2}}$

and $\dfrac{r_p}{r_a} = \dfrac{q_a}{q_p}\sqrt{\dfrac{m_p}{m_a}} = \dfrac{2\, q_p}{q_p}\sqrt{\dfrac{m_p}{4\, m_p}} = 1.$

**5.** *Singly charged magnesium* ($A = 24$) *ions are accelerated to kinetic energy* 2 keV *and are projected perpendicularly into a magnetic field B of magnitude* 0·6 T. *(a) Find the radius of the circle formed by the ions. (b) If there are also singly charged ions of the isotope magnesium*-26, *what would be the radius for these particles* ?

***Solution*** : The radius is given by

$$r = \frac{mv}{qB} = \frac{\sqrt{2\, mK}}{qB}$$

For $^{24}$Mg ions, $m = 24 \times m_p$ approximately and $q = 1{\cdot}6 \times 10^{-19}$ C.

Putting the values,

$$r = \frac{\sqrt{2 \times 24 \times 1{\cdot}67 \times 10^{-27}\ \text{kg} \times 2000 \times 1{\cdot}6 \times 10^{-19}\ \text{J}}}{1{\cdot}6 \times 10^{-19}\ \text{C} \times 0{\cdot}6\ \text{T}}$$

$= 0{\cdot}053$ m $= 5{\cdot}3$ cm.

For $^{26}$Mg, the radius $r'$ will be given by

$$r' = \frac{\sqrt{2\, m' K}}{qB}$$

or, $r' = r\sqrt{\dfrac{m'}{m}} = 5{\cdot}3\ \text{cm}\ \sqrt{\dfrac{26}{24}} = 5{\cdot}5$ cm.

**6.** *A particle having a charge* 20 μC *and mass* 20 μg *moves along a circle of radius* 5·0 cm *under the action of a magnetic field* $B = 1{\cdot}0$ T. *When the particle is at a point P, a uniform electric field is switched on and it is found that the particle continues on the tangent through P with a uniform velocity. Find the electric field.*

***Solution*** :

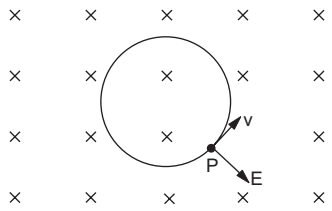



Figure 34-W3

When the particle moves along a circle in the magnetic field *B*, the magnetic force is radially inward. If an electric field of proper magnitude is switched on which is directed radially outwards, the particle may experience no force. It will then move along a straight line with uniform velocity. This will be the case when

$qE = qvB$ or, $E = vB$.

The radius of the circle in a magnetic field is given by

$$r = \frac{mv}{qB}$$

or, $v = \dfrac{rqB}{m}$

$$= \frac{(5{\cdot}0 \times 10^{-2}\ \text{m})\ (20 \times 10^{-6}\ \text{C})\ (1{\cdot}0\ \text{T})}{20 \times 10^{-9}\ \text{kg}} = 50\ \text{m s}^{-1}.$$

The required electric field is

$E = vB = (50\ \text{m s}^{-1})\ (1.0\ \text{T})$

$= 50\ \text{V m}^{-1}$.

This field will be in a direction which is radially outward at *P*.

**7.** *A particle of mass* $m = 1{\cdot}6 \times 10^{-27}$ kg *and charge* $q = 1{\cdot}6 \times 10^{-19}$ C *moves at a speed of* $1{\cdot}0 \times 10^{7}$ m s$^{-1}$. *It enters a region of uniform magnetic field at a point E, as shown in figure (34-W4). The field has a strength of* 1·0 T. *(a) The magnetic field is directed into the plane of the paper. The particle leaves the region of the field at the point F. Find the distance EF and the angle* θ. *(b) If the field is coming out of the paper, find the time spent by the particle in the region of the magnetic field after entering it at E.*

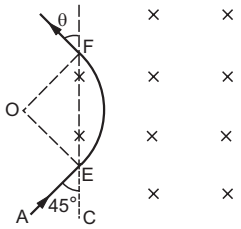



Figure 34-W4

***Solution*** : (a) As the particle enters the magnetic field, it will travel in a circular path. The centre will be on the line perpendicular to its velocity and the radius *r* will be $\dfrac{mv}{qB}$. The direction of the force $q\vec{v} \times \vec{B}$ shows that the centre will be outside the field as shown in figure (34-W4). As $\angle AEO = 90°$ (as *AE* is tangent and *OE* is radius) and $\angle AEC = 45°$, we have $\angle OEF = 45°$. As $OE = OF$ (they are radii of the circular arc), $\angle OFE = \angle OEF = 45°$. Also, *OF* is perpendicular to the velocity of the particle at *F*, so that $\theta = 45°$. From triangle *OEF*,

$EF = 2.OE \cos \angle OEF$

$$= 2 \cdot \frac{mv}{qB} \cdot \frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{2} \times (1{\cdot}6 \times 10^{-27}\ \text{kg}) \times (10^{7}\ \text{m s}^{-1})}{(1{\cdot}6 \times 10^{-19}\ \text{C}) \times 1{\cdot}0\ \text{T}}$$

$= \sqrt{2} \times 10^{-1}$ m $= 14$ cm.

(b)

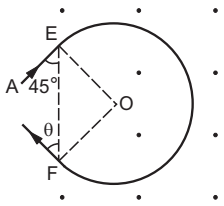



Figure 34-W5

If the magnetic field is coming out of the paper, the direction of the force $q\vec{v}\times\vec{B}$ shows that the centre $O$ will be inside the field region as shown in figure (34-W5). Again $\angle AEO = 90°$, giving

$$\angle OEF = \angle OFE = 45°.$$

Thus, the angle $EOF = 90°$. The particle describes three fourths of the complete circle inside the field. As the speed $v$ is uniform, the time spent in the magnetic field will be

$$\frac{3}{4}\times\frac{2\pi r}{v} = \frac{3\pi mv}{2vqB} = \frac{3\pi m}{2qB}$$

$$= \frac{3\times 3{\cdot}14\times 1{\cdot}6\times 10^{-27}\text{ kg}}{2\times 1{\cdot}6\times 10^{-19}\text{ C}\times 1{\cdot}0\text{ T}} = 4{\cdot}7\times 10^{-8}\text{ s}.$$

**8.** *A beam of protons with a velocity of* $4\times 10^5$ m s$^{-1}$ *enters a uniform magnetic field of* 0·3 T. *The velocity makes an angle of* 60° *with the magnetic field. Find the radius of the helical path taken by the proton beam and the pitch of the helix.*

***Solution*** : The components of the proton's velocity along and perpendicular to the magnetic field are

$$v_{||} = (4\times 10^5\text{ m s}^{-1})\cos 60° = 2\times 10^5\text{ m s}^{-1}.$$

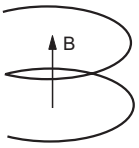



Figure 34-W6

and $\quad v_\perp = (4\times 10^5\text{ m s}^{-1})\sin 60° = 2\sqrt{3}\times 10^5\text{ m s}^{-1}.$

As the force $q\vec{v}\times\vec{B}$ is perpendicular to the magnetic field, the component $v_{||}$ will remain constant. In the plane perpendicular to the field, the proton will describe a circle whose radius is obtained from the equation

$$qv_\perp B = \frac{mv_\perp^2}{r}$$

or, $$r = \frac{mv_\perp}{qB} = \frac{(1{\cdot}67\times 10^{-27}\text{ kg})\times(2\sqrt{3}\times 10^5\text{ m s}^{-1})}{(1{\cdot}6\times 10^{-19}\text{ C})\times(0{\cdot}3\text{ T})}$$

$$\approx 0{\cdot}012\text{ m} = 1{\cdot}2\text{ cm}.$$

The time taken in one complete revolution in the plane perpendicular to $B$ is

$$T = \frac{2\pi r}{v_\perp} = \frac{2\times 3{\cdot}14\times 0{\cdot}012\text{ m}}{2\sqrt{3}\times 10^5\text{ m s}^{-1}}.$$

The distance moved along the field during this period, i.e., the pitch

$$= \frac{(2\times 10^5\text{ m s}^{-1})\times 2\times 3{\cdot}14\times 0{\cdot}012\text{ m}}{2\sqrt{3}\times 10^5\text{ m s}^{-1}}$$

$$= 0{\cdot}044\text{ m} = 4{\cdot}4\text{ cm}.$$

The qualitative nature of the path of the protons is shown in figure (34-W6).

**9.** *A rectangular coil of size* 3·0 cm × 4·0 cm *and having* 100 turns, *is pivoted about the z-axis as shown in figure (34-W7). The coil carries an electric current of* 2·0 A *and a magnetic field of* 1·0 T *is present along the y-axis. Find the torque acting on the coil if the side in the x–y plane makes an angle* θ = 37° *with the x-axis.*

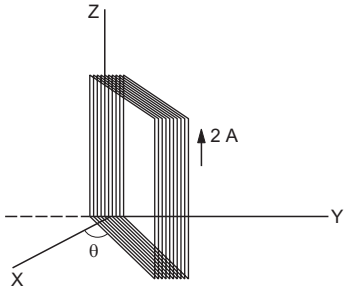



Figure 34-W7

***Solution*** : The magnetic moment of the loop is $\vec{\mu} = ni\vec{A}$ where $n$ is the number of turns, $i$ is the current and $\vec{A}$ is the area-vector. The direction of $\vec{A}$ is determined by the sense of the current and in this case it lies in the fourth quadrant making an angle $\theta = 37°$ with the negative $y$-axis.

Torque $\vec{\Gamma} = \vec{\mu}\times\vec{B} = ni\vec{A}\times\vec{B}$.

Thus, $\quad \Gamma = 100\times(2\text{ A})\times(12\times 10^{-4}\text{ m}^2)\times 1\text{ T}\times\sin 37°$

$$= 0{\cdot}14\text{ Nm}.$$

The torque is along the positive $z$-axis.

**10.** *An electron moves with a constant speed v along a circle of radius r. (a) Find the equivalent current through a point on its path. (b) Find the magnetic moment of the circulating electron. (c) Find the ratio of the magnetic moment to the angular momentum of the electron.*

***Solution*** : (a) Consider a point $P$ on the path of the electron. In one revolution of the electron, a charge $e$ crosses the point $P$. As the frequency of revolution is $v/(2\pi r)$, the charge crossing $P$ in unit time, i.e., the electric current is

$$i = \frac{ev}{2\pi r}.$$

(b) The area $A$ enclosed by this circular current is $\pi r^2$ so that the magnetic moment of the current is

$$\mu = iA = \left(\frac{ev}{2\pi r}\right)(\pi r^2) = \frac{evr}{2}$$

in a direction perpendicular to the loop.

(c) The angular momentum of the electron is $l = mvr$. Its direction is opposite to that of the magnetic moment. Thus,

$$\frac{\mu}{l} = \frac{-evr}{2\,mvr} = \frac{-e}{2\,m}.$$

**11.** *An electron is released from the origin at a place where a uniform electric field E and a uniform magnetic field B exist along the negative y-axis and the negative z-axis respectively. Find the displacement of the electron along the y-axis when its velocity becomes perpendicular to the electric field for the first time.*

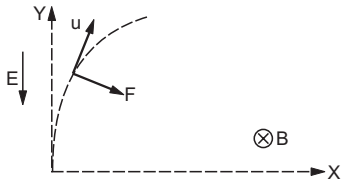



Figure 34-W8

***Solution*** **:** Let us take axes as shown in figure (34-W8). According to the right-handed system, the $z$-axis is upward in the figure and hence the magnetic field is shown downwards. At any time, the velocity of the electron may be written as

$$\vec{u} = u_x \vec{i} + u_y \vec{j}.$$

The electric and magnetic fields may be written as

$$\vec{E} = -E\vec{j}$$

and $$\vec{B} = -B\vec{k}$$

respectively. The force on the electron is

$$\vec{F} = -e(\vec{E} + \vec{u} \times \vec{B})$$
$$= eE\vec{j} + eB(u_y \vec{i} - u_x \vec{j}).$$

Thus, $$F_x = eu_y B$$

and $$F_y = e(E - u_x B).$$

The components of the acceleration are

$$a_x = \frac{du_x}{dt} = \frac{eB}{m} u_y \quad \ldots \text{(i)}$$

and $$a_y = \frac{du_y}{dt} = \frac{e}{m}(E - u_x B). \quad \ldots \text{(ii)}$$

We have,

$$\frac{d^2 u_y}{dt^2} = -\frac{eB}{m}\frac{du_x}{dt}$$
$$= -\frac{eB}{m} \cdot \frac{eB}{m} u_y$$
$$= -\omega^2 u_y$$

where $$\omega = \frac{eB}{m}. \quad \ldots \text{(iii)}$$

This equation is similar to that for a simple harmonic motion. Thus,

$$u_y = A \sin(\omega t + \delta) \quad \ldots \text{(iv)}$$

and hence,

$$\frac{du_y}{dt} = A\,\omega \cos(\omega t + \delta). \quad \ldots \text{(v)}$$

At $t = 0$, $u_y = 0$ and $\frac{du_y}{dt} = \frac{F_y}{m} = \frac{eE}{m}$.

Putting in (iv) and (v),

$$\delta = 0 \text{ and } A = \frac{eE}{m\omega} = \frac{E}{B}.$$

Thus, $$u_y = \frac{E}{B} \sin \omega t.$$

The path of the electron will be perpendicular to the $y$-axis when $u_y = 0$. This will be the case for the first time at $t$ where

$$\sin \omega t = 0$$

or, $$\omega t = \pi$$

or, $$t = \frac{\pi}{\omega} = \frac{\pi m}{eB}.$$

Also, $$u_y = \frac{dy}{dt} = \frac{E}{B} \sin \omega t$$

or, $$\int_0^y dy = \frac{E}{B}\int_0^t \sin \omega t\, dt$$

or, $$y = \frac{E}{B\omega}(1 - \cos \omega t).$$

At $$t = \frac{\pi}{\omega},$$

$$y = \frac{E}{B\omega}(1 - \cos \pi) = \frac{2E}{B\omega}.$$

Thus, the displacement along the $y$-axis is

$$\frac{2E}{B\omega} = \frac{2Em}{BeB} = \frac{2Em}{eB^2}.$$

□

## QUESTIONS FOR SHORT ANSWER

**1.** Suppose a charged particle moves with a velocity $v$ near a wire carrying an electric current. A magnetic force, therefore, acts on it. If the same particle is seen from a frame moving with velocity $v$ in the same direction, the charge will be found at rest. Will the magnetic force become zero in this frame ? Will the magnetic field become zero in this frame ?

**2.** Can a charged particle be accelerated by a magnetic field ? Can its speed be increased ?

**3.** Will a current loop placed in a magnetic field always experience a zero force ?

**4.** The free electrons in a conducting wire are in constant thermal motion. If such a wire, carrying no current, is placed in a magnetic field, is there a magnetic force on each free electron? Is there a magnetic force on the wire?

**5.** Assume that the magnetic field is uniform in a cubical region and is zero outside. Can you project a charged particle from outside into the field so that the particle describes a complete circle in the field ?

**6.** An electron beam projected along the positive $x$-axis deflects along the positive $y$-axis. If this deflection is caused by a magnetic field, what is the direction of the field ? Can we conclude that the field is parallel to the $z$-axis ?

**7.** Is it possible for a current loop to stay without rotating in a uniform magnetic field ? If yes, what should be the orientation of the loop ?

**8.** The net charge in a current-carrying wire is zero. Then, why does a magnetic field exert a force on it ?

**9.** The torque on a current loop is zero if the angle between the positive normal and the magnetic field is either $\theta = 0$ or $\theta = 180°$. In which of the two orientations, the equilibrium is stable ?

**10.** Verify that the units weber and volt second are the same.

## OBJECTIVE I

**1.** A positively charged particle projected towards east is deflected towards north by a magnetic field. The field may be
(a) towards west (b) towards south
(c) upward (d) downward.

**2.** A charged particle is whirled in a horizontal circle on a frictionless table by attaching it to a string fixed at one point. If a magnetic field is switched on in the vertical direction, the tension in the string
(a) will increase (b) will decrease
(c) will remain the same (d) may increase or decrease.

**3.** Which of the following particles will experience maximum magnetic force (magnitude) when projected with the same velocity perpendicular to a magnetic field ?
(a) Electron (b) Proton (c) $He^+$ (d) $Li^{++}$

**4.** Which of the following particles will describe the smallest circle when projected with the same velocity perpendicular to a magnetic field ?
(a) Electron (b) Proton (c) $He^+$ (d) $Li^+$

**5.** Which of the following particles will have minimum frequency of revolution when projected with the same velocity perpendicular to a magnetic field ?
(a) Electron (b) Proton (c) $He^+$ (d) $Li^+$

**6.** A circular loop of area 1 $cm^2$, carrying a current of 10 A, is placed in a magnetic field of 0·1 T perpendicular to the plane of the loop. The torque on the loop due to the magnetic field is
(a) zero (b) $10^{-4}$ N m (c) $10^{-2}$ N m (d) 1 N m

**7.** A beam consisting of protons and electrons moving at the same speed goes through a thin region in which there is a magnetic field perpendicular to the beam. The protons and the electrons
(a) will go undeviated
(b) will be deviated by the same angle and will not separate
(c) will be deviated by different angles and hence separate
(d) will be deviated by the same angle but will separate.

**8.** A charged particle moves in a uniform magnetic field. The velocity of the particle at some instant makes an acute angle with the magnetic field. The path of the particle will be
(a) a straight line (b) a circle
(c) a helix with uniform pitch
(d) a helix with nonuniform pitch.

**9.** A particle moves in a region having a uniform magnetic field and a parallel, uniform electric field. At some instant, the velocity of the particle is perpendicular to the field direction. The path of the particle will be
(a) a straight line (b) a circle
(c) a helix with uniform pitch
(d) a helix with nonuniform pitch.

**10.** An electric current $i$ enters and leaves a uniform circular wire of radius $a$ through diametrically opposite points. A charged particle $q$ moving along the axis of the circular wire passes through its centre at speed $v$. The magnetic force acting on the particle when it passes through the centre has a magnitude
(a) $qv\frac{\mu_0 i}{2a}$ (b) $qv\frac{\mu_0 i}{2\pi a}$ (c) $qv\frac{\mu_0 i}{a}$ (d) zero.

## OBJECTIVE II

**1.** If a charged particle at rest experiences no electromagnetic force,
(a) the electric field must be zero
(b) the magnetic field must be zero
(c) the electric field may or may not be zero
(d) the magnetic field may or may not be zero.

**2.** If a charged particle kept at rest experiences an electromagnetic force,
(a) the electric field must not be zero
(b) the magnetic field must not be zero
(c) the electric field may or may not be zero
(d) the magnetic field may or may not be zero.

**3.** If a charged particle projected in a gravity-free room deflects,
(a) there must be an electric field
(b) there must be a magnetic field
(c) both fields cannot be zero
(d) both fields can be nonzero.

**4.** A charged particle moves in a gravity-free space without change in velocity. Which of the following is/are possible ?
(a) $E = 0, B = 0$ (b) $E = 0, B \neq 0$
(c) $E \neq 0, B = 0$ (d) $E \neq 0, B \neq 0$

**5.** A charged particle moves along a circle under the action of possible constant electric and magnetic fields. Which of the following are possible ?
(a) $E = 0, B = 0$ (b) $E = 0, B \neq 0$
(c) $E \neq 0, B = 0$ (d) $E \neq 0, B \neq 0$

**6.** A charged particle goes undeflected in a region containing electric and magnetic field. It is possible that
(a) $\vec{E} \parallel \vec{B}$, $\vec{v} \parallel \vec{E}$ (b) $\vec{E}$ is not parallel to $\vec{B}$
(c) $\vec{v} \parallel \vec{B}$ but $\vec{E}$ is not parallel to $\vec{B}$
(d) $\vec{E} \parallel \vec{B}$ but $\vec{v}$ is not parallel to $\vec{E}$.

**7.** If a charged particle goes unaccelerated in a region containing electric and magnetic fields,
(a) $\vec{E}$ must be perpendicular to $\vec{B}$
(b) $\vec{v}$ must be perpendicular to $\vec{E}$
(c) $\vec{v}$ must be perpendicular to $\vec{B}$
(d) $E$ must be equal to $vB$.

**8.** Two ions have equal masses but one is singly-ionized and the other is doubly-ionized. They are projected from the same place in a uniform magnetic field with the same velocity perpendicular to the field.
(a) Both ions will go along circles of equal radii.
(b) The circle described by the singly-ionized charge will have a radius double that of the other circle.
(c) The two circles do not touch each other.
(d) The two circles touch each other.

**9.** An electron is moving along the positive $x$-axis. You want to apply a magnetic field for a short time so that the electron may reverse its direction and move parallel to the negative $x$-axis. This can be done by applying the magnetic field along
(a) $y$-axis (b) $z$-axis (c) $y$-axis only (d) $z$-axis only.

**10.** Let $\vec{E}$ and $\vec{B}$ denote electric and magnetic fields in a frame $S$ and $\vec{E}'$ and $\vec{B}'$ in another frame $S'$ moving with respect to $S$ at a velocity $\vec{v}$. Two of the following equations are wrong. Identify them.
(a) $B_y' = B_y + \frac{vE_z}{c^2}$ (b) $E_y' = E_y - \frac{vB_z}{c^2}$
(c) $B_y' = B_y + vE_z$ (d) $E_y' = E_y + vB_z$

## EXERCISES

**1.** An alpha particle is projected vertically upward with a speed of $3{\cdot}0 \times 10^4$ km s$^{-1}$ in a region where a magnetic field of magnitude 1·0 T exists in the direction south to north. Find the magnetic force that acts on the $\alpha$-particle.

**2.** An electron is projected horizontally with a kinetic energy of 10 keV. A magnetic field of strength $1{\cdot}0 \times 10^{-7}$ T exists in the vertically upward direction. (a) Will the electron deflect towards right or towards left of its motion ? (b) Calculate the sideways deflection of the electron in travelling through 1 m. Make appropriate approximations.

**3.** A magnetic field of $(4{\cdot}0 \times 10^{-3}\,\vec{k})$ T exerts a force of $(4{\cdot}0\,\vec{i} + 3{\cdot}0\,\vec{j}) \times 10^{-10}$ N on a particle having a charge of $1{\cdot}0 \times 10^{-9}$ C and going in the $x$–$y$ plane. Find the velocity of the particle.

**4.** An experimenter's diary reads as follows: "a charged particle is projected in a magnetic field of $(7{\cdot}0\,\vec{i} - 3{\cdot}0\,\vec{j}) \times 10^{-3}$ T. The acceleration of the particle is found to be $(\square\,\vec{i} + 7{\cdot}0\,\vec{j}) \times 10^{-6}$ m s$^{-2}$". The number to the left of $\vec{i}$ in the last expression was not readable. What can this number be ?

**5.** A 10 g bullet having a charge of 4·00 μC is fired at a speed of 270 m s$^{-1}$ in a horizontal direction. A vertical magnetic field of 500 μT exists in the space. Find the deflection of the bullet due to the magnetic field as it travels through 100 m. Make appropriate approximations.

**6.** When a proton is released from rest in a room, it starts with an initial acceleration $a_0$ towards west. When it is projected towards north with a speed $v_0$, it moves with an initial acceleration $3a_0$ towards west. Find the electric field and the minimum possible magnetic field in the room.

**7.** Consider a 10-cm long portion of a straight wire carrying a current of 10 A placed in a magnetic field of 0·1 T making an angle of 53° with the wire. What magnetic force does the wire experience ?

**8.** A current of 2 A enters at the corner $d$ of a square frame $abcd$ of side 20 cm and leaves at the opposite corner $b$. A magnetic field $B = 0{\cdot}1$ T exists in the space in a direction perpendicular to the plane of the frame as shown in figure

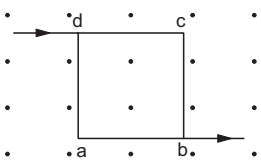


Figure 34-E1

(34-E1). Find the magnitude and direction of the magnetic forces on the four sides of the frame.

**9.** A magnetic field of strength 1·0 T is produced by a strong electromagnet in a cylindrical region of radius 4·0 cm as shown in figure (34-E2). A wire, carrying a current of 2·0 A, is placed perpendicular to and intersecting the axis of the cylindrical region. Find the magnitude of the force acting on the wire.

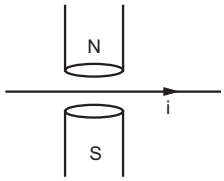



Figure 34-E2

**10.** A wire of length $l$ carries a current $i$ along the $x$-axis. A magnetic field exists which is given as $\vec{B} = B_0(\vec{i} + \vec{j} + \vec{k})$ T. Find the magnitude of the magnetic force acting on the wire.

**11.** A current of 5·0 A exists in the circuit shown in figure (34-E3). The wire $PQ$ has a length of 50 cm and the magnetic field in which it is immersed has a magnitude of 0·20 T. Find the magnetic force acting on the wire $PQ$.

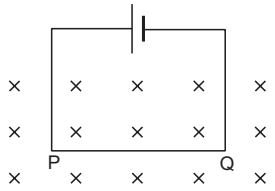



Figure 34-E3

**12.** A circular loop of radius $a$, carrying a current $i$, is placed in a two-dimensional magnetic field. The centre of the loop coincides with the centre of the field (figure 34-E4). The strength of the magnetic field at the periphery of the loop is $B$. Find the magnetic force on the wire.

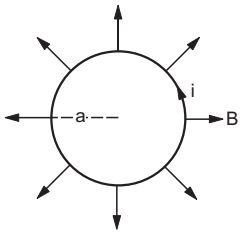



Figure 34-E4

**13.** A hypothetical magnetic field existing in a region is given by $\vec{B} = B_0 \vec{e}_r$, where $\vec{e}_r$ denotes the unit vector along the radial direction. A circular loop of radius $a$, carrying a current $i$, is placed with its plane parallel to the $x$–$y$ plane and the centre at (0, 0, $d$). Find the magnitude of the magnetic force acting on the loop.

**14.** A rectangular wire-loop of width $a$ is suspended from the insulated pan of a spring balance as shown in figure (34-E5). A current $i$ exists in the anticlockwise direction in the loop. A magnetic field $B$ exists in the lower region. Find the change in the tension of the spring if the current in the loop is reversed.

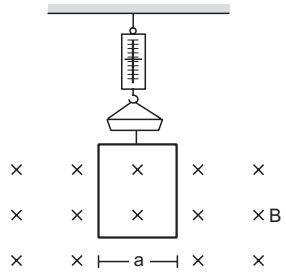



figure 34-E5

**15.** A current loop of arbitrary shape lies in a uniform magnetic field $B$. Show that the net magnetic force acting on the loop is zero.

**16.** Prove that the force acting on a current-carrying wire, joining two fixed points $a$ and $b$ in a uniform magnetic field, is independent of the shape of the wire.

**17.** A semicircular wire of radius 5·0 cm carries a current of 5·0 A. A magnetic field $B$ of magnitude 0·50 T exists along the perpendicular to the plane of the wire. Find the magnitude of the magnetic force acting on the wire.

**18.** A wire, carrying a current $i$, is kept in the $x$–$y$ plane along the curve $y = A \sin\left(\frac{2\pi}{\lambda} x\right)$. A magnetic field $B$ exists in the $z$-direction. Find the magnitude of the magnetic force on the portion of the wire between $x = 0$ and $x = \lambda$.

**19.** A rigid wire consists of a semicircular portion of radius $R$ and two straight sections (figure 34-E6). The wire is partially immersed in a perpendicular magnetic field $B$ as shown in the figure. Find the magnetic force on the wire if it carries a current $i$.

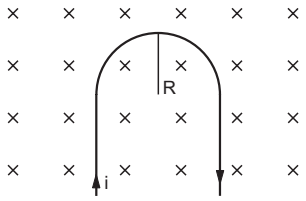



Figure 34-E6

**20.** A straight horizontal wire of mass 10 mg and length 1·0 m carries a current of 2·0 A. What minimum magnetic field $B$ should be applied in the region so that the magnetic force on the wire may balance its weight ?

**21.** Figure (34-E7) shows a rod $PQ$ of length 20·0 cm and mass 200 g suspended through a fixed point $O$ by two threads of lengths 20·0 cm each. A magnetic field of strength 0·500 T exists in the vicinity of the wire $PQ$ as shown in the figure. The wires connecting $PQ$ with the battery are loose and exert no force on $PQ$. (a) Find the tension in the threads when the switch $S$ is open. (b) A current of 2·0 A is established when the switch $S$ is closed. Find the tension in the threads now.

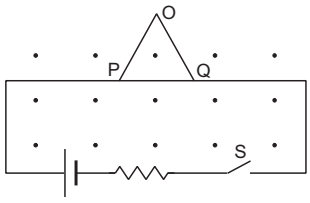



Figure 34-E7

**22.** Two metal strips, each of length $l$, are clamped parallel to each other on a horizontal floor with a separation $b$ between them. A wire of mass $m$ lies on them perpendicularly as shown in figure (34-E8). A vertically upward magnetic field of strength $B$ exists in the space. The metal strips are smooth but the coefficient of friction between the wire and the floor is $\mu$. A current $i$ is established when the switch $S$ is closed at the instant $t = 0$. Discuss the motion of the wire after the switch is closed. How far away from the strips will the wire reach ?

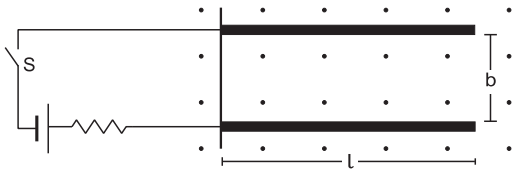



Figure 34-E8

**23.** A metal wire $PQ$ of mass 10 g lies at rest on two horizontal metal rails separated by 4·90 cm (figure 34-E9). A vertically downward magnetic field of magnitude 0·800 T exists in the space. The resistance of the circuit is slowly decreased and it is found that when the resistance goes below 20·0 Ω, the wire $PQ$ starts sliding on the rails. Find the coefficient of friction.

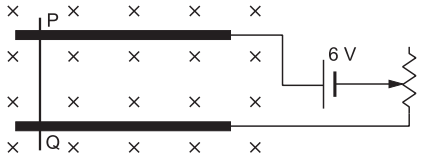



Figure 34-E9

**24.** A straight wire of length $l$ can slide on two parallel plastic rails kept in a horizontal plane with a separation $d$. The coefficient of friction between the wire and the rails is $\mu$. If the wire carries a current $i$, what minimum magnetic field should exist in the space in order to slide the wire on the rails.

**25.** Figure (34-E10) shows a circular wire-loop of radius $a$, carrying a current $i$, placed in a perpendicular magnetic field $B$. (a) Consider a small part $dl$ of the wire. Find the force on this part of the wire exerted by the magnetic field. (b) Find the force of compression in the wire.

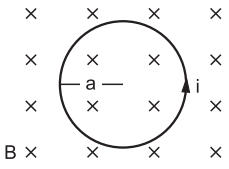



Figure 34-E10

**26**. Suppose that the radius of cross section of the wire used in the previous problem is $r$. Find the increase in the radius of the loop if the magnetic field is switched off. The Young modulus of the material of the wire is $Y$.

**27.** The magnetic field existing in a region is given by

$$\vec{B} = B_0\left(1 + \frac{x}{l}\right)\vec{k}.$$

A square loop of edge $l$ and carrying a current $i$, is placed with its edges parallel to the $x$–$y$ axes. Find the magnitude of the net magnetic force experienced by the loop.

**28.** A conducting wire of length $l$, lying normal to a magnetic field $B$, moves with a velocity $v$ as shown in figure (34-E11). (a) Find the average magnetic force on a free electron of the wire. (b) Due to this magnetic force, electrons concentrate at one end resulting in an electric field inside the wire. The redistribution stops when the electric force on the free electrons balances the magnetic force. Find the electric field developed inside the wire when the redistribution stops. (c) What potential difference is developed between the ends of the wire ?

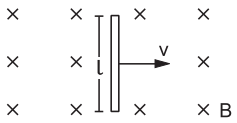



Figure 34-E11

**29.** A current $i$ is passed through a silver strip of width $d$ and area of cross section $A$. The number of free electrons per unit volume is $n$. (a) Find the drift velocity $v$ of the electrons. (b) If a magnetic field $B$ exists in the region as shown in figure (34-E12), what is the average magnetic force on the free electrons ? (c) Due to the magnetic force, the free electrons get accumulated on one side of the conductor along its length. This produces a transverse electric field in the conductor which opposes the magnetic force on the electrons. Find the magnitude of the electric field which will stop further accumulation of electrons. (d) What will be the potential difference developed across the width of the conductor due to the electron-accumulation ? The appearance of a transverse emf, when a current-carrying wire is placed in a magnetic field, is called *Hall effect.*

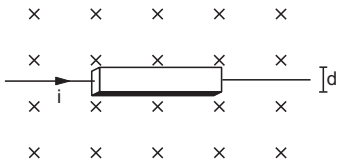



Figure 34-E12

**30.** A particle having a charge of $2{\cdot}0 \times 10^{-8}$ C and a mass of $2{\cdot}0 \times 10^{-10}$ g is projected with a speed of $2{\cdot}0 \times 10^{3}$ m s$^{-1}$ in a region having a uniform magnetic field of 0·10 T. The velocity is perpendicular to the field. Find the radius of the circle formed by the particle and also the time period.

**31.** A proton describes a circle of radius 1 cm in a magnetic field of strength 0·10 T. What would be the radius of the circle described by an $\alpha$-particle moving with the same speed in the same magnetic field ?

**32.** An electron having a kinetic energy of 100 eV circulates in a path of radius 10 cm in a magnetic field. Find the magnetic field and the number of revolutions per second made by the electron.

**33.** Protons having kinetic energy $K$ emerge from an accelerator as a narrow beam. The beam is bent by a perpendicular magnetic field so that it just misses a

plane target kept at a distance $l$ in front of the accelerator. Find the magnetic field.

**34.** A charged particle is accelerated through a potential difference of 12 kV and acquires a speed of $1{\cdot}0 \times 10^6$ m s$^{-1}$. It is then injected perpendicularly into a magnetic field of strength 0·2 T. Find the radius of the circle described by it.

**35.** Doubly ionized helium ions are projected with a speed of 10 km s$^{-1}$ in a direction perpendicular to a uniform magnetic field of magnitude 1·0 T. Find (a) the force acting on an ion, (b) the radius of the circle in which it circulates and (c) the time taken by an ion to complete the circle.

**36.** A proton is projected with a velocity of $3 \times 10^6$ m s$^{-1}$ perpendicular to a uniform magnetic field of 0·6 T. Find the acceleration of the proton.

**37.** (a) An electron moves along a circle of radius 1 m in a perpendicular magnetic field of strength 0·50 T. What would be its speed ? Is it reasonable ? (b) If a proton moves along a circle of the same radius in the same magnetic field, what would be its speed ?

**38.** A particle of mass $m$ and positive charge $q$, moving with a uniform velocity $v$, enters a magnetic field $B$ as shown in figure (34-E13). (a) Find the radius of the circular arc it describes in the magnetic field. (b) Find the angle subtended by the arc at the centre. (c) How long does the particle stay inside the magnetic field ? (d) Solve the three parts of the above problem if the charge $q$ on the particle is negative.

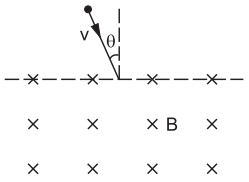



Figure 34-E13

**39.** A particle of mass $m$ and charge $q$ is projected into a region having a perpendicular magnetic field $B$. Find the angle of deviation (figure 34-E14) of the particle as it comes out of the magnetic field if the width $d$ of the region is very slightly smaller than

(a) $\frac{mv}{qB}$ (b) $\frac{mv}{2qB}$ (c) $\frac{2mv}{qB}$.

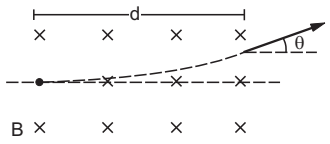



Figure 34-E14

**40.** A narrow beam of singly-charged carbon ions, moving at a constant velocity of $6{\cdot}0 \times 10^4$ m s$^{-1}$, is sent perpendicularly in a rectangular region having uniform magnetic field $B = 0{\cdot}5$ T (figure 34-E15). It is found that two beams emerge from the field in the backward direction, the separations from the incident beam being 3·0 cm and 3·5 cm. Identify the isotopes present in the ion beam. Take the mass of an ion = $A(1{\cdot}6 \times 10^{-27})$ kg, where $A$ is the mass number.

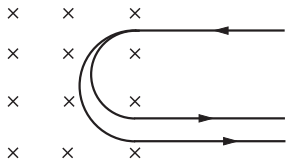


Figure 34-E15

**41.** $Fe^+$ ions are accelerated through a potential difference of 500 V and are injected normally into a homogeneous magnetic field $B$ of strength 20·0 mT. Find the radius of the circular paths followed by the isotopes with mass numbers 57 and 58. Take the mass of an ion = $A$ $(1{\cdot}6 \times 10^{-27})$ kg where $A$ is the mass number.

**42.** A narrow beam of singly charged potassium ions of kinetic energy 32 keV is injected into a region of width 1·00 cm having a magnetic field of strength 0·500 T as shown in figure (34-E16). The ions are collected at a screen 95·5 cm away from the field region. If the beam contains isotopes of atomic weights 39 and 41, find the separation between the points where these isotopes strike the screen. Take the mass of a potassium ion = $A$ $(1{\cdot}6 \times 10^{-27})$ kg where $A$ is the mass number.

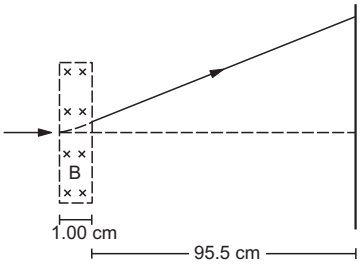



Figure 34-E16

**43.** Figure (34-E17) shows a convex lens of focal length 12 cm lying in a uniform magnetic field $B$ of magnitude 1·2 T parallel to its principal axis. A particle having a charge $2{\cdot}0 \times 10^{-3}$ C and mass $2{\cdot}0 \times 10^{-5}$ kg is projected perpendicular to the plane of the diagram with a speed of 4·8 m s$^{-1}$. The particle moves along a circle with its centre on the principal axis at a distance of 18 cm from the lens. Show that the image of the particle goes along a circle and find the radius of that circle.

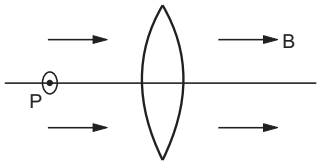



Figure 34-E17

**44.** Electrons emitted with negligible speed from an electron gun are accelerated through a potential difference $V$

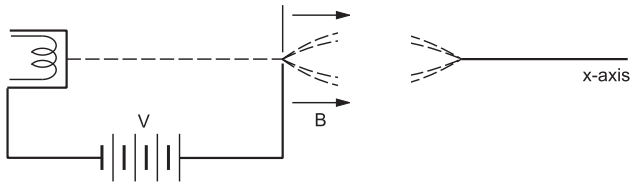



Figure 34-E18

along the $x$-axis. These electrons emerge from a narrow hole into a uniform magnetic field $B$ directed along this axis. However, some of the electrons emerging from the hole make slightly divergent angles as shown in figure (34-E18). Show that these paraxial electrons are refocussed on the $x$-axis at a distance

$$\sqrt{\frac{8\pi^2 mV}{eB^2}}.$$

**45.** Two particles, each having a mass $m$ are placed at a separation $d$ in a uniform magnetic field $B$ as shown in figure (34-E19). They have opposite charges of equal magnitude $q$. At time $t = 0$, the particles are projected towards each other, each with a speed $v$. Suppose the Coulomb force between the charges is switched off. (a) Find the maximum value $v_m$ of the projection speed so that the two particles do not collide. (b) What would be the minimum and maximum separation between the particles if $v = v_m/2$ ? (c) At what instant will a collision occur between the particles if $v = 2v_m$ ? (d) Suppose $v = 2v_m$ and the collision between the particles is completely inelastic. Describe the motion after the collision.

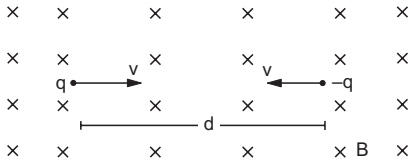



Figure 34-E19

**46.** A uniform magnetic field of magnitude 0·20 T exists in space from east to west. With what speed should a particle of mass 0·010 g and having a charge $1{\cdot}0 \times 10^{-5}$ C be projected from south to north so that it moves with a uniform velocity ?

**47.** A particle moves in a circle of diameter 1·0 cm under the action of a magnetic field of 0·40 T. An electric field of 200 V m$^{-1}$ makes the path straight. Find the charge/mass ratio of the particle.

**48.** A proton goes undeflected in a crossed electric and magnetic field (the fields are perpendicular to each other) at a speed of $2{\cdot}0 \times 10^5$ m s$^{-1}$. The velocity is perpendicular to both the fields. When the electric field is switched off, the proton moves along a circle of radius 4·0 cm. Find the magnitudes of the electric and the magnetic fields. Take the mass of the proton = $1{\cdot}6 \times 10^{-27}$ kg.

**49.** A particle having a charge of 5·0 μC and a mass of $5{\cdot}0 \times 10^{-12}$ kg is projected with a speed of 1·0 km s$^{-1}$ in a magnetic field of magnitude 5·0 mT. The angle between the magnetic field and the velocity is $\sin^{-1}(0{\cdot}90)$. Show that the path of the particle will be a helix. Find the diameter of the helix and its pitch.

**50.** A proton projected in a magnetic field of 0·020 T travels along a helical path of radius 5·0 cm and pitch 20 cm. Find the components of the velocity of the proton along and perpendicular to the magnetic field. Take the mass of the proton = $1{\cdot}6 \times 10^{-27}$ kg.

**51.** A particle having mass $m$ and charge $q$ is released from the origin in a region in which electric field and magnetic field are given by

$$\vec{B} = -B_0\vec{j} \text{ and } \vec{E} = E_0\vec{k}.$$

Find the speed of the particle as a function of its $z$-coordinate.

**52.** An electron is emitted with negligible speed from the negative plate of a parallel plate capacitor charged to a potential difference $V$. The separation between the plates is $d$ and a magnetic field $B$ exists in the space as shown in figure (34-E20). Show that the electron will fail to strike the upper plate if

$$d > \left(\frac{2m_e V}{eB_0^2}\right)^{\frac{1}{2}}.$$

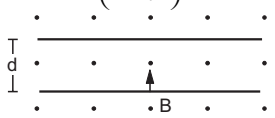



Figure 34-E20

**53.** A rectangular coil of 100 turns has length 5 cm and width 4 cm. It is placed with its plane parallel to a uniform magnetic field and a current of 2 A is sent through the coil. Find the magnitude of the magnetic field $B$, if the torque acting on the coil is 0·2 N m$^{-1}$.

**54.** A 50-turn circular coil of radius 2·0 cm carrying a current of 5·0 A is rotated in a magnetic field of strength 0·20 T. (a) What is the maximum torque that acts on the coil ? (b) In a particular position of the coil, the torque acting on it is half of this maximum. What is the angle between the magnetic field and the plane of the coil ?

**55.** A rectangular loop of sides 20 cm and 10 cm carries a current of 5·0 A. A uniform magnetic field of magnitude 0·20 T exists parallel to the longer side of the loop. (a) What is the force acting on the loop ? (b) What is the torque acting on the loop ?

**56.** A circular coil of radius 2·0 cm has 500 turns in it and carries a current of 1·0 A. Its axis makes an angle of 30° with the uniform magnetic field of magnitude 0·40 T that exists in the space. Find the torque acting on the coil.

**57.** A circular loop carrying a current $i$ has wire of total length $L$. A uniform magnetic field $B$ exists parallel to the plane of the loop. (a) Find the torque on the loop. (b) If the same length of the wire is used to form a square loop, what would be the torque ? Which is larger?

**58.** A square coil of edge $l$ having $n$ turns carries a current $i$. It is kept on a smooth horizontal plate. A uniform magnetic field $B$ exists in a direction parallel to an edge. The total mass of the coil is $M$. What should be the minimum value of $B$ for which the coil will start tipping over ?

**59.** Consider a nonconducting ring of radius $r$ and mass $m$ which has a total charge $q$ distributed uniformly on it. The ring is rotated about its axis with an angular speed ω. (a) Find the equivalent electric current in the ring. (b) Find the magnetic moment μ of the ring. (c) Show

that $\mu = \frac{q}{2m} l$ where $l$ is the angular momentum of the ring about its axis of rotation.

**60.** Consider a nonconducting plate of radius $r$ and mass $m$ which has a charge $q$ distributed uniformly over it. The plate is rotated about its axis with an angular speed $\omega$. Show that the magnetic moment $\mu$ and the angular momentum $l$ of the plate are related as $\mu = \frac{q}{2m} l$.

**61.** Consider a solid sphere of radius $r$ and mass $m$ which has a charge $q$ distributed uniformly over its volume. The sphere is rotated about a diameter with an angular speed $\omega$. Show that the magnetic moment $\mu$ and the angular momentum $l$ of the sphere are related as $\mu = \frac{q}{2m} l$.

□

## ANSWERS

### OBJECTIVE I

1. (d) 2. (d) 3. (d) 4. (a) 5. (d) 6. (a)
7. (c) 8. (c) 9. (d) 10. (d)

### OBJECTIVE II

1. (a), (d) 2. (a), (d) 3. (c), (d)
4. (a), (b), (d) 5. (b) 6. (a), (b)
7. (a), (b) 8. (b), (d) 9. (a), (b)
10. (b), (c)

### EXERCISES

1. $9{\cdot}6 \times 10^{-12}$ N towards west
2. (a) left (b) $\approx 1{\cdot}5$ cm
3. $(-75\,\vec{i} + 100\,\vec{j})$ m s$^{-1}$
4. 3·0
5. $3{\cdot}7 \times 10^{-6}$ m
6. $\frac{ma_0}{e}$ towards west, $\frac{2ma_0}{ev_0}$ downward
7. 0·08 N perpendicular to both the wire and the field
8. 0·02 N on each wire, on $da$ and $cb$ towards left and on $dc$ and $ab$ downward
9. 0·16 N
10. $\sqrt{2}\, B_0 il$
11. 0·50 N towards the inside of the circuit
12. $2\pi aiB$, perpendicular to the plane of the figure going into it
13. $\frac{2\pi a^2 iB_0}{\sqrt{a^2 + d^2}}$
14. $2iBa$
17. 0·25 N
18. $i\lambda B$
19. $2iRB$, upward in the figure
20. $4{\cdot}9 \times 10^{-5}$ T
21. (a) 1·13 N (b) 1·25 N
22. $\frac{ilbB}{\mu mg}$
23. 0·12
24. $\frac{\mu mg}{il}$
25. (a) $idlB$ towards the centre (b) $iaB$
26. $\frac{ia^2 B}{\pi r^2 Y}$
27. $iB_0 l$
28. (a) $evB$ (b) $vB$ (c) $lBv$
29. (a) $\frac{i}{Ane}$ (b) $\frac{iB}{An}$ upwards in the figure
(c) $\frac{iB}{Ane}$ (d) $\frac{iBd}{Ane}$
30. 20 cm, $6{\cdot}3 \times 10^{-4}$ s
31. 2 cm
32. $3{\cdot}4 \times 10^{-4}$ T, $9{\cdot}4 \times 10^{6}$
33. $\frac{\sqrt{2m_p K}}{el}$ where $m_p$ = mass of a proton
34. 12 cm
35. (a) $3{\cdot}2 \times 10^{-15}$ N (b) $2{\cdot}1 \times 10^{-4}$ m (c) $1{\cdot}31 \times 10^{-7}$ s
36. $1{\cdot}72 \times 10^{14}$ m s$^{-2}$
37. (a) $8{\cdot}8 \times 10^{10}$ m s$^{-1}$ (b) $4{\cdot}8 \times 10^{7}$ m s$^{-1}$
38. (a) $\frac{mv}{qB}$ (b) $\pi - 2\theta$
(c) $\frac{m}{qB}(\pi - 2\theta)$ (d) $\frac{mv}{qB}$, $\pi + 2\theta$, $\frac{m}{qB}(\pi + 2\theta)$
39. (a) $\pi/2$ (b) $\pi/6$ (c) $\pi$
40. $^{12}$C and $^{14}$C
41. 119 cm and 120 cm
42. 0·75 mm
43. 8 cm
45. (a) $\frac{qBd}{2\text{ m}}$ (b) $\frac{d}{2}$, $\frac{3d}{2}$ (c) $\frac{\pi m}{6qB}$ (d) the particles stick together and the combined mass moves with constant speed $v_m$ along the straight line drawn upward in the plane of figure through the point of collision

46. 49 m $s^{-1}$

47. $2{\cdot}5 \times 10^{5}$ C $kg^{-1}$

48. $1{\cdot}0 \times 10^{4}$ N $C^{-1}$, 0·05 T

49. 36 cm, 55 cm

50. $6{\cdot}4 \times 10^{4}$ m $s^{-1}$, $1{\cdot}0 \times 10^{5}$

51. $\sqrt{\frac{2qE_0 z}{m}}$

53. 0·5 T

54. (a) $6{\cdot}3 \times 10^{-2}$ N m (b) 60°

55. (a) zero (b) 0·02 N m parallel to the shorter side.

56. 0·13 N m

57. (a) $\frac{iL^2B}{4\pi}$ (b) $\frac{iL^2B}{16}$

58. $\frac{Mg}{2nil}$

59. (a) $\frac{q\omega}{2\pi}$ (b) $\frac{q\omega r^2}{2}$

□

CHAPTER 35

# MAGNETIC FIELD DUE TO A CURRENT

In the previous chapter, we defined magnetic field in terms of the force it exerts on a moving charge. In this chapter, we shall discuss how a magnetic field can be produced. A magnetic field can be produced by moving charges or electric currents. The basic equation governing the magnetic field due to a current distribution is the *Biot–Savart law.*

## 35.1 BIOT–SAVART LAW

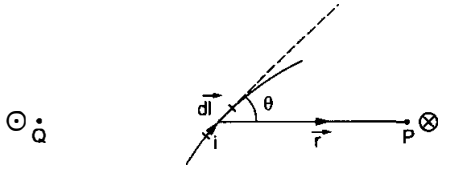



Figure 35.1

The magnetic field at a point $P$, due to a current element, is given by

$$d\vec{B} = \frac{1}{4\pi\varepsilon_0 c^2} i \frac{d\vec{l} \times \vec{r}}{r^3} \quad \ldots (35.1)$$

where $c$ is the speed of light, $i$ is the current, $d\vec{l}$ is the length-vector of the current element and $\vec{r}$ is the vector joining the current element to the point $P$. The quantity $\frac{1}{\varepsilon_0 c^2}$ is written as $\mu_0$ and is called the *permeability of vacuum.* Its value is $4\pi \times 10^{-7}$ T $mA^{-1}$. In terms of $\mu_0$, equation (35.1) becomes

$$d\vec{B} = \frac{\mu_0}{4\pi} i \frac{d\vec{l} \times \vec{r}}{r^3} \quad \ldots (35.2)$$

This equation is the mathematical form of Biot–Savart Law.

The magnitude of the field is

$$dB = \frac{\mu_0}{4\pi} \frac{idl \sin\theta}{r^2} \quad \ldots (35.3)$$

where $\theta$ is the angle between $d\vec{l}$ and $\vec{r}$. The direction of the field is perpendicular to the plane containing the current element and the point $P$ according to the rules of cross product. If we place the stretched right-hand palm along $d\vec{l}$ in such a way that the fingers curl towards $\vec{r}$, the cross product $d\vec{l} \times \vec{r}$ is along the thumb. Usually, the plane of the diagram contains both $d\vec{l}$ and $\vec{r}$. The magnetic field $d\vec{B}$ is then perpendicular to the plane of the diagram, either going into the plane or coming out of the plane. As usual, we denote the direction going into the plane by an encircled cross and the direction coming out of the plane by an encircled dot. In figure (35.1), the magnetic field at the point $P$ goes into the plane of the diagram and that at $Q$ comes out of this plane.

***Example 35.1***

*A wire placed along north–south direction carries a current of* 10 A *from south to north. Find the magnetic field due to a* 1 cm *piece of wire at a point* 200 cm *north-east from the piece.*

***Solution :***

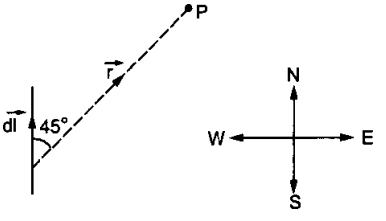



Figure 35.2

The situation is shown in figure (35.2). As the distance of $P$ from the wire is much larger than the length of the wire, we can treat the wire as a small element. The magnetic field is given by

$$d\vec{B} = \frac{\mu_0}{4\pi} i \frac{d\vec{l} \times \vec{r}}{r^3}$$

or,

$$dB = \frac{\mu_0}{4\pi} i \frac{dl \sin\theta}{r^2}$$

$$= (10^{-7}\ \text{T mA}^{-1})\,(10\ \text{A})\,\frac{(10^{-2}\ \text{m})\sin 45°}{(2\ \text{m})^2}$$

$$= 1{\cdot}8 \times 10^{-9}\ \text{T}.$$

The direction of $\vec{B}$ is the same as that of $d\vec{l} \times \vec{r}$. From the figure, it is vertically downward.

## 35.2 MAGNETIC FIELD DUE TO CURRENT IN A STRAIGHT WIRE

Let $MN$ (figure 35.3) be a portion of a straight wire carrying a current $i$. Let $P$ be a point at a distance $OP = d$ from it. The point $O$ is the foot of the perpendicular from $P$ to the wire.

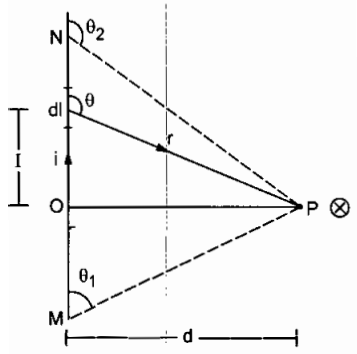



Figure 35.3

Let us consider an element $dl$ of the wire at a distance $l$ from the point $O$. The vector joining the element $dl$ with the point $P$ is $\vec{r}$. Let $\theta$ be the angle between $d\vec{l}$ and $\vec{r}$. The magnetic field at $P$ due to the element is

$$dB = \frac{\mu_0}{4\pi}\, i\, \frac{dl \sin\theta}{r^2} \qquad \ldots (i)$$

The direction of the field is determined by the vector $d\vec{l} \times \vec{r}$. It is perpendicular to the plane of the diagram and going into it. The direction of the field is the same for all elements of the wire and hence the net field due to the wire $MN$ is obtained by integrating equation (i) under proper limits.

From the figure, $l = -d\cot\theta$

or, $dl = d\,\text{cosec}^2\theta\, d\theta.$

Also, $r = d\,\text{cosec}\theta.$

Putting in (i),

$$dB = \frac{\mu_0 i}{4\pi d} \sin\theta\, d\theta$$

or, $$B = \frac{\mu_0 i}{4\pi d} [-\cos\theta]_{\theta_1}^{\theta_2}$$

$$= \frac{\mu_0 i}{4\pi d} (\cos\theta_1 - \cos\theta_2). \qquad \ldots (35.4)$$

Here $\theta_1$ and $\theta_2$ are the values of $\theta$ corresponding to the lower end and the upper end respectively.

### Field on a Perpendicular Bisector

Suppose, the length $MN = a$ and the point $P$ is on its perpendicular bisector. So,

$OM = ON = a/2$ and

$$\cos\theta_1 = \frac{a/2}{\sqrt{\frac{a^2}{4} + d^2}} = \frac{a}{\sqrt{a^2 + 4d^2}}$$

and $$\cos\theta_2 = -\frac{a}{\sqrt{a^2 + 4d^2}}.$$

Equation (35.4) then becomes

$$B = \frac{\mu_0 i}{4\pi d} \frac{2a}{\sqrt{a^2 + 4d^2}}$$

$$= \frac{\mu_0 ia}{2\pi d\sqrt{a^2 + 4d^2}}. \qquad \ldots (35.5)$$

### Field due to a Long, Straight Wire

In this case $\theta_1 = 0$ and $\theta_2 = \pi$. From equation (35.4), the magnetic field is

$$B = \frac{\mu_0 i}{2\pi d} \qquad \ldots (35.6)$$

The direction of the magnetic field at a point $P$ due to a long, straight wire can be found by a slight variation in the right-hand thumb rule. If we stretch the thumb of the right hand along the long current and curl our fingers to pass through the point $P$, the direction of the fingers at $P$ gives the direction of the magnetic field there.

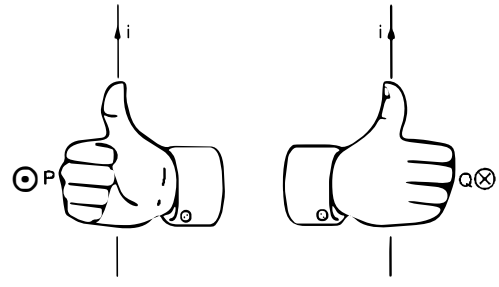



Figure 35.4

***Example 35.2***

*Figure (35.5) shows two long, straight wires carrying electric currents in opposite directions. The separation between the wires is* 5·0 cm. *Find the magnetic field at a point P midway between the wires.*

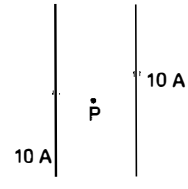



Figure 35.5

***Solution*** : The right-hand thumb rule shows that the magnetic field at $P$ due to each of the wires is

perpendicular to the plane of the diagram and is going into it. The magnitude of the field due to each wire is

$$B = \frac{\mu_0 i}{2\pi d}$$

$$= \frac{(2 \times 10^{-7}\ \text{T mA}^{-1})\ (10\ \text{A})}{2{\cdot}5 \times 10^{-2}\ \text{m}}$$

$$= 80\ \mu\text{T}.$$

The net field due to both the wires is $2 \times 80\ \mu\text{T} = 160\ \mu\text{T}$.

We can draw magnetic field lines on the pattern of electric field lines. A tangent to a magnetic field line gives the direction of the magnetic field existing at that point. For a long straight wire, the field lines are circles with their centres on the wire (figure 35.6).

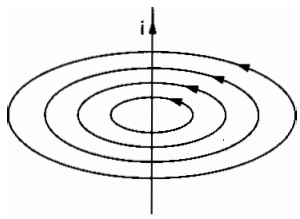



Figure 35.6

## 35.3 FORCE BETWEEN PARALLEL CURRENTS

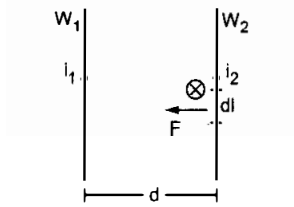



Figure 35.7

Consider two long wires $W_1$ and $W_2$ kept parallel to each other and carrying currents $i_1$ and $i_2$ respectively in the same direction (figure 35.7). The separation between the wires is $d$. Consider a small element $dl$ of the wire $W_2$. The magnetic field at $dl$ due to the wire $W_1$ is

$$B = \frac{\mu_0 i_1}{2\pi d} \qquad \ldots \text{(i)}$$

The field due to the portions of the wire $W_2$, above and below $dl$, is zero. Thus, (i) gives the net field at $dl$. The direction of this field is perpendicular to the plane of the diagram and going into it. The magnetic force at the element $dl$ is

$$d\vec{F} = i_2\, d\vec{l} \times \vec{B}$$

or,

$$dF = i_2\, dl\, \frac{\mu_0 i_1}{2\pi d}.$$

The vector product $d\vec{l} \times \vec{B}$ has a direction towards the wire $W_1$. Thus, the length $dl$ of wire $W_2$ is attracted towards the wire $W_1$. The force per unit length of the wire $W_2$ due to the wire $W_1$ is

$$\frac{dF}{dl} = \frac{\mu_0 i_1 i_2}{2\pi d}. \qquad \ldots \text{(35.7)}$$

If we take an element $dl$ in the wire $W_1$ and calculate the magnetic force per unit length of wire $W_1$ due to $W_2$, it is again given by (35.7).

If the parallel wires carry currents in opposite directions, the wires repel each other.

***Example 35.3***

*Two long, straight wires, each carrying an electric current of* 5·0 A, *are kept parallel to each other at a separation of* 2·5 cm. *Find the magnitude of the magnetic force experienced by* 10 cm *of a wire.*

***Solution*** : The field at the site of one wire due to the other is

$$B = \frac{\mu_0 i}{2\pi d} = \frac{(2 \times 10^{-7}\ \text{T mA}^{-1})\ (5{\cdot}0\ \text{A})}{2{\cdot}5 \times 10^{-2}\ \text{m}} = 4{\cdot}0 \times 10^{-5}\ \text{T}.$$

The force experienced by 10 cm of this wire due to the other is

$$F = i\, lB$$

$$= (5{\cdot}0\ \text{A})\ (10 \times 10^{-2}\ \text{m})\ (4{\cdot}0 \times 10^{-5}\ \text{T})$$

$$= 2{\cdot}0 \times 10^{-5}\ \text{N}.$$

### Definition of Ampere

Consider two parallel wires separated by 1 m and carrying a current of 1 A each. Then $i_1 = i_2 = 1$ A and $d = 1$ m, so that from equation (35.7),

$$\frac{dF}{dl} = 2 \times 10^{-7}\ \text{N m}^{-1}.$$

This is used to formally define the unit 'ampere' of electric current. *If two parallel, long wires, kept* 1 m *apart in vacuum, carry equal currents in the same direction and there is a force of attraction of* $2 \times 10^{-7}$ newton per metre *of each wire, the current in each wire is said to be* 1 ampere.

## 35.4 FIELD DUE TO A CIRCULAR CURRENT

### Field at the Centre

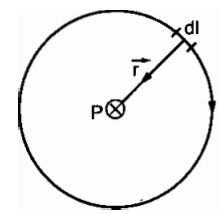



Figure 35.8

Consider a circular loop of radius $a$ carrying a current $i$. We have to find the magnetic field due to

this current at the centre of the loop. Consider any small element $dl$ of the wire (figure 35.8). The magnetic field at the centre $O$ due to the current element $i\vec{dl}$ is

$$d\vec{B} = \frac{\mu_0}{4\pi} i \frac{\vec{dl} \times \vec{r}}{r^3}$$

where $\vec{r}$ is the vector joining the element to the centre $O$. The direction of this field is perpendicular to the plane of the diagram and is going into it. The magnitude is

$$dB = \frac{\mu_0}{4\pi} \frac{idl}{a^2}. \quad \ldots \text{(i)}$$

As the fields due to all such elements have the same direction, the net field is also in this direction. It can, therefore, be obtained by integrating (i) under proper limits. Thus,

$$B = \int dB = \int \frac{\mu_0 i}{4\pi a^2} dl$$

$$= \frac{\mu_0 i}{4\pi a^2} \int dl = \frac{\mu_0 i}{4\pi a^2} \times 2\pi a = \frac{\mu_0 i}{2a}.$$

The direction of the magnetic field at the centre of a circular wire can be obtained using the right-hand thumb rule. If the fingers are curled along the current, the stretched thumb will point towards the magnetic field (figure 35.9).

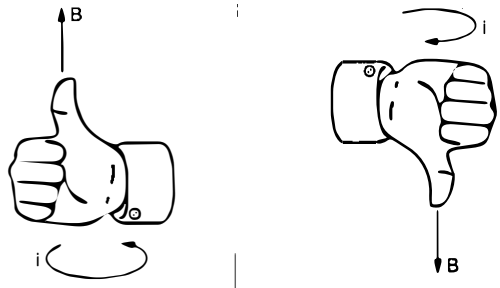



Figure 35.9

Another way to find the direction is to look into the loop along its axis. If the current is in anticlockwise direction, the magnetic field is towards the viewer. If the current is in clockwise direction, the field is away from the viewer.

In figure (35.8), the current is clockwise as seen by you. The magnetic field at the centre is away from you, i.e., is going into the plane of the diagram.

***Example 35.4***

*A circular coil of radius* 1·5 cm *carries a current of* 1·5 A. *If the coil has* 25 *turns, find the magnetic field at the centre.*

***Solution*** : The magnetic field at the centre due to each turn is

$$\frac{\mu_0 i}{2a}.$$

The net field due to all 25 turns is

$$B = \frac{\mu_0 i n}{2a} = \frac{(2\pi \times 10^{-7}\ \text{T mA}^{-1})(1{\cdot}5\ \text{A}) \times 25}{1{\cdot}5 \times 10^{-2}\ \text{m}}$$

$$= 1{\cdot}57 \times 10^{-3}\ \text{T}.$$

**Field at an Axial Point**

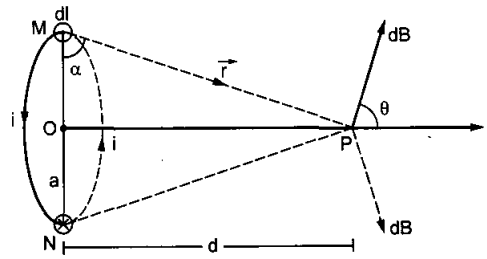



Figure 35.10

Consider a circular loop of radius $a$ carrying a current $i$. We have to find the magnetic field at a point $P$ on the axis of the loop at a distance $d$ from its centre $O$. In figure (35.10), the loop is perpendicular to the plane of the figure while its axis is in the plane of the figure. The current comes out of the plane at $M$ and goes into it at $N$. Consider a current element $i\vec{dl}$ of the wire at $M$. The vector joining the element to the point $P$ is $\vec{r} = \vec{MP}$. The magnetic field at $P$ due to this current element is

$$d\vec{B} = \frac{\mu_0}{4\pi} i \frac{\vec{dl} \times \vec{r}}{r^3}.$$

As $\vec{dl}$ is perpendicular to the plane of the figure, $\vec{dl} \times \vec{r}$ must be in the plane. The figure shows the direction of $\vec{dB}$ according to the rules of vector product. The magnitude of the field is

$$dB = \frac{\mu_0}{4\pi} \frac{i\,dl}{r^2} = \frac{\mu_0}{4\pi} \frac{i\ dl}{a^2 + d^2}.$$

The component along the axis is

$$dB \cos\theta = \frac{\mu_0\, ia\, dl}{4\pi (a^2 + d^2)^{3/2}}. \quad \ldots \text{(i)}$$

Now, consider the diametrically opposite current element at $N$. The field due to this element will have the same magnitude $dB$ and its direction will be along the dotted arrow shown in the figure. The two fields due to the elements at $M$ and at $N$ have a resultant along the axis of the loop. Dividing the loop in such pairs of diametrically opposite elements, we conclude that the resultant magnetic field at $P$ must be along the axis. The resultant field at $P$ can, therefore, be obtained by integrating the right-hand side of (i), i.e.,

$$B = \int \frac{\mu_0\, ia}{4\pi (a^2 + d^2)^{3/2}} dl$$

$$= \frac{\mu_0 ia}{4\pi(a^2 + d^2)^{3/2}} \times 2\pi a$$

$$= \frac{\mu_0 ia^2}{2(a^2 + d^2)^{3/2}}. \quad \ldots (35.8)$$

The right-hand thumb rule can be used to find the direction of the field.

**Field at a point far away from the centre**

If $d >> a$, equation (35.8) gives

$$B \approx \frac{\mu_0 ia^2}{2d^3} = \frac{2\mu_0 i(\pi a^2)}{4\pi d^3} \quad \ldots (ii)$$

Now, $\pi a^2$ is the area of the loop and $i(\pi a^2)$ is its magnetic dipole moment $\mu$. Using right-hand thumb rule, we see that the direction of the area-vector $\vec{A}$ and hence of the dipole moment is along the field $\vec{B}$. The magnetic field due to this small loop at an axial point is, therefore,

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{2\vec{\mu}}{d^3} \quad \ldots (35.9)$$

One can compare this with the expression for the electric field due to an electric dipole at a point on the dipole-axis. It is

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{2\vec{p}}{d^3}.$$

where $\vec{p}$ is the electric dipole moment.

The magnetic field at a point not on the axis is mathematically difficult to calculate. We show qualitatively in figure (35.11) the magnetic field lines due to a circular current which will give some idea of the field.

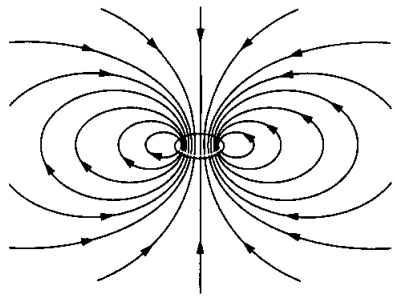


Figure 35.11

## 35.5 AMPERE'S LAW

Ampere's law gives another method to calculate the magnetic field due to a given current distribution. Consider any closed, plane curve (figure 35.12). Assign a sense to the curve by putting an arrow on the curve. Using the right-hand thumb rule, assign one side of the plane as positive and the other as negative. If you curl the fingers of the right hand along the arrow on the curve, the stretched thumb gives the positive side. The positive side can also be determined by looking into the loop along its axis. If the arrow on the loop is anticlockwise, the positive side is towards the viewer. If the arrow is clockwise, the positive side is away from the viewer. In figure (35.12), the positive side is going into the plane of the diagram.

Take a small length-element $d\vec{l}$ on the curve and let $\vec{B}$ be the resultant magnetic field at the position of $d\vec{l}$. Calculate the scalar product $\vec{B} \cdot d\vec{l}$ and integrate by varying $d\vec{l}$ on the closed curve. This integration is called *line integral* or *circulation* of $\vec{B}$ along the curve and is represented by the symbol

$$\oint \vec{B} \cdot d\vec{l}$$

Now look at the currents crossing the area bounded by the curve. A current directed towards the positive side of the plane area is taken as positive and a current directed towards the negative side is taken as negative. Ampere's law then states:

*The circulation* $\oint \vec{B} \cdot d\vec{l}$ *of the resultant magnetic field along a closed, plane curve is equal to* $\mu_0$ *times the total current crossing the area bounded by the closed curve provided the electric field inside the loop remains constant.* Thus,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 i. \quad \ldots (35.10)$$

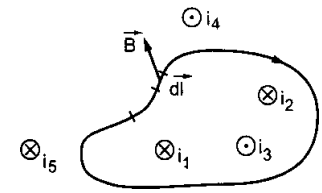



Figure 35.12

In figure (35.12), the positive side is going into the area of the diagram so that $i_1$ and $i_2$ are positive and $i_3$ is negative. Thus, the total current crossing the area is $i_1 + i_2 - i_3$. Any current outside the area is not included in writing the right-hand side of equation (35.10). The magnetic field $\vec{B}$ on the left-hand side is the resultant field due to all the currents existing anywhere.

Ampere's law may be derived from the Biot–Savart law and Biot–Savart law may be derived from the Ampere's law. Thus, the two are equivalent in scientific content. However, Ampere's law is more useful under certain symmetrical conditions. In such cases, the mathematics of finding the magnetic field becomes much simpler if we use the Ampere's law.

## 35.6 MAGNETIC FIELD AT A POINT DUE TO A LONG, STRAIGHT CURRENT

Figure (35.13a) shows a long, straight current $i$. We have to calculate the magnetic field at a point $P$

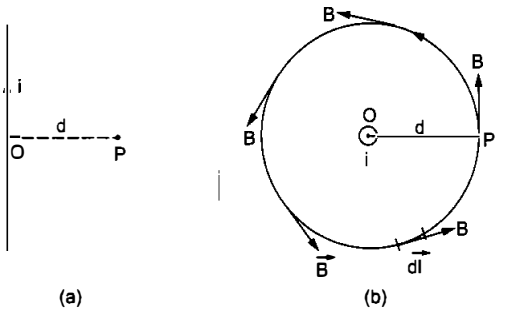



Figure 35.13

which is at a distance $OP = d$ from the wire. Figure (35.13b) shows the situation in the plane perpendicular to the wire and passing through $P$. The current is perpendicular to the plane of the diagram and is coming out of it.

Let us draw a circle passing through the point $P$ and with the centre at $O$. We put an arrow to show the positive sense of the circle. The radius of the circle is $OP = d$. The magnetic field due to the long, straight current at any point on the circle is along the tangent as shown in the figure. Same is the direction of the length-element $d\vec{l}$ there. By symmetry, all points of the circle are equivalent and hence the magnitude of the magnetic field should be the same at all these points. The circulation of magnetic field along the circle is

$$\oint \vec{B}\cdot d\vec{l} = \oint B\, dl$$

$$= B \oint dl = B\, 2\pi d.$$

The current crossing the area bounded by the circle is $i$. Thus, from Ampere's law,

$$B\, 2\pi d = \mu_0 i$$

or,
$$B = \frac{\mu_0 i}{2\pi d}.$$

We have already derived this equation from Biot–Savart law (equation 35.6).

## 35.7 SOLENOID

A solenoid is a wire wound closely in the form of a helix. The wire is coated with an insulating material so that although the adjacent turns physically touch each other, they are electrically insulated. Generally, the length of the solenoid is large as compared to the transverse dimension. For example, if the solenoid has circular turns, the length is large as compared to its radius. If it has rectangular turns, the length should be large as compared to the edges. The magnetic field due to a current-carrying solenoid can be easily pictured by examining the field due to a circular current. The field lines due to a circular current were drawn in figure (35.11). Figure (35.14a) shows the magnetic field lines due to a circular loop $A$ carrying a current and also due to a similar loop $B$ placed coaxially and carrying equal current. We see that at a point $P$, which is quite off the axis, the magnetic fields due to the two loops are in opposite directions. On the other hand, at a point $Q$ which is on the axis or close to the axis, the two fields are in the same direction. Figure (35.14b) shows the resultant field lines due to the two loops. A solenoid may be thought of as a stack of such circular currents placed coaxially one after the other. Figure (35.14c) represents the field due to a loosely wound solenoid. The fields at an outside point due to the neighbouring loops oppose each other, whereas at an inside point, the fields are in the same direction. These tendencies to have zero field outside and a uniform field inside become more and more effective as the solenoid is more and more tightly wound. The magnetic field inside a very tightly wound, long solenoid is uniform everywhere and is zero outside it (figure 35.14d).

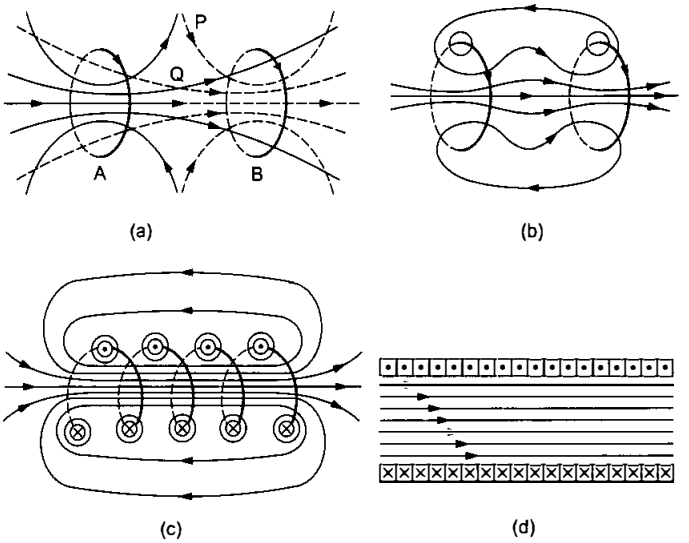



Figure 35.14

To calculate the magnetic field at a point $P$ inside the solenoid, let us draw a rectangle $PQRS$ as shown in figure (35.15). The line $PQ$ is parallel to the solenoid axis and hence parallel to the magnetic field $\vec{B}$ inside the solenoid. Thus,

$$\int_P^Q \vec{B}\cdot d\vec{l} = Bl.$$

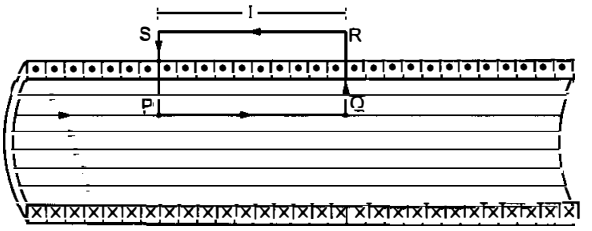



Figure 35.15

On the remaining three sides, $\vec{B}\cdot d\vec{l}$ is zero everywhere as $\vec{B}$ is either zero (outside the solenoid) or perpendicular to $d\vec{l}$ (inside the solenoid). Thus, the circulation of $\vec{B}$ along $PQRS$ is

$$\oint \vec{B}\cdot d\vec{l} = Bl.$$

Let $n$ be the number of turns per unit length along the length of the solenoid. A total of $nl$ turns cross the rectangle $PQRS$. Each turn carries a current $i$. So the net current crossing $PQRS = nli$.

Using Ampere's law,

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 nli$$

or, $$Bl = \mu_0 nli$$

or, $$B = \mu_0 ni. \quad \ldots (35.11)$$

***Example 35.5***

*A long solenoid is formed by winding* 20 turns $cm^{-1}$. *What current is necessary to produce a magnetic field of* 20 mT *inside the solenoid ?*

***Solution*** : The magnetic field inside the solenoid is

$$B = \mu_0 ni$$

or, $$20 \times 10^{-3}\ T = (4\pi \times 10^{-7}\ T\ mA^{-1}) \times (20 \times 10^{2}\ m^{-1})i$$

or, $$i = 8{\cdot}0\ A.$$

Equation (35.11) is strictly valid only for an infinitely long solenoid. In practice, if the length of the solenoid is more than about five to six times the diameter and we are looking at the field in the middle region, we can use this equation.

## 35.8 TOROID

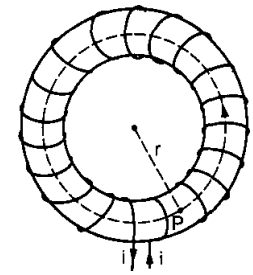



Figure 35.16

If a solenoid is bent in a circular shape and the ends are joined, we get a toroid. Alternatively, one can start with a nonconducting ring and wind a conducting wire closely on it. The magnetic field in such a toroid can be obtained by using Ampere's law.

Suppose, we have to find the field at a point $P$ inside the toroid. Let the distance of $P$ from the centre be $r$. Draw a circle through the point $P$ and concentric with the toroid. By symmetry, the field will have equal magnitude at all points of this circle. Also, the field is everywhere tangential to the circle. Thus,

$$\oint \vec{B}\cdot d\vec{l} = \int B\, dl = B \int dl = 2\pi rB.$$

If the total number of turns is $N$, the current crossing the area bounded by the circle is $Ni$ where $i$ is the current in the toroid. Using Ampere's law on this circle,

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 Ni$$

or, $$2\pi rB = \mu_0 Ni$$

or, $$B = \frac{\mu_0 Ni}{2\pi r} \quad \ldots (35.12)$$

### *Worked Out Examples*

**1.** *Two long wires a and b, carrying equal currents of* 10·0 A, *are placed parallel to each other with a separation of* 4·00 cm *between them as shown in figure (35-W1). Find the magnetic field B at each of the points P, Q and R.*

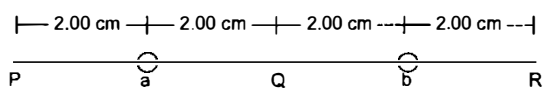



Figure 35-W1

***Solution*** : The magnetic field at $P$ due to the wire $a$ has magnitude

$$B_1 = \frac{\mu_0 i}{2\pi d} = \frac{4\pi \times 10^{-7}\ T\ mA^{-1} \times 10\ A}{2\pi \times 2 \times 10^{-2}\ m}$$

$$= 1{\cdot}00 \times 10^{-4}\ T.$$

Its direction will be perpendicular to the line shown and will point downward in the figure.

The field at this point due to the other wire has magnitude

$$B_2 = \frac{\mu_0 i}{2\pi d} = \frac{4\pi \times 10^{-7}\ T\ mA^{-1} \times 10\ A}{2\pi \times 6 \times 10^{-2}\ m}$$

$$= 0{\cdot}33 \times 10^{-4}\ T.$$

Its direction will be the same as that of $B_1$. Thus, the resultant field will be $1{\cdot}33 \times 10^{-4}$ T also along the same direction.

Similarly, the resultant magnetic field at $R$ will be $= 1{\cdot}33 \times 10^{-4}$ T along the direction pointing upward in the figure.

The magnetic field at point $Q$ due to the two wires will have equal magnitudes but opposite directions and hence the resultant field will be zero.

**2.** *Two parallel wires P and Q placed at a separation $d = 6$ cm carry electric currents $i_1 = 5$ A and $i_2 = 2$ A in opposite directions as shown in figure (35-W2a). Find the point on the line PQ where the resultant magnetic field is zero.*

Figure 35-W2

***Solution*** : At the desired point, the magnetic fields due to the two wires must have equal magnitude but opposite directions. The point should be either to the left of $P$ or to the right of $Q$. As the wire $Q$ has smaller current, the point should be closer to $Q$. Let this point $R$ be at a distance $x$ from $Q$ (figure 35-W2b).

The magnetic field at $R$ due to the current $i_1$ will have magnitude

$$B_1 = \frac{\mu_0 i_1}{2\pi(d+x)}$$

and will be directed downward in the plane of the figure. The field at the same point due to the current $i_2$ will be

$$B_2 = \frac{\mu_0 i_2}{2\pi x}$$

directed upward in the plane of the figure. If the resultant field at $R$ is zero, we should have $B_1 = B_2$, so that

$$\frac{i_1}{d+x} = \frac{i_2}{x}$$

giving,
$$x = \frac{i_2 d}{i_1 - i_2} = \frac{(2\text{ A})(6\text{ cm})}{(3\text{ A})} = 4\text{ cm}.$$

**3.** *Two long, straight wires a and b are 2·0 m apart, perpendicular to the plane of the paper as shown in figure (35-W3). The wire a carries a current of 9·6 A directed into the plane of the figure. The magnetic field at the point p at a distance of 10/11 m from the wire b is zero. Find (a) the magnitude and direction of the current in b, (b) the magnitude of the magnetic field B at the point s and (c) the force per unit length on the wire b.*

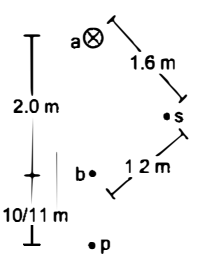



Figure 35-W3

***Solution*** :

(a) For the magnetic field at $p$ to be zero, the current in the wire $b$ should be coming out of the plane of the figure so that the fields due to $a$ and $b$ may be opposite at $p$. The magnitudes of these fields should be equal, so that

$$\frac{\mu_0(9{\cdot}6\text{ A})}{2\pi\left(2+\frac{10}{11}\right)\text{m}} = \frac{\mu_0 i}{2\pi\left(\frac{10}{11}\right)\text{m}}$$

or, $i = 3{\cdot}0$ A.

(b) $(ab)^2 = 4\text{ m}^2$

$(as)^2 = 2{\cdot}56\text{ m}^2$

and $(bs)^2 = 1{\cdot}44\text{ m}^2$

so that $(ab)^2 = (as)^2 + (bs)^2$ and angle $asb = 90°$.

The field at $s$ due to the wire $a$

$$= \frac{\mu_0(9{\cdot}6\text{ A})}{2\pi \times 1{\cdot}6\text{ m}} = \frac{\mu_0}{2\pi} \times 6\text{ Am}^{-1}$$

and that due to the wire $b$

$$= \frac{\mu_0}{2\pi}\frac{3\text{ A}}{1{\cdot}2\text{ m}} = \frac{\mu_0}{2\pi} \times 2{\cdot}5\text{ Am}^{-1}.$$

These fields are at 90° to each other so that their resultant will have a magnitude

$$\sqrt{\left(\frac{\mu_0}{2\pi} \times 6\text{ Am}^{-1}\right)^2 + \left(\frac{\mu_0}{2\pi} \times 2{\cdot}5\text{ Am}^{-1}\right)^2}$$

$$= \frac{\mu_0}{2\pi}\sqrt{36 + 6{\cdot}25}\text{ Am}^{-1}$$

$$= 2 \times 10^{-7}\text{ T mA}^{-1} \times 6{\cdot}5\text{ Am}^{-1}$$

$$= 1{\cdot}3 \times 10^{-6}\text{ T}.$$

(c) The force per unit length on the wire $b$

$$= \frac{\mu_0 i_1 i_2}{2\pi d} = (2 \times 10^{-7}\text{ T mA}^{-1}) \times \frac{(9{\cdot}6\text{ A})(3\text{ A})}{2{\cdot}0\text{ m}}$$

$$= 2{\cdot}9 \times 10^{-6}\text{ N}.$$

**4.** *A current of 2·00 A exists in a square loop of edge 10·0 cm. Find the magnetic field B at the centre of the square loop.*

***Solution*** : The magnetic fields at the centre due to the four sides will be equal in magnitude and direction. The field due to one side will be

$$B_1 = \frac{\mu_0 ia}{2\pi d\sqrt{a^2 + 4d^2}}.$$

Here, $a = 10$ cm and $d = a/2 = 5$ cm.

Thus,

$$B_1 = \frac{\mu_0(2\text{ A})}{2\pi(5\text{ cm})}\left[\frac{10\text{ cm}}{\sqrt{(10\text{ cm})^2 + 4(5\text{ cm})^2}}\right]$$

$$= 2 \times 10^{-7}\text{ T mA}^{-1} \times 2\text{ A} \times \frac{1}{5\sqrt{2}\text{ cm}}$$

$$= 5{\cdot}66 \times 10^{-6}\text{ T}.$$

Hence, the net field at the centre of the loop will be

$$4 \times 5{\cdot}66 \times 10^{-6}\ \text{T} = 22{\cdot}6 \times 10^{-6}\ \text{T}.$$

**5.** *Figure (35-W4) shows a square loop made from a uniform wire. Find the magnetic field at the centre of the square if a battery is connected between the points A and C.*

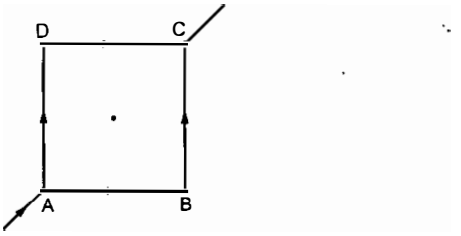



Figure 35-W4

***Solution*** : The current will be equally divided at *A*. The fields at the centre due to the currents in the wires *AB* and *DC* will be equal in magnitude and opposite in direction. The resultant of these two fields will be zero. Similarly, the resultant of the fields due to the wires *AD* and *BC* will be zero. Hence, the net field at the centre will be zero.

**6.** *Two long wires, carrying currents $i_1$ and $i_2$, are placed perpendicular to each other in such a way that they just avoid a contact. Find the magnetic force on a small length dl of the second wire situated at a distance l from the first wire.*

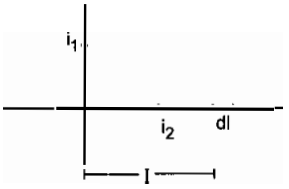



Figure 35-W5

***Solution*** : The situation is shown in figure (35-W5). The magnetic field at the site of *dl*, due to the first wire is,

$$B = \frac{\mu_0 i_1}{2\pi l}.$$

This field is perpendicular to the plane of the figure going into it. The magnetic force on the length *dl* is,

$$dF = i_2\, dl\, B \sin 90°$$

$$= \frac{\mu_0 i_1 i_2 dl}{2\pi l}$$

This force is parallel to the current $i_1$.

**7.** *Figure (35-W6) shows a part of an electric circuit. ABCD is a rectangular loop made of uniform wire. The length AD = BC = 1 cm. The sides AB and DC are long as compared to the other two sides. Find the magnetic force per unit length acting on the wire DC due to the wire AB if the ammeter reads 10 A.*

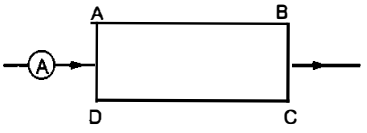



Figure 35-W6

***Solution*** : By symmetry, each of the wires *AB* and *DC* carries a current of 5 A. As the separation between them is 1 cm, the magnetic force per unit length of *DC* is

$$\frac{dF}{dl} = \frac{\mu_0 i_1 i_2}{2\pi d}$$

$$= \frac{(2 \times 10^{-7}\ \text{T mA}^{-1})(5\ \text{A})(5\ \text{A})}{1 \times 10^{-2}\ \text{m}}$$

$$= 5 \times 10^{-4}\ \text{TA} = 5 \times 10^{-4}\ \text{N m}^{-1}.$$

**8.** *Figure (35-W7) shows a current loop having two circular arcs joined by two radial lines. Find the magnetic field B at the centre O.*

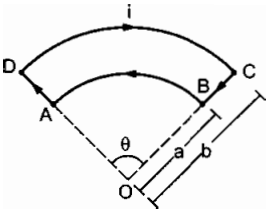



Figure 35-W7

***Solution*** : As the point *O* is on the line *AD*, the magnetic field at *O* due to *AD* is zero. Similarly, the field at *O* due to *BC* is also zero. The field at the centre of a circular current loop is given by $B = \frac{\mu_0 i}{2a}$. The field due to the circular arc *BA* will be

$$B_1 = \left(\frac{\theta}{2\pi}\right)\left(\frac{\mu_0 i}{2a}\right).$$

The right-hand thumb rule shows that the field is coming out of the plane of the figure. The field due to the circular arc *DC* is

$$B_2 = \left(\frac{\theta}{2\pi}\right)\left(\frac{\mu_0 i}{2b}\right)$$

going into the plane of the figure. The resultant field at *O* is

$$B = B_1 - B_2 = \frac{\mu_0 i\theta(b-a)}{4\pi ab}$$

coming out of the plane.

**9.** *Find the magnetic field at the point P in figure (35-W8). The curved portion is a semicircle and the straight wires are long.*

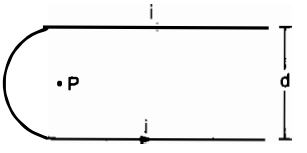



Figure 35-W8

***Solution*** : The magnetic field at *P* due to any current element in the figure is perpendicular to the plane of the figure and coming out of it. The field due to the

upper straight wire is

$$B_1 = \frac{1}{2} \times \frac{\mu_0 i}{2\pi\left(\frac{d}{2}\right)} = \frac{\mu_0 i}{2\pi d}.$$

Same is the field $B_2$ due to the lower straight wire. The field due to the semicircle of radius $(d/2)$ is

$$B_3 = \frac{1}{2} \times \frac{\mu_0 i}{2\left(\frac{d}{2}\right)} = \frac{\mu_0 i}{2d}.$$

The net field is $B = B_1 + B_2 + B_3 = \frac{\mu_0 i}{2d}\left(1 + \frac{2}{\pi}\right)$.

**10.** *The magnetic field B due to a current-carrying circular loop of radius* 12 cm *at its centre is* $0{\cdot}50 \times 10^{-4}$ T. *Find the magnetic field due to this loop at a point on the axis at a distance of* 5·0 cm *from the centre.*

***Solution*** : The magnetic field at the centre of a circular loop is

$$B_0 = \frac{\mu_0 i}{2a}$$

and that at an axial point is

$$B = \frac{\mu_0 i a^2}{2(a^2 + x^2)^{3/2}}.$$

Thus, $\frac{B}{B_0} = \frac{a^3}{(a^2 + x^2)^{3/2}}$

or, $B = (0{\cdot}50 \times 10^{-4}\ \text{T}) \times \frac{(12\ \text{cm})^3}{(144\ \text{cm}^2 + 25\ \text{cm}^2)^{3/2}}$

$= 3{\cdot}9 \times 10^{-5}$ T.

**11.** *Consider a coaxial cable which consists of an inner wire of radius a surrounded by an outer shell of inner and outer radii b and c respectively. The inner wire carries an electric current* $i_0$ *and the outer shell carries an equal current in opposite direction. Find the magnetic field at a distance x from the axis where (a)* $x < a$, *(b)* $a < x < b$ *(c)* $b < x < c$ *and (d)* $x > c$. *Assume that the current density is uniform in the inner wire and also uniform in the outer shell.*

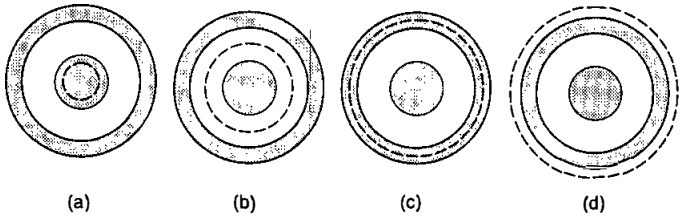



Figure 35-W9

***Solution*** :

A cross section of the cable is shown in figure (35-W9). Draw a circle of radius $x$ with the centre at the axis of the cable. The parts $a$, $b$, $c$ and $d$ of the figure correspond to the four parts of the problem. By symmetry, the magnetic field at each point of a circle will have the same magnitude and will be tangential to it. The circulation of $B$ along this circle is, therefore,

$$\oint \vec{B} \cdot d\vec{l} = B\, 2\pi x$$

in each of the four parts of the figure.

(a) The current enclosed within the circle in part $a$ is

$$\frac{i_0}{\pi a^2} \cdot \pi x^2 = \frac{i_0}{a^2} x^2.$$

Ampere's law $\oint \vec{B} \cdot d\vec{l} = \mu_0 i$ gives

$$B\, 2\pi x = \frac{\mu_0 i_0 x^2}{a^2} \quad \text{or,} \quad B = \frac{\mu_0 i_0 x}{2\pi a^2}$$

The direction will be along the tangent to the circle.

(b) The current enclosed within the circle in part $b$ is $i_0$ so that

$$B\, 2\pi x = \mu_0 i_0 \quad \text{or,} \quad B = \frac{\mu_0 i_0}{2\pi x}.$$

(c) The area of cross section of the outer shell is $\pi c^2 - \pi b^2$. The area of cross section of the outer shell within the circle in part $c$ of the figure is $\pi x^2 - \pi b^2$. Thus, the current through this part is $\frac{i_0(x^2 - b^2)}{c^2 - b^2}$. This is in the opposite direction to the current $i_0$ in the inner wire. Thus, the net current enclosed by the circle is

$$i_0 - \frac{i_0(x^2 - b^2)}{c^2 - b^2} = \frac{i_0(c^2 - x^2)}{c^2 - b^2}$$

From Ampere's law,

$$B\, 2\pi x = \frac{\mu_0 i_0 (c^2 - x^2)}{c^2 - b^2}$$

or, $$B = \frac{\mu_0 i_0 (c^2 - x^2)}{2\pi x (c^2 - b^2)}.$$

(d) The net current enclosed by the circle in part $d$ of the figure is zero and hence

$$B\, 2\pi x = 0 \quad \text{or,} \quad B = 0.$$

**12.** *Figure (35-W10) shows a cross section of a large metal sheet carrying an electric current along its surface. The current in a strip of width dl is Kdl where K is a constant. Find the magnetic field at a point P at a distance x from the metal sheet.*

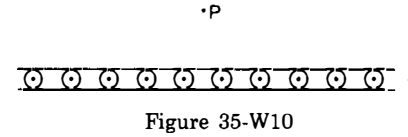



Figure 35-W10

***Solution*** : Consider two strips $A$ and $C$ of the sheet situated symmetrically on the two sides of $P$ (figure 35-W11a). The magnetic field at $P$ due to the strip $A$ is $B_a$ perpendicular to $AP$ and that due to the strip $C$ is $B_c$ perpendicular to $CP$. The resultant of these two is parallel to the width $AC$ of the sheet. The field due to

the whole sheet will also be in this direction. Suppose this field has magnitude $B$.

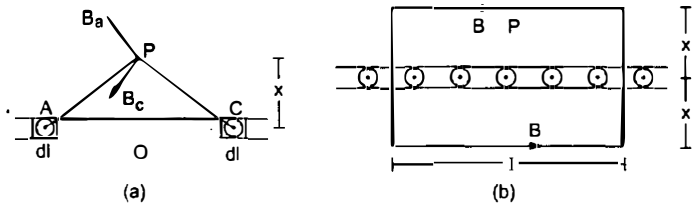



Figure 35-W11

The field on the opposite side of the sheet at the same distance will also be $B$ but in opposite direction. Applying Ampere's law to the rectangle shown in figure (35-W11b),

$$2Bl = \mu_0 Kl$$

or, $$B = \frac{1}{2}\mu_0 K.$$

Note that it is independent of $x$.

**13.** *Consider the situation described in the previous example. A particle of mass m having a charge q is placed at a distance d from the metal sheet and is projected towards it. Find the maximum velocity of projection for which the particle does not hit the sheet.*

***Solution*** : As the magnetic field is uniform and the particle is projected in a direction perpendicular to the field, it will describe a circular path. The particle will not hit the metal sheet if the radius of this circle is smaller than $d$. For the maximum velocity, the radius is just equal to $d$. Thus,

$$qvB = \frac{mv^2}{d}$$

or, $$qv\frac{\mu_0 K}{2} = \frac{mv^2}{d}$$

or, $$v = \frac{\mu_0 qKd}{2m}.$$

**14.** *Three identical long solenoids P, Q and R are connected to each other as shown in figure (35-W12). If the magnetic field at the centre of P is 2·0 T, what would be the field at the centre of Q? Assume that the field due to any solenoid is confined within the volume of that solenoid only.*

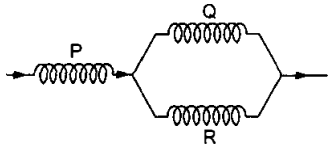



Figure 35-W12

***Solution*** : As the solenoids are identical, the currents in $Q$ and $R$ will be the same and will be half the current in $P$. The magnetic field within a solenoid is given by $B = \mu_0 ni$. Hence the field in $Q$ will be equal to the field in $R$ and will be half the field in $P$, i.e., will be 1·0 T.

**15.** *A long, straight wire carries a current i. A particle having a positive charge q and mass m, kept at a distance $x_0$ from the wire is projected towards it with a speed v. Find the minimum separation between the wire and the particle.*

***Solution*** :

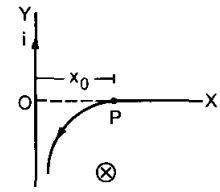



Figure 35-W13

Let the particle be initially at $P$ (figure 35-W13). Take the wire as the $y$-axis and the foot of perpendicular from $P$ to the wire as the origin. Take the line $OP$ as the $x$-axis. We have, $OP = x_0$. The magnetic field $B$ at any point to the right of the wire is along the negative $z$-axis. The magnetic force on the particle is, therefore, in the $x$–$y$ plane. As there is no initial velocity along the $z$-axis, the motion will be in the $x$–$y$ plane. Also, its speed remains unchanged. As the magnetic field is not uniform, the particle does not go along a circle.

The force at time $t$ is $\vec{F} = q\vec{v} \times \vec{B}$

$$= q(\vec{i}\, v_x + \vec{j}\, v_y) \times \left(-\frac{\mu_0 i}{2\pi x}\vec{k}\right)$$

$$= \vec{j}\, q v_x \frac{\mu_0 i}{2\pi x} - \vec{i}\, q v_y \frac{\mu_0 i}{2\pi x}$$

Thus $$a_x = \frac{F_x}{m} = -\frac{\mu_0 qi}{2\pi m}\frac{v_y}{x} = -\lambda\frac{v_y}{x} \quad \ldots \text{(i)}$$

where $$\lambda = \frac{\mu_0 qi}{2\pi m}$$

Also, $$a_x = \frac{dv_x}{dt} = \frac{dv_x}{dx}\frac{dx}{dt} = \frac{v_x dv_x}{dx} \quad \ldots \text{(ii)}$$

As $$v_x^2 + v_y^2 = v^2,$$

$$2v_x\, dv_x + 2v_y\, dv_y = 0$$

giving $$v_x\, dv_x = -v_y\, dv_y. \quad \ldots \text{(iii)}$$

From (i), (ii) and (iii),

$$\frac{v_y\, dv_y}{dx} = \frac{\lambda v_y}{x}$$

or, $$\frac{dx}{x} = \frac{dv_y}{\lambda}$$

Initially $x = x_0$ and $v_y = 0$. At minimum separation from the wire, $v_x = 0$ so that $v_y = -v$.

Thus $$\int_{x_0}^{x}\frac{dx}{x} = \int_0^{-v}\frac{dv_y}{\lambda}$$

or, $$\ln\frac{x}{x_0} = -\frac{v}{\lambda}$$

or, $$x = x_0 e^{-v/\lambda} = x_0 e^{-\frac{2\pi mv}{\mu_0 qi}}.$$

## QUESTIONS FOR SHORT ANSWER

1. An electric current flows in a wire from north to south. What will be the direction of the magnetic field due to this wire at a point (a) east of the wire, (b) west of the wire, (c) vertically above the wire and (d) vertically below the wire?
2. The magnetic field due to a long straight wire has been derived in terms of $\mu_0$, $i$ and $d$. Express this in terms of $\varepsilon_0$, $c$, $i$ and $d$.
3. You are facing a circular wire carrying an electric current. The current is clockwise as seen by you. Is the field at the centre coming towards you or going away from you ?
4. In Ampere's law $\oint \vec{B}\cdot d\vec{l} = \mu_0 i$, the current outside the curve is not included on the right hand side. Does it mean that the magnetic field $B$ calculated by using Ampere's law, gives the contribution of only the currents crossing the area bounded by the curve ?
5. The magnetic field inside a tightly wound, long solenoid is $B = \mu_0 ni$. It suggests that the field does not depend on the total length of the solenoid, and hence if we add more loops at the ends of a solenoid the field should not increase. Explain qualitatively why the extra-added loops do not have a considerable effect on the field inside the solenoid.
6. A long, straight wire carries a current. Is Ampere's law valid for a loop that does not enclose the wire, or that encloses the wire but is not circular ?
7. A straight wire carrying an electric current is placed along the axis of a uniformly charged ring. Will there be a magnetic force on the wire if the ring starts rotating about the wire ? If yes, in which direction ?
8. Two wires carrying equal currents $i$ each, are placed perpendicular to each other, just avoiding a contact. If one wire is held fixed and the other is free to move under magnetic forces, what kind of motion will result ?
9. Two proton beams going in the same direction repel each other whereas two wires carrying currents in the same direction attract each other. Explain.
10. In order to have a current in a long wire, it should be connected to a battery or some such device. Can we obtain the magnetic field due to a straight, long wire by using Ampere's law without mentioning this other part of the circuit ?
11. Quite often, connecting wires carrying currents in opposite directions are twisted together in using electrical appliances. Explain how it avoids unwanted magnetic fields.
12. Two current-carrying wires may attract each other. In absence of other forces, the wires will move towards each other increasing the kinetic energy. Does it contradict the fact that the magnetic force cannot do any work and hence cannot increase the kinetic energy ?

## OBJECTIVE I

1. A vertical wire carries a current in upward direction. An electron beam sent horizontally towards the wire will be deflected
   (a) towards right (b) towards left
   (c) upwards (d) downwards.
2. A current-carrying, straight wire is kept along the axis of a circular loop carrying a current. The straight wire
   (a) will exert an inward force on the circular loop
   (b) will exert an outward force on the circular loop
   (c) will not exert any force on the circular loop
   (d) will exert a force on the circular loop parallel to itself.
3. A proton beam is going from north to south and an electron beam is going from south to north. Neglecting the earth's magnetic field, the electron beam will be deflected
   (a) towards the proton beam
   (b) away from the proton beam
   (c) upwards (d) downwards.
4. A circular loop is kept in that vertical plane which contains the north–south direction. It carries a current that is towards north at the topmost point. Let $A$ be a point on the axis of the circle to the east of it and $B$ a point on this axis to the west of it. The magnetic field due to the loop
   (a) is towards east at $A$ and towards west at $B$
   (b) is towards west at $A$ and towards east at $B$
   (c) is towards east at both $A$ and $B$
   (d) is towards west at both $A$ and $B$.
5. Consider the situation shown in figure (35-Q1). The straight wire is fixed but the loop can move under magnetic force. The loop will
   (a) remain stationary
   (b) move towards the wire
   (c) move away from the wire
   (d) rotate about the wire.

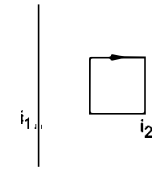



Figure 35-Q1

6. A charged particle is moved along a magnetic field line. The magnetic force on the particle is
   (a) along its velocity (b) opposite to its velocity
   (c) perpendicular to its velocity (d) zero.
7. A moving charge produces
   (a) electric field only (b) magnetic field only
   (c) both of them (d) none of them.

8. A particle is projected in a plane perpendicular to a uniform magnetic field. The area bounded by the path described by the particle is proportional to
(a) the velocity (b) the momentum
(c) the kinetic energy (d) none of these.

9. Two particles $X$ and $Y$ having equal charge, after being accelerated through the same potential difference enter a region of uniform magnetic field and describe circular paths of radii $R_1$ and $R_2$ respectively. The ratio of the mass of $X$ to that of $Y$ is
(a) $(R_1/R_2)^{1/2}$ (b) $R_1/R_2$ (c) $(R_1/R_2)^2$ (d) $R_1R_2$.

10. Two parallel wires carry currents of 20 A and 40 A in opposite directions. Another wire carrying a current antiparallel to 20 A is placed midway between the two wires. The magnetic force on it will be
(a) towards 20 A (b) towards 40 A (c) zero
(d) perpendicular to the plane of the currents.

11. Two parallel, long wires carry currents $i_1$ and $i_2$ with $i_1 > i_2$. When the currents are in the same direction, the magnetic field at a point midway between the wires is 10 μT. If the direction of $i_2$ is reversed, the field becomes 30 μT. The ratio $i_1/i_2$ is
(a) 4 (b) 3 (c) 2 (d) 1.

12. Consider a long, straight wire of cross-sectional area $A$ carrying a current $i$. Let there be $n$ free electrons per unit volume. An observer places himself on a trolley moving in the direction opposite to the current with a speed $v = \frac{i}{nAe}$ and separated from the wire by a distance $r$. The magnetic field seen by the observer is very nearly
(a) $\frac{\mu_0 i}{2\pi r}$ (b) zero (c) $\frac{\mu_0 i}{\pi r}$ (d) $\frac{2\mu_0 i}{\pi r}$.

## OBJECTIVE II

1. The magnetic field at the origin due to a current element $i\,d\vec{l}$ placed at a position $\vec{r}$ is
(a) $\frac{\mu_0 i}{4\pi}\frac{d\vec{l}\times\vec{r}}{r^3}$ (b) $-\frac{\mu_0 i}{4\pi}\frac{\vec{r}\times d\vec{l}}{r^3}$
(c) $\frac{\mu_0 i}{4\pi}\frac{\vec{r}\times d\vec{l}}{r^3}$ (d) $-\frac{\mu_0 i}{4\pi}\frac{d\vec{l}\times\vec{r}}{r^3}$.

2. Consider three quantities $x = E/B$, $y = \sqrt{1/\mu_0\varepsilon_0}$ and $z = \frac{l}{CR}$. Here, $l$ is the length of a wire, $C$ is a capacitance and $R$ is a resistance. All other symbols have standard meanings.
(a) $x$, $y$ have the same dimensions.
(b) $y$, $z$ have the same dimensions.
(c) $z$, $x$ have the same dimensions.
(d) None of the three pairs have the same dimensions.

3. A long, straight wire carries a current along the $z$-axis. One can find two points in the $x$–$y$ plane such that
(a) the magnetic fields are equal
(b) the directions of the magnetic fields are the same
(c) the magnitudes of the magnetic fields are equal
(d) the field at one point is opposite to that at the other point.

4. A long, straight wire of radius $R$ carries a current distributed uniformly over its cross section. The magnitude of the magnetic field is
(a) maximum at the axis of the wire
(b) minimum at the axis of the wire
(c) maximum at the surface of the wire
(d) minimum at the surface of the wire.

5. A hollow tube is carrying an electric current along its length distributed uniformly over its surface. The magnetic field
(a) increases linearly from the axis to the surface
(b) is constant inside the tube
(c) is zero at the axis
(d) is zero just outside the tube.

6. In a coaxial, straight cable, the central conductor and the outer conductor carry equal currents in opposite directions. The magnetic field is zero
(a) outside the cable
(b) inside the inner conductor
(c) inside the outer conductor
(d) in between the two conductors.

7. A steady electric current is flowing through a cylindrical conductor.
(a) The electric field at the axis of the conductor is zero.
(b) The magnetic field at the axis of the conductor is zero.
(c) The electric field in the vicinity of the conductor is zero.
(d) The magnetic field in the vicinity of the conductor is zero.

## EXERCISES

1. Using the formulae $\vec{F} = q\vec{v}\times\vec{B}$ and $B = \frac{\mu_0 i}{2\pi r}$, show that the SI units of the magnetic field B and the permeability constant $\mu_0$ may be written as $\mathrm{N\,mA^{-1}}$ and $\mathrm{N\,A^{-2}}$ respectively.

2. A current of 10 A is established in a long wire along the positive $z$-axis. Find the magnetic field $\vec{B}$ at the point (1 m, 0, 0).

3. A copper wire of diameter 1·6 mm carries a current of 20 A. Find the maximum magnitude of the magnetic field $\vec{B}$ due to this current.
4. A transmission wire carries a current of 100 A. What would be the magnetic field $B$ at a point on the road if the wire is 8 m above the road ?
5. A long, straight wire carrying a current of 1·0 A is placed horizontally in a uniform magnetic field $B = 1{\cdot}0 \times 10^{-5}$ T pointing vertically upward (figure 35-E1). Find the magnitude of the resultant magnetic field at the points $P$ and $Q$, both situated at a distance of 2·0 cm from the wire in the same horizontal plane.

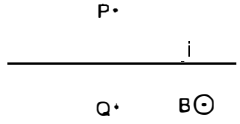



Figure 35-E1

6. A long, straight wire of radius $r$ carries a current $i$ and is placed horizontally in a uniform magnetic field $B$ pointing vertically upward. The current is uniformly distributed over its cross section. (a) At what points will the resultant magnetic field have maximum magnitude ? What will be the maximum magnitude ? (b) What will be the minimum magnitude of the resultant magnetic field ?
7. A long, straight wire carrying a current of 30 A is placed in an external, uniform magnetic field of $4{\cdot}0 \times 10^{-4}$ T parallel to the current. Find the magnitude of the resultant magnetic field at a point 2·0 cm away from the wire.
8. A long, vertical wire carrying a current of 10 A in the upward direction is placed in a region where a horizontal magnetic field of magnitude $2{\cdot}0 \times 10^{-3}$ T exists from south to north. Find the point where the resultant magnetic field is zero.
9. Figure (35-E2) shows two parallel wires separated by a distance of 4·0 cm and carrying equal currents of 10 A along opposite directions. Find the magnitude of the magnetic field $B$ at the points $A_1$, $A_2$, $A_3$ and $A_4$.

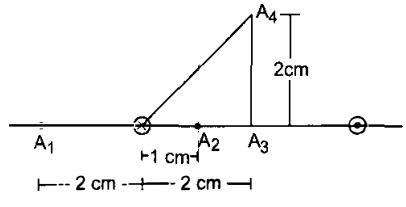



Figure 35-E2

10. Two parallel wires carry equal currents of 10 A along the same direction and are separated by a distance of 2·0 cm. Find the magnetic field at a point which is 2·0 cm away from each of these wires.
11. Two long, straight wires, each carrying a current of 5 A, are placed along the $x$- and $y$-axes respectively. The currents point along the positive directions of the axes. Find the magnetic fields at the points (a) (1 m, 1 m), (b) (– 1 m, 1 m), (c) (– 1 m, – 1 m) and (d) ( 1 m, – 1 m).
12. Four long, straight wires, each carrying a current of 5·0 A, are placed in a plane as shown in figure (35-E3). The points of intersection form a square of side 5·0 cm. (a) Find the magnetic field at the centre $P$ of the square. (b) $Q_1$, $Q_2$, $Q_3$ and $Q_4$ are points situated on the diagonals of the square and at a distance from $P$ that is equal to the length of the diagonal of the square. Find the magnetic fields at these points.

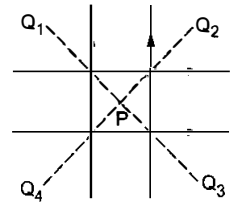



Figure 35-E3

13. Figure (35-E4) shows a long wire bent at the middle to form a right angle. Show that the magnitudes of the magnetic fields at the points $P$, $Q$, $R$ and $S$ are equal and find this magnitude.

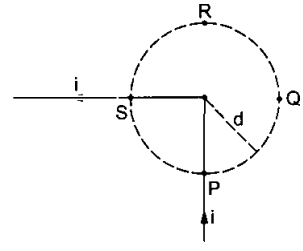



Figure 35-E4

14. Consider a straight piece of length $x$ of a wire carrying a current $i$. Let $P$ be a point on the perpendicular bisector of the piece, situated at a distance $d$ from its middle point. Show that for $d >> x$, the magnetic field at $P$ varies as $1/d^2$ whereas for $d << x$, it varies as $1/d$.
15. Consider a 10-cm long piece of a wire which carries a current of 10 A. Find the magnitude of the magnetic field due to the piece at a point which makes an equilateral triangle with the ends of the piece.
16. A long, straight wire carries a current $i$. Let $B_1$ be the magnetic field at a point $P$ at a distance $d$ from the wire. Consider a section of length $l$ of this wire such that the point $P$ lies on a perpendicular bisector of the section. Let $B_2$ be the magnetic field at this point due to this section only. Find the value of $d/l$ so that $B_2$ differs from $B_1$ by 1%.
17. Figure (35-E5) shows a square loop $ABCD$ with edge-length $a$. The resistance of the wire $ABC$ is $r$ and that of $ADC$ is $2r$. Find the magnetic field $B$ at the centre of the loop assuming uniform wires.

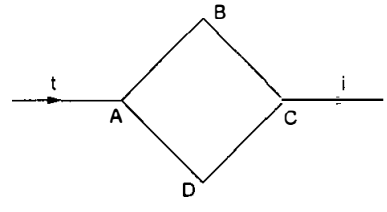



Figure 35-E5

18. Figure (35-E6) shows a square loop of edge $a$ made of a uniform wire. A current $i$ enters the loop at the point $A$ and leaves it at the point $C$. Find the magnetic field at

the point $P$ which is on the perpendicular bisector of $AB$ at a distance $a/4$ from it.

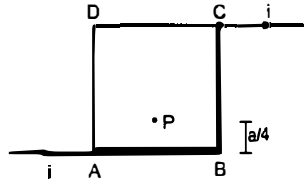



Figure 35-E6

**19.** Consider the situation described in the previous problem. Suppose the current $i$ enters the loop at the point $A$ and leaves it at the point $B$. Find the magnetic field at the centre of the loop.

**20.** The wire $ABC$ shown in figure (35-E7) forms an equilateral triangle. Find the magnetic field $B$ at the centre $O$ of the triangle assuming the wire to be uniform.

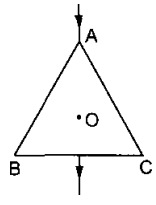



Figure 35-E7

**21.** A wire of length $l$ is bent in the form of an equilateral triangle and carries an electric current $i$. (a) Find the magnetic field $B$ at the centre. (b) If the wire is bent in the form of a square, what would be the value of $B$ at the centre ?

**22.** A long wire carrying a current $i$ is bent to form a plane angle $\alpha$. Find the magnetic field $B$ at a point on the bisector of this angle situated at a distance $x$ from the vertex.

**23.** Find the magnetic field $B$ at the centre of a rectangular loop of length $l$ and width $b$, carrying a current $i$.

**24.** A regular polygon of $n$ sides is formed by bending a wire of total length $2\pi r$ which carries a current $i$. (a) Find the magnetic field $B$ at the centre of the polygon. (b) By letting $n \to \infty$, deduce the expression for the magnetic field at the centre of a circular current.

**25.** Each of the batteries shown in figure (35-E8) has an emf equal to 5 V. Show that the magnetic field $B$ at the point $P$ is zero for any set of values of the resistances.

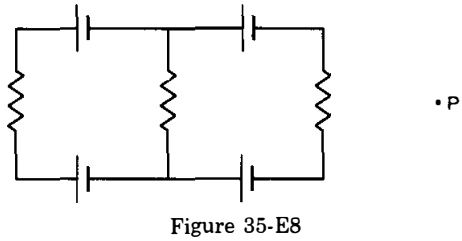



Figure 35-E8

**26.** A straight, long wire carries a current of 20 A. Another wire carrying equal current is placed parallel to it. If the force acting on a length of 10 cm of the second wire is $2{\cdot}0 \times 10^{-5}$ N, what is the separation between them ?

**27.** Three coplanar parallel wires, each carrying a current of 10 A along the same direction, are placed with a separation 5·0 cm between the consecutive ones. Find the magnitude of the magnetic force per unit length acting on the wires.

**28.** Two parallel wires separated by a distance of 10 cm carry currents of 10 A and 40 A along the same direction. Where should a third current be placed so that it experiences no magnetic force ?

**29.** Figure (35-E9) shows a part of an electric circuit. The wires $AB$, $CD$ and $EF$ are long and have identical resistances. The separation between the neighbouring wires is 1·0 cm. The wires $AE$ and $BF$ have negligible resistance and the ammeter reads 30 A. Calculate the magnetic force per unit length of $AB$ and $CD$.

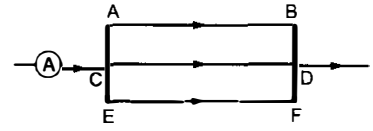



Figure 35-E9

**30.** A long, straight wire is fixed horizontally and carries a current of 50·0 A. A second wire having linear mass density $1{\cdot}0 \times 10^{-4}$ kg m$^{-1}$ is placed parallel to and directly above this wire at a separation of 5·0 mm. What current should this second wire carry such that the magnetic repulsion can balance its weight ?

**31.** A square loop $PQRS$ carrying a current of 6·0 A is placed near a long wire carrying 10 A as shown in figure (35-E10). (a) Show that the magnetic force acting on the part $PQ$ is equal and opposite to that on the part $RS$. (b) Find the magnetic force on the square loop.

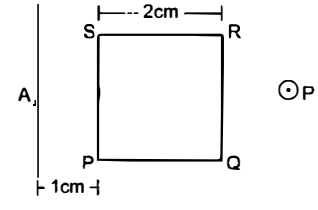



Figure 35-E10

**32.** A circular loop of one turn carries a current of 5·00 A. If the magnetic field $B$ at the centre is 0·200 mT, find the radius of the loop.

**33.** A current-carrying circular coil of 100 turns and radius 5·0 cm produces a magnetic field of $6{\cdot}0 \times 10^{-5}$ T at its centre. Find the value of the current.

**34.** An electron makes $3 \times 10^{5}$ revolutions per second in a circle of radius 0·5 angstrom. Find the magnetic field $B$ at the centre of the circle.

**35.** A conducting circular loop of radius $a$ is connected to two long, straight wires. The straight wires carry a current $i$ as shown in figure (35-E11). Find the magnetic field $B$ at the centre of the loop.

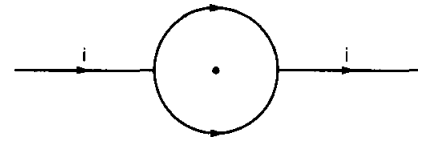



Figure 35-E11

**36.** Two circular coils of radii 5·0 cm and 10 cm carry equal currents of 2·0 A. The coils have 50 and 100 turns respectively and are placed in such a way that their planes as well as the centres coincide. Find the magnitude of the magnetic field $B$ at the common centre of the coils if the currents in the coils are (a) in the same sense (b) in the opposite sense.

**37.** If the outer coil of the previous problem is rotated through 90° about a diameter, what would be the magnitude of the magnetic field $B$ at the centre?

**38.** A circular loop of radius 20 cm carries a current of 10 A. An electron crosses the plane of the loop with a speed of $2{\cdot}0 \times 10^6$ m s$^{-1}$. The direction of motion makes an angle of 30° with the axis of the circle and passes through its centre. Find the magnitude of the magnetic force on the electron at the instant it crosses the plane.

**39.** A circular loop of radius $R$ carries a current $I$. Another circular loop of radius $r$ ($<< R$) carries a current $i$ and is placed at the centre of the larger loop. The planes of the two circles are at right angle to each other. Find the torque acting on the smaller loop.

**40.** A circular loop of radius $r$ carrying a current $i$ is held at the centre of another circular loop of radius $R(>> r)$ carrying a current $I$. The plane of the smaller loop makes an angle of 30° with that of the larger loop. If the smaller loop is held fixed in this position by applying a single force at a point on its periphery, what would be the minimum magnitude of this force?

**41.** Find the magnetic field $B$ due to a semicircular wire of radius 10·0 cm carrying a current of 5·0 A at its centre of curvature.

**42.** A piece of wire carrying a current of 6·00 A is bent in the form of a circular arc of radius 10·0 cm, and it subtends an angle of 120° at the centre. Find the magnetic field $B$ due to this piece of wire at the centre.

**43.** A circular loop of radius $r$ carries a current $i$. How should a long, straight wire carrying a current $4i$ be placed in the plane of the circle so that the magnetic field at the centre becomes zero?

**44.** A circular coil of 200 turns has a radius of 10 cm and carries a current of 2·0 A. (a) Find the magnitude of the magnetic field $\vec{B}$ at the centre of the coil. (b) At what distance from the centre along the axis of the coil will the field $B$ drop to half its value at the centre? ($\sqrt[3]{4} = 1{\cdot}5874$ ..)

**45.** A circular loop of radius 4·0 cm is placed in a horizontal plane and carries an electric current of 5·0 A in the clockwise direction as seen from above. Find the magnetic field (a) at a point 3·0 cm above the centre of the loop (b) at a point 3·0 cm below the centre of the loop.

**46.** A charge of $3{\cdot}14 \times 10^{-6}$ C is distributed uniformly over a circular ring of radius 20·0 cm. The ring rotates about its axis with an angular velocity of 60.0 rad s$^{-1}$. Find the ratio of the electric field to the magnetic field at a point on the axis at a distance of 5·00 cm from the centre.

**47.** A thin but long, hollow, cylindrical tube of radius $r$ carries a current $i$ along its length. Find the magnitude of the magnetic field at a distance $r/2$ from the surface (a) inside the tube (b) outside the tube.

**48.** A long, cylindrical tube of inner and outer radii $a$ and $b$ carries a current $i$ distributed uniformly over its cross section. Find the magnitude of the magnetic field at a point (a) just inside the tube (b) just outside the tube.

**49.** A long, cylindrical wire of radius $b$ carries a current $i$ distributed uniformly over its cross section. Find the magnitude of the magnetic field at a point inside the wire at a distance $a$ from the axis.

**50.** A solid wire of radius 10 cm carries a current of 5·0 A distributed uniformly over its cross section. Find the magnetic field $B$ at a point at a distance (a) 2 cm (b) 10 cm and (c) 20 cm away from the axis. Sketch a graph of $B$ versus $x$ for $0 < x < 20$ cm.

**51.** Sometimes we show an idealised magnetic field which is uniform in a given region and falls to zero abruptly. One such field is represented in figure (35-E12). Using Ampere's law over the path $PQRS$, show that such a field is not possible.

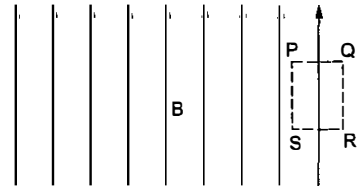



Figure 35-E12

**52.** Two large metal sheets carry surface currents as shown in figure (35-E13). The current through a strip of width $dl$ is $Kdl$ where $K$ is a constant. Find the magnetic field at the points $P$, $Q$ and $R$.

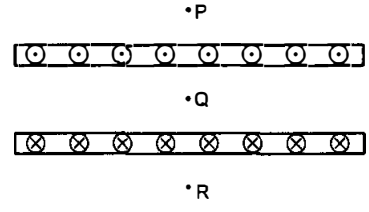



Figure 35-E13

**53.** Consider the situation of the previous problem. A particle having charge $q$ and mass $m$ is projected from the point $Q$ in a direction going into the plane of the diagram. It is found to describe a circle of radius $r$ between the two plates. Find the speed of the charged particle.

**54.** The magnetic field $B$ inside a long solenoid, carrying a current of 5·00 A, is $3{\cdot}14 \times 10^{-2}$ T. Find the number of turns per unit length of the solenoid.

**55.** A long solenoid is fabricated by closely winding a wire of radius 0·5 mm over a cylindrical nonmagnetic frame so that the successive turns nearly touch each other. What would be the magnetic field $B$ at the centre of the solenoid if it carries a current of 5 A?

**56.** A copper wire having resistance 0·01 ohm in each metre is used to wind a 400-turn solenoid of radius 1·0 cm and length 20 cm. Find the emf of a battery which when

connected across the solenoid will cause a magnetic field of $1{\cdot}0 \times 10^{-2}$ T near the centre of the solenoid.

**57.** A tightly-wound solenoid of radius $a$ and length $l$ has $n$ turns per unit length. It carries an electric current $i$. Consider a length $dx$ of the solenoid at a distance $x$ from one end. This contains $n\,dx$ turns and may be approximated as a circular current $i\,n\,dx$. (a) Write the magnetic field at the centre of the solenoid due to this circular current. Integrate this expression under proper limits to find the magnetic field at the centre of the solenoid. (b) Verify that if $l >> a$, the field tends to $B = \mu_0 ni$ and if $a >> l$, the field tends to $B = \frac{\mu_0 nil}{2a}$. Interpret these results.

**58.** A tightly-wound, long solenoid carries a current of 2·00 A. An electron is found to execute a uniform circular motion inside the solenoid with a frequency of $1{\cdot}00 \times 10^8$ rev $s^{-1}$. Find the number of turns per metre in the solenoid.

**59.** A tightly-wound, long solenoid has $n$ turns per unit length, a radius $r$ and carries a current $i$. A particle having charge $q$ and mass $m$ is projected from a point on the axis in a direction perpendicular to the axis. What can be the maximum speed for which the particle does not strike the solenoid?

**60.** A tightly-wound, long solenoid is kept with its axis parallel to a large metal sheet carrying a surface current. The surface current through a width $dl$ of the sheet is $Kdl$ and the number of turns per unit length of the solenoid is $n$. The magnetic field near the centre of the solenoid is found to be zero. (a) Find the current in the solenoid. (b) If the solenoid is rotated to make its axis perpendicular to the metal sheet, what would be the magnitude of the magnetic field near its centre?

**61.** A capacitor of capacitance 100 μF is connected to a battery of 20 volts for a long time and then disconnected from it. It is now connected across a long solenoid having 4000 turns per metre. It is found that the potential difference across the capacitor drops to 90% of its maximum value in 2·0 seconds. Estimate the average magnetic field produced at the centre of the solenoid during this period.

□

## ANSWERS

### OBJECTIVE I

1. (c) 2. (c) 3. (a) 4. (d) 5. (b) 6. (d)
7. (c) 8. (c) 9. (c) 10. (b) 11. (c) 12. (a)

### OBJECTIVE II

1. (c), (d) 2. (a), (b), (c) 3. (b), (c), (d)
4. (b), (c) 5. (b), (c) 6. (a)
7. (b), (c)

### EXERCISES

2. 2 μT along the positive $y$-axis
3. 5·0 mT
4. 2·5 μT
5. 20 μT, zero
6. (a) looking along the current, at the leftmost points on the wire's surface, $B + \frac{\mu_0 i}{2\pi r}$

   (b) zero if $r \le \frac{\mu_0 i}{2\pi B}$, $B - \frac{\mu_0 i}{2\pi r}$ if $r > \frac{\mu_0 i}{2\pi B}$
7. $5 \times 10^{-4}$ T
8. 1·0 mm west to the wire
9. (a) $0{\cdot}67 \times 10^{-4}$ T (b) $2{\cdot}7 \times 10^{-4}$ T
   (c) $2{\cdot}0 \times 10^{-4}$ T (d) $1{\cdot}0 \times 10^{-4}$ T
10. $1{\cdot}7 \times 10^{-4}$ T in a direction parallel to the plane of the wires and perpendicular to the wires
11. (a) zero (b) 2 μT along the $z$-axis
    (c) zero and (d) 2 μT along the negative $z$-axis
12. (a) zero (b) $Q_1$ : $1{\cdot}1 \times 10^{-4}$ T, ⊙, $Q_2$ : zero,
    $Q_3$ : $1{\cdot}1 \times 10^{-4}$ T, ⊗, and $Q_4$ : zero
13. $\frac{\mu_0 i}{4\pi d}$
15. 11·5 μT
16. 0·07
17. $\frac{\sqrt{2}\,\mu_0 i}{3\pi a}$, ⊗
18. $\frac{2\mu_0 i}{\pi a}\left(\frac{1}{\sqrt{5}} - \frac{1}{3\sqrt{13}}\right)$
19. zero
20. zero
21. (a) $\frac{27\mu_0 i}{2\pi l}$ (b) $\frac{8\sqrt{2}\mu_0 i}{\pi l}$
22. $\frac{\mu_0 i}{2\pi x}\cot\frac{\alpha}{4}$
23. $\frac{2\mu_0 i\sqrt{l^2 + b^2}}{\pi l b}$
24. (a) $\frac{\mu_0 i n^2 \sin\frac{\pi}{n}\tan\frac{\pi}{n}}{2\pi^2 r}$

26. 40 cm

27. zero on the middle wire and $6{\cdot}0 \times 10^{-4}$ N towards the middle wire on each of the rest two

28. 2 cm from the 10 A current and 8 cm from the other

29. $3 \times 10^{-3}$ N m$^{-1}$, downward zero

30. 0·49 A in opposite directiom

31. (b) $1{\cdot}6 \times 10^{-5}$ N towards right

32. 1·57 cm

33. 48 mA

34. $6 \times 10^{-10}$ T

35. zero

36. (a) $8\pi \times 10^{-4}$ T (b) zero

37. 1·8 mT

38. $16\pi \times 10^{-19}$ N

39. $\dfrac{\mu_0 \pi i I r^2}{2R}$

40. $\dfrac{\mu_0 \pi i I r}{4R}$

41. $1{\cdot}6 \times 10^{-5}$ T

42. $1{\cdot}26 \times 10^{-5}$ T

43. at a distance of $4r/\pi$ from the centre in such a way that the direction of the current in it is opposite to that in the nearest part of the circular wire

44. (a) 2·51 mT (b) 7·66 cm

45. $4{\cdot}0 \times 10^{-5}$ T, downwards in both the cases

46. $1{\cdot}88 \times 10^{15}$ m s$^{-1}$

47. (a) zero (b) $\dfrac{\mu_0 i}{3\pi r}$

48. (a) zero (b) $\dfrac{\mu_0 i}{2\pi b}$

49. $\dfrac{\mu_0 ia}{2\pi b^2}$

50. (a) 2·0 μT (b) 10 μT (c) 5·0 μT

52. 0, $\mu_0 K$ towards right in the figure, 0

53. $\dfrac{\mu_0 Kqr}{m}$

54. 5000 turns m$^{-1}$

55. $2\pi \times 10^{-3}$ T

56. 1 V

57. (a) $\dfrac{\mu_0 ni}{\sqrt{1+\left(\dfrac{2a}{l}\right)^2}}$

58. 1420 turns m$^{-1}$

59. $\dfrac{\mu_0 qrni}{2m}$

60. (a) $\dfrac{K}{2n}$ (b) $\dfrac{\mu_0 K}{\sqrt{2}}$

61. $16\pi \times 10^{-8}$ T

□

## CHAPTER 36

# PERMANENT MAGNETS

### 36.1 MAGNETIC POLES AND BAR MAGNETS

We have seen that a small current loop carrying a current $i$, produces a magnetic field

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{2\vec{\mu}}{d^3} \quad \ldots \text{(i)}$$

at an axial point. Here $\vec{\mu} = i\vec{A}$ is the magnetic dipole moment of the current loop. The vector $\vec{A}$ represents the area-vector of the current loop. Also, a current loop placed in a magnetic field $\vec{B}$ experiences a torque

$$\vec{\Gamma} = \vec{\mu} \times \vec{B}. \quad \ldots \text{(ii)}$$

We also know that an electric dipole produces an electric field

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{2\vec{p}}{d^3} \quad \ldots \text{(iii)}$$

at an axial point and it experiences a torque

$$\vec{\Gamma} = \vec{p} \times \vec{E} \quad \ldots \text{(iv)}$$

when placed in an electric field. Equations (i) and (ii) for a current loop are similar in structure to the equations (iii) and (iv) for an electric dipole with $\vec{\mu}$ taking the role of $\vec{p}$ and $\frac{\mu_0}{4\pi}$ taking the role of $\frac{1}{4\pi\varepsilon_0}$. The similarity suggests that the behaviour of a current loop can be described by the following hypothetical model:

(a) There are two types of magnetic charges, positive magnetic charge and negative magnetic charge. A magnetic charge $m$ placed in a magnetic field $\vec{B}$ experiences a force

$$\vec{F} = m\vec{B}. \quad \ldots \text{(36.1)}$$

The force on a positive magnetic charge is along the field and the force on a negative magnetic charge is opposite to the field.

(b) A magnetic charge $m$ produces a magnetic field

$$B = \frac{\mu_0}{4\pi}\frac{m}{r^2} \quad \ldots \text{(36.2)}$$

at a distance $r$ from it. The field is radially outward if the magnetic charge is positive and is inward if it is negative.

(c) A magnetic dipole is formed when a negative magnetic charge $-m$ and a positive magnetic charge $+m$ are placed at a small separation $d$. The magnetic dipole moment is $\mu = md$ and its direction is from $-m$ to $+m$. The line joining $-m$ and $+m$ is called the *axis* of the dipole.

(d) A current loop of area $A$ carrying a current $i$ may be replaced by a magnetic dipole of dipole moment $\mu = md = iA$ placed along the axis of the loop. The area-vector $\vec{A}$ points in the direction $-m$ to $+m$.

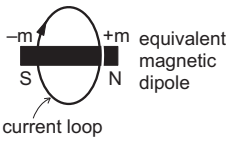



Figure 36.1

The model is very useful in studying magnetic effects and is widely used. It is customary to call a positive magnetic charge a *north pole* and a negative magnetic charge a *south pole*. They are represented by the letters $N$ and $S$ respectively. The quantity $m$ is called *pole strength*. From the equation $md = iA$ or $F = mB$, we can easily see that the unit of pole strength is A–m. We can find the magnetic field due to a magnetic dipole at any point $P$ using equation (36.2) for both the poles.

A solenoid very closely resembles a combination of circular loops placed side by side. If $i$ be the current through it and $A$ be the area of cross-section, the dipole moment of each turn is $\mu = iA$. In our model, each turn may be replaced by a small dipole placed at the centre of the loop along its axis. Suppose, each turn is

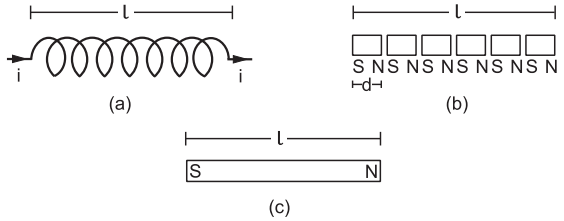



Figure 36.2

replaced by a magnetic dipole with pole strength $m$ and separation $d$ between the north and south poles. We have $md = iA$. Figure (36.2a) and (36.2b) show a current-carrying solenoid and its equivalent in terms of magnetic poles.

Suppose we take the value of $d$ in such a way that the north pole of one dipole touches the south pole of the adjacent one. South poles and north poles, then, neutralise each other except at the ends. Thus, a current-carrying solenoid can be replaced by just a single south pole and a single north pole of pole strength $m$ each, placed at a separation equal to the length of the solenoid (figure 36.2c).

We can obtain the field outside the solenoid using the above model. Each pole produces a field given by equation (36.2). The resultant field is the vector sum of the fields produced by the south pole and the north pole. Figure (36.3) shows the magnetic field lines due to a current-carrying solenoid.

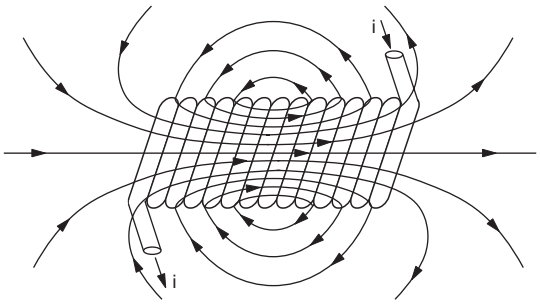



Figure 36.3

Note that the magnetic field inside the solenoid is opposite in direction from what one expects from the pole picture. The magnetic field lines are closed curves. They do not start or end at a point as is the case with electric field lines.

***Example 36.1***

*A solenoid of length* 10 cm *and radius* 1 cm *contains* 200 turns *and carries a current of* 10 A. *Find the magnetic field at a point on the axis at a distance of* 10 cm *from the centre.*

***Solution*** : The dipole moment of each turn is

$$\mu = iA = (10\text{ A})(\pi\text{ cm}^2)$$
$$= \pi \times 10^{-3}\text{ A m}^2.$$

If each current loop is replaced by a dipole having pole strength $m$ and separation between the poles $d$, we have

$$\mu = md.$$

As there are 200 turns,

$$200\, d = 10\text{ cm}$$

or,
$$d = 5 \times 10^{-4}\text{ m}.$$

Thus,

$$m = \frac{\mu}{d} = \frac{\pi \times 10^{-3}\text{ A m}^2}{5 \times 10^{-4}\text{ m}} = 2\pi\text{ A m}.$$

We can replace the solenoid by a south pole and a north pole of equal pole strength $2\pi$ A m, separated by 10 cm. The equivalent picture is shown in figure (36.4).

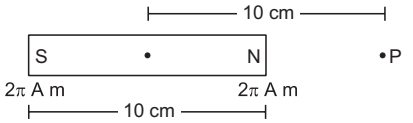



Figure 36.4

The magnetic field at $P$ due to the north pole is

$$B_N = \frac{\mu_0}{4\pi}\frac{2\pi\text{ A m}}{(5\text{ cm})^2} = 2{\cdot}5 \times 10^{-4}\text{ T}.$$

The magnetic field at $P$ due to the south pole is

$$B_S = \frac{\mu_0}{4\pi}\frac{2\pi\text{ A m}}{(15\text{ cm})^2} = 0{\cdot}3 \times 10^{-4}\text{ T}.$$

The field $B_N$ is away from the poles and $B_S$ is towards the poles. The resultant field at $P$ is

$$B = B_N - B_S$$
$$= 2{\cdot}2 \times 10^{-4}\text{ T}$$

away from the solenoid.

In nature, we find certain objects whose magnetic behaviour may be described by assuming that there is a south pole placed at a certain point in the object and a north pole placed at a different point. Such an object is called a *magnet*. A magnet in the shape of a rod or a bar is called a *bar magnet*. The poles appear at points which are slightly inside the two ends. The line joining the positions of the assumed poles is called the *magnetic axis* of the bar magnet. The magnetic field lines due to a bar magnet are similar to those shown in figure (36.3). How can a rod produce magnetic field when no electric current is passed through it ? Let us now discuss this question.

A simple model tells us that matter is made of atoms and each atom contains electrons circulating around its nucleus. These moving electrons constitute electric currents at the atomic level. The actual description of these atomic currents is quite complicated but we can assume that these atomic currents are equivalent to small, circular current loops. In magnets, these loops are arranged nearly parallel to each other and have currents in the same sense.

Figure (36.5) shows the currents in a cross section of a cylindrical bar magnet. At any point inside the

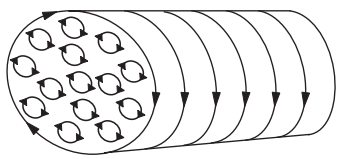


Figure 36.5

magnet, the net current is zero because the currents from the adjacent loops cancel each other. However, there is a net current along the surface as there is no cancellation of currents there. Due to such a surface current, the cylindrical magnet is equivalent to a closely-wound, current-carrying solenoid and hence produces a magnetic field similar to the solenoid. We can, therefore, treat the bar magnet as having a north pole and a south pole separated by a length $l$. Suppose the surface current is $I$ per unit length of the magnet. The total current at the surface of the magnet of length $l$ is $Il$. If the cross-sectional area is $A$, the magnetic dipole moment is

$$\mu = \text{current} \times \text{area} = IlA.$$

If the pole strength is $m$, the magnetic moment may also be written as $\mu = ml$.

Thus, $ml = IlA$

or, $m = IA.$ … (36.3)

In the above discussion we have not considered the end effect. At the two ends of the magnet, the currents behave differently from those inside the magnet. Because of this effect, the magnetic poles appear slightly inside the bar. The distance between the locations of the assumed poles is called the *magnetic length* of the magnet. The distance between the ends is called the *geometrical length*. It is found that

$$\frac{\text{magnetic length}}{\text{geometrical length}} \approx 0{\cdot}84.$$

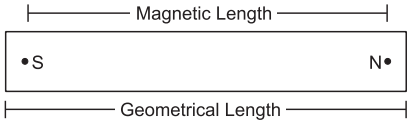



Figure 36.6

The magnetic moment of a bar magnet is conventionally denoted by $M$. Also, the magnetic length of a bar magnet is written as $2l$. If $m$ be the pole strength and $2l$ the magnetic length of a bar magnet, its magnetic moment is

$$M = 2ml. \quad \text{… (36.4)}$$

## 36.2 TORQUE ON A BAR MAGNET PLACED IN A MAGNETIC FIELD

Suppose a bar magnet of magnetic length $2l$ and pole strength $m$ is placed in a uniform magnetic field $\vec{B}$ (figure 36.7). The angle between the magnet and the magnetic field is $\theta$. The force on the north pole is $mB$ along the field and that on the south pole is $mB$ opposite to the field. The torque of these two forces is

$$\Gamma = mB\,l\sin\theta + mB\,l\sin\theta$$
$$= 2\,mB\,l\sin\theta = MB\sin\theta$$

where $M$ is the magnetic moment of the magnet. This torque tries to rotate the magnet so as to align it with the field. We can write the torque as

$$\vec{\Gamma} = \vec{M} \times \vec{B}.$$

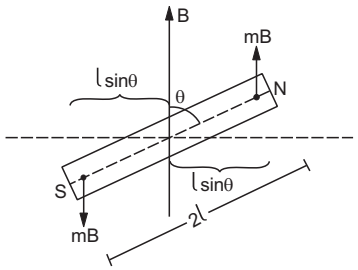



Figure 36.7

This equation is the same as that obtained earlier for a current loop. If an external agent rotates the magnet slowly, the agent has to exert a torque $MB\sin\theta$ opposite to that exerted by the field. The work done by the agent in changing the angle from $\theta$ to $\theta + d\theta$ is $dW = (MB\sin\theta)d\theta$. The work done in rotating the magnet from an angle $\theta_0$ to $\theta$ is

$$W = \int_{\theta_0}^{\theta} MB\sin\theta\, d\theta = MB(\cos\theta_0 - \cos\theta).$$

This work is stored as the potential energy of the field–magnet system. Thus,

$$U(\theta) - U(\theta_0) = MB(\cos\theta_0 - \cos\theta).$$

If we take the potential energy at $\theta = 90°$ to be zero, the potential energy at an angle $\theta$ is

$$U(\theta) = U(\theta) - U(90°)$$
$$= -MB\cos\theta = -\vec{M}\cdot\vec{B}. \quad \text{… (36.5)}$$

We see from this equation that the SI unit for the magnetic moment $M$ may also be written as J $T^{-1}$.

***Example 36.2***

*A bar magnet having a magnetic moment of* $1{\cdot}0 \times 10^{4}$ J $T^{-1}$ *is free to rotate in a horizontal plane. A horizontal magnetic field* $B = 4 \times 10^{-5}$ T *exists in the space. Find the work done in rotating the magnet slowly from a direction parallel to the field to a direction* 60° *from the field.*

***Solution*** : The work done by the external agent = change in potential energy

$$= (-MB\cos\theta_2) - (-MB\cos\theta_1)$$
$$= -MB(\cos 60° - \cos 0°) = \frac{1}{2}MB$$
$$= \frac{1}{2} \times (1{\cdot}0 \times 10^{4}\ \text{J T}^{-1})\,(4 \times 10^{-5}\ \text{T}) = 0{\cdot}2\ \text{J}.$$

## 36.3 MAGNETIC FIELD DUE TO A BAR MAGNET

### (a) End-on Position

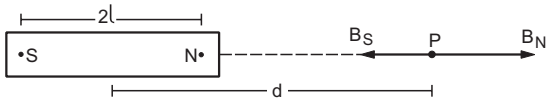



Figure 36.8

A position on the magnetic axis of a bar magnet is called an *end-on position.* Suppose *SN* is a bar magnet of magnetic length 2*l* and pole strength *m*. Let *P* be a point in end-on position at a distance *d* from the centre of the magnet. The magnetic field at *P* due to the north pole is

$$B_N = \frac{\mu_0}{4\pi} \frac{m}{(d-l)^2}$$

directed away from the magnet. The field due to the south pole is

$$B_S = \frac{\mu_0}{4\pi} \frac{m}{(d+l)^2}$$

directed towards the magnet. The resultant field is

$$\begin{aligned} B &= B_N - B_S \\ &= \frac{\mu_0 m}{4\pi}\left[\frac{1}{(d-l)^2} - \frac{1}{(d+l)^2}\right] \\ &= \frac{\mu_0}{4\pi} \frac{2Md}{(d^2 - l^2)^2} \qquad \ldots (36.6) \end{aligned}$$

where $M = 2ml$ is the magnetic moment of the magnet. If $d >> l$, the magnet may be called a magnetic dipole and the field at an end-on position is

$$B = \frac{\mu_0}{4\pi} \frac{2M}{d^3}. \qquad \ldots (36.7)$$

***Example 36.3***

*A magnet is* 10 cm *long and its pole strength is* 120 CGS units (1 CGS unit *of pole strength* = 0·1 A m). *Find the magnitude of the magnetic field B at a point on its axis at a distance* 20 cm *from it.*

***Solution*** :

The pole strength is $m$ = 120 CGS units = 12 A m.

Magnetic length is $2l$ = 10 cm or $l$ = 0·05 m.

Distance from the magnet is $d$ = 20 cm = 0·2 m. The field $B$ at a point in end-on position is

$$\begin{aligned} B &= \frac{\mu_0}{4\pi} \frac{2Md}{(d^2 - l^2)^2} \\ &= \frac{\mu_0}{4\pi} \frac{4mld}{(d^2 - l^2)^2} \\ &= \left(10^{-7} \frac{\text{T m}}{\text{A}}\right) \frac{4 \times (12 \text{ A } m) \times (0{\cdot}05 \text{ m}) \times (0{\cdot}2 \text{ m})}{[(0{\cdot}2 \text{ m})^2 - (0{\cdot}05 \text{ m})^2]^2} \\ &= 3{\cdot}4 \times 10^{-5} \text{ T}. \end{aligned}$$

### (b) Broadside-on Position

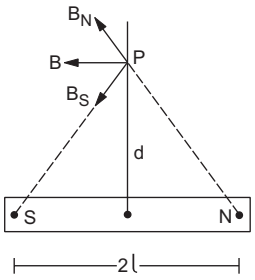



Figure 36.9

A position on a perpendicular bisector of the bar magnet is called *broadside-on position*. Let *P* be a point in the broadside-on position of the bar magnet at a distance *d* from its centre. The pole strength of the magnet is *m* and its magnetic length *SN* is 2*l*. The field at *P* due to the north pole may be written as

$$\vec{B}_N = \frac{\mu_0}{4\pi} \frac{m\,\vec{NP}}{NP^3}.$$

This gives the magnitude as well as the direction of the field due to the north pole. The field due to the south pole is

$$\vec{B}_S = \frac{\mu_0}{4\pi} \frac{m\,\vec{PS}}{PS^3}.$$

Now, $NP = PS = (d^2 + l^2)^{1/2}$ so that the resultant field at $P$ is

$$\begin{aligned} \vec{B} &= \vec{B}_N + \vec{B}_S \\ &= \frac{\mu_0}{4\pi} \frac{m}{(d^2 + l^2)^{3/2}} (\vec{NP} + \vec{PS}) \\ &= \frac{\mu_0}{4\pi} \frac{m\,\vec{NS}}{(d^2 + l^2)^{3/2}}. \end{aligned}$$

The magnitude of the field is

$$B = \frac{\mu_0}{4\pi} \frac{m\,2l}{(d^2 + l^2)^{3/2}} = \frac{\mu_0}{4\pi} \frac{M}{(d^2 + l^2)^{3/2}} \qquad \ldots (36.8)$$

where $M = 2ml$ is the magnetic moment of the magnet. The direction of the field is parallel to the axis, from the north pole to the south pole.

If $d >> l$, the magnet may be called a magnetic dipole and the magnetic field at a point in broadside-on position is

$$B = \frac{\mu_0}{4\pi} \frac{M}{d^3}. \qquad \ldots (36.9)$$

## 36.4 MAGNETIC SCALAR POTENTIAL

Magnetic scalar potential is defined in the same way as gravitational or electrostatic potential. We define the change in potential $V(\vec{r}_2) - V(\vec{r}_1)$ by the equation

$$V(\vec{r_2}) - V(\vec{r_1}) = -\int_{\vec{r_1}}^{\vec{r_2}} \vec{B}\cdot d\vec{r}. \qquad \ldots \quad (36.10)$$

Generally, the potential at infinity (a point far away from all sources of magnetic field) is taken to be zero. Taking $\vec{r_1}$ equal to ∞ and $\vec{r_2} = \vec{r}$, equation (36.10) gives

$$V(\vec{r}) = -\int_{\infty}^{\vec{r}} \vec{B}\cdot d\vec{r}.$$

The component of the magnetic field in any direction is given by

$$B_l = -\frac{dV}{dl} \qquad \ldots \quad (36.11)$$

where $dl$ is a small distance along the given direction.

For a pole of pole strength $m$, the field at a distance $r$ is

$$B = \frac{\mu_0}{4\pi}\frac{m}{r^2}$$

radially away from the pole. So the potential at a distance $r$ is

$$V(r) = -\int_{\infty}^{r} \frac{\mu_0}{4\pi} m \frac{dr}{r^2}$$

$$= \frac{\mu_0}{4\pi}\frac{m}{r}. \qquad \ldots \quad (36.12)$$

**Magnetic Scalar Potential due to a Magnetic Dipole**

Suppose, $SN$ is a magnetic dipole of length $2l$ and pole strength $m$ (figure 36.10). The magnetic scalar potential is needed at a point $P$ at a distance $OP = r(>> l)$ from the centre of the dipole. The angle $PON = \theta$.

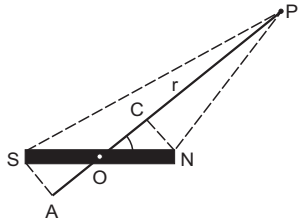



Figure 36.10

Let $SA$ be the perpendicular from $S$ to $OP$ and $NC$ be the perpendicular from $N$ to $OP$. As $r >> l$,

$$PS \approx PA = PO + OA = r + l\cos\theta.$$

Similarly, $PN \approx PC = PO - OC = r - l\cos\theta$.

The magnetic scalar potential at $P$ due to the north pole is

$$V_N = \frac{\mu_0}{4\pi}\frac{m}{NP}$$

$$= \frac{\mu_0}{4\pi}\frac{m}{r - l\cos\theta}$$

and that due to the south pole is

$$V_S = -\frac{\mu_0}{4\pi}\frac{m}{SP}$$

$$= -\frac{\mu_0}{4\pi}\frac{m}{r + l\cos\theta}.$$

The net potential at $P$ due to the dipole is

$$V = V_N + V_S$$

$$= \frac{\mu_0 m}{4\pi}\left(\frac{1}{r - l\cos\theta} - \frac{1}{r + l\cos\theta}\right)$$

$$= \frac{\mu_0}{4\pi}\frac{m(2l\cos\theta)}{(r^2 - l^2\cos^2\theta)}$$

$$\approx \frac{\mu_0}{4\pi}\frac{M\cos\theta}{r^2}. \qquad \ldots \quad (36.13)$$

**Magnetic Field due to a Dipole**

Let $SN$ be a magnetic dipole and $P$ be a point far away from the dipole (figure 36.11). The distance $OP = r$ and the angle $PON = \theta$. If we move a small distance $PQ$ in the direction of $OP$, the value of $r$ is changed to $r + dr$ while θ remains unchanged. Similarly, if we move a small distance $PR$ in the direction perpendicular to $OP$, θ is changed from θ to θ + dθ while $r$ remains very nearly constant. The distance moved is $rd\theta$.

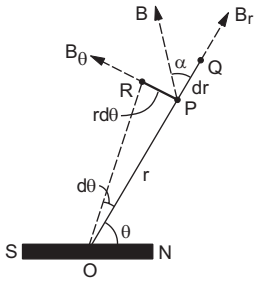



Figure 36.11

The component of magnetic field along $OP$ is

$$B_r = -\frac{dV}{PQ} = -\left[\frac{dV}{dr}\right]_{\theta = \text{constant}}$$

$$= -\frac{\partial V}{\partial r}$$

$$= -\frac{\partial}{\partial r}\left(\frac{\mu_0}{4\pi}\frac{M\cos\theta}{r^2}\right)$$

$$= \frac{\mu_0}{4\pi}\frac{2M\cos\theta}{r^3}. \qquad \ldots \quad (i)$$

The component perpendicular to $OP$ is

$$B_\theta = -\frac{dV}{PR} = -\left[\frac{dV}{rd\theta}\right]_{r = \text{constant}}$$

$$= -\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\mu_0}{4\pi}\frac{M\cos\theta}{r^2}\right)$$

$$= \frac{\mu_0}{4\pi} \frac{M \sin\theta}{r^3} . \quad \ldots \text{(ii)}$$

The resultant magnetic field at $P$ is

$$B = \sqrt{B_r^2 + B_\theta^2}$$

$$= \frac{\mu_0}{4\pi} \frac{M}{r^3} \sqrt{(2\cos\theta)^2 + (\sin\theta)^2}$$

$$= \frac{\mu_0}{4\pi} \frac{M}{r^3} \sqrt{1 + 3\cos^2\theta} . \quad \ldots \text{(36.14)}$$

If it makes an angle $\alpha$ with $OP$,

$$\tan\alpha = \frac{B_\theta}{B_r} .$$

From (i) and (ii),

$$\tan\alpha = \frac{\sin\theta}{2\cos\theta} = \frac{\tan\theta}{2} . \quad \ldots \text{(36.15)}$$

***Example 36.4***

*Find the magnetic field due to a dipole of magnetic moment* $1{\cdot}2$ A m$^2$ *at a point* 1 m *away from it in a direction making an angle of* $60°$ *with the dipole-axis.*

***Solution*** : The magnitude of the field is

$$B = \frac{\mu_0}{4\pi} \frac{M}{r^3} \sqrt{1 + 3\cos^2\theta}$$

$$= \left(10^{-7} \frac{\text{T m}}{\text{A}}\right) \frac{1{\cdot}2 \text{ A m}^2}{1 \text{ m}^3} \sqrt{1 + 3\cos^2 60°}$$

$$= 1{\cdot}6 \times 10^{-7} \text{ T.}$$

The direction of the field makes an angle $\alpha$ with the radial line where

$$\tan\alpha = \frac{\tan\theta}{2} = \frac{\sqrt{3}}{2} .$$

## 36.5 TERRESTRIAL MAGNETISM

Earth is a natural source of magnetic field. We have magnetic field present everywhere near the earth's surface. The magnitude and direction of this field can be obtained approximately by assuming that the earth has a magnetic dipole of dipole moment about $8{\cdot}0 \times 10^{22}$ J T$^{-1}$ located at its centre (figure 36.12). The axis of this dipole makes an angle of about $11{\cdot}5°$ with the earth's axis of rotation. The dipole-axis cuts the earth's surface at two points, one near the geographical north pole and the other near the geographical south pole. The first of these points is called *geomagnetic north pole* and the other is called *geomagnetic south pole.*

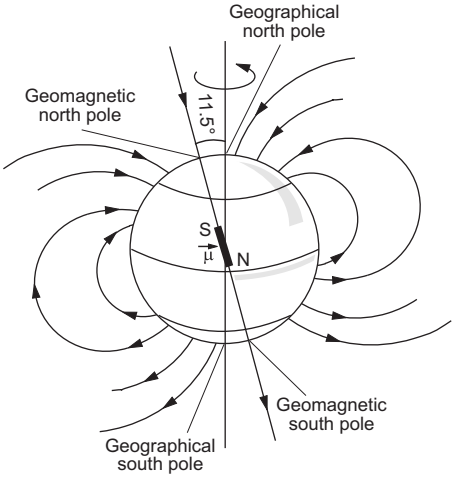



Figure 36.12

If we suspend a bar magnet freely at a point near the earth's surface, it will stay along the magnetic field there. The north pole will point towards the direction of the magnetic field. At the geomagnetic poles, the magnetic field is vertical. If we suspend the bar magnet near the geomagnetic north pole, it will become vertical with its north pole towards the earth's surface. Similarly, if we suspend a bar magnet near the geomagnetic south pole, it will become vertical with its south pole pointing towards the earth's surface. Geomagnetic poles may, therefore, be defined as "the points where a freely suspended bar magnet becomes vertical".

If we treat the assumed magnetic dipole inside the earth as a pair of north and south poles (figure 36.12), the south pole will be towards the geomagnetic north pole and the north pole will be towards the geomagnetic south pole. This may be easily remembered by using the fact that the north pole of the suspended magnet should be attracted by the south pole of the assumed dipole.

Earth's magnetic field changes both in magnitude and direction as time passes. It is fairly constant over a span of a few days, but noticeable changes occur in say, ten years. Studies of magnetic rocks have revealed that the magnetic field may even reverse its direction. It appears that in the past $7{\cdot}6 \times 10^7$ years, already 171 such reversals have taken place. The latest reversal in earth's magnetic field is believed to have occurred around 10,000 years ago.

The theory of earth's magnetic field is not yet well-understood. At present, it seems that the field results mainly due to circulating electric currents induced in the molten liquid and other conducting material inside the earth.

### Elements of the Earth's Magnetic Field

The earth's magnetic field at a point on its surface is usually characterised by three quantities: (a) declination (b) inclination or dip and (c) horizontal component of the field. These are known as the *elements of the earth's magnetic field.*

**Declination**

A plane passing through the geographical poles (that is, through the axis of rotation of the earth) and a given point $P$ on the earth's surface is called the *geographical meridian* at the point $P$. Similarly, the plane passing through the geomagnetic poles (that is, through the dipole-axis of the earth) and the point $P$ is called the *magnetic meridian* at the point $P$. In other words, the magnetic meridian is a vertical plane through the point $P$ that contains the geomagnetic poles. The magnetic field due to the earth at $P$ must be in this plane (magnetic meridian).

The angle made by the magnetic meridian at a point with the geographical meridian is called the *declination* at that point. The knowledge of declination fixes the vertical plane in which the earth's magnetic field lies.

Navigators often use a magnetic compass needle to locate direction. A compass needle is a short and light magnetic needle, free to rotate about a vertical axis. The needle is enclosed in a small case with a glass-top. The needle stays in equilibrium when it is in magnetic meridian. Hence the north direction shown by the needle makes an angle equal to the declination with the true north and navigators have to take care of it.

**Inclination or dip**

The angle made by the earth's magnetic field with the horizontal direction in the magnetic meridian, is called the *inclination* or *dip* at that point.

In the magnetic nothern hemisphere, the vertical component of the earth's magnetic field points downwards. The north pole of a freely suspended magnet, therefore, *dips* (goes down).

The knowledge of declination and inclination completely specifies the direction of the earth's magnetic field.

**Horizontal component of the earth's magnetic field**

As the name indicates, the *horizontal component* is component of the earth's magnetic field in the horizontal direction in the magnetic meridian. This direction is towards the magnetic north.

Figure (36.13) shows the three elements. Starting from the geographical meridian we draw the magnetic meridian at an angle θ (declination). In the magnetic meridian we draw the horizontal direction specifying magnetic north. The magnetic field is at an angle δ (dip) from this direction. The horizontal component $B_H$ and the total field $B$ are related as

$$B_H = B\cos\delta$$

or, $$B = B_H/\cos\delta.$$

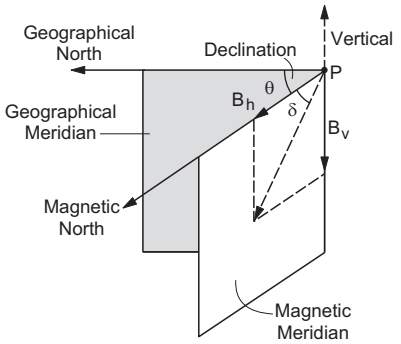



Figure 36.13

Thus, from the knowledge of the three elements, both the magnitude and direction of the earth's magnetic field can be obtained.

***Example 36.5***

*The horizontal component of the earth's magnetic field is* $3{\cdot}6 \times 10^{-5}$ T *where the dip is* 60°. *Find the magnitude of the earth's magnetic field.*

***Solution*** : We have $B_H = B\cos\delta$

or, $$B = \frac{B_H}{\cos\delta} = \frac{3{\cdot}6\times10^{-5}\ \text{T}}{\cos 60^\circ} = 7{\cdot}2\times10^{-5}\ \text{T}.$$

## 36.6 DETERMINATION OF DIP AT A PLACE

**Dip Circle**

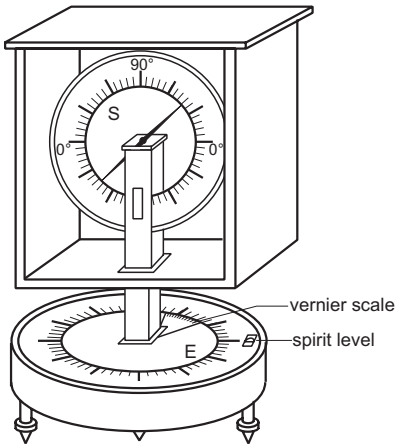



Figure 36.14

The dip at a place can be determined by an apparatus known as *dip circle*. It consists of a vertical circular scale $S$ and a magnetic needle (a small pointed permanent magnet) pivoted at the centre of the scale. The needle can rotate freely in the vertical plane of the scale. The pointed ends move over the graduations on the scale which are marked 0°–0° in the horizontal and 90°–90° in the vertical direction. The scale $S$ together with the needle is enclosed in a glass cover which can be rotated about a vertical axis. The angle rotated can be read from a horizontal angular scale $E$, fixed with the base, and a vernier scale fixed with the

stand supporting the glass cover. The base can be made horizontal by levelling screws fixed with it. A spirit-level fixed to the apparatus helps in levelling.

**Determination of Dip**

**Determination of magnetic meridian**

At the beginning of the experiment, the base of the dip circle is made horizontal with the help of the levelling screws and the magnetic needle is pivoted in its place. The glass cover containing the vertical scale $S$ and the needle is rotated about the vertical axis till the needle becomes vertical and reads 90°–90° on the vertical scale. In this condition, the plane of the circular scale $S$ is perpendicular to the magnetic meridian. The horizontal component $B_H$ is perpendicular to this plane and hence does not take part in rotating the needle. The needle is aligned with the vertical component $B_V$ and hence reads 90°–90°. The reading of the vernier is noted and the glass cover is rotated exactly through 90° from this position. The plane of the circular scale $S$ is now the same as the magnetic meridian.

**Measurement of dip**

When the plane of the vertical scale $S$ is the same as the magnetic meridian, the earth's magnetic field $\vec{B}$ is in this same plane. In this case, the needle rests in the direction of $\vec{B}$. The readings of the ends of the needle on the vertical scale now directly give the value of the dip.

**Possible errors and their remedies**

Errors may occur because of several imperfections in the instrument. Some of the possible errors and their remedies are given below.

(a) *The centre of the needle is not at the centre of the vertical scale.*

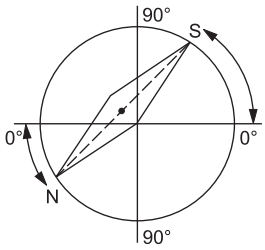



Figure 36.15

If the centre of the needle does not coincide with the centre of the scale, the readings do not represent the true dip. The reading of one end of the needle is less than the true value of the dip and the reading of the other end is greater by the same amount. Thus, both ends are read and the average is taken.

(b) 0°–0° *line is not horizontal.*

If the 0°–0° line on the scale is not horizontal, the value of dip will have some error. This error may be removed as described below.

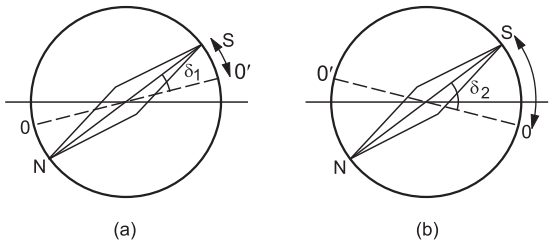



Figure 36.16

Bring the vertical scale in the magnetic meridian and note the readings of the ends of the needle (figure 36.16a). Now, rotate the circle through 180° about the vertical axis and again note the readings (figure 36.16b). The average of these readings will not have the error due to 0°–0° line.

(c) *The magnetic and the geometrical axes of the needle are different.*

In the experiment, we read the angles corresponding to the ends of the needle. If the magnetic axis is inclined at an angle with the line joining the ends, the dip obtained is in error. This error can be removed by inverting the needle on its bearing and repeating the previous readings. The average of these readings is free of this error.

(d) *Centre of mass of the needle does not coincide with the pivot.*

If the centre of mass of the needle is not at the pivot, its weight $mg$ will have a torque and will affect the equilibrium position. To remove this error, one has to read the dip and then take out the needle. The needle should be demagnetized and then remagnetized in opposite direction. Thus, the position of the north and south poles are interchanged. The centre of mass now appears on the other side of the pivot and hence the effect of $mg$ is also reversed. The dip is again determined with this needle and the average is taken.

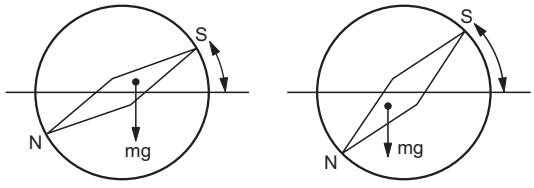



Figure 36.17

Thus, one should take 16 readings of dip and an average of all these gives the true dip.

**Apparent Dip**

If the dip circle is not kept in the magnetic meridian, the needle will not show the correct direction of earth's magnetic field. The angle made by the needle

with the horizontal is called the *apparent dip* for this plane. If the dip circle is at an angle $\alpha$ to the meridian, the effective horizontal component in this plane is $B'_H = B_H \cos\alpha$. The vertical component is still $B_v$. If $\delta'$ is the apparent dip and $\delta$ is the true dip, we have

$$\tan\delta' = \frac{B_v}{B'_H}$$

or, $$\cot\delta' = \frac{B'_H}{B_v} = \frac{B_H \cos\alpha}{B_v}$$

or, $$\cot\delta' = \cot\delta \cos\alpha. \quad \ldots \text{(i)}$$

Now suppose, the dip circle is rotated through an angle of 90° from this position. It will now make an angle $(90° - \alpha)$ with the meridian. The effective horizontal component in this plane is $B''_H = B_H \sin\alpha$. If $\delta''$ be the apparent dip, we shall have

$$\cot\delta'' = \cot\delta \sin\alpha. \quad \ldots \text{(ii)}$$

Squaring and adding (i) and (ii),

$$\cot^2\delta' + \cot^2\delta'' = \cot^2\delta. \quad \ldots (36.16)$$

Thus, one can get the true dip $\delta$ without locating the magnetic meridian.

***Example 36.6***

*At 45° to the magnetic meridian, the apparent dip is 30°. Find the true dip.*

***Solution*** : At 45° to the magnetic meridian, the effective horizontal component of the earth's magnetic field is $B'_H = B_H \cos 45° = \frac{1}{\sqrt{2}} B_H$. The apparent dip $\delta'$ is given by

$$\tan\delta' = \frac{B_v}{B'_H} = \frac{\sqrt{2}\, B_v}{B_H} = \sqrt{2}\tan\delta$$

where $\delta$ is the true dip. Thus,

$$\tan 30° = \sqrt{2}\tan\delta$$

or, $$\delta = \tan^{-1}\sqrt{1/6}.$$

## 36.7 NEUTRAL POINT

Suppose at a point, the horizontal component of the magnetic field due to a magnet is equal and opposite to the earth's horizontal magnetic field. The net horizontal field is zero at such a point. If a compass needle is placed at such a point, it can stay in any position. Such a point is called a *neutral point*.

## 36.8 TANGENT GALVANOMETER

Tangent galvanometer is an instrument to measure an electric current. The essential parts are a vertical circular coil $C$ of conducting wire and a small compass needle $A$ pivoted at the centre of the coil (figure 36.18). The coil $C$ together with its frame is fixed to a horizontal base $B$ provided with levelling screws. Terminals $T_1$ and $T_2$ connected to the coil are

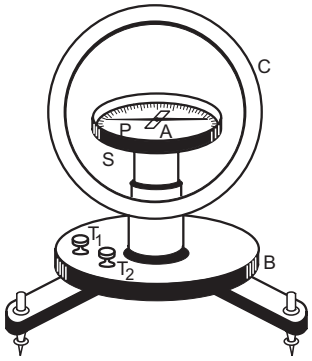



Figure 36.18

provided on this base for connecting the galvanometer to an external circuit. An aluminium pointer $P$ is rigidly attached with the compass needle and perpendicular to it. The compass needle together with the pointer can rotate freely about the vertical axis. The ends of the pointer move over a graduated, horizontal circular scale. The graduations are marked from 0° to 90° in each quadrant. The scale, the pointer and the compass needle are enclosed in a closed cylindrical box which is placed with its centre coinciding with the centre of the coil. The box can also be rotated about the vertical axis. The upper surface of the box is made of glass so that the things inside it are visible. To avoid the errors due to parallax, a plane mirror is fixed at the lower surface of the box. While noting the reading of the pointer, the eye should be properly positioned so that the image of the pointer is just below the pointer.

When there is no current through the galvanometer, the compass needle is in magnetic north–south direction. To measure a current with the tangent galvanometer, the base is rotated in such a way that the plane of the coil is parallel to the compass needle. The plane then coincides with the magnetic meridian. The box containing the needle is rotated so that the aluminium pointer reads 0°–0° on the scale.

The current to be measured is passed through the coil. The current through the coil produces a magnetic field at the centre and the compass needle deflects under its action. The pointer deflects through the same angle and the deflection of both the ends are read from the horizontal scale. The average of these two is calculated.

Suppose the current through the coil is $i$. The radius of the coil is $r$ and the number of turns in it is $n$. The magnetic field produced at the centre is

$$B = \frac{\mu_0 in}{2r}. \quad \ldots \text{(i)}$$

This field is perpendicular to the plane of the coil. This direction is horizontal and perpendicular to the magnetic meridian and hence to the horizontal component $B_H$ of the earth's magnetic field. The

resultant horizontal magnetic field is

$$B_r = \sqrt{B^2 + B_H^2}$$

in a direction making an angle θ with $B_H$ (figure 36.19) where

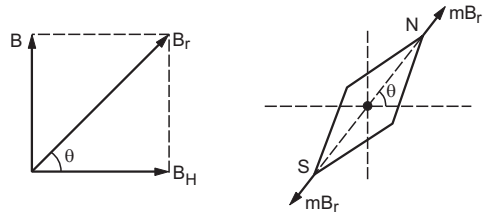



Figure 36.19

$$\tan\theta = B/B_H. \qquad \ldots \text{ (ii)}$$

If $m$ be the pole strength of the needle, the force on the north pole of the needle is $mB_r$ along $B_r$ and on the south pole is $mB_r$, opposite to $B_r$. The needle will stay in equilibrium when its length is parallel to $B_r$, because then no torque is produced by the two forces. Thus, the deflection of the needle from its original position is θ as given by (ii). Using (i) and (ii),

$$B_H \tan\theta = \frac{\mu_0 in}{2r}$$

or, $$i = \frac{2r\,B_H}{\mu_0\, n}\tan\theta$$

or, $$i = K\tan\theta, \qquad \ldots \text{ (36.17)}$$

where $K = \frac{2r\,B_H}{\mu_0\, n}$ is a constant for the given galvanometer at a given place. This constant is called the *reduction factor* of the galvanometer. The reduction factor may be obtained by passing a known current $i$ through the galvanometer, measuring θ and then using (36.17).

Equation (i) is strictly valid only at the centre of the coil. The poles of the needle are slightly away from the centre. Thus, the length of the needle should be small as compared to the radius of the coil.

**Sensitivity**

Good sensitivity means that the change in deflection is large for a given fractional change in current.

We have

$$i = K\tan\theta$$

or, $$di = K\sec^2\theta\, d\theta$$

or, $$\frac{di}{i} = \frac{K\sec^2\theta\, d\theta}{K\tan\theta} = \frac{2\, d\theta}{\sin 2\theta}$$

or, $$d\theta = \frac{1}{2}\sin 2\theta\left(\frac{di}{i}\right).$$

Thus, for good sensitivity, sin 2θ should be as large as possible. This is the case when θ = 45°. So, the tangent galvanometer is most sensitive when the deflection is around 45°.

***Example 36.7***

*A tangent galvanometer has* 66 turns *and the diameter of its coil is* 22 cm. *It gives a deflection of* 45° *for* 0·10 A *current. What is the value of the horizontal component of the earth's magnetic field* ?

***Solution*** : For a tangent galvanometer

$$i = K\tan\theta$$

$$= \frac{2r\,B_H}{\mu_0\, n}\tan\theta$$

or, $$B_H = \frac{\mu_0\, ni}{2r\tan\theta}$$

$$= \frac{\left(4\pi\times 10^{-7}\,\frac{\text{T m}}{\text{A}}\right)\times 66\times(0{\cdot}1\text{ A})}{(22\times 10^{-2}\text{ m})\,(\tan 45°)}$$

$$= 3{\cdot}8\times 10^{-5}\text{ T}.$$

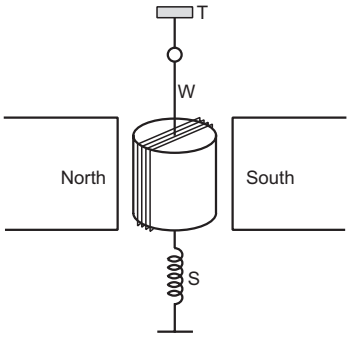



Figure 36.20

## 36.9 MOVING-COIL GALVANOMETER

The main parts of a moving-coil galvanometer are shown in figure (36.20). A rectangular coil of several turns is wound over a soft-iron core. The wire of the coil is coated with an insulating material so that each turn is insulated from the other and from the iron core. The coil is suspended between the two pole pieces of a strong permanent magnet. A fine strip $W$ of phosphor bronze is used to suspend the coil. The upper end of this strip is attached to a torsion head $T$. The lower end of the coil is attached to a spring $S$ also made of phosphor bronze. A small mirror is fixed on the suspension strip and is used to measure the deflection of the coil with the help of a lamp–scale arrangement. Terminals are connected to the suspension strip $W$ and the spring $S$. These terminals are used to pass current through the galvanometer.

The current to be measured is passed through the galvanometer. As the coil is in the magnetic field $\vec{B}$ of the permanent magnet, a torque $\vec{\Gamma} = ni\vec{A}\times\vec{B}$ acts on

the coil. Here $n$ = number of turns, $i$ = current in the coil, $\vec{A}$ = area-vector of the coil and $\vec{B}$ = magnetic field at the site of the coil. This torque deflects the coil from

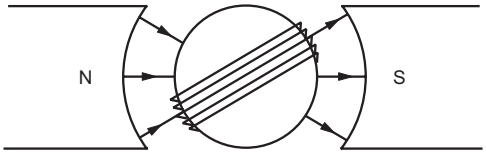



Figure 36.21

its equilibrium position.

The pole pieces are made cylindrical. As a result, the magnetic field at the arms of the coil remains parallel to the plane of the coil everywhere even as the coil rotates. The deflecting torque is then $\Gamma = niAB$. As the upper end of the suspension strip $W$ is fixed, the strip gets twisted when the coil rotates. This produces a restoring torque acting on the coil. If the deflection of the coil is θ and the torsional constant of the suspension strip is $k$, the restoring torque is $k\theta$. The coil will stay at a deflection θ where

$$niAB = k\theta$$

or, $$i = \frac{k}{nAB}\theta. \quad \ldots (36.18)$$

Hence, the current is proportional to the deflection. The constant $\frac{k}{nAB}$ is called the *galvanometer constant* and may be found by passing a known current, measuring the deflection θ and putting these values in equation (36.18).

**Sensitivity**

The sensitivity of a moving-coil galvanometer is defined as $\theta/i$. From equation (36.18), the sensitivity is $\frac{nAB}{k}$. For large sensitivity, the field $B$ should be large. The presence of soft-iron core increases the magnetic field. We shall discuss in a later chapter how soft iron increases magnetic field.

## 36.10 SHUNT

A galvanometer is usually a delicate and sensitive instrument and only a small current is sufficient to deflect the coil to its maximum allowed value. If a current larger than this permissible value is passed through the galvanometer, it may get damaged. A small resistance $R_s$ called *shunt*, is connected in parallel with the galvanometer to save it from such accidents. The main current $i$ is divided in two parts,

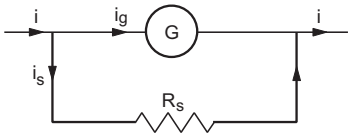



Figure 36.22

$i_g$ through the galvanometer and remaining $i_s = i - i_g$ through the shunt. If the resistance of the galvanometer coil is $R_g$, we have

$$i_g = \frac{R_s}{R_s + R_g}\, i. \quad \ldots (36.19)$$

As $R_s$ is much smaller than $R_g$, only a small fraction goes through the galvanometer.

***Example 36.8***

*A galvanometer having a coil of resistance* 20 Ω *needs* 20 mA *current for full-scale deflection. In order to pass a maximum current of* 2 A *through the galvanometer, what resistance should be added as a shunt* ?

***Solution*** : Out of the main current of 2 A, only 20 mA should go through the coil. The current through the coil is

$$i_g = \frac{R_s}{R_s + R_g}\, i$$

or, $$20 \text{ mA} = \frac{R_s}{R_s + (20\ \Omega)}\, 2 \text{ A}$$

or, $$\frac{20 \text{ mA}}{2 \text{ A}} = \frac{R_s}{R_s + (20\ \Omega)}$$

or, $$(10^{-2})\,(R_s + 20\ \Omega) = R_s$$

or, $$R_s = \frac{20}{99}\ \Omega \approx 0{\cdot}2\ \Omega.$$

## 36.11 TANGENT LAW OF PERPENDICULAR FIELDS

When a compass needle is placed in the earth's magnetic field, it stays along the horizontal component $B_H$ of the field. The magnetic forces $mB_H$ and $-mB_H$ on the poles do not produce any torque in this case. If an external horizontal magnetic field $B$ is produced which is perpendicular to $B_H$, the needle deflects from its position. The situation is the same as that shown in figure (36.19). The resultant of $B$ and $B_H$ is

$$B_r = \sqrt{B^2 + B_H^2}$$

making an angle θ with $B_H$ so that

$$\tan\theta = \frac{B}{B_H}. \quad \ldots (i)$$

The forces on the poles are $mB_r$, $-mB_r$ which may produce a torque and deflect the needle. The needle can stay in a position parallel to the resultant horizontal field $B_r$. Thus, the deflection of the needle in equilibrium is θ. Using (i), the external magnetic field $B$ may be written in terms of $B_H$ and θ as

$$B = B_H \tan\theta. \quad \ldots (36.20)$$

This is known as the *tangent law of perpendicular fields.*

## 36.12 DEFLECTION MAGNETOMETER

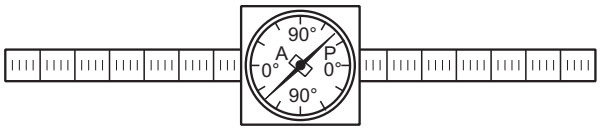



Figure 36.23

A deflection magnetometer (figure 36.23) consists of a small compass needle $A$ pivoted at the centre of a graduated circular scale. The graduations are marked from 0° to 90° in each quadrant. An aluminium pointer $P$ is rigidly fixed with the needle and perpendicular to it. The ends of the pointer move on the circular scale. These are enclosed in a cylindrical box known as the magnetometer box. The upper cover of the box is made of glass so that the things inside are visible. A plane mirror is fixed on the lower surface so that the pointer may be read without parallax. This arrangement is the same as that used in a tangent galvanometer.

The magnetometer box is kept in a wooden frame having two long arms. Metre scales are fitted on the two arms. The reading of a scale at any point directly gives the distance of that point from the centre of the compass needle.

The basic use of a deflection magnetometer is to determine $M/B_H$ for a permanent bar magnet. Here $M$ is the magnetic moment of the magnet and $B_H$ is the horizontal component of the earth's magnetic field. This quantity $M/B_H$ can be measured in two standard positions of the magnetometer. One is called Tan-$A$ position of Gauss and the other is called Tan-$B$ position of Gauss.

### Tan-*A* Position

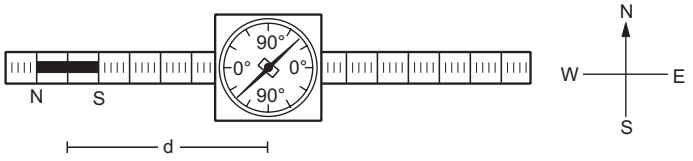



Figure 36.24

In this, the arms of the magnetometer are kept along the magnetic east–west direction. The aluminium pointer shows this direction when no extra magnets or magnetic materials are present nearby. The magnetometer box is rotated in its plane till the pointer reads 0°–0°. The magnet is now kept on one of the arms, parallel to its length (figure 36.24). The needle deflects and the deflection θ in its equilibrium position is read from the circular scale. The distance of the centre of the magnet from the centre of the compass is calculated from the linear scale on the arm.

Let $d$ be this distance and $2l$ be the magnetic length of the magnet. In Tan-$A$ position of the magnetometer, the compass needle is in end-on position of the bar magnet. The magnetic field due to the bar magnet at the site of the needle is, therefore,

$$B = \frac{\mu_0}{4\pi} \frac{2Md}{(d^2 - l^2)^2}.$$

This field is along the length of the magnet, that is, towards east or towards west. It is, therefore, perpendicular to the earth's field $B_H$. From the tangent law, we have,

$$B = B_H \tan\theta$$

or,
$$\frac{\mu_0}{4\pi} \frac{2Md}{(d^2 - l^2)^2} = B_H \tan\theta$$

or,
$$\frac{M}{B_H} = \frac{4\pi}{\mu_0} \frac{(d^2 - l^2)^2}{2d} \tan\theta. \quad \ldots \quad (36.21)$$

Knowing all the quantities on the right-hand side, one gets $M/B_H$.

#### Possible errors and their remedies

Errors may occur due to various reasons.

(a) The pivot of the needle may not be at the centre of the circular scale (figure 36.25). To remove this error both ends of the pointer are read and the mean is taken.

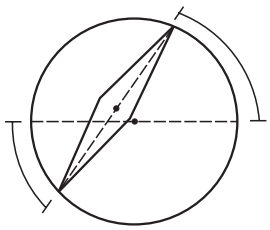


Figure 36.25

(b) The magnetic centre of the bar magnet may not coincide with its geometrical centre (figure 36.26). The measured distance $d$ is more or less than the actual distance $d_1$ of the centre of the needle from the magnetic centre of the magnet. To remove the error due to this, the magnet is rotated through 180° about the vertical so that the positions of the north pole and the south pole are interchanged. The deflections are again noted with both ends of the pointer. These readings correspond to the distance $d_2$. The average of the sets of deflections corresponding to the distances $d_1$ and $d_2$ give the correct value approximately.

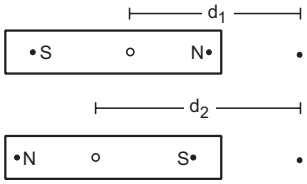



Figure 36.26

(c) The geometrical axis of the bar magnet may not coincide with the magnetic axis (figure 36.27). To

Figure 36.27

avoid error due to this, the magnet is put upside down at the same position and the readings are taken.

(d) The zero of the linear scale may not coincide with the centre of the circular scale. To remove the error due to this, the magnet is kept on the other arm of the magnetometer at the same distance from the needle and all the readings are repeated.

Thus, one gets sixteen values of θ for the same distance $d$. The mean of these sixteen values gives the correct value of θ.

**Tan-*B* Position**

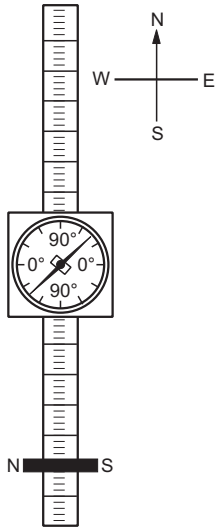



Figure 36.28

In this position, the arms of the magnetometer are kept in the magnetic north to south direction. The box is rotated so that the pointer reads 0°–0°. The bar magnet is placed on one of the arms symmetrically and at right angles to it (figure 36.28). The distance $d$ of the centre of the magnet from the centre of the compass needle is calculated from the linear scale. The deflection θ of the needle is noted from the circular scale. To remove the errors due to the reasons described above, the deflections are read in various situations mentioned below. Both ends of the pointer are read. The magnet is put upside down and readings are taken. The bar magnet is rotated through 180° to interchange the positions of north and south poles and again both ends of the pointer are read. The magnet is again put upside down and readings are taken for both ends of the pointer. The magnet is kept on the other arm at the same distance and the corresponding readings are taken. The mean of these sixteen values gives correct θ for this $d$.

In tan-$B$ position, the compass is in broadside-on position of the magnet. The magnetic field at the compass due to the magnet is, therefore,

$$B = \frac{\mu_0}{4\pi} \frac{M}{(d^2 + l^2)^{3/2}}.$$

The field is parallel to the axis of the magnet and hence it is towards east or towards west. The earth's magnetic field is from south to north. Using tangent law,

$$B = B_H \tan\theta$$

or, $$\frac{\mu_0}{4\pi} \frac{M}{(d^2 + l^2)^{3/2}} = B_H \tan\theta$$

or, $$\frac{M}{B_H} = \frac{4\pi}{\mu_0} (d^2 + l^2)^{3/2} \tan\theta. \quad \ldots (36.22)$$

**Applications of a Deflection Magnetometer**

A variety of quantities may be obtained from the basic measurement of $M/B_H$ using a deflection magnetometer. Here are some of the examples.

**(a) Comparison of the magnetic moments $M_1$ and $M_2$ of two magnets**

One can find $M_1/B_H$ and $M_2/B_H$ separately for the two magnets and then get the ratio $M_1/M_2$. There is another simple method known as *null method* to get $M_1/M_2$. The experiment can be done either in Tan-$A$ position or in Tan-$B$ position. The two magnets are placed on the two arms of the magnetometer. The distances of the magnets from the centre of the magnetometer are so adjusted that the deflection of the needle is zero. In this case, the magnetic field at the needle due to the first magnet is equal in magnitude to the field due to the other magnet. If the magnetometer is used in Tan-$A$ position,

$$\frac{\mu_0}{4\pi} \frac{2 M_1 d_1}{(d_1^2 - l_1^2)^2} = \frac{\mu_0}{4\pi} \frac{2 M_2 d_2}{(d_2^2 - l_2^2)^2}$$

or, $$\frac{M_1}{M_2} = \frac{d_2 (d_1^2 - l_1^2)^2}{d_1 (d_2^2 - l_2^2)^2}.$$

If the magnetometer is used in Tan-$B$ position,

$$\frac{\mu_0}{4\pi} \frac{M_1}{(d_1^2 + l_1^2)^{3/2}} = \frac{\mu_0}{4\pi} \frac{M_2}{(d_2^2 + l_2^2)^{3/2}}$$

or, $$\frac{M_1}{M_2} = \frac{(d_1^2 + l_1^2)^{3/2}}{(d_2^2 + l_2^2)^{3/2}}.$$

Null method is easier and better than finding $M_1/B_H$ and $M_2/B_H$ separately and then calculating $M_1/M_2$. This is because, here the deflection remains zero and the possible errors in the measurement of θ do not occur. Also, it is more sensitive, because even a small deflection from 0°–0° gives the indication that the adjustment is not perfect.

**(b) Verification of inverse square law for magnetic field due to a magnetic pole**

Equations (36.21) and (36.22) for $M/B_H$ are deduced from the basic equation (36.2) giving the magnetic field due to a magnetic pole. Equation (36.2) shows that the magnetic field due to a magnetic pole is inversely proportional to the square of the distance. Thus, if we verify equation (36.21) or (36.22), inverse square law is verified.

For Tan-*A* position, from equation (36.21),

$$\cot\theta = \frac{4\pi}{\mu_0}\frac{B_H}{M}\frac{(d^2 - l^2)^2}{2d}. \quad \ldots \text{(i)}$$

A magnet is placed at a distance $d$ in Tan-*A* position of the magnetometer and the corresponding value of deflection $\theta$ is noted. The experiment is repeated for different values of $d$ and a graph between $\cot\theta$ and $\frac{(d^2 - l^2)^2}{2d}$ is plotted. The graph turns out to be a straight line passing through the origin (figure 36.29a). This is consistent with equation (i) above and hence the inverse square law is verified.

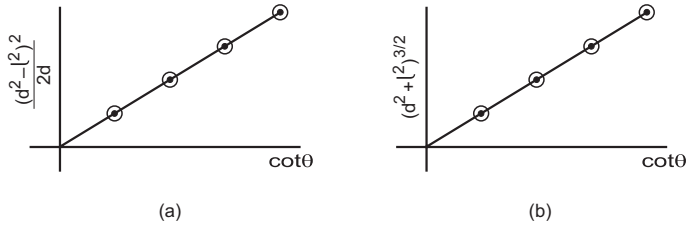



Figure 36.29

One can also do the experiment in Tan-*B* position. The deflection $\theta$ and the distance $d$ are related through equation (36.22). We have,

$$\cot\theta = \frac{4\pi}{\mu_0}\frac{B_H}{M}(d^2 + l^2)^{3/2}. \quad \ldots \text{(ii)}$$

The values of deflection $\theta$ are noted for different values of the distance $d$. A graph is drawn between $\cot\theta$ and $(d^2 + l^2)^{3/2}$. The graph turns out to be a straight line passing through the origin (figure 36.29b). This verifies the inverse square law.

**(c) Comparison of the horizontal components of the earth's magnetic field at two places**

Suppose the horizontal component of the earth's magnetic field is $B_{H1}$ at the first place and $B_{H2}$ at the second. A deflection magnetometer is taken and a bar magnet is kept at a distance $d$ in Tan-*A* position. The deflection $\theta_1$ of the needle is noted. The magnetometer is now taken to the second place and the same magnet is kept at the same distance $d$ in Tan-*A* position. The deflection $\theta_2$ is noted. We have from equation (36.21),

$$\frac{M}{B_{H1}} = \frac{4\pi}{\mu_0}\frac{(d^2 - l^2)^2}{2d}\tan\theta_1$$

and $$\frac{M}{B_{H2}} = \frac{4\pi}{\mu_0}\frac{(d^2 - l^2)^2}{2d}\tan\theta_2.$$

Thus, $$\frac{B_{H1}}{B_{H2}} = \frac{\tan\theta_2}{\tan\theta_1}.$$

The experiment can also be done in Tan-*B* position. Using equation (36.22), we again get the same relation for $\frac{B_{H1}}{B_{H2}}$.

## 36.13 OSCILLATION MAGNETOMETER

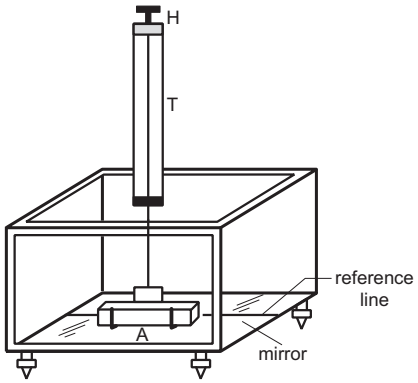



Figure 36.30

The design of an oscillation magnetometer is shown in figure (36.30). It consists of a rectangular wooden box having glass walls at the sides and at the top. Three of the walls are fixed and the fourth may be slid in and out. This is used to close or open the box. A plane mirror is fixed on the inner surface of the bottom of the box. A line parallel to the length of the box is drawn on the mirror in the middle. This is known as the *reference line*. There are levelling screws at the bottom on which the box rests. A vertical cylindrical glass tube *T*, having a torsion head *H* at its top, is fitted to the top plate of the box. A hanger *A* is suspended in the box through an unspun silk thread. The upper end of the thread is attached to the torsion head.

**Measuring $MB_H$ for a Bar Magnet**

The basic use of an oscillation magnetometer is to measure $MB_H$ for a bar magnet. To start with, the instrument is made horizontal with the help of a spirit level and the levelling screws. A compass needle is placed on the reference line and the box is rotated to make the reference line parallel to the needle. The reference line is now in the magnetic meridian. The needle is removed. A nonmagnetic heavy piece (generally made of brass) is put on the hanger. Due to the twist in the thread, the bar rotates and finally stays in equilibrium when the twist is removed. With the help of the torsion head, the bar is made parallel to the reference line in equilibrium.

The magnet is now gently placed in the hanger after removing the heavy piece. The north pole should be towards the north. The magnet is set into angular oscillation about the thread by giving it a slight angular deflection. This may be done by bringing another magnet close to the box and then removing it. The time period of oscillation is obtained by measuring the time required for, say 20 oscillations. To measure the time period accurately, one should look at the magnet through the top glass cover of the box. The eye should be positioned in such a way that the image of the magnet in the bottom mirror should be just below the magnet. Oscillations are counted as the magnet crosses the reference line. The box is kept closed at the time of oscillations, so that air currents do not disturb the oscillations.

**Expression for time period**

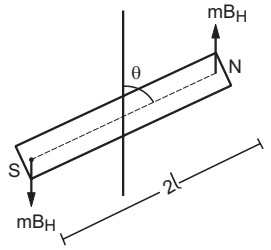



Figure 36.31

Figure (36.31) shows the position of the magnet when it is deflected through an angle θ from the mean position during its oscillation. The magnetic field in the horizontal direction is $B_H$ from south to north. If $m$ be the pole strength, the force on the north pole is $mB_H$ towards north and on the south pole it is $mB_H$ towards south. The length of the magnet is $2l$.

The torque of each of the two forces about the vertical axis is $mB_H l \sin\theta$ and it tries to bring the magnet towards the equilibrium position. The net torque about the vertical axis is

$$\Gamma = -2\, mB_H\, l \sin\theta$$
$$= -MB_H \sin\theta. \quad \ldots \text{(i)}$$

We neglect the torque due to the small twist produced in the thread as the magnet rotates. If the angular amplitude of oscillations is small, $\sin\theta \approx \theta$ and equation (i) becomes

$$\Gamma = -\, MB_H \theta.$$

Also, $\Gamma = I\alpha$

where $I$ is the moment of inertia of the magnet about the vertical axis and α is the angular acceleration. Thus,

$$\alpha = \frac{\Gamma}{I} = -\frac{MB_H}{I}\theta$$

or, $\alpha = -\omega^2\theta$

where $\omega = \sqrt{\frac{MB_H}{I}}$. This is an equation of angular simple harmonic motion. The time period is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{MB_H}}. \quad \ldots \text{(36.23)}$$

**Calculation of $MB_H$**

From equation (36.23), $MB_H = \frac{4\pi^2 I}{T^2}$. … (36.24)

For a magnet of rectangular cross section, the moment of inertia about the axis of rotation is

$$I = \frac{W(a^2 + b^2)}{12}$$

where $a$ is the geometrical length, $b$ is the breadth and $W$ is the mass of the magnet. Measuring these quantities and the time period, $MB_H$ can be obtained from equation (36.24).

***Example 36.9***

*A compass needle oscillates* 20 times per minute *at a place where the dip is* 45° *and* 30 times per minute *where the dip is* 30°. *Compare the total magnetic field due to the earth at the two places.*

***Solution*** : The time period of oscillation is given by

$$T = 2\pi\sqrt{\frac{I}{MB_H}}.$$

The time period at the first place is $T_1$ = 1/20 minute = 3·0 s and at the second place it is $T_2$ = 1/30 minute = 2·0 s.

If the total magnetic field at the first place is $B_1$, the horizontal component of the field is

$$B_{H1} = B_1 \cos 45° = B_1/\sqrt{2}.$$

Similarly, if the total magnetic field at the second place is $B_2$, the horizontal component is

$$B_{H2} = B_2 \cos 30° = B_2\sqrt{3}/2.$$

We have,

$$T_1 = 2\pi\sqrt{\frac{I}{MB_{H1}}} \text{ and } T_2 = 2\pi\sqrt{\frac{I}{MB_{H2}}}.$$

Thus,

$$\frac{T_1}{T_2} = \sqrt{\frac{B_{H2}}{B_{H1}}} \quad \text{or,} \quad \frac{B_{H2}}{B_{H1}} = \frac{T_1^2}{T_2^2}$$

or, $$\frac{B_2\sqrt{3}/2}{B_1/\sqrt{2}} = \frac{T_1^2}{T_2^2}$$

or, $$\frac{B_2}{B_1} = \sqrt{\frac{2}{3}}\frac{T_1^2}{T_2^2} = \sqrt{\frac{2}{3}} \times \frac{9}{4} = 1{\cdot}83.$$

Once we know how to measure $MB_H$, we can easily compare magnetic moments $M_1$ and $M_2$ by measuring

$M_1B_H$ and $M_2B_H$. Similarly, we can compare the horizontal components of the earth's magnetic field at two places.

### 36.14 DETERMINATION OF $M$ AND $B_H$

The magnetic moment of a bar magnet can be obtained by measuring $M/B_H$ using a deflection magnetometer and $MB_H$ using an oscillation magnetometer. If

$$M/B_H = X \text{ and } MB_H = Y,$$

we have

$$M = \sqrt{\left(\frac{M}{B_H}\right)(MB_H)} = \sqrt{XY}.$$

One can also determine the horizontal component $B_H$ of earth's magnetic field at any place. Using any magnet one can find $M/B_H$ and $MB_H$ as above.

We have,

$$B_H = \sqrt{\frac{MB_H}{M/B_H}} = \sqrt{\frac{Y}{X}}.$$

If $M/B_H$ is measured in Tan-$A$ position,

$$X = \frac{M}{B_H} = \frac{4\pi}{\mu_0}\frac{(d^2 - l^2)^2}{2d}\tan\theta$$

and $$Y = MB_H = \frac{4\pi^2 I}{T^2}.$$

Then $$M = \sqrt{XY} = \frac{2\pi(d^2 - l^2)}{T}\sqrt{\frac{4\pi}{\mu_0}\frac{I\tan\theta}{2d}}$$

and $$B_H = \sqrt{\frac{Y}{X}} = \frac{2\pi}{T(d^2 - l^2)}\sqrt{\frac{\mu_0}{4\pi}\frac{2Id}{\tan\theta}}.$$

We can also measure $M/B_H$ and $MB_H$ in Tan-$B$ position and get $M$ and $B_H$ as above.

### 36.15 GAUSS'S LAW FOR MAGNETISM

From Coulomb's law

$$E = \frac{q}{4\pi\varepsilon_0 r^2},$$

we can derive the Gauss's law for electric field, i.e.,

$$\oint \vec{E}\cdot d\vec{S} = \frac{q_{\text{inside}}}{\varepsilon_0}$$

where $\oint \vec{E}\cdot d\vec{S}$ is the electric flux and $q_{\text{inside}}$ is the "net charge" enclosed by the closed surface.

Similarly, from the equation

$$B = \frac{\mu_0}{4\pi}\frac{m}{r^2}$$

we can derive Gauss's law for magnetism as

$$\oint \vec{B}\cdot d\vec{S} = \mu_0 m_{\text{inside}},$$

where $\oint \vec{B}\cdot d\vec{S}$ is the magnetic flux and $m_{\text{inside}}$ is the "net pole strength" inside the closed surface. However, there is an additional feature for the magnetic case. We do not have an isolated magnetic pole in nature. At least none has been found to exist till date. The smallest unit of the source of magnetic field is a magnetic dipole where the "net magnetic pole" is zero. Hence, the net magnetic pole enclosed by any closed surface is always zero. Correspondingly, the flux of the magnetic field through any closed surface is zero. *Gauss's law for magnetism,* therefore, states that

$$\oint \vec{B}\cdot d\vec{S} = 0. \quad \ldots (36.25)$$

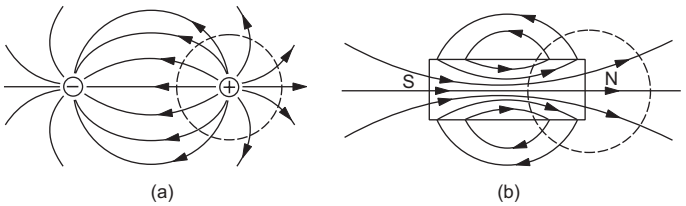



Figure 36.32

Figure (36.32a) shows the electric field lines through a closed surface which encloses one charge of an electric dipole. Figure (36.32b) shows the magnetic field lines through a closed surface which encloses one end of a bar magnet. One can see that the electric field lines only go out of the surface, giving a nonzero flux. On the other hand, the number of magnetic field lines going out of the surface is equal to the number going into it. The magnetic flux is positive at some places on the surface and is negative at others giving the total flux equal to zero.

### *Worked Out Examples*

**1.** *A bar magnet has a pole strength of* 3·6 A m *and magnetic length* 8 cm. *Find the magnetic field at (a) a point on the axis at a distance of* 6 cm *from the centre towards the north pole and (b) a point on the perpendicular bisector at the same distance.*

***Solution*** :

(a) The point in question is in end-on position, so the magnetic field is,

$$B = \frac{\mu_0}{4\pi}\frac{2Md}{(d^2 - l^2)^2}$$

$$= 10^{-7}\,\frac{\text{T m}}{\text{A}} \times \frac{2 \times 3{\cdot}6\ \text{A m} \times 0{\cdot}08\ \text{m} \times 0{\cdot}06\ \text{m}}{[(0{\cdot}06\ \text{m})^2 - (0{\cdot}04\ \text{m})^2]^2}$$

$$= 8{\cdot}6 \times 10^{-4}\ \text{T}.$$

The field will be away from the magnet.

(b) In this case the point is in broadside-on position so that the field is

$$B = \frac{\mu_0}{4\pi}\,\frac{M}{(d^2 + l^2)^{3/2}}$$

$$= 10^{-7}\,\frac{\text{T m}}{\text{A}} \times \frac{3{\cdot}6\ \text{A m} \times 0{\cdot}08\ \text{m}}{[(0{\cdot}06\ \text{m})^2 + (0{\cdot}04\ \text{m})^2]^{3/2}}$$

$$= 7{\cdot}7 \times 10^{-5}\ \text{T}.$$

The field will be parallel to the magnet.

**2.** *A magnet is suspended by a vertical string attached to its middle point. Find the position in which the magnet can stay in equilibrium. The horizontal component of the earth's magnetic field* = 25 μT *and its vertical component* = 40 μT. *Assume that the string makes contact with the magnet only at a single point.*

***Solution*** **:** The magnetic field of earth is in the north–south plane (magnetic meridian) making an angle θ with the horizontal such that

$$\tan\theta = \frac{B_V}{B_H} = \frac{40}{25}$$

or, $$\theta = 58°.$$

As the tension and the force of gravity act through the centre, their torque about the centre is zero. To make the net torque acting on the magnet zero, it must stay in the direction of the resultant magnetic field. Hence, it stays in the magnetic meridian making an angle of 58° with the horizontal.

**3.** *A magnetic needle having magnetic moment* 10 A m$^2$ *and length* 2·0 cm *is clamped at its centre in such a way that it can rotate in the vertical east–west plane. A horizontal force towards east is applied at the north pole to keep the needle fixed at an angle of* 30° *with the vertical. Find the magnitude of the applied force. The vertical component of the earth's magnetic field is* 40 μT.

***Solution*** **:**

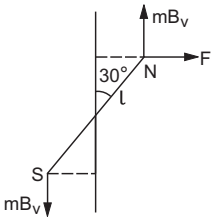



Figure 36-W1

The situation is shown in figure (36-W1). As the needle is in equilibrium, the torque of all the forces about the centre should be zero. As the needle can rotate in the vertical east–west plane, the horizontal component of earth's magnetic field is ineffective. This gives,

$$mB_V\, l \sin 30° + mB_V\, l \sin 30° = Fl \cos 30°$$

or, $$F = 2\, mB_V \tan 30°$$

$$= 2\,\frac{M}{2l}\,B_V \tan 30°$$

$$= \frac{(10\ \text{A–m}^2)\,(40 \times 10^{-6}\ \text{T})}{(1{\cdot}0 \times 10^{-2}\ \text{m})\,\sqrt{3}}$$

$$= 2{\cdot}3 \times 10^{-2}\ \text{N}.$$

**4.** *The magnetic scalar potential due to a magnetic dipole at a point on its axis situated at a distance of* 20 cm *from its centre is found to be* 1·2 × 10$^{-5}$ T m. *Find the magnetic moment of the dipole.*

***Solution*** **:** The magnetic potential due to a dipole is

$$V = \frac{\mu_0}{4\pi}\,\frac{M \cos\theta}{r^2}$$

or, $$1{\cdot}2 \times 10^{-5}\ \text{T m} = \left(10^{-7}\,\frac{\text{T m}}{\text{A}}\right)\frac{M}{(0{\cdot}2\ \text{m})^2}$$

or, $$M = 4{\cdot}8\ \text{A m}^2.$$

**5.** *A bar magnet of magnetic moment* 2·0 A m$^2$ *is free to rotate about a vertical axis through its centre. The magnet is released from rest from the east–west position. Find the kinetic energy of the magnet as it takes the north–south position. The horizontal component of the earth's magnetic field is* $B$ = 25 μT.

***Solution*** **:** The magnetic potential energy of the dipole in a uniform magnetic field is given by $U = -MB\cos\theta$. As the earth's magnetic field is from south to north, the initial value of θ is π/2 and final value of θ is 0. Thus, the decrease in magnetic potential energy during the rotation is

$$U_i - U_f = -MB\cos(\pi/2) + MB\cos 0$$

$$= 2{\cdot}0\ \text{A m}^2 \times 25\ \mu\text{T} = 50\ \mu\text{J}.$$

Thus, the kinetic energy in the north–south position is 50 μJ.

**6.** *Figure (36-W2a) shows two identical magnetic dipoles a and b of magnetic moments M each, placed at a separation d, with their axes perpendicular to each other. Find the magnetic field at the point P midway between the dipoles.*

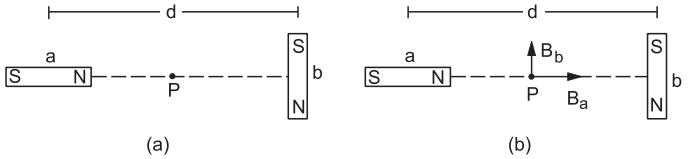



Figure 36-W2

***Solution*** **:** The point $P$ is in end-on position for the dipole $a$ and in broadside-on position for the dipole $b$. The magnetic field at $P$ due to $a$ is $B_a = \dfrac{\mu_0}{4\pi}\,\dfrac{2M}{(d/2)^3}$ along the

axis of $a$, and that due to $b$ is $B_b = \frac{\mu_0 \quad M}{4\pi\,(d/2)^3}$ parallel to the axis of $b$ as shown in figure (36-W2b). The resultant field at $P$ is, therefore,

$$B = \sqrt{B_a^2 + B_b^2}$$
$$= \frac{\mu_0 M}{4\pi (d/2)^3}\sqrt{1^2 + 2^2}$$
$$= \frac{2\sqrt{5}\,\mu_0 M}{\pi d^3}.$$

The direction of this field makes an angle $\alpha$ with $B_a$ such that $\tan\alpha = B_b/B_a = 1/2$.

**7.** *A bar magnet of length* 8 cm *and having a pole strength of* 1·0 A m *is placed vertically on a horizontal table with its south pole on the table. A neutral point is found on the table at a distance of* 6·0 cm *north of the magnet. Calculate the earth's horizontal magnetic field.*

***Solution :***

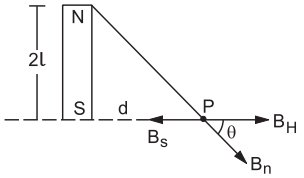



Figure 36-W3

The situation is shown in figure (36-W3). The magnetic field at $P$ due to the south pole is

$$B_s = \frac{\mu_0}{4\pi}\frac{m}{d^2}$$

towards south and that due to the north pole is

$$B_n = \frac{\mu_0}{4\pi}\frac{m}{(d^2 + 4l^2)}$$

along $NP$. The horizontal component of this field will be towards north and will have the magnitude

$$= \frac{\mu_0}{4\pi}\frac{m}{(d^2 + 4l^2)} \cdot \frac{d}{(d^2 + 4l^2)^{1/2}}.$$

The resultant horizontal field due to the magnet is, therefore,

$$\frac{\mu_0}{4\pi}\frac{m}{d^2} - \frac{\mu_0}{4\pi}\frac{md}{(d^2 + 4l^2)^{3/2}}$$

towards south. As $P$ is a neutral point, this field should be equal in magnitude to the earth's magnetic field $B_H$ which is towards north. Thus,

$$B_H = \frac{\mu_0 m}{4\pi}\left[\frac{1}{d^2} - \frac{d}{(d^2 + 4l^2)^{3/2}}\right]$$
$$= \frac{\mu_0 m}{4\pi}\left[\frac{1}{36\text{ cm}^2} - \frac{6\text{ cm}}{(36\text{ cm}^2 + 64\text{ cm}^2)^{3/2}}\right]$$
$$= 10^{-7}\text{ T m A}^{-1} \times (1{\cdot}0\text{ A m}) \times \left[\frac{1}{36\text{ cm}^2} - \frac{6}{1000\text{ cm}^2}\right]$$
$$= (10^{-7}\text{ T m}^2)\,[0{\cdot}028 - 0{\cdot}006] \times 10^4\text{ m}^{-2}$$
$$= 22 \times 10^{-6}\text{ T} = 22\ \mu\text{T}.$$

**8.** *The magnetic field at a point on the magnetic equator is found to be* $3{\cdot}1 \times 10^{-5}$ T. *Taking the earth's radius to be* 6400 km, *calculate the magnetic moment of the assumed dipole at the earth's centre.*

***Solution :***

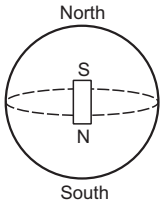



Figure 36-W4

A point on the magnetic equator is in broadside-on position of the earth's assumed dipole (figure 36-W4). The field is, therefore,

$$B = \frac{\mu_0}{4\pi}\frac{M}{R^3}$$

or, $$M = \frac{4\pi}{\mu_0}BR^3$$
$$= 10^7\text{ A m}^{-1}\text{ T}^{-1} \times 3{\cdot}1 \times 10^{-5}\text{ T} \times (6400 \times 10^3)^3\text{ m}^3$$
$$= 8{\cdot}1 \times 10^{22}\text{ A m}^2.$$

**9.** *The earth's magnetic field at geomagnetic poles has a magnitude* $6{\cdot}2 \times 10^{-5}$ T. *Find the magnitude and the direction of the field at a point on the earth's surface where the radius makes an angle of* 135° *with the axis of the earth's assumed magnetic dipole. What is the inclination (dip) at this point* ?

***Solution :***

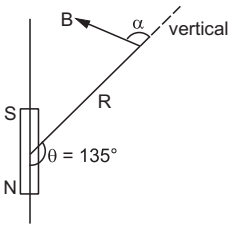



Figure 36-W5

Assuming the earth's field to be due to a dipole at the centre, geomagnetic poles are in end-on position (figure 36-W5).

The magnetic field $B$ at geomagnetic poles is

$$B_p = \frac{\mu_0}{4\pi}\frac{2M}{R^3}.$$

The magnetic field due to a dipole at a distance $R$ away from its centre has a magnitude

$$B = \frac{\mu_0}{4\pi}\frac{M}{R^3}(1 + 3\cos^2\theta)^{1/2} = \frac{1}{2}B_p(1 + 3\cos^2\theta)^{1/2}.$$

This field is in a direction making an angle $\alpha$ with the radial direction such that $\tan\alpha = (\tan\theta)/2$, as shown in

the figure. At the point given, θ = 135° and thus the field $B$ is

$$B = \frac{B_p}{2}(1 + 3\cos^2 135°)^{1/2}$$

$$= \frac{1}{2} \times 6{\cdot}2 \times 10^{-5}\ \text{T} \times 1{\cdot}58$$

$$= 4{\cdot}9 \times 10^{-5}\ \text{T}.$$

The angle α of this field with the vertical is given by

$$\tan\alpha = \frac{\tan\theta}{2} = \frac{\tan 135°}{2} = -0{\cdot}5$$

giving $\alpha = 153°$.

The inclination (dip) is the angle made by the earth's magnetic field with the horizontal plane. Here it is 153° – 90° = 63° below the horizontal.

**10.** *A magnetic needle free to rotate in a fixed vertical plane stays in a direction making an angle of* 60° *with the horizontal. If the dip at that place is* 37°*, find the angle of the fixed vertical plane with the meridian.*

***Solution*** **:** If the vertical plane makes an angle θ with the meridian, the horizontal component of the earth's field in that plane will be $B_H\cos\theta$. Thus the apparent dip $\delta_1$, i.e., the angle between the needle in equilibrium and the horizontal will be given by

$$\tan\delta_1 = \frac{B_V}{B_H\cos\theta} = \frac{\tan\delta}{\cos\theta}$$

or, $$\cos\theta = \frac{\tan\delta}{\tan\delta_1}$$

$$= \frac{\tan 37°}{\tan 60°} = \frac{3}{4\sqrt{3}} = \frac{\sqrt{3}}{4}$$

or, $\theta = 64°$.

**11.** *A dip circle shows an apparent dip of* 60° *at a place where the true dip is* 45°*. If the dip circle is rotated through* 90°*, what apparent dip will it show ?*

***Solution*** **:** If $\delta_1$ and $\delta_2$ be the apparent dips shown by the dip circle in the two perpendicular positions, the true dip δ is given by

$$\cot^2\delta = \cot^2\delta_1 + \cot^2\delta_2$$

or, $$\cot^2 45° = \cot^2 60° + \cot^2\delta_2$$

or, $$\cot^2\delta_2 = 2/3$$

or, $$\cot\delta_2 = 0{\cdot}816 \text{ giving } \delta_2 = 51°.$$

**12.** *A magnetic needle of length* 10 cm*, suspended at its middle point through a thread, stays at an angle of* 45° *with the horizontal. The horizontal component of the earth's magnetic field is* 18 μT*. (a) Find the vertical component of this field. (b) If the pole strength of the needle is* 1·6 A–m*, what vertical force should be applied to an end so as to keep it in horizontal position?*

***Solution*** **:**

(a)

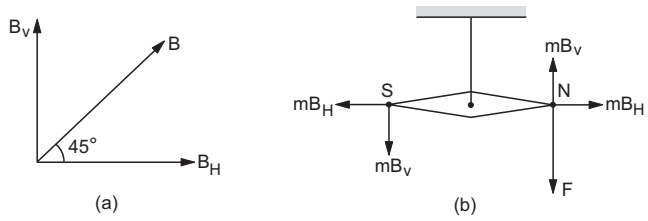



Figure 36-W6

Without the applied force, the needle will stay in the direction of the resultant magnetic field of the earth. Thus, the dip δ at the place is 45°. From figure (36-W6a),

$$\tan 45° = B_V / B_H$$

or, $$B_V = B_H = 18\ \mu\text{T}.$$

(b) When the force $F$ is applied (figure 36-W6b), the needle stays in horizontal position. Taking torque about the centre of the magnet,

$$2mB_V \times l = F \times l$$

or, $$F = 2mB_V$$

$$= 2 \times (1{\cdot}6\ \text{A m}) \times (18 \times 10^{-6}\ \text{T})$$

$$= 5{\cdot}8 \times 10^{-5}\ \text{N}.$$

**13.** *A tangent galvanometer has a coil of* 50 *turns and a radius of* 20 cm*. The horizontal component of the earth's magnetic field is* $B_H = 3 \times 10^{-5}$ T*. Find the current which gives a deflection of* 45°*.*

***Solution*** **:** We have

$$i = K\tan\theta = \frac{2rB_H}{\mu_0 n}\tan\theta$$

$$= \frac{2 \times (0{\cdot}20\ \text{m}) \times (3 \times 10^{-5}\ \text{T})}{4\pi \times 10^{-7}\ \text{T m A}^{-1} \times 50}\tan 45° = 0{\cdot}19\ \text{A}.$$

**14.** *A moving-coil galvanometer has* 100 *turns and each turn has an area* 2·0 $\text{cm}^2$*. The magnetic field produced by the magnet is* 0·01 T*. The deflection in the coil is* 0·05 radian *when a current of* 10 mA *is passed through it. Find the torsional constant of the suspension wire.*

***Solution*** **:** We have

$$i = \frac{K}{nAB}\theta$$

or, $$K = \frac{inAB}{\theta}$$

$$= \frac{(10 \times 10^{-3}\ \text{A}) \times 100 \times (2{\cdot}0 \times 10^{-4}\ \text{m}^2) \times 0{\cdot}01\ \text{T}}{0{\cdot}05\ \text{rad}}$$

$$= 4{\cdot}0 \times 10^{-5}\ \text{N m rad}^{-1}.$$

**15.** *A galvanometer coil has a resistance of* 100 Ω*. When a current passes through the galvanometer,* 1% *of the current goes through the coil and the rest through the shunt. Find the resistance of the shunt.*

***Solution*** :

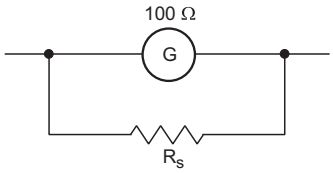



Figure 36-W7

The situation is shown in figure (36-W7). As the potential differences across the 100 Ω coil and across the shunt $R_s$ are the same,

$$0{\cdot}01\, i \times 100\ \Omega = 0{\cdot}99\, i \times R_s$$

or, $$R_s = \frac{0{\cdot}01 \times 100}{0{\cdot}99}\ \Omega = \frac{100}{99}\ \Omega\ .$$

**16.** *The needle of a deflection magnetometer deflects through* 45° *from north to south when the instrument is used in Tan-A position with a magnet of length* 10 cm *placed at a distance of* 25 cm. *(a) Find the magnetic moment of the magnet if the earth's horizontal magnetic field is* 20 μT. *(b) If the magnetometer is used in Tan-B position with the same magnet at the same separation from the needle, what will be the deflection* ?

***Solution*** : (a) In Tan-*A* position, the needle is in end-on position of the magnet so that the field at the needle due to the magnet is

$$B = \frac{\mu_0}{4\pi}\frac{2Md}{(d^2 - l^2)^2}\ .$$

Thus, $$\frac{\mu_0}{4\pi}\frac{2Md}{(d^2 - l^2)^2} = B_H \tan\theta$$

or, $$M = \frac{4\pi}{\mu_0}\frac{B_H \tan\theta (d^2 - l^2)^2}{2d}$$

$$= 10^7\ \text{A m}^{-1}\ \text{T}^{-1} \times \frac{20 \times 10^{-6}\ \text{T} \times 1 \times (625 - 25)^2 \times 10^{-8}\ \text{m}^4}{2 \times 25 \times 10^{-2}\ \text{m}}$$

$$= 1{\cdot}44\ \text{A m}^2.$$

(b) In Tan-*B* position, the needle is in broadside-on position of the manget, so that

$$B = \frac{\mu_0}{4\pi}\frac{M}{(d^2 + l^2)^{3/2}} = B_H \tan\theta$$

or, $$\tan\theta = \frac{10^{-7}\ \text{T m A}^{-1} \times 1{\cdot}44\ \text{A m}^2}{(625 + 25)^{3/2} \times 10^{-6}\ \text{m}^3 \times 20 \times 10^{-6}\ \text{T}}$$

$$= 0{\cdot}43$$

or, $$\theta = 23{\cdot}5°.$$

**17.** *Figure (36-W8) shows a short magnet executing small oscillations in an oscillation magnetometer in earth's magnetic field having horizontal component* 24 μT. *The time period of oscillation is* 0·10 s. *An upward electric current of* 18 A *is established in the vertical wire placed* 20 cm *east of the magnet by closing the switch S. Find the new time period.*

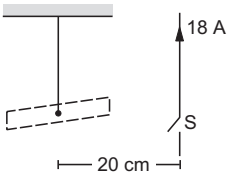



Figure 36-W8

***Solution*** : The magnetic field at the site of the short magnet due to the vertical current is

$$B = \frac{\mu_0\, i}{2\pi d} = (2 \times 10^{-7}\ \text{T m A}^{-1})\ \frac{18\ \text{A}}{0{\cdot}20\ \text{m}}$$

$$= 18\ \mu\text{T}.$$

As the wire is east of the magnet, this magnetic field will be from north to south according to the right-hand thumb rule. The earth's magnetic field has horizontal component 24 μT from south to north. Thus, the resultant field will be 6·0 μT from south to north. If $T_1$ and $T_2$ be the time periods without and with the current,

$$T_1 = \sqrt{\frac{I}{MB_H}}\ \text{ and }\ T_2 = \sqrt{\frac{I}{M(B_H - B)}}$$

or, $$\frac{T_2}{T_1} = \sqrt{\frac{B_H}{B_H - B}} = \sqrt{\frac{24\ \mu\text{T}}{6\ \mu\text{T}}} = 2$$

or, $$T_2 = 2T_1 = 0{\cdot}20\ \text{s}.$$

**18.** *The frequency of oscillation of the magnet in an oscillation magnetometer in the earth's magnetic field is* 40 oscillations per minute. *A short bar magnet is placed to the north of the magnetometer, at a separation of* 20 cm *from the oscillating magnet, with its north pole pointing towards north (figure 36-W9). The frequency of oscillation is found to increase to* 60 oscillations per minute. *Calculate the magnetic moment of this short bar magnet. Horizontal component of the earth's magnetic field is* 24 μT.

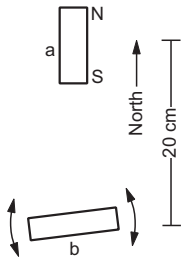



Figure 36-W9

***Solution*** : Let the magnetic field due to the short magnet have magnitude $B$ at the site of the oscillating magnet. From the figure, this magnetic field will be towards north and hence the resultant horizontal field will be $B_H + B$. Let $M$ and $M'$ denote the magnetic moments of the oscillating magnet and the other magnet respectively. If ν and ν′ be the frequencies without and with the other magnet, we have

$$\nu = \frac{1}{2\pi}\sqrt{\frac{MB_H}{I}}\ \text{ and}$$

$$\nu' = \frac{1}{2\pi}\sqrt{\frac{M(B_H + B)}{I}}$$

or, $$\frac{\nu'^{\,2}}{\nu^{\,2}} = \frac{B_H + B}{B_H}$$

or, $$\left(\frac{60}{40}\right)^2 = 1 + \frac{B}{B_H}$$

or, $$\frac{B}{B_H} = 1{\cdot}25$$

or, $$B = 1{\cdot}25 \times 24\ \mu\text{T} = 30 \times 10^{-6}\ \text{T}.$$

The oscillating magnet is in end-on position of the short magnet. Thus, the field $B$ can be written as

$$B = \frac{\mu_0}{4\pi}\frac{2M'}{d^3}$$

or, $$M' = \frac{2\pi}{\mu_0} Bd^3$$

$$= 0{\cdot}5 \times 10^7\ \text{A m}^{-1}\ \text{T}^{-1} \times (30 \times 10^{-6}\ \text{T}) \times (20 \times 10^{-2}\ \text{m})^3$$

$$= 1{\cdot}2\ \text{A m}^2.$$

**19.** *A bar magnet of mass* 100 g, *length* 7·0 cm, *width* 1·0 cm *and height* 0·50 cm *takes* π/2 seconds *to complete an oscillation in an oscillation magnetometer placed in a horizontal magnetic field of* 25 μT. *(a) Find the magnetic moment of the magnet. (b) If the magnet is put in the magnetometer with its* 0·50 cm *edge horizontal, what would be the time period* ?

***Solution*** : (a) The moment of inertia of the magnet about the axis of rotation is

$$I = \frac{m'}{12}(L^2 + b^2)$$

$$= \frac{100 \times 10^{-3}}{12}[(7 \times 10^{-2})^2 + (1 \times 10^{-2})^2]\ \text{kg m}^2$$

$$= \frac{25}{6} \times 10^{-5}\ \text{kg m}^2.$$

We have,

$$T = 2\pi\sqrt{\frac{I}{MB}} \qquad \ldots \text{(i)}$$

or, $$M = \frac{4\pi^2 I}{BT^2} = \frac{4\pi^2 \times 25 \times 10^{-5}\ \text{kg m}^2}{6 \times (25 \times 10^{-6}\ \text{T}) \times \frac{\pi^2}{4}\ \text{s}^2}$$

$$\approx 27\ \text{A m}^2.$$

(b) In this case the moment of inertia becomes

$$I' = \frac{m'}{12}(L^2 + b'^2) \quad \text{where} \quad b' = 0{\cdot}5\ \text{cm}.$$

The time period would be

$$T' = \sqrt{\frac{I'}{MB}}. \qquad \ldots \text{(ii)}$$

Dividing by equation (i),

$$\frac{T'}{T} = \sqrt{\frac{I'}{I}}$$

$$= \frac{\sqrt{\frac{m'}{12}(L^2 + b'^2)}}{\sqrt{\frac{m'}{12}(L^2 + b^2)}} = \frac{\sqrt{(7\ \text{cm})^2 + (0{\cdot}5\ \text{cm})^2}}{\sqrt{(7\ \text{cm})^2 + (1{\cdot}0\ \text{cm})^2}}$$

$$= 0{\cdot}992$$

or, $$T' = \frac{0{\cdot}992 \times \pi}{2}\ \text{s} = 0{\cdot}496\pi\ \text{s}.$$

□

## QUESTIONS FOR SHORT ANSWER

**1.** Can we have a single north pole, or a single south pole?

**2.** Do two distinct poles actually exist at two nearby points in a magnetic dipole ?

**3.** An iron needle is attracted to the ends of a bar magnet but not to the middle region of the magnet. Is the material making up the ends of a bar magnet different from that of the middle region ?

**4.** Compare the direction of the magnetic field inside a solenoid with that of the field there if the solenoid is replaced by its equivalent combination of north pole and south pole.

**5.** Sketch the magnetic field lines for a current-carrying circular loop near its centre. Replace the loop by an equivalent magnetic dipole and sketch the magnetic field lines near the centre of the dipole. Identify the difference.

**6.** The force on a north pole, $\vec{F} = m\vec{B}$, is parallel to the field $\vec{B}$. Does it contradict our earlier knowledge that a magnetic field can exert forces only perpendicular to itself ?

**7.** Two bar magnets are placed close to each other with their opposite poles facing each other. In absence of other forces, the magnets are pulled towards each other and their kinetic energy increases. Does it contradict our earlier knowledge that magnetic forces cannot do any work and hence cannot increase kinetic energy of a system ?

**8.** Magnetic scalar potential is defined as

$$U(\vec{r}_2) - U(\vec{r}_1) = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{B}\cdot d\vec{l}.$$

Apply this equation to a closed curve enclosing a long

straight wire. The RHS of the above equation is then $-\mu_0 i$ by Ampere's law. We see that $U(\vec{r}_2) \neq U(\vec{r}_1)$ even when $\vec{r}_2 = \vec{r}_1$. Can we have a magnetic scalar potential in this case ?

**9.** Can the earth's magnetic field be vertical at a place ? What will happen to a freely suspended magnet at such a place ? What is the value of dip here ?

**10.** Can the dip at a place be (a) zero (b) 90° ?

**11.** The reduction factor $K$ of a tangent galvanometer is written on the instrument. The manual says that the current is obtained by multiplying this factor to $\tan\theta$. The procedure works well at Bhuwaneshwar. Will the procedure work if the instrument is taken to Nepal ? If there is some error, can it be corrected by correcting the manual or the instrument will have to be taken back to the factory ?

## OBJECTIVE I

**1.** A circular loop carrying a current is replaced by an equivalent magnetic dipole. A point on the axis of the loop is in
(a) end-on position (b) broadside-on position
(c) both (d) none.

**2.** A circular loop carrying a current is replaced by an equivalent magnetic dipole. A point on the loop is in
(a) end-on position (b) broadside-on position
(c) both (d) none.

**3.** When a current in a circular loop is equivalently replaced by a magnetic dipole,
(a) the pole strength $m$ of each pole is fixed
(b) the distance $d$ between the poles is fixed
(c) the product $md$ is fixed
(d) none of the above.

**4.** Let $r$ be the distance of a point on the axis of a bar magnet from its centre. The magnetic field at such a point is proportional to
(a) $\frac{1}{r}$ (b) $\frac{1}{r^2}$ (c) $\frac{1}{r^3}$ (d) none of these.

**5.** Let $r$ be the distance of a point on the axis of a magnetic dipole from its centre. The magnetic field at such a point is proportional to
(a) $\frac{1}{r}$ (b) $\frac{1}{r^2}$ (c) $\frac{1}{r^3}$ (d) none of these.

**6.** Two short magnets of equal dipole moments $M$ are fastened perpendicuarly at their centres (figure 36-Q1). The magnitude of the magnetic field at a distance $d$ from the centre on the bisector of the right angle is
(a) $\frac{\mu_0}{4\pi}\frac{M}{d^3}$ (b) $\frac{\mu_0}{4\pi}\frac{\sqrt{2}M}{d^3}$ (c) $\frac{\mu_0}{4\pi}\frac{2\sqrt{2}M}{d^3}$ (d) $\frac{\mu_0}{4\pi}\frac{2M}{d^3}$.

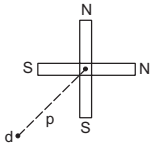



Figure 36-Q1

**7.** Magnetic meridian is
(a) a point (b) a line along north–south
(c) a horizontal plane (d) a vertical plane.

**8.** A compass needle which is allowed to move in a horizontal plane is taken to a geomagnetic pole. It
(a) will stay in north–south direction only
(b) will stay in east–west direction only
(c) will become rigid showing no movement
(d) will stay in any position.

**9.** A dip circle is taken to geomagnetic equator. The needle is allowed to move in a vertical plane perpendicular to the magnetic meridian. The needle will stay
(a) in horizontal direction only
(b) in vertical direction only
(c) in any direction except vertical and horizontal
(d) in any direction it is released.

**10.** Which of the following four graphs may best represent the current–deflection relation in a tangent galvanometer ?

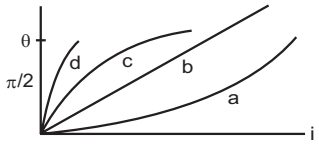



Figure 36-Q2

**11.** A tangent galvanometer is connected directly to an ideal battery. If the number of turns in the coil is doubled, the deflection will
(a) increase (b) decrease
(c) remain unchanged (d) either increase or decrease.

**12.** If the current is doubled, the deflection is also doubled in
(a) a tangent galvanometer
(b) a moving-coil galvanometer
(c) both (d) none.

**13.** A very long bar magnet is placed with its north pole coinciding with the centre of a circular loop carrying an electric current $i$. The magnetic field due to the magnet at a point on the periphery of the wire is $B$. The radius of the loop is $a$. The force on the wire is
(a) very nearly $2\pi aiB$ perpendicular to the plane of the wire
(b) $2\pi aiB$ in the plane of the wire
(c) $\pi aiB$ along the magnet (d) zero.

## OBJECTIVE II

**1.** Pick the correct options.
(a) Magnetic field is produced by electric charges only.
(b) Magnetic poles are only mathematical assumptions having no real existence.
(c) A north pole is equivalent to a clockwise current and a south pole is equivalent to an anticlockwise current.
(d) A bar magnet is equivalent to a long, straight current.

**2.** A horizontal circular loop carries a current that looks clockwise when viewed from above. It is replaced by an equivalent magnetic dipole consisting of a south pole $S$ and a north pole $N$.
(a) The line $SN$ should be along a diameter of the loop.
(b) The line $SN$ should be perpendicular to the plane of the loop.
(c) The south pole should be below the loop.
(d) The north pole should be below the loop.

**3.** Consider a magnetic dipole kept in the north to south direction. Let $P_1$, $P_2$, $Q_1$, $Q_2$ be four points at the same distance from the dipole towards north, south, east and west of the dipole respectively. The directions of the magnetic field due to the dipole are the same at
(a) $P_1$ and $P_2$ (b) $Q_1$ and $Q_2$
(c) $P_1$ and $Q_1$ (d) $P_2$ and $Q_2$.

**4.** Consider the situation of the previous problem. The directions of the magnetic field due to the dipole are opposite at
(a) $P_1$ and $P_2$ (b) $Q_1$ and $Q_2$
(c) $P_1$ and $Q_1$ (d) $P_2$ and $Q_2$.

**5.** To measure the magnetic moment of a bar magnet, one may use
(a) a tangent galvanometer
(b) a deflection galvanometer if the earth's horizontal field is known
(c) an oscillation magnetometer if the earth's horizontal field is known
(d) both deflection and oscillation magnetometer if the earth's horizontal field is not known.

## EXERCISES

**1.** A long bar magnet has a pole strength of 10 Am. Find the magnetic field at a point on the axis of the magnet at a distance of 5 cm from the north pole of the magnet.

**2.** Two long bar magnets are placed with their axes coinciding in such a way that the north pole of the first magnet is 2·0 cm from the south pole of the second. If both the magnets have a pole strength of 10 Am, find the force exerted by one magnet on the other.

**3.** A uniform magnetic field of $0{\cdot}20 \times 10^{-3}$ T exists in the space. Find the change in the magnetic scalar potential as one moves through 50 cm along the field.

**4.** Figure (36-E1) shows some of the equipotential surfaces of the magnetic scalar potential. Find the magnetic field $B$ at a point in the region.

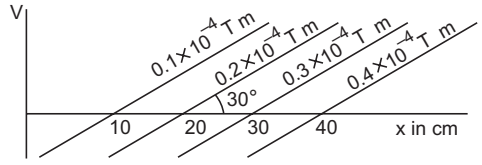



Figure 36-E1

**5.** The magnetic field at a point, 10 cm away from a magnetic dipole, is found to be $2{\cdot}0 \times 10^{-4}$ T. Find the magnetic moment of the dipole if the point is (a) in end-on position of the dipole and (b) in broadside-on position of the dipole.

**6.** Show that the magnetic field at a point due to a magnetic dipole is perpendicular to the magnetic axis if the line joining the point with the centre of the dipole makes an angle of $\tan^{-1}(\sqrt{2})$ with the magnetic axis.

**7.** A bar magnet has a length of 8 cm. The magnetic field at a point at a distance 3 cm from the centre in the broadside-on position is found to be $4 \times 10^{-6}$ T. Find the pole strength of the magnet.

**8.** A magnetic dipole of magnetic moment 1·44 A m$^2$ is placed horizontally with the north pole pointing towards north. Find the position of the neutral point if the horizontal component of the earth's magnetic field is 18 μT.

**9.** A magnetic dipole of magnetic moment 0·72 A m$^2$ is placed horizontally with the north pole pointing towards south. Find the position of the neutral point if the horizontal component of the earth's magnetic field is 18 μT.

**10.** A magnetic dipole of magnetic moment $0{\cdot}72\sqrt{2}$ A m$^2$ is placed horizontally with the north pole pointing towards east. Find the position of the neutral point if the horizontal component of the earth's magnetic field is 18 μT.

**11.** The magnetic moment of the assumed dipole at the earth's centre is $8{\cdot}0 \times 10^{22}$ A m$^2$. Calculate the magnetic field $B$ at the geomagnetic poles of the earth. Radius of the earth is 6400 km.

**12.** If the earth's magnetic field has a magnitude $3{\cdot}4 \times 10^{-5}$ T at the magnetic equator of the earth, what would be its value at the earth's geomagnetic poles ?

**13.** The magnetic field due to the earth has a horizontal component of 26 μT at a place where the dip is 60°. Find the vertical component and the magnitude of the field.

**14.** A magnetic needle is free to rotate in a vertical plane which makes an angle of 60° with the magnetic meridian. If the needle stays in a direction making an angle of $\tan^{-1}(2/\sqrt{3})$ with the horizontal, what would be the dip at that place ?

**15.** The needle of a dip circle shows an apparent dip of 45° in a particular position and 53° when the circle is rotated through 90°. Find the true dip.

**16.** A tangent galvanometer shows a deflection of 45° when 10 mA of current is passed through it. If the horizontal component of the earth's magnetic field is $B_H = 3{\cdot}6 \times 10^{-5}$ T and radius of the coil is 10 cm, find the number of turns in the coil.

**17.** A moving-coil galvanometer has a 50-turn coil of size 2 cm × 2 cm. It is suspended between the magnetic poles producing a magnetic field of 0·5 T. Find the torque on the coil due to the magnetic field when a current of 20 mA passes through it.

**18.** A short magnet produces a deflection of 37° in a deflection magnetometer in Tan-*A* position when placed at a separation of 10 cm from the needle. Find the ratio of the magnetic moment of the magnet to the earth's horizontal magnetic field.

**19.** The magnetometer of the previous problem is used with the same magnet in Tan-*B* position. Where should the magnet be placed to produce a 37° deflection of the needle ?

**20.** A deflection magnetometer is placed with its arms in north–south direction. How and where should a short magnet having $M/B_H = 40$ A m$^2$ T$^{-1}$ be placed so that the needle can stay in any position ?

**21.** A bar magnet takes π/10 second to complete one oscillation in an oscillation magnetometer. The moment of inertia of the magnet about the axis of rotation is $1{\cdot}2 \times 10^{-4}$ kg m$^2$ and the earth's horizontal magnetic field is 30 μT. Find the magnetic moment of the magnet.

**22.** The combination of two bar magnets makes 10 oscillations per second in an oscillation magnetometer when like poles are tied together and 2 oscillations per second when unlike poles are tied together. Find the ratio of the magnetic moments of the magnets. Neglect any induced magnetism.

**23.** A short magnet oscillates in an oscillation magnetometer with a time period of 0·10 s where the earth's horizontal magnetic field is 24 μT. A downward current of 18 A is established in a vertical wire placed 20 cm east of the magnet. Find the new time period.

**24.** A bar magnet makes 40 oscillations per minute in an oscillation magnetometer. An identical magnet is demagnetized completely and is placed over the magnet in the magnetometer. Find the time taken for 40 oscillations by this combination. Neglect any induced magnetism.

**25.** A short magnet makes 40 oscillations per minute when used in an oscillation magnetometer at a place where the earth's horizontal magnetic field is 25 μT. Another short magnet of magnetic moment 1·6 A m$^2$ is placed 20 cm east of the oscillating magnet. Find the new frequency of oscillation if the magnet has its north pole (a) towards north and (b) towards south.

□

## ANSWERS

### OBJECTIVE I

1. (a) 2. (b) 3. (c) 4. (d) 5. (c) 6. (c)
7. (d) 8. (d) 9. (d) 10. (c) 11. (c) 12. (b)
13. (a)

### OBJECTIVE II

1. (a), (b) 2. (b), (d) 3. (a), (b)
4. (c), (d) 5. (b), (c), (d)

### EXERCISES

1. $4 \times 10^{-4}$ T
2. $2{\cdot}5 \times 10^{-2}$ N
3. decreases by $0{\cdot}10 \times 10^{-3}$ T m
4. $2{\cdot}0 \times 10^{-4}$ T
5. (a) 1·0 A m$^2$ and (b) 2·0 A m$^2$
7. $6 \times 10^{-5}$ A m
8. at a distance of 20 cm in the plane bisecting the dipole
9. 20 cm south of the dipole
10. 20 cm from the dipole, $\tan^{-1}\sqrt{2}$ south of east
11. 60 μT
12. $6{\cdot}8 \times 10^{-5}$ T
13. 45 μT, 52 μT
14. 30°
15. 39°
16. 570
17. $2 \times 10^{-4}$ N m
18. $3{\cdot}75 \times 10^3 \frac{\text{A m}^2}{\text{T}}$
19. 7·9 cm from the centre
20. 2·0 cm from the needle, north pole pointing towards south
21. 1600 A m$^2$
22. 13 : 12
23. 0·076 s
24. $\sqrt{2}$ minutes
25. (a) 18 oscillations/min (b) 54 oscillations/min

□

CHAPTER 37

# MAGNETIC PROPERTIES OF MATTER

## 37.1 MAGNETIZATION OF MATERIALS: INTENSITY OF MAGNETIZATION

Matter is made of atoms and atoms are made of nuclei and electrons. The electrons in an atom move about the nucleus in closed paths and hence constitute electric current loops. As a current loop has a magnetic dipole moment, each electron in an atom has a magnetic moment due to its orbital motion. Besides this, each electron has a permanent angular momentum which is present even if it is at rest. This permanent angular momentum is called the *spin angular momentum* of the electron and can be understood only through quantum mechanics. Corresponding to its spin, each electron has a permanent magnetic moment. This magnetic moment has a fixed magnitude $\mu_s = 9{\cdot}285 \times 10^{-24}$ J T$^{-1}$. The magnetic moment due to the motion of the electron is over and above this. The nucleus may also have a magnetic moment but it is about several thousand times smaller than the magnetic moment of an electron. The resultant magnetic moment of an atom is the vector sum of all such magnetic moments.

The magnetic moments of the electrons of an atom have a tendency to cancel in pairs. Thus, the magnetic moments of the two electrons of a helium atom cancel each other. In a number of atoms and ions, the resultant magnetic moment is zero. But in some cases, the magnetic moment of an atom is not zero. Such an atom may be represented by a magnetic dipole having a permanent magnetic moment. We shall first discuss materials made of such atoms.

Any object of finite size contains a large number of atoms. In general, the magnetic moments of these atoms are randomly oriented and there is no net magnetic moment in any volume of the material that contains more than several thousand atoms (figure 37.1a). This volume is still quite small at macroscopic scale. However, when the material is kept in an external magnetic field, torques act on the atomic dipoles and these torques try to align them parallel to the field (figure 37.1b). The alignment is only partial, because, the thermal motion of the atoms frequently changes the orientation of the atoms and hence tries to randomize the magnetic moments. The degree of alignment increases if the strength of the applied field is increased and also if the temperature is decreased. With sufficiently strong fields, the alignment is near perfect. We then say that the material is magnetically *saturated.*

When the atomic dipoles are aligned, partially or fully, there is a net magnetic moment in the direction of the field in any small volume of the material. We define the *magnetization vector* $\vec{I}$ as the magnetic moment per unit volume. It is also called the *intensity of magnetization* or simply *magnetization.* Thus,

$$\vec{I} = \frac{\text{magnetic moment}}{\text{volume}} = \frac{\vec{M}}{V}. \qquad \ldots \quad (37.1)$$

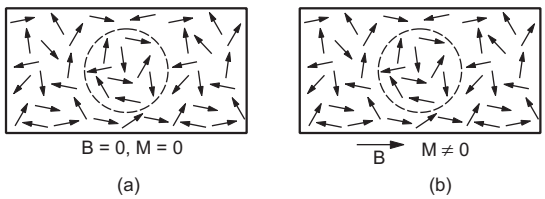



Figure 37.1

The unit of magnetic moment is ampere metre$^2$ so that from equation (37.1) the unit of $I$ is ampere metre$^{-1}$.

Consider a bar magnet of pole strength $m$, length $2l$ and area of cross-section $A$. The magnetic moment of the bar magnet is $M = 2ml$. The intensity of magnetization is

$$I = \frac{M}{V} = \frac{2ml}{A(2l)} = \frac{m}{A}.$$

Thus, for a bar magnet, the intensity of magnetization may be defined as the *pole strength per unit face area.*

***Example 37.1***

*A bar magnet made of steel has a magnetic moment of* $2{\cdot}5$ A m$^2$ *and a mass of* $6{\cdot}6 \times 10^{-3}$ kg. *If the density of*

*steel is* $7{\cdot}9 \times 10^{3}$ kg m$^{-3}$, *find the intensity of magnetization of the magnet.*

***Solution*** : The volume of the bar magnet is

$$V = \frac{\text{mass}}{\text{density}} = \frac{6{\cdot}6 \times 10^{-3}\ \text{kg}}{7{\cdot}9 \times 10^{3}\ \text{kg m}^{-3}}$$

$$= 8{\cdot}3 \times 10^{-7}\ \text{m}^{3}.$$

The intensity of magnetization is

$$I = \frac{M}{V} = \frac{2{\cdot}5\ \text{A m}^{2}}{8{\cdot}3 \times 10^{-7}\ \text{m}^{3}} \approx 3{\cdot}0 \times 10^{6}\ \text{A m}^{-1}.$$

## 37.2 PARAMAGNETISM, FERROMAGNETISM AND DIAMAGNETISM

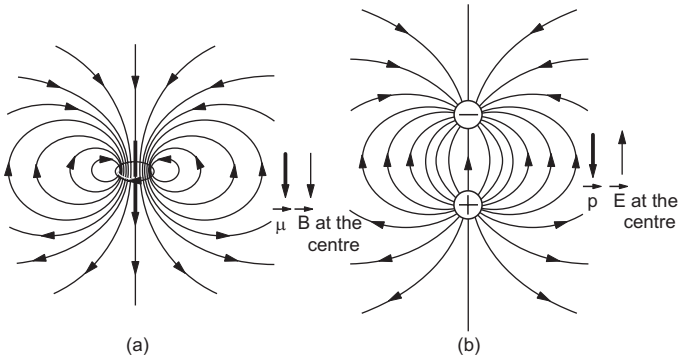



Figure 37.2

Figure (37.2a) shows a current loop and the magnetic field lines associated with it. The dipole moment $\vec{\mu} = i\vec{A}$ is also shown in the figure. The magnetic field at the centre of the dipole is in the direction of $\vec{\mu}$. This behaviour is opposite to that in the case of an electric dipole. The electric field at the centre of an electric dipole is opposite to the dipole moment (figure 37.2b).

Thus, when a magnetic field aligns the atomic dipoles in its direction, the magnetized material produces an extra magnetic field in the material in the direction of the applied field. The resultant magnetic field in the material is then greater than the applied field. The tendency to increase the magnetic field due to magnetization of material is called *paramagnetism* and materials which exhibit paramagnetism are called *paramagnetic materials.*

In some materials, the permanent atomic magnetic moments have strong tendency to align themselves even without any external field. These materials are called ferromagnetic materials and permanent magnets are made from them. The force between neighbouring atoms, responsible for their alignment, is called *exchange coupling* and it can only be explained by quantum mechanics. In normal unmagnetized state, the atoms form domains inside the material as suggested by figure (37.3). The atoms in any domain have magnetic moments in the same direction giving a net large magnetic moment to the domain. Different domains, however, have different directions of magnetic moment and hence, the material remains unmagnetized. Different domains have different sizes, the size may be as large as a millimetre in linear dimension. Remember, a volume of 1 mm$^{3}$ contains about $10^{20}$ atoms !

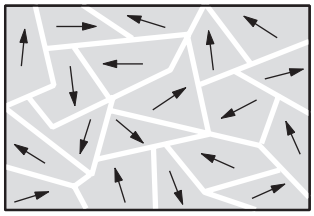


Figure 37.3

If a magnetic field is applied, the domains which are aligned along the direction of the field grow in size and those opposite to it get reduced. This happens because walls of the domains move across the sample. Also, domains may orient themselves in favour of the applied field. Figure (37.4) shows a qualitative description of the processes of domain-growing and domain-alignment.

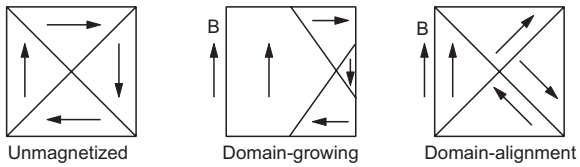



Figure 37.4

Because of the domain character of ferromagnetic materials, even if a small magnetic field is applied, it gives rise to large magnetization. The resultant field is much larger than the applied field in such a material.

Now suppose, the individual atoms of a material do not have a net magnetic dipole moment. When such a substance is placed in a magnetic field, dipole moments are induced in the atoms by the applied field. From Lenz's law, the magnetic field due to the induced magnetic moments opposes the original field. The resultant field in such materials is, therefore, smaller than the applied field. This phenomenon is called *diamagnetism* and such materials are called *diamagnetic* materials.

Magnetic moments are induced in all materials whenever a magnetic field is applied. Thus, all materials have the property of diamagnetism. However, if there is a permanent atomic magnetic moment, paramagnetism or ferromagnetism is much stronger than diamagnetism and the material does not show diamagnetic properties.

### 37.3 MAGNETIC INTENSITY *H*

When a magnetic field is applied to a material, the material gets magnetized. The actual magnetic field inside the material is the sum of the applied magnetic field and the magnetic field due to magnetization. It is convenient to define a new vector field

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{I} \qquad \ldots (37.2)$$

where $\vec{B}$ is the resultant magnetic field and $\vec{I}$ is the intensity of magnetization. This quantity $\vec{H}$ is called *magnetic intensity* or *magnetizing field intensity*. The unit of $H$ is the same as that of $I$, that is, ampere metre $^{-1}$. If no material is present (vacuum), $\vec{I} = 0$ and we have

$$\vec{H} = \frac{\vec{B}}{\mu_0}. \qquad \ldots (37.3)$$

Thus, the magnetic intensity due to a current element $i\,d\vec{l}$ is, from Biot–Savart law,

$$d\vec{H} = \frac{1}{4\pi} \frac{i\,d\vec{l} \times \vec{r}}{r^3}. \qquad \ldots (37.4)$$

The magnetic intensity due to a magnetic pole of pole strength $m$ at a distance $r$ from it is

$$H = \frac{m}{4\pi r^2}. \qquad \ldots (37.5)$$

Whenever the end effects of a magnetized material can be neglected, the magnetic intensity due to magnetization is zero. This may be the case with a ring-shaped material or in the middle portion of a long rod. The magnetic intensity in a material is then determined by the external sources only, even if the material is magnetized.

***Example 37.2***

*Find the magnetic intensity H at the centre of a long solenoid having n turns per unit length and carrying a current i (a) when no material is kept in it and (b) when a long copper rod is inserted in the solenoid.*

***Solution*** : (a) When there is no rod, the magnetic field at the centre of the solenoid is given by

$$B = \mu_0 ni.$$

The magnetic intensity is

$$H = \frac{B}{\mu_0} = ni.$$

(b) As the solenoid and the rod are long and we are interested in the magnetic intensity at the centre, the end effects may be neglected. There is no effect of the rod on the magnetic intensity at the centre. Its value in both cases are the same. Thus $H = ni$.

### 37.4 MAGNETIC SUSCEPTIBILITY

For paramagnetic and diamagnetic substances, the intensity of magnetization of a material is directly proportional to the magnetic intensity. Thus,

$$\vec{I} = \chi \vec{H}. \qquad \ldots (37.6)$$

The proportionality constant $\chi$ is called the *susceptibility* of the material. Table (37.1) gives susceptibility of some chosen materials. As $I$ and $H$ have the same dimensions, the susceptibility $\chi$ is a dimensionless constant. As there can be no magnetization in vacuum, $I = 0$ and hence $\chi = 0$. The materials with positive value of $\chi$ are paramagnetic and those with negative value of $\chi$ are diamagnetic.

**Table 37.1** : ***Susceptibilities of some materials***

| Material | Temperature in °C | $\chi$ in $10^{-5}$ |
|---|---|---|
| Vacuum | | Zero |
| Air | STP | 0·04 |
| Oxygen (gas) | STP | 0·18 |
| Magnesium | 20 | 1·2 |
| Aluminium | 20 | 2·1 |
| Tungsten | 20 | 6·8 |
| Titanium | 20 | 7·06 |
| Cerium | 18 | 130 |
| Ferric chloride | 20 | 306 |
| Oxygen (liquid) | – 219 | 490 |
| Carbon dioxide (1 atm) | 20 | $-2{\cdot}3 \times 10^{-4}$ |
| Nitrogen (1 atm) | 20 | $-5{\cdot}0 \times 10^{-4}$ |
| Hydrogen (1 atm) | 20 | $-9{\cdot}9 \times 10^{-4}$ |
| Sodium | 20 | – 0·24 |
| Copper | 18 | – 0·96 |
| Lead | 18 | – 1·6 |
| Carbon (diamond) | 20 | – 2·2 |
| Silver | 20 | – 2·6 |
| Mercury | 18 | – 2·8 |
| Gold | 20 | – 3·6 |
| Carbon (graphite) | 20 | – 9·9 |
| Bismuth | 20 | – 16·6 |

### 37.5 PERMEABILITY

The magnetic intensity is given by

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{I}$$

or,
$$\vec{B} = \mu_0(\vec{H} + \vec{I})$$
$$= \mu_0(\vec{H} + \chi\vec{H})$$
$$= \mu_0(1 + \chi)\vec{H}.$$

We can write, $\vec{B} = \mu\vec{H}$ ... (37.7)

where $\mu = \mu_0(1 + \chi)$ is a constant called the *permeability* of the material. The permeability of vacuum is $\mu_0$ as $\chi = 0$ for vacuum. The constant

$$\mu_r = \frac{\mu}{\mu_0} = 1 + \chi \qquad \ldots \quad (37.8)$$

is called the *relative permeability* of the material.

The significance of relative permeability may be understood by the following simple description. Consider a long solenoid (or a toroid) having $n$ turns per unit length and carrying a current $i$. The magnetic field $B_0$ inside the solenoid is

$$B_0 = \mu_0 ni$$

and the magnetic intensity is $H = ni$.

Now suppose, a material is inserted into the solenoid. The magnetic field now becomes

$$B = \mu H = \mu ni$$

or,
$$\frac{B}{B_0} = \frac{\mu}{\mu_0} = \mu_r.$$

Thus, $\mu_r$ is the factor by which the magnetic field $B$ is increased when a material is brought in the field.

***Example 37.3***

*Find the per cent increase in the magnetic field B when the space within a current-carrying toroid is filled with aluminium. The susceptibility of aluminium is* $2{\cdot}1 \times 10^{-5}$.

***Solution*** : In absence of aluminium, the magnetic field is

$$B_0 = \mu_0 H.$$

As the space inside the toroid is filled with aluminium, the field becomes

$$B = \mu H = \mu_0(1 + \chi)H.$$

The increase in the field is

$$B - B_0 = \mu_0 \chi H.$$

The per cent increase is

$$\frac{B - B_0}{B_0} \times 100 = \frac{\mu_0 \chi H}{\mu_0 H} \times 100$$
$$= \chi \times 100$$
$$= 2{\cdot}1 \times 10^{-3}.$$

## 37.6 CURIE'S LAW

As the temperature is increased, the randomization of individual atomic magnetic moments increases, decreasing the magnetization $I$ for a given magnetic intensity $H$. The resultant magnetic field $B$ decreases, which means $\chi$ decreases as $T$ increases. *Curie's law* states that far away from saturation, the susceptibility of a paramagnetic substance is inversely proportional to the absolute temperature:

$$\chi = \frac{c}{T} \qquad \ldots \quad (37.9)$$

where $c$ is a constant called the *Curie constant*. When a ferromagnetic material is heated, it becomes paramagnetic at a certain temperature. This temperature is called *Curie point* or *Curie temperature*. After this, the susceptibility varies with temperature as

$$\chi = \frac{c'}{T - T_c} \qquad \ldots \quad (37.10)$$

where $T_c$ is the Curie point and $c'$ is a constant. The Curie point of iron is 1043 K.

**Table 37.2 :** ***Curie temperatures for some ferromagnetic substances***

| Substance | $T_c$ (K) |
|---|---|
| Iron | 1043 |
| Cobalt | 1394 |
| Nickel | 631 |
| Gadolinium | 317 |
| $Fe_2O_3$ | 893 |

## 37.7 PROPERTIES OF DIA-, PARA- AND FERROMAGNETIC SUBSTANCES

Suppose a material is placed in an external magnetic field. If the material is paramagnetic, a small magnetization occurs in the direction of the field. If it is ferromagnetic, a large magnetization occurs in the direction of the field and if the material is diamagnetic, a small magnetization occurs opposite to the field. The lines of magnetic field $\vec{B}$, thus, become more dense in a paramagnetic or ferromagnetic material but become less dense in a diamagnetic material.

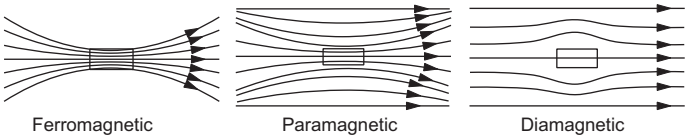



Figure 37.5

The magnetic susceptibility is a small but positive quantity ($\approx 10^{-3}$ to $10^{-5}$) for paramagnetic substances; of the order of several thousands (positive) for ferromagnetic substances and small but negative for diamagnetic substances. The relative permeability $\mu_r = 1 + \chi$ is slightly more than 1 for paramagnetic, of the order of thousands for ferromagnetic and slightly less than 1 for diamagnetic substances.

Paramagnetism or diamagnetism may be found in solids, liquids or gases but ferromagnetism is normally found only in solids.

A paramagnetic substance is weakly attracted by a magnet, a ferromagnetic substance is strongly attracted by a magnet and a diamagnetic substance is weakly repelled by a magnet. Thus, when a rod is

suspended in a magnetic field (figure 37.6), the rod becomes perpendicular to the field if it is diamagnetic and parallel to the field if it is paramagnetic or ferromagnetic.

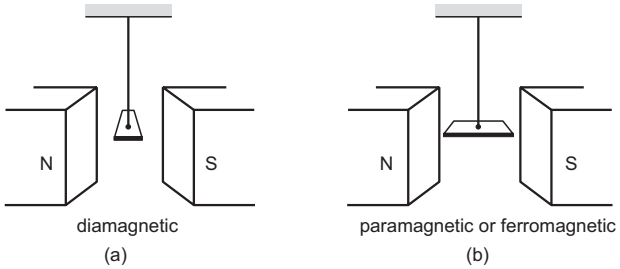



Figure 37.6

The susceptibility is inversely proportional to the absolute temperature for paramagnetic substances but it varies in a complicated way for ferromagnetic substances. For diamagnetic substances, the variation of $\chi$ with temperature is very small.

One can make permanent magnets from ferromagnetic substances only. Hysteresis (explained below) is also shown by ferromagnetic substances only.

### 37.8 HYSTERESIS

The magnetization in a ferromagnetic material not only depends on the magnetic intensity $H$ but also on the previous history of the specimen. Suppose a ferromagnetic material is formed in the shape of a ring and is placed inside a toroid having $n$ turns per unit length. A current $i$ can be passed through the toroid to produce a magnetic intensity $H$ in it. The magnetic field produced by the current is

$$B_0 = \mu_0 ni$$

and hence $$H = \frac{B_0}{\mu_0} = ni. \quad \ldots \text{(i)}$$

Note that $B_0$ is the field produced by the toroid-current only. The ring gets magnetized and produces an extra field due to magnetization. The total field in the ring is

$$B = \mu_0 (H + I)$$

or, $$I = \frac{B}{\mu_0} - H$$

$$= \frac{B}{\mu_0} - ni. \quad \ldots \text{(ii)}$$

One can measure the total field $B$ inside the ring by using an apparatus known as Rowland's ring. The intensity of magnetization $I$ can then be obtained from equation (ii). Thus, from (i) and (ii) one can obtain $H$ and $I$ for any current.

Figure (37.7) shows a typical magnetization curve when the current is changed. In the beginning, the current is zero and the sample has no magnetization.

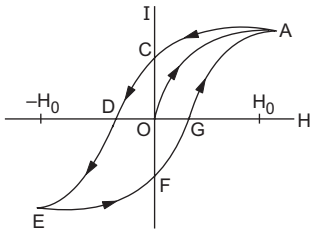



Figure 37.7

Thus, $H = 0$ and $I = 0$. This corresponds to the point $O$. As the current is increased, $H$ increases and the magnetization increases. As the current is increased to a maximum, $H$ becomes $H_0$ and the magnetization $I$ becomes nearly saturated. In the whole process, the magnetization varies along the path $OA$. Now suppose, the current is gradually decreased. The magnetization decreases but the path $OA$ is not retraced. As the current reduces to zero, $H$ also becomes zero. But, there is still some magnetization left in the ring. The domains, that were aligned at the time of increasing $H$, are not completely randomized as the magnetic intensity $H$ is reduced to zero. The remaining value of $I$ at the point $C$ is called the *retentivity* of the material. To reduce $I$ to zero, a current must be passed in the opposite direction so as to disalign the domains forcibly. The value of $H$ needed to make $I = 0$ is called *coercive force*. In figure (37.7), the coercive force is represented by the magnitude of $H$ corresponding to $OD$. As the current is increased further in the opposite direction, the material gets magnetized in the opposite direction. The magnetization $I$ follows the path $DE$ as the magnetic intensity becomes $-H_0$. If the current is now reduced to zero, the magnetization $I$ follows the path $EF$. Finally, if the current is increased in the original direction, the point $A$ is reached via $FGA$. If we repeat the current cycle so that $H$ changes from $H_0$ to $-H_0$ to $H_0$, the curve $ACDEFGA$ is retraced.

As $H$ is increased and then decreased to its original value, the magnetization $I$, in general, does not return to its original value. This fact is called *hysteresis*. The curve $ACDEFGA$ is called the *hysteresis loop*. The area of the hysteresis loop is proportional to the thermal energy developed per unit volume of the material as it goes through the hysteresis cycle.

### 37.9 SOFT IRON AND STEEL

Figure (37.8) shows hysteresis loops for soft iron and steel. The retentivity and the coercive force are larger for steel than for soft iron. The area of the hysteresis loop is also larger for steel than for soft iron. Soft iron is, therefore, easily magnetized by a magnetizing field but only a small magnetization is retained when the field is removed. Also, the loss of energy, as the material is taken through periodic variations in magnetizing fields, is small. Materials

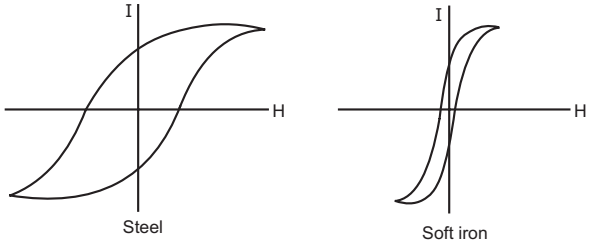



Figure 37.8

like soft iron are suitable for making electromagnets and cores inside current-carrying coils to increase the magnetic field. In transformers, moving-coil galvanometers, etc., soft-iron core is used in the coils.

On the other hand, steel and similar materials are suitable for making permanent magnets. Large magnetizing fields are needed to appreciably magnetize the material. But once magnetized, the magnetization is retained to a large extent even when the magnetizing field is removed (retentivity is large). The magnetization is not easily destroyed even if the material is exposed to stray reverse fields (coercive force is large).

## *Worked Out Examples*

**1.** *A tightly-wound, long solenoid having* 50 turns $cm^{-1}$, *carries a current of* 4·00 A. *Find the magnetic intensity H and the magnetic field B at the centre of the solenoid. What will be the values of these quantities if an iron core is inserted in the solenoid and the magnetization I in the core is* $4{\cdot}00 \times 10^6$ A $m^{-1}$ ?

***Solution*** : The magnetic intensity $H$ at the centre of a long solenoid is

$$H = ni$$
$$= 50 \times 10^2 \text{ m}^{-1} \times 4 \text{ A} = 2 \times 10^4 \text{ A m}^{-1}.$$

The magnetic field $B = \mu_0 H$

$$= 4\pi \times 10^{-7} \text{ Tm A}^{-1} \times 2 \times 10^4 \text{ A m}^{-1}$$
$$= 8\pi \times 10^{-3} \text{ T} \approx 25 \text{ mT}.$$

The value of $H$ does not change as the iron core is inserted and remains $2 \times 10^4$ A $m^{-1}$. The magnetic field $B$ becomes

$$B = \mu_0(H + I)$$
$$= (4\pi \times 10^{-7} \text{ Tm A}^{-1})\ (2 \times 10^4 + 4 \times 10^6) \text{ A m}^{-1} = 5{\cdot}05 \text{ T}.$$

It should be noted that the magnetic intensity $H$ is very small as compared to the magnetization $I$ in presence of the iron core.

**2.** *A long, cylindrical iron core of cross-sectional area* 5·00 $cm^2$ *is inserted into a long solenoid having* 2000 turns $m^{-1}$ *and carrying a current* 2·00 A. *The magnetic field inside the core is found to be* 1·57 T. *Neglecting the end effects, find the magnetization I of the core and the pole strength developed.*

***Solution*** : The magnetic intensity $H$ inside the solenoid is

$$H = ni = 2000 \text{ m}^{-1} \times 2 \text{ A} = 4000 \text{ A m}^{-1}.$$

Also $B = \mu_0(H + I)$

or, $$I = \frac{B}{\mu_0} - H$$

$$= \frac{1{\cdot}57 \text{ T}}{4\pi \times 10^{-7} \text{ Tm A}^{-1}} - 4000 \text{ A m}^{-1}$$

$$= (1{\cdot}25 \times 10^6 - 4000) \text{ A m}^{-1} \approx 1{\cdot}25 \times 10^6 \text{ A m}^{-1}.$$

Note again that the magnetization $I >> H$ for iron core. The pole strength developed at the ends is

$$m = IA$$
$$= (1{\cdot}25 \times 10^6 \text{ A m}^{-1}) \times (5 \times 10^{-4} \text{ m}^2) = 625 \text{ A m}.$$

**3.** *An ideal solenoid having* 40 turns $cm^{-1}$ *has an aluminium core and carries a current of* 2·0 A. *Calculate the magnetization I developed in the core and the magnetic field B at the centre. The susceptibility* $\chi$ *of aluminium* = $2{\cdot}3 \times 10^{-5}$.

***Solution*** : The magnetic intensity $H$ at the centre of the solenoid is

$$H = ni = 4000 \text{ turns m}^{-1} \times 2{\cdot}0 \text{ A}$$
$$= 8000 \text{ A m}^{-1}.$$

The magnetization is $I = \chi H$

$$= 2{\cdot}3 \times 10^{-5} \times 8000 \text{ A m}^{-1} = 0{\cdot}18 \text{ A m}^{-1}.$$

The magnetic field is $B = \mu_0(H + I)$

$$= (4\pi \times 10^{-7} \text{ Tm A}^{-1})\ [800 + 0{\cdot}18] \text{ A m}^{-1} \approx 3{\cdot}2\pi \times 10^{-4} \text{ T}.$$

Note that $H >> I$ in case of a paramagnetic core.

**4.** *Find (a) the magnetization I, (b) the magnetic intensity H and (c) the magnetic field B at the centre of a bar magnet having pole strength* 3·6 A m, *magnetic length* 12 cm *and cross-sectional area* 0·90 $cm^2$.

***Solution*** : (a) Magnetization $I = \frac{m}{A} = \frac{3{\cdot}6 \text{ A m}^{-1}}{0{\cdot}90 \times 10^{-4} \text{ m}^2}$

$$= 4 \times 10^4 \text{ A m}^{-1}.$$

The direction will be from the south pole to the north pole at the centre of the magnet.

(b) Magnetic intensity $H_n$ due to the north pole is

$$H_n = \frac{1}{4\pi} \frac{m}{d^2} = \frac{3{\cdot}6 \text{ A m}^{-1}}{4\pi \times (6 \times 10^{-2} \text{ m})^2} = 79{\cdot}6 \text{ A m}^{-1}.$$

The direction will be towards the south pole. The magnetic intensity $H_s$ at this point due to the south pole is also 79.6 A $m^{-1}$ in the same direction. The resultant

magnetic intensity is

$$H = H_n + H_s$$

$$= 159{\cdot}2 \text{ A m}^{-1} \text{ towards the south pole.}$$

(c) The magnetic field $\vec{B}$ at the centre is

$$\vec{B} = \mu_0(\vec{H} + \vec{I})$$

or, $$B = (4\pi \times 10^{-7} \text{ Tm A}^{-1})\ (4 \times 10^4 - 159{\cdot}2) \text{ A m}^{-1}$$

$$= 5{\cdot}0 \times 10^{-2} \text{ T.}$$

The field is towards the north pole.

**5.** *The maximum value of the permeability of* μ*-metal* (77% *Ni*, 16% *Fe*, 5% *Cu*, 2% *Cr*) *is* $0{\cdot}126 \text{ Tm A}^{-1}$. *Find the maximum relative permeability and susceptibility.*

***Solution*** : Relative permeability is

$$\mu_r = \frac{\mu}{\mu_0} = \frac{0{\cdot}126 \text{ Tm A}^{-1}}{4\pi \times 10^{-7} \text{ Tm A}^{-1}}$$

$$= 1{\cdot}00 \times 10^5.$$

Susceptibility $\chi = \mu_r - 1 \approx 1{\cdot}00 \times 10^5$.

**6.** *A toroid has a mean radius R equal to* $20/\pi$ cm, *and a total of* 400 *turns of wire carrying a current of* 2·0 A. *An aluminium ring at temperature* 280 K *inside the toroid provides the core. (a) If the magnetization I is* $4{\cdot}8 \times 10^{-2} \text{ A m}^{-1}$, *find the susceptibility of aluminium at* 280 K. *(b) If the temperature of the aluminium ring is raised to* 320 K, *what will be the magnetization* ?

***Solution*** : (a) The number of turns per unit length of the toroid is

$$n = \frac{400}{2\pi R}.$$

The magnetic intensity $H$ in the core is

$$H = ni$$

$$= \frac{400 \times 2{\cdot}0 \text{ A}}{2\pi \times \frac{20}{\pi} \times 10^{-2} \text{ m}} = 2000 \text{ A m}^{-1}.$$

The susceptibility is

$$\chi = I/H$$

$$= \frac{4{\cdot}8 \times 10^{-2} \text{ A m}^{-1}}{2000 \text{ A m}^{-1}} = 2{\cdot}4 \times 10^{-5}.$$

(b) The susceptibility $\chi$ of a paramagnetic substance varies with absolute temperature as $\chi = c/T$.

Thus, $\chi_2 / \chi_1 = T_1 / T_2$.

The susceptibility of aluminium at temperature 320 K is, therefore,

$$\chi = \frac{280}{320} \times 2{\cdot}4 \times 10^{-5} = 2{\cdot}1 \times 10^{-5}.$$

Thus, the magnetization at 320 K is

$$I = \chi H$$

$$= 2{\cdot}1 \times 10^{-5} \times 2000 \text{ A m}^{-1}.$$

$$= 4{\cdot}2 \times 10^{-2} \text{ A m}^{-1}.$$

□

## QUESTIONS FOR SHORT ANSWER

**1.** When a dielectric is placed in an electric field, it gets polarized. The electric field in a polarized material is less than the applied field. When a paramagnetic substance is kept in a magnetic field, the field in the substance is more than the applied field. Explain the reason of this opposite behaviour.

**2**. The property of diamagnetism is said to be present in all materials. Then, why are some materials paramagnetic or ferromagnetic ?

**3.** Do permeability and relative permeability have the same dimensions ?

**4.** A rod when suspended in a magnetic field stays in east–west direction. Can we be sure that the field is in the east–west direction ? Can it be in the north–south direction ?

**5.** Why cannot we make permanent magnets from paramagnetic materials ?

**6.** Can we have magnetic hysteresis in paramagnetic or diamagnetic substances ?

**7.** When a ferromagnetic material goes through a hysteresis loop, its thermal energy is increased. Where does this energy come from ?

**8.** What are the advantages of using soft iron as a core, instead of steel, in the coils of galvanometers ?

**9.** To keep valuable instruments away from the earth's magnetic field, they are enclosed in iron boxes. Explain.

## OBJECTIVE I

**1.** A paramagnetic material is placed in a magnetic field. Consider the following statements:

(A) If the magnetic field is increased, the magnetization is increased.

(B) If the temperature is increased, the magnetization is increased.
(a) Both A and B are true.
(b) A is true but B is false.
(c) B is true but A is false.
(d) Both A and B are false.

**2.** A paramagnetic material is kept in a magnetic field. The field is increased till the magnetization becomes constant. If the temperature is now decreased, the magnetization
(a) will increase (b) decrease
(c) remain constant (d) may increase or decrease.

**3.** A ferromagnetic material is placed in an external magnetic field. The magnetic domains
(a) increase in size (b) decrease in size
(c) may increase or decrease in size
(d) have no relation with the field.

**4.** A long, straight wire carries a current $i$. The magnetizing field intensity $H$ is measured at a point $P$ close to the wire. A long, cylindrical iron rod is brought close to the wire so that the point $P$ is at the centre of the rod. The value of $H$ at $P$ will
(a) increase many times (b) decrease many times
(c) remain almost constant (d) become zero.

**5.** The magnetic susceptibility is negative for
(a) paramagnetic materials only
(b) diamagnetic materials only
(c) ferromagnetic materials only
(d) paramagnetic and ferromagnetic materials.

**6.** The desirable properties for making permanent magnets are
(a) high retentivity and high coercive force
(b) high retentivity and low coercive force
(c) low retentivity and high coercive force
(d) low retentivity and low coercive force.

**7.** Electromagnets are made of soft iron because soft iron has
(a) high retentivity and high coercive force
(b) high retentivity and low coercive force
(c) low retentivity and high coercive force
(d) low retentivity and low coercive force.

## OBJECTIVE II

**1.** Pick the correct options.
(a) All electrons have magnetic moment.
(b) All protons have magnetic moment.
(c) All nuclei have magnetic moment.
(d) All atoms have magnetic moment.

**2.** The permanent magnetic moment of the atoms of a material is not zero. The material
(a) must be paramagnetic (b) must be diamagnetic
(c) must be ferromagnetic (d) may be paramagnetic.

**3.** The permanent magnetic moment of the atoms of a material is zero. The material
(a) must be paramagnetic (b) must be diamagnetic
(c) must be ferromagnetic (d) may be paramagnetic.

**4.** Which of the following pairs has quantities of the same dimensions ?
(a) Magnetic field $B$ and magnetizing field intensity $H$
(b) Magnetic field $B$ and intensity of magnetization $I$
(c) Magnetizing field intensity $H$ and intensity of magnetization $I$
(d) Longitudinal strain and magnetic susceptibility.

**5.** When a ferromagnetic material goes through a hysteresis loop, the magnetic susceptibility
(a) has a fixed value (b) may be zero
(c) may be infinity (d) may be negative.

**6.** Mark out the correct options.
(a) Diamagnetism occurs in all materials.
(b) Diamagnetism results from the partial alignment of permanent magnetic moment.
(c) The magnetizing field intensity $H$ is always zero in free space.
(d) The magnetic field of induced magnetic moment is opposite to the applied field.

## EXERCISES

**1.** The magnetic intensity $H$ at the centre of a long solenoid carrying a current of 2·0 A, is found to be 1500 A m$^{-1}$. Find the number of turns per centimetre of the solenoid.

**2.** A rod is inserted as the core in the current-carrying solenoid of the previous problem. (a) What is the magnetic intensity $H$ at the centre ? (b) If the magnetization $I$ of the core is found to be 0·12 A m$^{-1}$, find the susceptibility of the material of the rod. (c) Is the material paramagnetic, diamagnetic or ferromagnetic ?

**3.** The magnetic field inside a long solenoid having 50 turns cm$^{-1}$ is increased from $2{\cdot}5 \times 10^{-3}$ T to 2·5 T when an iron core of cross-sectional area 4 cm$^2$ is inserted into it. Find (a) the current in the solenoid, (b) the magnetization $I$ of the core and (c) the pole strength developed in the core.

**4.** A bar magnet of length 1 cm and cross-sectional area 1·0 cm$^2$ produces a magnetic field of $1{\cdot}5 \times 10^{-4}$ T at a point in end-on position at a distance 15 cm away from the centre. (a) Find the magnetic moment $M$ of the magnet. (b) Find the magnetization $I$ of the magnet. (c) Find the magnetic field $B$ at the centre of the magnet.

**5.** The susceptibility of annealed iron at saturation is 5500. Find the permeability of annealed iron at saturation.

**6.** The magnetic field $B$ and the magnetic intensity $H$ in a material are found to be 1·6 T and 1000 A m$^{-1}$

respectively. Calculate the relative permeability $\mu_r$ and the susceptibility $\chi$ of the material.

**7.** The susceptibility of magnesium at 300 K is $1{\cdot}2 \times 10^{-5}$. At what temperature will the susceptibility increase to $1{\cdot}8 \times 10^{-5}$?

**8.** Assume that each iron atom has a permanent magnetic moment equal to 2 Bohr magnetons (1 Bohr magneton equals $9{\cdot}27 \times 10^{-24}$ A m$^2$). The density of atoms in iron is $8{\cdot}52 \times 10^{28}$ atoms m$^{-3}$. (a) Find the maximum magnetization $I$ in a long cylinder of iron. (b) Find the maximum magnetic field $B$ on the axis inside the cylinder.

**9.** The coercive force for a certain permanent magnet is $4{\cdot}0 \times 10^4$ A m$^{-1}$. This magnet is placed inside a long solenoid of 40 turns/cm and a current is passed in the solenoid to demagnetize it completely. Find the current.

□

## ANSWERS

### OBJECTIVE I

1. (b) 2. (c) 3. (c) 4. (c) 5. (b) 6. (a)
7. (d)

### OBJECTIVE II

1. (a), (b) 2. (d) 3. (b) 4. (c), (d)
5. (b), (c), (d) 6. (a), (d)

### EXERCISES

1. 7·5
2. (a) 1500 A m$^{-1}$ (b) $8{\cdot}0 \times 10^{-5}$ (c) paramagnetic
3. (a) 0·4 A (b) $2{\cdot}0 \times 10^6$ A m$^{-1}$ (c) 800 A m
4. (a) 2·5 A m$^2$ (b) $2{\cdot}5 \times 10^6$ A m$^{-1}$ (c) 1·2 T
5. $6{\cdot}9 \times 10^{-3}$
6. $1{\cdot}3 \times 10^3$ each
7. 200 K
8. (a) $1{\cdot}58 \times 10^6$ A m$^{-1}$ (b) 2·0 T
9. 10 A

□

## CHAPTER 38

# ELECTROMAGNETIC INDUCTION

### 38.1 FARADAY'S LAW OF ELECTROMAGNETIC INDUCTION

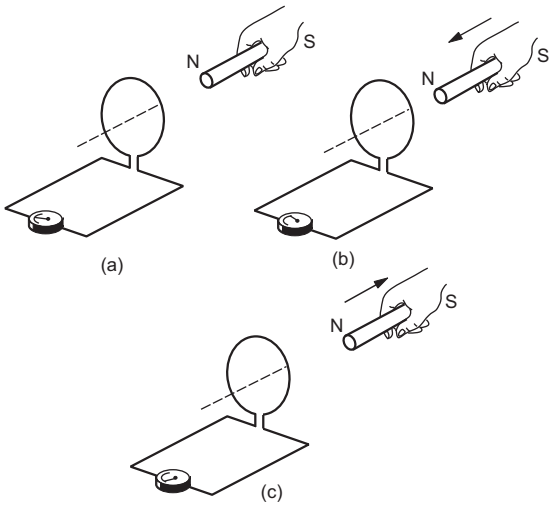



Figure 38.1

Figure (38.1a) shows a bar magnet placed along the axis of a conducting loop containing a galvanometer. There is no current in the loop and correspondingly no deflection in the galvanometer. If we move the magnet towards the loop (figure 38.1b), there is a deflection in the galvanometer showing that there is an electric current in the loop. If the magnet is moved away from the loop (figure 38.1c), again there is a current but the current is in the opposite direction. The current exists as long as the magnet is moving. Faraday studied this behaviour in detail by performing a number of experiments and discovered the following law of nature:

*Whenever the flux of magnetic field through the area bounded by a closed conducting loop changes, an emf is produced in the loop. The emf is given by*

$$\mathcal{E} = -\frac{d\Phi}{dt} \qquad \ldots \quad (38.1)$$

*where* $\Phi = \int \vec{B}\cdot d\vec{S}$ *is the flux of the magnetic field through the area.*

We shall call the quantity $\Phi$ *magnetic flux*. The SI unit of magnetic flux is called weber which is equivalent to tesla metre$^2$.

The law described by equation (38.1) is called *Faraday's law of electromagnetic induction.* The flux may be changed in a number of ways. One can change the magnitude of the magnetic field $\vec{B}$ at the site of the loop, the area of the loop or the angle between the area-vector $d\vec{S}$ and the magnetic field $\vec{B}$. In any case, as long as the flux keeps changing, the emf is present. The emf so produced drives an electric current through the loop. If the resistance of the loop is $R$, the current is

$$i = \frac{\mathcal{E}}{R} = -\frac{1}{R}\frac{d\Phi}{dt}. \qquad \ldots \quad (38.2)$$

The emf developed by a changing flux is called *induced emf* and the current produced by this emf is called *induced current.*

**Direction of Induced Current**

The direction of the induced current in a loop may be obtained using equation (38.1) or (38.2). The procedure to decide the direction is as follows:

Put an arrow on the loop to choose the positive sense of current. This choice is arbitrary. Using right-hand thumb rule find the positive direction of the normal to the area bounded by the loop. If the fingers curl along the loop in the positive sense, the thumb represents the positive direction of the normal. Calculate the flux $\Phi = \int \vec{B}\cdot d\vec{S}$ through the area bounded by the loop. If the flux increases with time, $\frac{d\Phi}{dt}$ is positive and $\mathcal{E}$ is negative from equation (38.1). Correspondingly, the current is negative. It is, therefore, in the direction opposite to the arrow put on the loop. If $\Phi$ decreases with time, $\frac{d\Phi}{dt}$ is negative, $\mathcal{E}$ is positive and the current is along the arrow.

***Example 38.1***

*Figure (38.2) shows a conducting loop placed near a long, straight wire carrying a current i as shown. If the current increases continuously, find the direction of the induced current in the loop.*

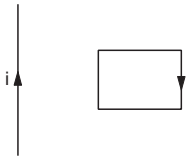



Figure 38.2

***Solution*** **:** Let us put an arrow on the loop as shown in the figure. The right-hand thumb rule shows that the positive normal to the loop is going into the plane of the diagram. Also, the same rule shows that the magnetic field at the site of the loop due to the current is also going into the plane of the diagram. Thus, $\vec{B}$ and $d\vec{S}$ are along the same direction everywhere so that the flux $\Phi = \int \vec{B}\cdot d\vec{S}$ is positive. If $i$ increases, the magnitude of $\Phi$ increases. Since $\Phi$ is positive and its magnitude increases, $\frac{d\Phi}{dt}$ is positive. Thus, $\mathcal{E}$ is negative and hence, the current is negative. The current is, therefore, induced in the direction opposite to the arrow.

## 38.2 LENZ'S LAW

Another way to find the direction of the induced current in a conducting loop is to use Lenz's law. The current is induced by the changing magnetic flux. The induced current itself produces a magnetic field and hence a magnetic flux. This magnetic flux may have the same sign as the original flux or it may have the opposite sign. It strengthens the original flux if it has the same sign and weakens it otherwise. Lenz's law states:

*The direction of the induced current is such that it opposes the change that has induced it.*

If a current is induced by an increasing flux, it will weaken the original flux. If a current is induced by a decreasing flux, it will strengthen the original flux.

Figure (38.3) shows some situations. In figure (38.3a), a magnet is brought towards a circular loop. The north pole faces the loop. As the magnet gets closer to the loop, the magnetic field increases and hence, the flux of the magnetic field through the area of the loop increases. The induced current should weaken the flux. The original field is away from the magnet, so the induced field should be towards the magnet. Using the right-hand thumb rule, we can find the direction of the current that produces a field towards the magnet.

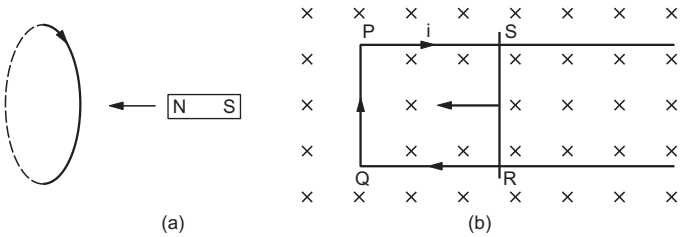



Figure 38.3

In figure (38.3b), the wire $RS$ slides towards left so that the area of $PQRS$ decreases. As a result, the magnetic flux through the area $PQRS$ decreases. The induced current should strengthen the original flux. The induced current should produce a magnetic field along the original field which is going into the plane of the diagram. Using the right-hand thumb rule, we find that the induced current should be clockwise as shown in the figure.

## 38.3 THE ORIGIN OF INDUCED EMF

An electric current is established in a conducting wire when an electric field exists in it. The flow of charges tend to destroy the field and some external mechanism is needed to maintain the electric field in the wire. It is the work done per unit charge by this external mechanism that we call emf. When the magnetic flux through a closed loop changes, an electric current results. What is the external mechanism that maintains the electric field in the loop to drive the current ? In other words, what is the mechanism to produce an emf ? Let us now investigate this question.

The flux $\int \vec{B}\cdot d\vec{S}$ can be changed by

(a) keeping the magnetic field constant as time passes and moving whole or part of the loop

(b) keeping the loop at rest and changing the magnetic field

(c) combination of (a) and (b), that is, by moving the loop (partly or wholely) as well as by changing the field.

The mechanism by which emf is produced is different in the two basic processes (a) and (b). We now study them under the headings *motional emf* and *induced electric field.*

### Motional Emf

Figure (38.4) shows a rod $PQ$ of length $l$ moving in a magnetic field $\vec{B}$ with a constant velocity $\vec{v}$. The length of the rod is perpendicular to the magnetic field and the velocity is perpendicular to both the magnetic

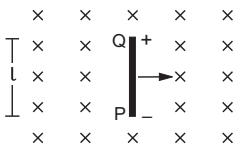



Figure 38.4

field and the rod. The free electrons of the wire also move with this velocity $\vec{v}$ together with the random velocity they have in the rod. The magnetic force due to the random velocity is zero on the average. Thus, the magnetic field exerts an average force $\vec{F}_b = q\vec{v} \times \vec{B}$ on each free electron where $q = -1{\cdot}6 \times 10^{-19}$ C is the charge on the electron. This force is towards $QP$ and hence the free electrons will move towards $P$. Negative charge is accumulated at $P$ and positive charge appears at $Q$. An electrostatic field $E$ is developed within the wire from $Q$ to $P$. This field exerts a force $\vec{F}_e = q\vec{E}$ on each free electron. The charge keeps on accumulating until a situation comes when $F_b = F_e$

or, $|q\vec{v} \times \vec{B}| = |q\vec{E}|$ or, $vB = E$.

After this, there is no resultant force on the free electrons of the wire $PQ$. The potential difference between the ends $Q$ and $P$ is

$$V = El = vBl.$$

Thus, it is the magnetic force on the moving free electrons that maintains the potential difference $V = vBl$ and hence produces an emf

$$\mathcal{E} = vBl. \qquad \ldots (38.3)$$

As this emf is produced due to the motion of a conductor, it is called *motional emf.*

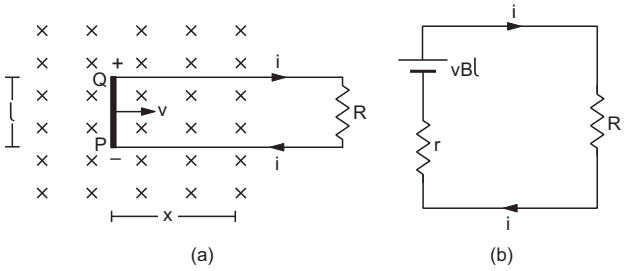



Figure 38.5

If the ends $P$ and $Q$ are connected by an external resistor (figure 38.5a), an electric field is produced in this resistor due to the potential difference. A current is established in the circuit. The electrons flow from $P$ to $Q$ via the external circuit and this tries to neutralise the charges accumulated at $P$ and $Q$. The magnetic force $qvB$ on the free electrons in the wire $QP$, however, drives the electrons back from $Q$ to $P$ to maintain the potential difference and hence the current.

Thus, we can replace the moving rod $QP$ by a battery of emf $vBl$ with the positive terminal at $Q$ and the negative terminal at $P$. The resistance $r$ of the rod $QP$ may be treated as the internal resistance of the battery. Figure (38.5b) shows the equivalent circuit. The current is $i = \frac{vBl}{R + r}$ in the clockwise direction (induced current).

We can also find the induced emf and the induced current in the loop in figure (38.5a) from Faraday's law of electromagnetic induction. If $x$ be the length of the circuit in the magnetic field at time $t$, the magnetic flux through the area bounded by the loop is

$$\Phi = Blx.$$

The magnitude of the induced emf is

$$\mathcal{E} = \left|\frac{d\Phi}{dt}\right| = \left|Bl\,\frac{dx}{dt}\right|$$

$$= vBl.$$

The current is $i = \frac{vBl}{R + r}$. The direction of the current can be worked out from Lenz's law.

***Example 38.2***

*Figure (38.6a) shows a rectangular loop MNOP being pulled out of a magnetic field with a uniform velocity v by applying an external force F. The length MN is equal to l and the total resistance of the loop is R. Find (a) the current in the loop, (b) the magnetic force on the loop, (c) the external force F needed to maintain the velocity, (d) the power delivered by the external force and (e) the thermal power developed in the loop.*

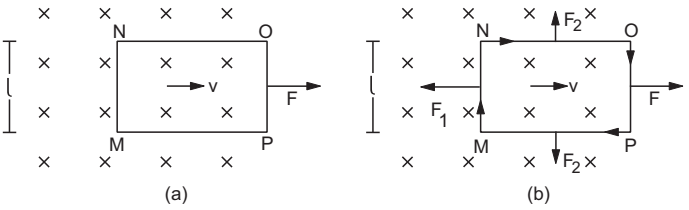



Figure 38.6

***Solution*** : (a) The emf induced in the loop is due to the motion of the wire $MN$. The emf is $\mathcal{E} = vBl$ with the positive end at $N$ and the negative end at $M$. The current is

$$i = \frac{\mathcal{E}}{R} = \frac{vBl}{R}$$

in the clockwise direction (figure 38.6b).

(b) The magnetic force on the wire $MN$ is $\vec{F}_1 = i\vec{l} \times \vec{B}$. The magnitude is $F_1 = ilB = \frac{vB^2l^2}{R}$ and is opposite to the velocity. The forces on the parts of the wire $NO$ and $PM$, lying in the field, cancel each other. The resultant magnetic force on the loop is, therefore, $F_1 = \frac{B^2l^2v}{R}$ opposite to the velocity.

(c) To move the loop at a constant velocity, the resultant force on it should be zero. Thus, one should pull the loop with a force

$$F = F_1 = \frac{vB^2l^2}{R}.$$

(d) The power delivered by the external force is

$$P = F v = \frac{v^2 B^2 l^2}{R}.$$

(e) The thermal power developed is

$$P = i^2 R = \left(\frac{vBl}{R}\right)^2 R = \frac{v^2 B^2 l^2}{R}.$$

We see that the power delivered by the external force is equal to the thermal power developed in the loop. This is consistent with the principle of conservation of energy.

**Induced Electric Field**

Consider a conducting loop placed at rest in a magnetic field $\vec{B}$. Suppose, the field is constant till $t = 0$ and then changes with time. An induced current starts in the loop at $t = 0$.

The free electrons were at rest till $t = 0$ (we are not interested in the random motion of the electrons). The magnetic field cannot exert force on electrons at rest. Thus, the magnetic force cannot start the induced current. The electrons may be forced to move only by an electric field and hence we conclude that an electric field appears at $t = 0$. This electric field is produced by the changing magnetic field and not by charged particles according to the Coulomb's law or the Gauss's law. The electric field produced by the changing magnetic field is nonelectrostatic and nonconservative in nature. We cannot define a potential corresponding to this field. We call it *induced electric field*. The lines of induced electric field are closed curves. There are no starting and terminating points of the field lines.

If $\vec{E}$ be the induced electric field, the force on a charge $q$ placed in the field is $q\vec{E}$. The work done per unit charge as the charge moves through $d\vec{l}$ is $\vec{E}\cdot d\vec{l}$. The emf developed in the loop is, therefore,

$$\mathcal{E} = \oint \vec{E}\cdot d\vec{l}.$$

Using Faraday's law of induction,

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

or, $$\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi}{dt}. \quad \ldots (38.4)$$

The presence of a conducting loop is not necessary to have an induced electric field. As long as $\vec{B}$ keeps changing, the induced electric field is present. If a loop is there, the free electrons start drifting and consequently an induced current results.

**38.4 EDDY CURRENT**

Consider a solid plate of metal which enters a region having a magnetic field (figure 38.7a). Consider a loop drawn on the plate, a part of which is in the field. As the plate moves, the magnetic flux through the area bounded by the loop changes and hence a current is induced. There may be a number of such loops on the plate and hence currents are induced on the surface along a variety of paths. Such currents are called *eddy currents*. The basic idea is that we do not have a definite conducting loop to guide the induced current. The system itself looks for the loops on the surface along which eddy currents are induced. Because of the eddy currents in the metal plate, thermal energy is produced in it. This energy comes at the cost of the kinetic energy of the plate and the plate slows down. This is known as *electromagnetic damping*. To reduce electromagnetic damping, one can cut slots in the plate (figure 38.7b). This reduces the possible paths of the eddy current considerably.

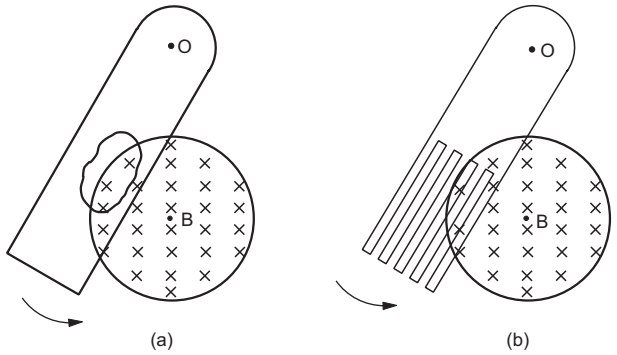



Figure 38.7

**38.5 SELF-INDUCTION**

When a current is established in a closed conducting loop, it produces a magnetic field. This magnetic field has its flux through the area bounded by the loop. If the current changes with time, the flux through the loop changes and hence an emf is induced in the loop. This process is called *self-induction*. The name is so chosen because the emf is induced in the loop by changing the current in the same loop.

The magnetic field at any point due to a current is proportional to the current. The magnetic flux through the area bounded by a current-carrying loop is, therefore, proportional to the current. We can write

$$\Phi = Li \quad \ldots (38.5)$$

where $L$ is a constant depending on the geometrical construction of the loop. This constant is called *self-inductance* of the loop. The induced emf $\mathcal{E}$, when the current in the coil changes, is given by

$$\mathcal{E} = -\frac{d\Phi}{dt}.$$

Using equation (38.5),

$$\mathcal{E} = -L\frac{di}{dt}. \quad \ldots (38.6)$$

The SI unit of self-inductance $L$ is weber ampere$^{-1}$ from equation (38.5) or volt second ampere$^{-1}$ from (38.6). It is given a special name henry and is abbreviated as H.

If we have a coil or a solenoid of $N$ turns, the flux through each turn is $\int \vec{B}\cdot d\vec{S}$. If this flux changes, an emf is induced in each turn. The net emf induced between the ends of the coil is the sum of all these. Thus,

$$\mathcal{E} = -N\frac{d}{dt}\int \vec{B}\cdot d\vec{S}.$$

One can compare this with equation (38.6) to get the inductance.

***Example 38.3***

*An average induced emf of* 0·20 V *appears in a coil when the current in it is changed from* 5·0 A *in one direction to* 5·0 A *in the opposite direction in* 0·20 s. *Find the self-inductance of the coil.*

***Solution*** :

$$\text{Average } \frac{di}{dt} = \frac{(-5{\cdot}0\text{ A}) - (5{\cdot}0\text{ A})}{0{\cdot}20\text{ s}} = -50\text{ A/s}.$$

Using $\mathcal{E} = -L\frac{di}{dt}$,

$$0{\cdot}2\text{ V} = L(50\text{ A/s})$$

or, $$L = \frac{0{\cdot}2\text{ V}}{50\text{ A/s}} = 4{\cdot}0\text{ mH}.$$

**Self-inductance of a Long Solenoid**

Consider a long solenoid of radius $r$ having $n$ turns per unit length. Suppose a current $i$ is passed through the solenoid. The magnetic field produced inside the solenoid is $B = \mu_0 ni$. The flux through each turn of the solenoid is

$$\Phi = \int \vec{B}\cdot d\vec{S} = (\mu_0 ni)\pi r^2.$$

The emf induced in each turn is

$$-\frac{d\Phi}{dt} = -\mu_0 n\pi r^2 \frac{di}{dt}.$$

As there are $nl$ turns in a length $l$ of the solenoid, the net emf across a length $l$ is

$$\mathcal{E} = -(nl)(\mu_0 n\pi r^2)\frac{di}{dt}.$$

Comparing with $\varepsilon = -L\frac{di}{dt}$, the self-inductance is

$$L = \mu_0 n^2 \pi r^2 l. \quad \ldots (38.7)$$

We see that the self-inductance depends only on geometrical factors.

A coil or a solenoid made from thick wire has negligible resistance but a considerable self-inductance. Such an element is called an *ideal inductor* and is indicated by the symbol  .

The self-induced emf in a coil opposes the change in the current that has induced it. This is in accordance with the Lenz's law. If the current is increasing, the induced current will be opposite to the original current. If the current is decreasing, the induced current will be along the original current.

***Example 38.4***

*Consider the circuit shown in figure (38.8). The sliding contact is being pulled towards right so that the resistance in the circuit is increasing. Its value at the instant shown is* 12 Ω. *Will the current be more than* 0·50 A *or less than it at this instant* ?

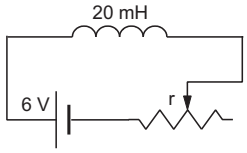



Figure 38.8

***Solution*** : As the sliding contact is being pulled, the current in the circuit changes. An induced emf $\mathcal{E} = -L\frac{di}{dt}$ is produced across the inductor. The net emf in the circuit is $6\text{ V} - L\frac{di}{dt}$ and hence the current is

$$i = \frac{6\text{ V} - L\frac{di}{dt}}{12\ \Omega} \quad \ldots \text{(i)}$$

at the instant shown. Now the resistance in the circuit is increasing, the current is decreasing and so $\frac{di}{dt}$ is negative. Thus, the numerator of (i) is more than 6 V and hence $i$ is greater than $\frac{6\text{ V}}{12\ \Omega} = 0{\cdot}50$ A.

## 38.6 GROWTH AND DECAY OF CURRENT IN AN LR CIRCUIT

**Growth of Current**

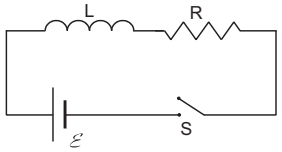



Figure 38.9

Figure (38.9) shows an inductance $L$, a resistance $R$ and a source of emf $\mathcal{E}$ connected in series through a switch $S$. Initially, the switch is open and there is no current in the circuit. At $t = 0$, the switch is closed and the circuit is completed. As the current increases in the inductor, a self-induced emf $\left(-L\frac{di}{dt}\right)$ is produced. Using Kirchhoff's loop law,

$$\mathcal{E} - L\frac{di}{dt} = Ri$$

or, $$L\frac{di}{dt} = \mathcal{E} - Ri$$

or, $$\frac{di}{\mathcal{E} - Ri} = \frac{dt}{L}.$$

At $t = 0, i = 0$ and at time $t$ the current is $i$. Thus,

$$\int_0^i \frac{di}{\mathcal{E} - Ri} = \int_0^t \frac{dt}{L}$$

or, $$-\frac{1}{R}\ln\frac{\mathcal{E} - Ri}{\mathcal{E}} = \frac{t}{L}$$

or, $$\frac{\mathcal{E} - Ri}{\mathcal{E}} = e^{-tR/L}$$

or, $$\mathcal{E} - Ri = \mathcal{E}\, e^{-tR/L}$$

or, $$i = \frac{\mathcal{E}}{R}(1 - e^{-tR/L}). \quad \ldots (38.8)$$

The constant $L/R$ has dimensions of time and is called the *time constant* of the $LR$ circuit. Writing $L/R = \tau$ and $\mathcal{E}/R = i_0$, equation (38.8) becomes

$$i = i_0(1 - e^{-t/\tau}). \quad \ldots (38.9)$$

Figure (38.10) shows the plot of the current versus time. The current gradually rises from $t = 0$ and attains the maximum value $i_0$ after a long time. At $t = \tau$, the current is

$$i = i_0\left(1 - \frac{1}{e}\right) = 0{\cdot}63\ i_0.$$

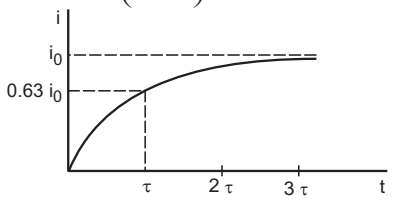



Figure 38.10

Thus, in one time constant, the current reaches 63% of the maximum value. The time constant tells us how fast will the current grow. If the time constant is small, the growth is steep. Equation (38.9) shows that $i = i_0$ at $t = \infty$. In principle, it takes infinite time for the current to attain its maximum value. In practice, however, a small number of time constants may be sufficient for the current to reach almost the maximum value.

***Example 38.5***

*An inductor* ($L = 20$ mH), *a resistor* ($R = 100\ \Omega$) *and a battery* ($\mathcal{E} = 10$ V) *are connected in series. Find (a) the time constant, (b) the maximum current and (c) the time elapsed before the current reaches* 99% *of the maximum value.*

***Solution*** : (a) The time constant is

$$\tau = \frac{L}{R} = \frac{20\text{ mH}}{100\ \Omega} = 0{\cdot}20\text{ ms}.$$

(b) The maximum current is

$$i = \mathcal{E}/R = \frac{10\text{ V}}{100\ \Omega} = 0{\cdot}10\text{ A}.$$

(c) Using $i = i_0(1 - e^{-t/\tau})$,

$$0{\cdot}99\, i_0 = i_0(1 - e^{-t/\tau})$$

or, $$e^{-t/\tau} = 0{\cdot}01$$

or, $$\frac{t}{\tau} = -\ln(0{\cdot}01)$$

or, $$t = (0{\cdot}20\text{ ms})\ln(100) = 0{\cdot}92\text{ ms}.$$

**Decay of Current**

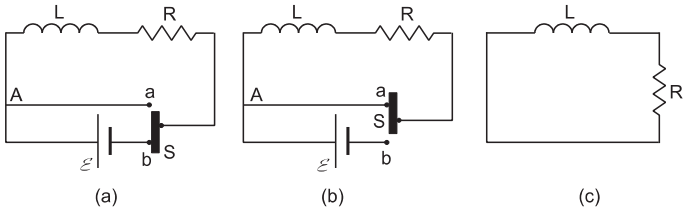



Figure 38.11

Consider the arrangement shown in figure (38.11a). The sliding switch $S$ can be slid up and down. The circuit is complete and a steady current $i = i_0$ is maintained through the circuit. Suddenly at $t = 0$, the switch $S$ is moved to connect the point $a$. This completes the circuit through the wire $Aa$ and disconnects the battery from the circuit (figure 38.11b). The special arrangement of the switch ensures that the circuit through the wire $Aa$ is completed before the battery is disconnected. The equivalent circuit is redrawn in figure (38.11c).

As the battery is disconnected, the current decreases in the circuit. This induces an emf $\left(-L\frac{di}{dt}\right)$ in the inductor. As this is the only emf in the circuit,

$$-L\frac{di}{dt} = Ri$$

or, $$\frac{di}{i} = -\frac{R}{L}dt.$$

At $t = 0, i = i_0$. If the current at time $t$ be $i$,

$$\int_{i_0}^{i} \frac{di}{i} = \int_0^t -\frac{R}{L}dt$$

or, $$\ln\frac{i}{i_0} = -\frac{R}{L}t$$

or, $$i = i_0\, e^{-tR/L} \quad \ldots (38.10)$$

or, $$i = i_0\, e^{-t/\tau} \quad \ldots (38.11)$$

where $\tau = L/R$ is the time constant of the circuit.

We see that the current does not suddenly fall to zero. It gradually decreases as time passes. At $t = \tau$,

$$i = i_0 \ /e = 0{\cdot}37\ i_0.$$

The current reduces to 37% of the initial value in one time constant, i.e., 63% of the decay is complete. If the time constant is small, the decay is steep. Figure (38.12) shows the plot of the current against time.

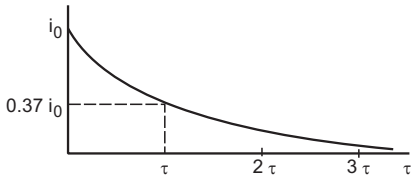



Figure 38.12

***Example 38.6***

*An inductor* ($L = 20$ mH), *a resistor* ($R = 100\ \Omega$) *and a battery* ($\mathcal{E} = 10$ V) *are connected in series. After a long time the circuit is short-circuited and then the battery is disconnected. Find the current in the circuit* 1 ms *after short-circuiting.*

***Solution :***

The initial current is $i = i_0 = \mathcal{E}/R = \dfrac{10\ \text{V}}{100\ \Omega} = 0{\cdot}10\ \text{A}.$

The time constant is $\tau = L/R = \dfrac{20\ \text{mH}}{100\ \Omega} = 0{\cdot}20\ \text{ms}.$

The current at $t = 1$ ms is

$$i = i_0\, e^{-t/\tau}$$
$$= (0{\cdot}10\ \text{A})\, e^{-(1\ \text{ms}/\,0{\cdot}20\ \text{ms})}$$
$$= (0{\cdot}10\ \text{A})\, e^{-5} = 6{\cdot}7 \times 10^{-4}\ \text{A}.$$

## 38.7 ENERGY STORED IN AN INDUCTOR

When a capacitor is charged, electric field builds up between its plates and energy is stored in it. Similarly, when an inductor carries a current, a magnetic field builds up in it and magnetic energy is stored in it.

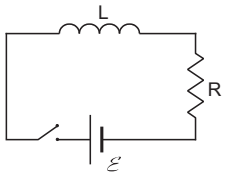



Figure 38.13

Consider the circuit shown in figure (38.13). As the connections are made, the current grows in the circuit and the magnetic field increases in the inductor. Part of the work done by the battery during the process is stored in the inductor as magnetic field energy and the rest appears as thermal energy in the resistor. After sufficient time, the current and hence the magnetic field becomes constant and further work done by the battery appears completely as thermal energy. If $i$ be the current in the circuit at time $t$, we have

$$\mathcal{E} - L\frac{di}{dt} = iR$$

or, $$\mathcal{E}\ i\,dt = i^2 R\,dt + L\,i\,di$$

or, $$\int_0^t \mathcal{E}\ i\,dt = \int_0^t i^2 R\,dt + \int_0^i L\,i\,di$$

or, $$\int_0^t \mathcal{E}\ i\,dt = \int_0^t i^2 R\,dt + \frac{1}{2}L\,i^2. \qquad \ldots \text{(i)}$$

Now ($i\,dt$) is the charge flowing through the circuit during the time $t$ to $t + dt$. Thus, ($\mathcal{E}\ i\,dt$) is the work done by the battery in this period. The quantity on the left-hand side of equation (i) is, therefore, the total work done by the battery in time 0 to $t$. Similarly, the first term on the right-hand side of equation (i) is the total thermal energy (Joule heat) developed in the resistor in time $t$. Thus $\frac{1}{2}Li^2$ is the energy stored in the inductor as the current in it increases from 0 to $i$. As the energy is zero when the current is zero, the energy stored in an inductor, carrying a current $i$, is

$$U = \frac{1}{2}L\,i^2. \qquad \ldots \text{(38.12)}$$

### Energy Density in Magnetic Field

Consider a long solenoid of radius $r$, length $l$ and having $n$ turns per unit length. If it carries a current $i$, the magnetic field within it is

$$B = \mu_0\, ni.$$

Neglecting the end effects, the field outside is zero. The self-inductance of this solenoid is, from equation (38.7),

$$L = \mu_0\, n^2 \pi r^2 l.$$

The magnetic energy is, therefore,

$$U = \frac{1}{2}Li^2 = \frac{1}{2}\mu_0\, n^2 \pi r^2 l\, i^2$$
$$= \frac{1}{2\,\mu_0}(\mu_0\, ni)^2\, V = \frac{B^2}{2\mu_0}V$$

where $V = \pi r^2 l$ is the volume enclosed by the solenoid. As the field is assumed uniform throughout the volume of the solenoid and zero outside, the energy per unit volume in the magnetic field, i.e., the energy density, is

$$u = \frac{U}{V} = \frac{B^2}{2\mu_0}. \qquad \ldots \text{(38.13)}$$

In deriving this equation, we have assumed that there is no magnetic material at the site of the field.

***Example 38.7***

*Calculate the energy stored in an inductor of inductance* 50 mH *when a current of* 2·0 A *is passed through it.*

***Solution*** : The energy stored is

$$U = \frac{1}{2} Li^2 = \frac{1}{2} (50 \times 10^{-3} \text{ H}) (2{\cdot}0 \text{ A})^2 = 0{\cdot}10 \text{ J}.$$

## 38.8 MUTUAL INDUCTION

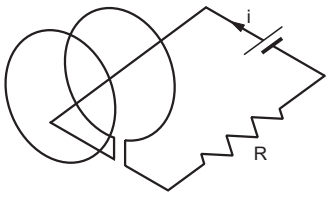



Figure 38.14

Suppose two closed circuits are placed close to each other and a current $i$ is passed in one. It produces a magnetic field and this field has a flux $\Phi$ through the area bounded by the other circuit. As the magnetic field at a point is proportional to the current producing it, we can write

$$\Phi = Mi \qquad \ldots \quad (38.14)$$

where $M$ is a constant depending on the geometrical shapes of the two circuits and their placing. This constant is called *mutual inductance* of the given pair of circuits. If the same current $i$ is passed in the second circuit and the flux is calculated through the area bounded by the first circuit, the same proportionality constant $M$ appears. If there are more than one turns in a circuit, one has to add the flux through each turn before applying equation (38.14).

If the current $i$ in one circuit changes with time, the flux through the area bounded by the second circuit also changes. Thus, an emf is induced in the second circuit. This phenomenon is called *mutual induction.* From equation (38.14), the induced emf is

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

$$= -M \frac{di}{dt}. \qquad \ldots \quad (38.15)$$

***Example 38.8***

*A solenoid $S_1$ is placed inside another solenoid $S_2$ as shown in figure (38.15). The radii of the inner and the outer solenoids are $r_1$ and $r_2$ respectively and the numbers of turns per unit length are $n_1$ and $n_2$ respectively. Consider a length $l$ of each solenoid. Calculate the mutual inductance between them.*

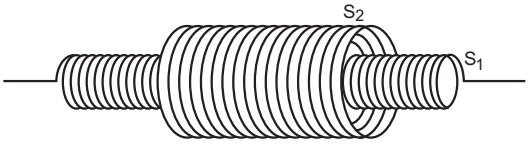



Figure 38.15

***Solution*** : Suppose a current $i$ is passed through the inner solenoid $S_1$. A magnetic field $B = \mu_0 n_1 i$ is produced inside $S_1$ whereas the field outside it is zero. The flux through each turn of $S_2$ is

$$B\pi r_1^2 = \mu_0 n_1 i \pi r_1^2.$$

The total flux through all the turns in a length $l$ of $S_2$ is

$$\Phi = (\mu_0 n_1 i \pi r_1^2) n_2 l = (\mu_0 n_1 n_2 \pi r_1^2 l)\, i.$$

Thus, $$M = \mu_0 n_1 n_2 \pi r_1^2 l. \qquad \ldots \quad \text{(i)}$$

## 38.9 INDUCTION COIL

An induction coil is used to produce a large emf from a source of low emf. The schematic design of an induction coil known as Ruhmkorff's induction coil is shown in figure (38.16). It consists of a primary coil $T$ wound over a laminated soft-iron core and a secondary coil $S$ wound coaxially over the primary coil. The secondary coil is connected to two rods $G_1$ and $G_2$. The separation between the rods may be adjusted.

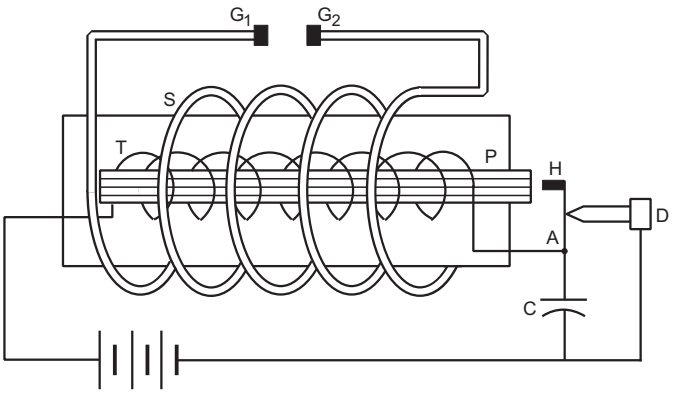



Figure 38.16

The primary circuit contains a battery and a circuit interrupter. The circuit interrupter may be formed as follows. One end of the primary coil is connected to a thin metallic strip $A$ with a soft-iron hammer $H$ at one end. The hammer is close to the soft-iron core of the primary. The other end of the primary is connected to a screw $D$ through the battery. The screw just touches the metallic strip. This arrangement forms the circuit interrupter. A capacitor $C$ is connected in parallel to the circuit interrupter as shown in the figure.

**Working**

(a) *Make and break*: When the screw $D$ touches the strip $A$, the primary circuit is completed and a current is established in the primary circuit. Because of the current in the primary coil, the soft-iron core becomes magnetized. It attracts the iron hammer $H$ and the

contact between the screw $D$ and the strip $A$ is broken. The current stops in the primary circuit, the iron core is demagnetized and the strip acquires its natural position. The screw $D$ again touches the strip and the current is established. This process of successive 'make and break' continues.

(b) *Growth and decay of current in the primary*: When the contact between the screw and the strip is made, the current grows in the primary circuit. Because of the self-inductance of the primary coil, the growth will be slow. When the contact is broken, the current suddenly decreases to zero. The rate of decay of the current when the circuit is broken is quite high as compared to the rate of growth of the current when the circuit is closed.

(c) *Induced emf in the secondary*: As the current $i$ in the primary changes, the magnetic flux linked with the secondary also changes. This induces an emf across the ends of the secondary coil. This emf appears between the rods $G_1$ and $G_2$. When the circuit breaks, $\left|\frac{di}{dt}\right|$ is very large and hence the emf induced in the secondary is also very large. With suitable separation between the rods, one can see sparks jumping from one rod to the other. One can easily produce emf of the order of 50000 V starting from a 12 V battery using the above arrangement. The secondary emf induced at the time of break is in the opposite direction to that induced at the time of make. The high emf produced between the rods can be used to operate equipments like a discharge tube.

(d) *Role of the capacitor*: When the circuit is broken, a large emf is produced due to the self-induction in the primary which tends to drive a current in the direction of the original current. A large potential difference appears between the screw $D$ and the strip $A$. This may cause sparks to jump and hence the current to continue in the same direction. This reduces the rate of decay of the current thereby reducing the emf across the secondary. Secondly, repeated sparks between the screw and the strip damage the surfaces. The capacitor provides an alternative path to the current when the circuit is broken. The current charges the capacitor. Thus, sparks do not occur across the screw–strip gap and the current drops more rapidly. This increases $\left|\frac{di}{dt}\right|$ and hence the induced emf across the secondary. Not only this, the charged capacitor soon gets discharged by sending a current through the primary in the opposite direction. Thus, the currrent changes not only from $i$ to 0 but from $i$ to almost $-i$. The change in flux is almost doubled and correspondingly, the induced emf across the secondary is also increased.

## *Worked Out Examples*

**1.** *A conducting circular loop is placed in a uniform magnetic field* $B = 0{\cdot}020$ T *with its plane perpendicular to the field. Somehow, the radius of the loop starts shrinking at a constant rate of* $1{\cdot}0$ mm s$^{-1}$. *Find the induced emf in the loop at an instant when the radius is* 2 cm.

***Solution*** : Let the radius be $r$ at time $t$. The flux of the magnetic field at this instant is $\Phi = \pi r^2 B$.

Thus, $$\frac{d\Phi}{dt} = 2\pi r B \frac{dr}{dt}.$$

The induced emf when $r = 2{\cdot}0$ cm is, therefore,

$$\mathcal{E} = 2\pi(2\text{ cm})\,(0{\cdot}02\text{ T})\,(1{\cdot}0\text{ mm s}^{-1}) = 2{\cdot}5\ \mu\text{V}.$$

**2.** *A uniform magnetic field* $B$ *exists in a direction perpendicular to the plane of a square frame made of copper wire. The wire has a diameter of* 2 mm *and a total length of* 40 cm. *The magntic field changes with time at a steady rate* $dB/dt = 0{\cdot}02$ T s$^{-1}$. *Find the current induced in the frame. Resistivity of copper* $= 1{\cdot}7 \times 10^{-8}$ Wm.

***Solution*** :

The area $A$ of the loop $= \left(\frac{40\text{ cm}}{4}\right)\left(\frac{40\text{ cm}}{4}\right) = 0{\cdot}01\text{ m}^2$.

If the magnetic field at an instant is $B$, the flux through the frame at that instant will be $\Phi = BA$. As the area remains constant, the magnitude of the emf induced will be

$$\mathcal{E} = \frac{d\Phi}{dt} = A\frac{dB}{dt}$$

$$= (0{\cdot}01\text{ m}^2)\,(0{\cdot}02\text{ T/s}) = 2 \times 10^{-4}\text{ V}.$$

The resistance of the loop is $R = \rho\frac{l}{A}$

$$= \frac{(1{\cdot}7 \times 10^{-8}\ \Omega\text{m})\,(40\text{ cm})}{\pi \times 1\text{ mm}^2}$$

$$= \frac{(1{\cdot}7 \times 10^{-8}\ \Omega\text{m})\,(40 \times 10^{-2}\text{ m})}{3{\cdot}14 \times 1 \times 10^{-6}\text{ m}^2} = 2{\cdot}16 \times 10^{-3}\ \Omega.$$

Hence, the current induced in the loop will be

$$i = \frac{2 \times 10^{-4}\text{ V}}{2{\cdot}16 \times 10^{-3}\ \Omega} = 9{\cdot}3 \times 10^{-2}\text{ A}.$$

**3.** *A conducting circular loop of face area* $2{\cdot}5 \times 10^{-3}$ m$^2$ *is placed perpendicular to a magnetic field which varies as* $B = (0{\cdot}20\text{ T})\sin[(50\pi\text{ s}^{-1})t]$. *(a) Find the charge flowing through any cross-section during the time* $t = 0$ *to* $t = 40$ ms. *(b) If the resistance of the loop is* 10 Ω, *find the thermal energy developed in the loop in this period.*

***Solution*** **:** The face area of the loop is $A = 2{\cdot}5 \times 10^{-3}\ \text{m}^2$ and the magnetic field changes as $B = B_0 \sin \omega t$ where $B_0 = 0{\cdot}20$ T and $\omega = 50\pi\ \text{s}^{-1}$. The resistance of the loop is $R = 10\ \Omega$.

The flux through the loop at time $t$ is

$$\Phi = B_0 A \sin \omega t.$$

The emf induced is

$$\mathcal{E} = -\frac{d\Phi}{dt} = -B_0 A\omega \cos \omega t.$$

The current is $i = \dfrac{\mathcal{E}}{R} = -\dfrac{B_o A\omega}{R} \cos \omega t = -i_0 \cos \omega t.$

The current changes sinusoidally with the time period

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{50\pi\ \text{s}^{-1}} = 40\ \text{ms}.$$

(a) The charge flowing through any cross-section in 40 ms is

$$Q = \int_0^T i dt = -i_0 \int_0^T \cos \omega t\, dt$$

$$= -\frac{i_0}{\omega} [\sin \omega t]_0^T = 0.$$

(b) The thermal energy produced in 40 ms is

$$H = \int_0^T i^2 R dt = i_0^2 R \int_0^T \cos^2 \omega t\, dt$$

$$= \frac{i_0^2 R}{2} \int_0^T (1 + \cos 2\omega t)\, dt$$

$$= \frac{i_0^2 R}{2} \left[ t + \frac{\sin 2\omega t}{2\omega} \right]_0^T$$

$$= \frac{i_0^2 RT}{2} = \frac{B_0^2 A^2 \omega^2}{2R^2} R\left(\frac{2\pi}{\omega}\right) = \frac{\pi B_0^2 A^2 \omega}{R}$$

$$= \frac{\pi \times (0{\cdot}20\ \text{T})^2 \times (2{\cdot}5 \times 10^{-3}\ \text{m}^2)^2 \times (50\pi\ \text{s}^{-1})}{10\ \Omega}$$

$$= 1{\cdot}25 \times 10^{-5}\ \text{J}.$$

*4. A long solenoid of radius* 2 cm *has* 100 turns/cm *and is surrounded by a* 100-turn *coil of radius* 4 cm *having a total resistance of* 20 Ω. *The coil is connected to a galvanometer as shown in figure (38-W1). If the current in the solenoid is changed from* 5 A *in one direction to* 5 A *in the opposite direction, find the charge which flows through the galvanometer.*

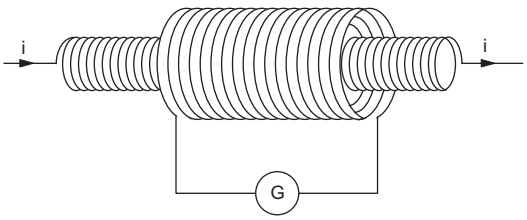



Figure 38-W1

***Solution*** **:** If the current in the solenoid is $i$, the magnetic field inside the solenoid is $B = \mu_0 ni$ parallel to its axis. Outside the solenoid, the field will be zero. The flux of the magnetic field through the coil will be $\Phi = B\pi r^2 N$ where $r$ is the radius of the solenoid and $N$ is the number of turns in the coil. The induced emf will have magnitude

$$\frac{d\Phi}{dt} = N\pi r^2 \frac{dB}{dt} = \pi r^2 N \mu_0 n \frac{di}{dt}.$$

If $R$ denotes the resistance of the coil, the current through the galvanometer is

$$I = \frac{\pi r^2 N}{R} \mu_0 n \frac{di}{dt}$$

or, $$I\, dt = \frac{\pi r^2 N}{R} \mu_0 n\, di.$$

The total charge passing through the galvanometer is

$$\Delta Q = \int I\, dt = \frac{\pi r^2 N}{R} \mu_0 n \int di$$

$$= \frac{\pi r^2 N \mu_0 n}{R} \Delta i$$

$$= \frac{\pi(2\ \text{cm})^2 \times 100 \times 4\pi \times 10^{-7}\ \text{TmA}^{-1} \times 100\ \text{cm}^{-1} \times 10\ \text{A}}{20\ \Omega}$$

$$\approx 8 \times 10^{-4}\ \text{C} = 800\ \mu\text{C}.$$

**5.** *The magnetic field B shown in figure (38-W2) is directed into the plane of the paper. ACDA is a semicircular conducting loop of radius r with the centre at O. The loop is now made to rotate clockwise with a constant angular velocity* ω *about an axis passing through O and perpendicular to the plane of the paper. The resistance of the loop is R. Obtain an expression for the magnitude of the induced current in the loop. Plot a graph between the induced current i and* ω*t, for two periods of rotation.*

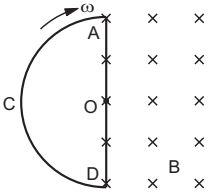



Figure 38-W2

***Solution*** **:** When the loop rotates through an angle θ, which is less than π (figure 38-W3a), the area inside the field region is

$$A = \frac{\theta}{\pi} \frac{\pi r^2}{2} = \frac{\theta r^2}{2} = \frac{\omega t r^2}{2}.$$

The flux of the magnetic field at time $t$ is

$$\Phi = BA = B \frac{\omega t r^2}{2}.$$

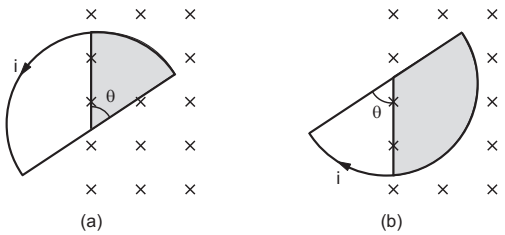



Figure 38-W3

The induced emf $= -\dfrac{d\Phi}{dt} = -\dfrac{B\omega r^2}{2}$.

The magnitude of the induced current will be

$$i = \frac{B\omega r^2}{2R}.$$

As the flux is increasing, the direction of the induced current will be anticlockwise so that the field due to the induced current is opposite to the original field.

After half a rotation, the area in the field region will start decreasing (figure 38-W3b) and will be given by

$$A(t) = \frac{\pi r^2}{2} - \frac{\omega t r^2}{2}.$$

Hence, the induced current will have the same magnitude but opposite sense. The plot for two time periods is shown in figure (38-W4).

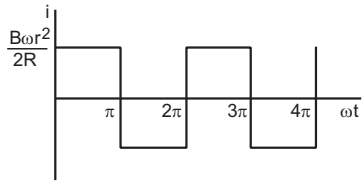



Figure 38-W4

**6.** *Figure (38-W5) shows a square loop having* 100 *turns, an area of* $2{\cdot}5 \times 10^{-3}$ m$^2$ *and a resistance of* 100 Ω. *The magnetic field has a magnitude* $B = 0{\cdot}40$ T. *Find the work done in pulling the loop out of the field, slowly and uniformly in* 1·0 s.

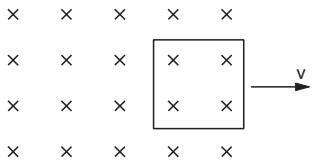



Figure 38-W5

***Solution*** :

The side of the square is

$$l = \sqrt{2{\cdot}5 \times 10^{-3}\ \text{m}^2} = 0{\cdot}05\ \text{m}.$$

As it is uniformly pulled out in 1·0 s, the speed of the loop is

$$v = 0{\cdot}05\ \text{m s}^{-1}.$$

The emf induced in the left arm of the loop is

$$\mathcal{E} = NvBl$$
$$= 100 \times (0{\cdot}05\ \text{m s}^{-1}) \times (0{\cdot}40\ \text{T}) \times (0{\cdot}05\ \text{m})$$
$$= 0{\cdot}1\ \text{V}.$$

The current in the loop is

$$i = \frac{0{\cdot}1\ \text{V}}{100\ \Omega} = 1{\cdot}0 \times 10^{-3}\ \text{A}.$$

The force on the left arm due to the magnetic field is

$$F = NilB = 100 \times (1{\cdot}0 \times 10^{-3}\ \text{A}) \times (0{\cdot}05\ \text{m}) \times (0{\cdot}40\ \text{T})$$
$$= 2{\cdot}0 \times 10^{-3}\ \text{N}.$$

This force is towards left in the figure. To pull the loop uniformly, an external force of $2{\cdot}0 \times 10^{-3}$ N towards right must be applied. The work done by this force is

$$W = (2{\cdot}0 \times 10^{-3}\ \text{N}) \times (0{\cdot}05\ \text{m}) = 1{\cdot}0 \times 10^{-4}\ \text{J}.$$

**7.** *Magadh Express takes* 16 hours *to cover the distance of* 960 km *between Patna and Gaziabad. The rails are separated by* 130 cm *and the vertical component of the earth's magnetic field is* $4{\cdot}0 \times 10^{-5}$ T. *(a) Find the average emf induced across the width of the train. (b) If the leakage resistance between the rails is* 100 Ω, *find the retarding force on the train due to the magnetic field.*

***Solution*** : As the train moves in a magnetic field, a motional emf $\mathcal{E} = vBl$ is produced across its width. Here $B$ is the component of the magnetic field in a direction perpendicular to the plane of the motion, i.e., the vertical component.

The speed of the train is $v = \dfrac{960\ \text{km}}{16\ \text{h}} = 16{\cdot}67\ \text{m s}^{-1}$.

Thus, $\mathcal{E} = (16{\cdot}67\ \text{m s}^{-1})\ (4{\cdot}0 \times 10^{-5}\ \text{T})\ (1{\cdot}30\ \text{m})$

$$= 8{\cdot}6 \times 10^{-4}\ \text{V}.$$

The leakage current is $i = \mathcal{E}/R$ and the retarding force is

$$F = ilB = \frac{8{\cdot}6 \times 10^{-4}\ \text{V}}{100\ \Omega} \times 1{\cdot}3\ \text{m} \times 4{\cdot}0 \times 10^{-5}\ \text{T}$$
$$= 4{\cdot}47 \times 10^{-10}\ \text{N}.$$

**8.** *A square loop of edge a having n turns is rotated with a uniform angular velocity* ω *about one of its diagonals which is kept fixed in a horizontal position (figure 38-W6). A uniform magnetic field B exists in the vertical direction. Find the emf induced in the coil.*

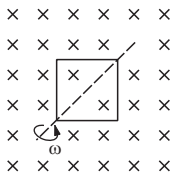



Figure 38-W6

***Solution*** : The area of the square frame is $A = a^2$. If the normal to the frame makes an angle $\theta = 0$ with the magnetic field at $t = 0$, this angle will become $\theta = \omega t$ at time $t$. The flux of the magnetic field at this time is

$$\Phi = nBA\cos\theta = nBa^2\cos\omega t.$$

The induced emf is

$$\mathcal{E} = -\frac{d\Phi}{dt} = nBa^2\omega\sin\omega t.$$

Thus, an alternating emf is induced in the coil.

**9.** *A conducting circular loop of radius r is rotated about its diameter at a constant angular velocity* ω *in a*

*magnetic field B perpendicular to the axis of rotation. In what position of the loop is the induced emf zero ?*

***Solution*** **:** Suppose, the normal to the loop is parallel to the magnetic field at $t = 0$. At time $t$, the normal will make an angle $\theta = \omega t$ with this position. The flux of the magnetic field through the loop is

$$\Phi = B\pi r^2 \cos \omega t$$

so that the induced emf at time $t$ is

$$\mathcal{E} = \omega B\pi r^2 \sin \omega t.$$

This is zero when $\omega t = n\pi$, i.e., when $\theta = 0, \pi, 2\pi, ...$, etc. These are the positions when the plane of the loop is normal to the magnetic field. It may be noted that at these positions, the flux has the maximum magnitude.

**10.** *Figure (38-W7) shows a horizontal magnetic field which is uniform above the dotted line and is zero below it. A long, rectangular, conducting loop of width L, mass m and resistance R is placed partly above and partly below the dotted line with the lower edge parallel to it. With what velocity should it be pushed downwards so that it may continue to fall without any acceleration ?*

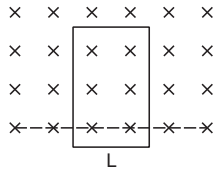



Figure 38-W7

***Solution*** **:** Let the uniform velocity of fall be $v$. The emf is induced across the upper wire and its magnitude is $\mathcal{E} = vBl$. The current induced in the frame is

$$i = \frac{vBl}{R}$$

so that, the magnetic force on the upper arm is $F = ilB = \frac{vB^2l^2}{R}$. This force is in the upward direction. As the frame falls uniformly, this force should balance its weight. Thus,

$$mg = \frac{vB^2l^2}{R}$$

or,
$$v = \frac{mgR}{B^2l^2}.$$

**11.** *Figure (38-W8a) shows a wire of length l which can slide on a U-shaped rail of negligible resistance. The resistance of the wire is R. The wire is pulled to the right with a constant speed v. Draw an equivalent circuit diagram representing the induced emf by a battery. Find the current in the wire using this diagram.*

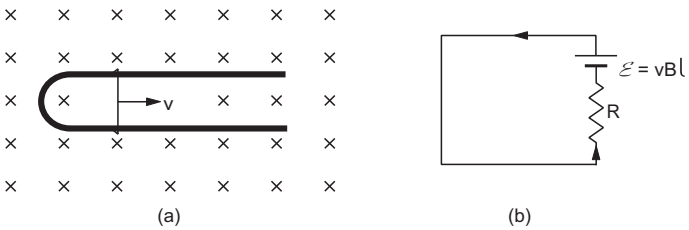



Figure 38-W8

***Solution*** **:** The emf is induced due to the moving wire. The magnitude of this emf is $\mathcal{E} = vBl$. As the wire moves towards right, the force $q\vec{v} \times \vec{B}$ on a positive charge acts in the upward direction in the figure. The positive terminal of the equivalent battery appears upwards. The resistance of the wire acts as the internal resistance of the equivalent battery. The equivalent circuit is drawn in figure (38-W8b).

The current in the circuit is, from Ohm's law,

$$i = \frac{\mathcal{E}}{R} = \frac{vBl}{R}.$$

It is in the upward direction along the wire.

**12.** *A rod of length l is translating at a velocity v making an angle θ with its length. A uniform magnetic field B exists in a direction perpendicular to the plane of motion. Calculate the emf induced in the rod. Draw a figure representing the induced emf by an equivalent battery.*

***Solution*** **:**

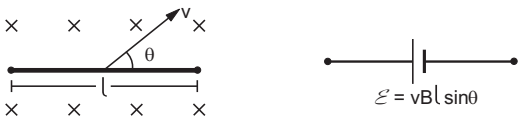



Figure 38-W9

The situation is shown in figure (38-W9a). The component of the velocity perpendicular to the length of the rod is $v_\perp = v \sin\theta$. Only this component is effective in producing the emf in the rod. As the magnetic field is perpendicular to the plane of motion, the emf induced across the ends is

$$\mathcal{E} = v_\perp Bl = vBl \sin\theta.$$

In the figure shown, the positive charges of the rod shift towards left due to the force $q\vec{v} \times \vec{B}$. Thus, the left side of the rod is electrically positive. Figure (38-W9b) shows the equivalent battery.

**13.** *The horizontal component of the earth's magnetic field at a place is* $3{\cdot}0 \times 10^{-4}$ T *and the dip is* 53°. *A metal rod of length* 25 cm *is placed in the north–south direction and is moved at a constant speed of* 10 cm $s^{-1}$ *towards east. Calculate the emf induced in the rod.*

***Solution*** **:** The induced emf is $\mathcal{E} = vBl$ where $l$ is the length of the rod, $v$ is its speed in the perpendicular direction and $B$ is the component of the magnetic field perpendicular to both $l$ and $v$. In the present case, $B$ is the vertical component of the earth's magnetic field.

The dip at a place is given by

$$\tan\delta = \frac{B_V}{B_H}$$

or, $B_V = B_H \tan\delta$

$= (3{\cdot}0 \times 10^{-4}\ \text{T}) \tan 53^\circ = 4{\cdot}0 \times 10^{-4}\ \text{T}.$

The emf induced is $\mathcal{E} = vBl$

$= (0{\cdot}10\ \text{m s}^{-1}) \times (4{\cdot}0 \times 10^{-4}\ \text{T}) \times (0{\cdot}25\ \text{m})$

$= 1{\cdot}0 \times 10^{-5}\ \text{V} = 10\ \mu\text{V}.$

**14.** *An angle aob made of a conducting wire moves along its bisector through a magnetic field B as suggested by figure (38-W10a). Find the emf induced between the two free ends if the magnetic field is perpendicular to the plane of the angle.*

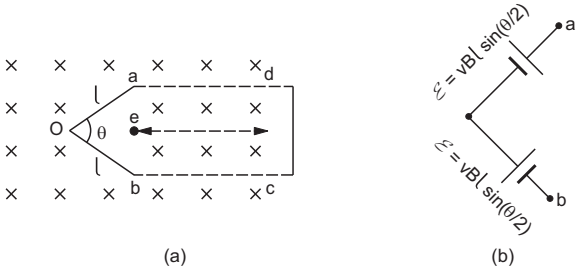



Figure 38-W10

***Solution*** **:** Consider the circuit as being closed externally, as shown in the figure. If $ad = x$, the area of the rectangular part $abcd$ is $A = 2xl \sin(\theta/2)$. As the angle moves towards right, the flux of the magnetic field through this rectangular area decreases at the rate

$$\frac{d\Phi}{dt} = B\frac{dA}{dt} = 2Blv \sin(\theta/2).$$

This is also the rate of decrease of the flux through the closed circuit shown in the figure. So the induced emf is

$$\mathcal{E} = 2Blv \sin(\theta/2).$$

As the emf is induced solely because of the movement of the angle, this is the emf induced between its ends.

**Alternative method**

The rod $oa$ is equivalent to a battery of emf $vBl \sin(\theta/2)$. The positive charges of $oa$ shift towards $a$ due to the force $q\vec{v} \times \vec{B}$. The positive terminal of the battery appears towards $a$. Similarly, the rod $ob$ is equivalent to a battery of emf $vBl \sin(\theta/2)$ with the positive terminal towards $o$. The equivalent circuit is shown in figure (38-W10b). Clearly, the emf between the points $a$ and $b$ is $2\,Blv \sin(\theta/2)$.

**15.** *Figure (38-W11a) shows a wire ab of length l and resistance R which can slide on a smooth pair of rails.*

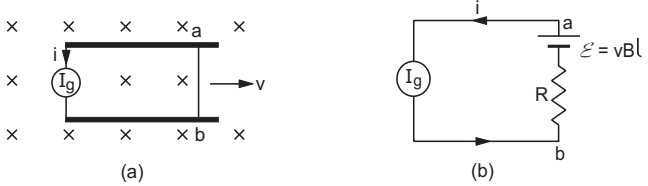



Figure 38-W11

*$I_g$ is a current generator which supplies a constant current i in the circuit. If the wire ab slides at a speed v towards right, find the potential difference between a and b.*

***Solution*** **:** The moving wire $ab$ is equivalent to a battery of emf $vBl$ having a resistance $R$. If it moves towards right and the magnetic field is going into the plane of the figure, the force $q\vec{v} \times \vec{B}$ will push the positive charges towards $a$. Thus the positive terminal of the equivalent battery is towards $a$. An equivalent circuit is shown in figure (38-W13b). The potential difference between $a$ and $b$ is

$$V_a - V_b = vBl - iR.$$

**16.** *A square loop of side* 10 cm *and resistance* 1 Ω *is moved towards right with a constant velocity $v_0$ as shown in figure (38-W12). The left arm of the loop is in a uniform magnetic field of* 2 T. *The field is perpendicular to the plane of the drawing and is going into it. The loop is connected to a network of resistors each of value* 3 Ω. *With what speed should the loop be moved so that a steady current of* 1 mA *flows in the loop.*

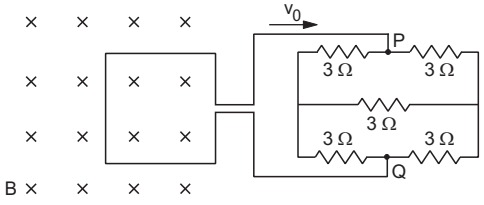



Figure 38-W12

***Solution*** **:** The equivalent resistance of the network of the resistors, between $P$ and $Q$ will be 3 Ω. The total resistance of the circuit is 1 Ω + 3 Ω = 4 Ω.

The emf induced in the loop is

$$\mathcal{E} = vBl = v_0(2\ \text{T})\ (10\ \text{cm}).$$

The current in the loop will be $i = \frac{\mathcal{E}}{R}$

or, $$1 \times 10^{-3}\ \text{A} = \frac{v_0(2\ \text{T})\ (0{\cdot}1\ \text{m})}{4\ \Omega}$$

giving $v_0 = \dfrac{(4\ \Omega)\ (1 \times 10^{-3}\ \text{A})}{0{\cdot}2\ \text{Tm}} = 2\ \text{cm s}^{-1}.$

**17.** *A metal rod of length l rotates about an end with a uniform angular velocity* ω. *A uniform magnetic field $\vec{B}$ exists in the direction of the axis of rotation. Calculate the emf induced between the ends of the rod. Neglect the centripetal force acting on the free electrons as they move in circular paths.*

***Solution*** **:**

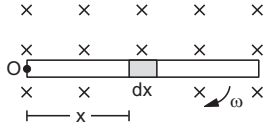



Figure 38-W13

Consider an element $dx$ of the rod at a distance $x$ from the axis of rotation. The linear speed of this element is $\omega x$. The element moves in a direction perpendicular to its length as well as perpendicular to the magnetic field. The emf induced between the ends of this element is

$$d\mathcal{E} = B\,\omega x\,dx.$$

The emfs of all such elements will add to give the net emf between the ends of the rod. This emf is, therefore,

$$\mathcal{E} = \int d\mathcal{E} = \int_0^l B\,\omega x\,dx = \frac{1}{2}B\omega l^2.$$

**18.** *Figure (38-W14a) shows a conducting circular loop of radius a placed in a uniform, perpendicular magnetic field B. A metal rod OA is pivoted at the centre O of the loop. The other end A of the rod touches the loop. The rod OA and the loop are resistanceless but a resistor having a resistance R is connected between O and a fixed point C on the loop. The rod OA is made to rotate anticlockwise at a small but uniform angular speed ω by an external force. Find (a) the current in the resistance R and (b) the torque of the external force needed to keep the rod rotating with the constant angular velocity ω.*

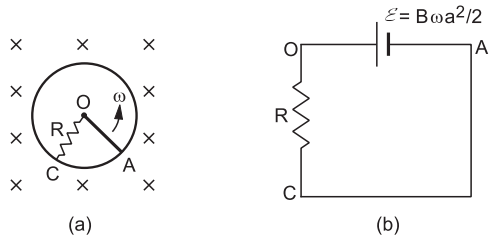



Figure 38-W14

***Solution*** **:** The emf between the ends of the rotating rod is

$$\mathcal{E} = \int d\mathcal{E} = \int_0^a B\omega\,x\,dx = \frac{1}{2}B\omega\,a^2.$$

The positive charges of the rod will be pushed towards $O$ by the magnetic field. Thus, the rod may be replaced by a battery of emf $= \frac{1}{2}B\omega\,a^2$ with the positive terminal towards $O$. The equivalent circuit diagram is shown in figure (38-W14b). The circular loop joins $A$ to $C$ by a resistanceless path.

(a) The current in the resistance $R$ is

$$i = \frac{\mathcal{E}}{R} = \frac{B\omega\,a^2}{2R}.$$

(b) The force on the rod due to the magnetic field is $F = iaB$. As the force is uniformly distributed over $OA$, it may be assumed to act at the middle point of $OA$. The torque is, therefore,

$$\Gamma = (iaB)\frac{a}{2} = \frac{B^2\omega\,a^4}{4R}$$

in clockwise direction. To keep the rod rotating at uniform angular velocity, an external torque $\frac{B^2\omega\,a^4}{4R}$ in anticlockwise direction is needed.

**19.** *Figure (38-W15) shows a conducting loop abcdefa made of six segments ab, bc, cd, de, ef and fa, each of length l. Each segment makes a right angle with the next so that abc is in the x–z plane, cde in the x–y plane and efa is in the y–z plane. A uniform magnetic field B exists along the x-axis. If the magnetic field changes at a rate $\frac{dB}{dt}$, find the emf induced in the loop.*

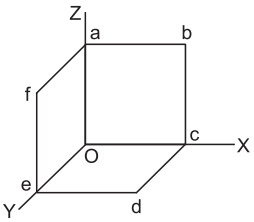



Figure 38-W15

***Solution*** **:** As the magnetic field is along the $x$-axis, the flux through the loop is equal to the magnetic field multiplied by the area of projection of the loop on the $y$–$z$ plane. This projection on the $y$–$z$ plane will be *aoef* which has an area $l^2$. Thus, the flux is $\Phi = Bl^2$. The induced emf is $\mathcal{E} = \frac{dB}{dt}l^2$.

**20.** *A wire of mass m and length l can freely slide on a pair of parallel, smooth, horizontal rails placed in a vertical magnetic field B (figure 38-W16). The rails are connected by a capacitor of capacitance C. The electric resistance of the rails and the wire is zero. If a constant force F acts on the wire as shown in the figure, find the acceleration of the wire.*

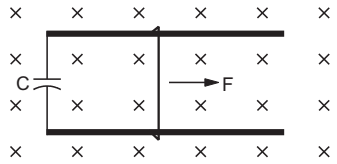



Figure 38-W16

***Solution*** **:** Suppose the velocity of the wire is $v$ at time $t$. The induced emf is $\mathcal{E} = vBl$. As there is no resistance anywhere, the charge on the capacitor will be

$$q = C\mathcal{E} = CvBl$$

at time $t$. The current in the circuit will be

$$i = \frac{dq}{dt} = CBl\frac{dv}{dt} = CBla.$$

Because of this current through the wire, there will be a magnetic force

$$F' = ilB = CB^2l^2a$$

towards left. The net force on the wire is $F - F'$.

From Newton's law,

$$F - F' = ma$$

or, $F - CB^2 l^2 a = ma$

or, $a = \dfrac{F}{m + CB^2 l^2}$.

**21.** *An inductor coil stores* 32 J *of magnetic field energy and dissipates energy as heat at the rate of* 320 W *when a current of* 4 A *is passed through it. Find the time constant of the circuit when this coil is joined across an ideal battery.*

***Solution*** : The magnetic field energy stored in an inductor is $U = \frac{1}{2} Li^2$.

Thus, $32 \text{ J} = \frac{1}{2} L (4 \text{ A})^2$

or, $L = 4 \text{ H}$.

The power dissipated as heat is given by

$$P = i^2 R$$

or, $320 \text{ W} = (4 \text{ A})^2 R$, giving $R = 20 \ \Omega$.

The time constant of the circuit is

$$\tau = \frac{L}{R} = \frac{4 \text{ H}}{20 \ \Omega} = 0{\cdot}2 \text{ s}.$$

**22.** *A* 12 V *battery connected to a* 6 Ω, 10 H *coil through a switch drives a constant current in the circuit. The switch is suddenly opened. Assuming that it took* 1 ms *to open the switch, calculate the average emf induced across the coil.*

***Solution*** : The steady-state current is $\frac{12 \text{ V}}{6 \ \Omega} = 2 \text{ A}$. The final current is zero. Thus,

$$\frac{di}{dt} = -\frac{2 \text{ A}}{1 \text{ ms}} = -2 \times 10^3 \text{ A s}^{-1}.$$

The induced emf is $\mathcal{E} = -\frac{d\Phi}{dt} = -L \frac{di}{dt}$

$$= -(10 \text{ H}) \times (-2 \times 10^3 \text{ A s}^{-1}) = 20000 \text{V}.$$

Such a high emf may cause sparks across the open switch.

**23.** *A solenoid of inductance* 50 mH *and resistance* 10 Ω *is connected to a battery of* 6 V. *Find the time elapsed before the current acquires half of its steady-state value.*

***Solution*** : The time constant of the circuit is

$$\tau = L/R = 50 \text{ mH}/10 \ \Omega = 5 \text{ ms}.$$

The current at time $t$ is given by

$$i = i_0 (1 - e^{-t/\tau}).$$

For $i = i_0/2$,

$$i_0/2 = i_0 (1 - e^{-t/\tau})$$

or, $e^{-t/\tau} = \frac{1}{2}$

or, $\frac{t}{\tau} = \ln 2$

giving $t = \tau \ln 2 = (5 \text{ ms})(0{\cdot}693) = 3{\cdot}5 \text{ ms}$.

**24.** *An LR circuit having* $L = 4{\cdot}0$ H, $R = 1{\cdot}0 \ \Omega$ *and* $\mathcal{E} = 6{\cdot}0$ V *is switched on at* $t = 0$. *Find the power dissipated in Joule heating at* $t = 4{\cdot}0$ s.

***Solution*** : The time constant of the circuit is

$$\tau = L/R = 4{\cdot}0 \text{ H}/1{\cdot}0 \ \Omega = 4{\cdot}0 \text{ s}.$$

The current at $t = 4{\cdot}0$ s is, therefore,

$$i = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau}) = (6 \text{ A}) \left(1 - \frac{1}{e}\right)$$

$$= (6 \text{ A}) \times (0{\cdot}63) = 3{\cdot}8 \text{ A}.$$

The power dissipated in Joule heating $= i^2 R$

$$= (3{\cdot}8 \text{ A})^2 \times 10 \ \Omega \approx 140 \ \Omega.$$

**25.** *An LR combination is connected to an ideal battery. If* $L = 10$ mH, $R = 2{\cdot}0 \ \Omega$ *and* $\mathcal{E} = 2{\cdot}0$ V, *how much time will it take for the current to reach* 0·63 A ?

***Solution*** : The steady-state current in the $LR$ circuit is

$$i_0 = \mathcal{E}/R = 2{\cdot}0 \text{ V}/2{\cdot}0 \ \Omega = 1 \text{ A}.$$

Thus, 0·63 A is 63% of the steady-state current. As we know, it takes one time constant for the current to reach 63% of its steady-state value. Hence the required time

$$= L/R = 10 \text{ mH}/2{\cdot}0 \ \Omega = 5{\cdot}0 \text{ ms}.$$

**26.** *An inductor–resistance–battery circuit is switched on at* $t = 0$. *If the emf of the battery is* $\mathcal{E}$, *find the charge which passes through the battery in one time constant* τ.

***Solution*** : The current at time $t$ is given by

$$i = i_0 (1 - e^{-t/\tau}) \text{ where } i_0 = \mathcal{E}/R.$$

The charge passed through the battery during the period $t$ to $t + dt$ is $i \, dt$. Thus, the total charge passed during 0 to τ is

$$Q = \int_0^\tau i \, dt = i_0 \int_0^\tau (1 - e^{-t/\tau}) \, dt = i_0 \left[ t - \frac{e^{-t/\tau}}{-1/\tau} \right]_0^\tau$$

$$= i_0 [\tau + \tau (e^{-1} - 1)] = i_0 \tau / e.$$

**27.** *A coil of inductance* 1·0 H *and resistance* 100 Ω *is connected to a battery of emf* 12 V. *Find the energy stored in the magnetic field associated with the coil at an instant* 10 ms *after the circuit is switched on.*

***Solution*** : The energy in the magnetic field associated with the coil is

$$U = \frac{1}{2} Li^2 = \frac{1}{2} L \left[ \frac{\mathcal{E}}{R} (1 - e^{-t/\tau}) \right]^2. \quad \ldots \text{(i)}$$

The time constant of the circuit is

$$\tau = \frac{L}{R} = \frac{1{\cdot}0 \text{ H}}{100 \ \Omega} = 10 \text{ ms}.$$

Putting the numerical values in (i), the energy at $t = 10$ ms is

$$\frac{1}{2} \times (1{\cdot}0 \text{ H}) \times [0{\cdot}12 \text{ A}(1 - 1/e)]^2$$

$$= 2{\cdot}8 \text{ mJ}.$$

**28.** *An inductance L and a resistance R are connected in series with a battery of emf $\mathcal{E}$. Find the maximum rate at which the energy is stored in the magnetic field.*

***Solution :***

The energy stored in the magnetic field at time $t$ is

$$U = \frac{1}{2} Li^2 = \frac{1}{2} Li_0^2 (1 - e^{-t/\tau})^2.$$

The rate at which the energy is stored is

$$P = \frac{dU}{dt} = Li_0^2 (1 - e^{-t/\tau})(-e^{-t/\tau})\left(-\frac{1}{\tau}\right)$$

$$= \frac{Li_0^2}{\tau}(e^{-t/\tau} - e^{-2t/\tau}). \quad \ldots \text{(i)}$$

This rate will be maximum when $\frac{dP}{dt} = 0$

or, $$\frac{Li_0^2}{\tau}\left(-\frac{1}{\tau}e^{-t/\tau} + \frac{2}{\tau}e^{-2t/\tau}\right) = 0$$

or, $$e^{-t/\tau} = \frac{1}{2}.$$

Putting in (i),

$$P_{\max} = \frac{Li_0^2}{\tau}\left(\frac{1}{2} - \frac{1}{4}\right)$$

$$= \frac{L\mathcal{E}^2}{4R^2(L/R)} = \frac{\mathcal{E}^2}{4R}.$$

**29.** *Two conducting circular loops of radii $R_1$ and $R_2$ are placed in the same plane with their centres coinciding. Find the mutual inductance between them assuming $R_2 << R_1$.*

***Solution :*** Suppose a current $i$ is established in the outer loop. The magnetic field at the centre will be

$$B = \frac{\mu_0 i}{2R_1}.$$

As the radius $R_2$ of the inner coil is small compared to $R_1$, the flux of magnetic field through it will be approximately

$$\Phi = \frac{\mu_0 i}{2R_1} \pi R_2^2$$

so that the mutual inductance is

$$M = \frac{\Phi}{i} = \frac{\mu_0 \pi R_2^2}{2R_1}.$$

□

## QUESTIONS FOR SHORT ANSWER

**1.** A metallic loop is placed in a nonuniform magnetic field. Will an emf be induced in the loop ?

**2.** An inductor is connected to a battery through a switch. Explain why the emf induced in the inductor is much larger when the switch is opened as compared to the emf induced when the switch is closed.

**3.** The coil of a moving-coil galvanometer keeps on oscillating for a long time if it is deflected and released. If the ends of the coil are connected together, the oscillation stops at once. Explain.

**4.** A short magnet is moved along the axis of a conducting loop. Show that the loop repels the magnet if the magnet is approaching the loop and attracts the magnet if it is going away from the loop.

**5.** Two circular loops are placed coaxially but separated by a distance. A battery is suddenly connected to one of the loops establishing a current in it. Will there be a current induced in the other loop ? If yes, when does the current start and when does it end ? Do the loops attract each other or do they repel ?

**6.** The battery discussed in the previous question is suddenly disconnected. Is a current induced in the other loop ? If yes, when does it start and when does it end ? Do the loops attract each other or repel ?

**7.** If the magnetic field outside a copper box is suddenly changed, what happens to the magnetic field inside the box ? Such low-resistivity metals are used to form enclosures which shield objects inside them against varying magnetic fields.

**8.** Metallic (nonferromagnetic) and nonmetallic particles in a solid waste may be separated as follows. The waste is allowed to slide down an incline over permanent magnets. The metallic particles slow down as compared to the nonmetallic ones and hence are separated. Discuss the role of eddy currents in the process.

**9.** A pivoted aluminium bar falls much more slowly through a small region containing a magnetic field than a similar bar of an insulating material. Explain.

**10.** A metallic bob $A$ oscillates through the space between the poles of an electromagnet (figure 38-Q1). The oscillations are more quickly damped when the circuit

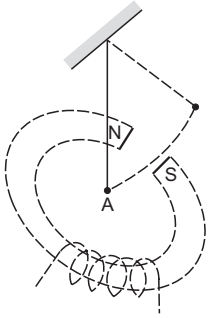



Figure 38-Q1

is on, as compared to the case when the circuit is off. Explain.

**11.** Two circular loops are placed with their centres separated by a fixed distance. How would you orient the loops to have (a) the largest mutual inductance (b) the smallest mutual inductance ?

**12.** Consider the self-inductance per unit length of a solenoid at its centre and that near its ends. Which of the two is greater ?

**13.** Consider the energy density in a solenoid at its centre and that near its ends. Which of the two is greater ?

## OBJECTIVE I

**1.** A rod of length $l$ rotates with a small but uniform angular velocity ω about its perpendicular bisector. A uniform magnetic field $B$ exists parallel to the axis of rotation. The potential difference between the centre of the rod and an end is

(a) zero (b) $\frac{1}{8}\omega Bl^2$ (c) $\frac{1}{2}\omega Bl^2$ (d) $B\omega l^2$.

**2.** A rod of length $l$ rotates with a uniform angular velocity ω about its perpendicular bisector. A uniform magnetic field $B$ exists parallel to the axis of rotation. The potential difference between the two ends of the rod is

(a) zero (b) $\frac{1}{2}Bl\omega^2$ (c) $Bl\omega^2$ (d) $2Bl\omega^2$.

**3.** Consider the situation shown in figure (38-Q2). If the switch is closed and after some time it is opened again, the closed loop will show

(a) an anticlockwise current-pulse
(b) a clockwise current-pulse
(c) an anticlockwise current-pulse and then a clockwise current-pulse
(d) a clockwise current-pulse and then an anticlockwise current-pulse.

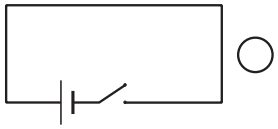


Figure 38-Q2

**4.** Solve the previous question if the closed loop is completely enclosed in the circuit containing the switch.

**5.** A bar magnet is released from rest along the axis of a very long, vertical copper tube. After some time the magnet

(a) will stop in the tube
(b) will move with almost contant speed
(c) will move with an acceleration $g$
(d) will oscillate.

**6.** Figure (38-Q3) shows a horizontal solenoid connected to a battery and a switch. A copper ring is placed on a frictionless track, the axis of the ring being along the axis of the solenoid. As the switch is closed, the ring will

(a) remain stationary
(b) move towards the solenoid
(c) move away from the solenoid
(d) move towards the solenoid or away from it depending on which terminal (positive or negative) of the battery is connected to the left end of the solenoid.

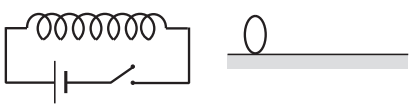


Figure 38-Q3

**7.** Consider the following statements:

(A) An emf can be induced by moving a conductor in a magnetic field.
(B) An emf can be induced by changing the magnetic field.

(a) Both A and B are true. (b) A is true but B is false.
(c) B is true but A is false. (d) Both A and B are false.

**8.** Consider the situation shown in figure (38-Q4). The wire $AB$ is slid on the fixed rails with a constant velocity. If the wire $AB$ is replaced by a semicircular wire, the magnitude of the induced current will

(a) increase (b) remain the same (c) decrease
(d) increase or decrease depending on whether the semi-circle bulges towards the resistance or away from it.

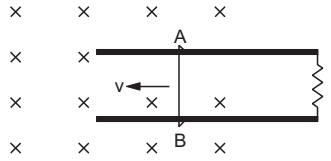



Figure 38-Q4

**9.** Figure (38-Q5a) shows a conducting loop being pulled out of a magnetic field with a speed $v$. Which of the four plots shown in figure (38-Q5b) may represent the power delivered by the pulling agent as a function of the speed $v$ ?

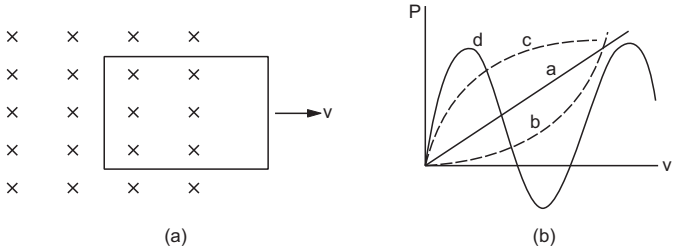



Figure 38-Q5

**10.** Two circular loops of equal radii are placed coaxially at some separation. The first is cut and a battery is inserted in between to drive a current in it. The current changes slightly because of the variation in resistance with temperature. During this period, the two loops

(a) attract each other (b) repel each other
(c) do not exert any force on each other
(d) attract or repel each other depending on the sense of the current.

**11.** A small, conducting circular loop is placed inside a long solenoid carrying a current. The plane of the loop contains the axis of the solenoid. If the current in the solenoid is varied, the current induced in the loop is
(a) clockwise (b) anticlockwise (c) zero
(d) clockwise or anticlockwise depending on whether the resistance is increased or decreased.

**12.** A conducting square loop of side $l$ and resistance $R$ moves in its plane with a uniform velocity $v$ perpendicular to one of its sides. A uniform and constant magnetic field $B$ exists along the perpendicular to the plane of the loop as shown in figure (38-Q6). The current induced in the loop is
(a) $Blv/R$ clockwise (b) $Blv/R$ anticlockwise
(c) $2Blv/R$ anticlockwise (d) zero.

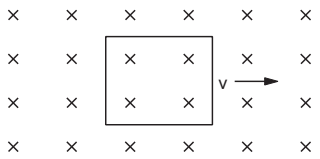


Figure 38-Q6

## OBJECTIVE II

**1.** A bar magnet is moved along the axis of a copper ring placed far away from the magnet. Looking from the side of the magnet, an anticlockwise current is found to be induced in the ring. Which of the following may be true ?
(a) The south pole faces the ring and the magnet moves towards it.
(b) The north pole faces the ring and the magnet moves towards it.
(c) The south pole faces the ring and the magnet moves away from it.
(d) The north pole faces the ring and the magnet moves away from it.

**2.** A conducting rod is moved with a constant velocity $v$ in a magnetic field. A potential difference appears across the two ends
(a) if $\vec{v} \,||\, \vec{l}$ (b) if $\vec{v} \,||\, \vec{B}$ (c) if $\vec{l} \,||\, \vec{B}$
(d) none of these.

**3.** A conducting loop is placed in a uniform magnetic field with its plane perpendicular to the field. An emf is induced in the loop if
(a) it is translated
(b) it is rotated about its axis
(c) it is rotated about a diameter
(d) it is deformed.

**4.** A metal sheet is placed in front of a strong magnetic pole. A force is needed to
(a) hold the sheet there if the metal is magnetic
(b) hold the sheet there if the metal is nonmagnetic
(c) move the sheet away from the pole with uniform velocity if the metal is magnetic
(d) move the sheet away from the pole with uniform velocity if the metal is nonmagnetic.
Neglect any effect of paramagnetism, diamagnetism and gravity.

**5.** A constant current $i$ is maintained in a solenoid. Which of the following quantities will increase if an iron rod is inserted in the solenoid along its axis ?
(a) magnetic field at the centre
(b) magnetic flux linked with the solenoid
(c) self-inductance of the solenoid
(d) rate of Joule heating.

**6.** Two solenoids have identical geometrical construction but one is made of thick wire and the other of thin wire. Which of the following quantities are different for the two solenoids ?
(a) self-inductance
(b) rate of Joule heating if the same current goes through them
(c) magnetic field energy if the same current goes through them
(d) time constant if one solenoid is connected to one battery and the other is connected to another battery.

**7.** An $LR$ circuit with a battery is connected at $t = 0$. Which of the following quantities is not zero just after the connection ?
(a) Current in the circuit
(b) Magnetic field energy in the inductor
(c) Power delivered by the battery
(d) Emf induced in the inductor

**8.** A rod $AB$ moves with a uniform velocity $v$ in a uniform magnetic field as shown in figure (38-Q7).
(a) The rod becomes electrically charged.
(b) The end $A$ becomes positively charged.
(c) The end $B$ becomes positively charged.
(d) The rod becomes hot because of Joule heating.

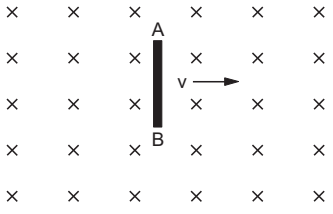


Figure 38-Q7

**9.** $L$, $C$ and $R$ represent the physical quantities inductance, capacitance and resistance respectively. Which of the following combinations have dimensions of frequency ?
(a) $\frac{1}{RC}$ (b) $\frac{R}{L}$ (c) $\frac{1}{\sqrt{LC}}$ (d) $C/L$.

**10.** The switches in figure (38-Q8a) and (38-8b) are closed at $t = 0$ and reopened after a long time at $t = t_0$.

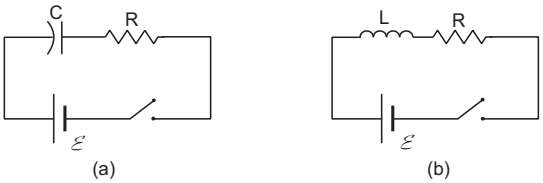


Figure 38-Q8

(a) The charge on $C$ just after $t = 0$ is $\mathcal{E}C$.

(b) The charge on $C$ long after $t = 0$ is $\mathcal{E}C$.

(c) The current in $L$ just before $t = t_0$ is $\mathcal{E}/R$.

(d) The current in $L$ long after $t = t_0$ is $\mathcal{E}/R$.

## EXERCISES

**1.** Calculate the dimensions of (a) $\int \vec{E}\cdot d\vec{l}$, (b) $vBl$ and (c) $\frac{d\Phi_B}{dt}$. The symbols have their usual meanings.

**2.** The flux of magnetic field through a closed conducting loop changes with time according to the equation, $\Phi = at^2 + bt + c$. (a) Write the SI units of $a$, $b$ and $c$. (b) If the magnitudes of $a$, $b$ and $c$ are 0·20, 0·40 and 0·60 respectively, find the induced emf at $t = 2$ s.

**3.** (a) The magnetic field in a region varies as shown in figure (38-E1). Calculate the average induced emf in a conducting loop of area $2{\cdot}0 \times 10^{-3}\ \text{m}^2$ placed perpendicular to the field in each of the 10 ms intervals shown. (b) In which intervals is the emf not constant ? Neglect the behaviour near the ends of 10 ms intervals.

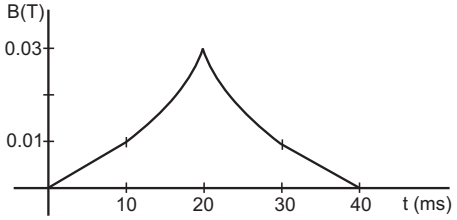



Figure 38-E1

4. A conducting circular loop having a radius of 5·0 cm, is placed perpendicular to a magnetic field of 0·50 T. It is removed from the field in 0·50 s. Find the average emf produced in the loop during this time.

**5.** A conducting circular loop of area 1 $\text{mm}^2$ is placed coplanarly with a long, straight wire at a distance of 20 cm from it. The straight wire carries an electric current which changes from 10 A to zero in 0·1 s. Find the average emf induced in the loop in 0·1 s.

**6.** A square-shaped copper coil has edges of length 50 cm and contains 50 turns. It is placed perpendicular to a 1·0 T magnetic field. It is removed from the magnetic field in 0·25 s and restored in its original place in the next 0·25 s. Find the magnitude of the average emf induced in the loop during (a) its removal, (b) its restoration and (c) its motion.

7. Suppose the resistance of the coil in the previous problem is 25 Ω. Assume that the coil moves with uniform velocity during its removal and restoration. Find the thermal energy developed in the coil during (a) its removal, (b) its restoration and (c) its motion.

**8.** A conducting loop of area 5·0 $\text{cm}^2$ is placed in a magnetic field which varies sinusoidally with time as $B = B_0 \sin \omega t$ where $B_o = 0{\cdot}20$ T and $\omega = 300\ \text{s}^{-1}$. The normal to the coil makes an angle of 60° with the field. Find (a) the maximum emf induced in the coil, (b) the emf induced at $\tau = (\pi/900)$s and (c) the emf induced at $t = (\pi/600)$ s.

**9.** Figure (38-E2) shows a conducting square loop placed parallel to the pole-faces of a ring magnet. The pole-faces have an area of 1 $\text{cm}^2$ each and the field between the poles is 0·10 T. The wires making the loop are all outside the magnetic field. If the magnet is removed in 1·0 s, what is the average emf induced in the loop ?

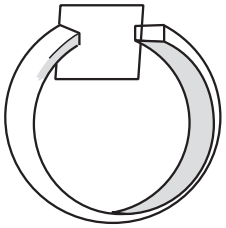


Figure 38-E2

**10.** A conducting square loop having edges of length 2·0 cm is rotated through 180° about a diagonal in 0·20 s. A magnetic field $B$ exists in the region which is perpendicular to the loop in its initial position. If the average induced emf during the rotation is 20 mV, find the magnitude of the magnetic field.

**11.** A conducting loop of face-area $A$ and resistance $R$ is placed perpendicular to a magnetic field $B$. The loop is withdrawn completely from the field. Find the charge which flows through any cross-section of the wire in the process. Note that it is independent of the shape of the loop as well as the way it is withdrawn.

**12.** A long solenoid of radius 2 cm has 100 turns/cm and carries a current of 5 A. A coil of radius 1 cm having 100 turns and a total resistance of 20 Ω is placed inside the solenoid coaxially. The coil is connected to a galvanometer. If the current in the solenoid is reversed in direction, find the charge flown through the galvanometer.

**13.** Figure (38-E3) shows a metallic square frame of edge $a$ in a vertical plane. A uniform magnetic field $B$ exists in the space in a direction perpendicular to the plane of the figure. Two boys pull the opposite corners of the square to deform it into a rhombus. They start pulling the corners at $t = 0$ and displace the corners at a uniform speed $u$. (a) Find the induced emf in the frame at the instant when the angles at these corners reduce to 60°. (b) Find the induced current in the frame at this instant if the total resistance of the frame is $R$. (c) Find the total charge which flows through a side of the frame by the time the square is deformed into a straight line.

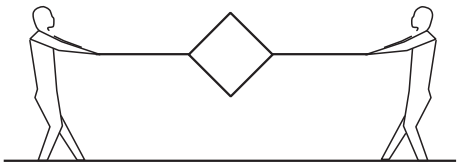


Figure 38-E3

**14.** The north pole of a magnet is brought down along the axis of a horizontal circular coil (figure 38-E4). As a result, the flux through the coil changes from 0·35 weber to 0·85 weber in an interval of half a second. Find the average emf induced during this period. Is the induced current clockwise or anticlockwise as you look into the coil from the side of the magnet ?

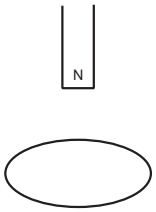



Figure 38-E4

**15.** A wire-loop confined in a plane is rotated in its own plane with some angular velocity. A uniform magnetic field exists in the region. Find the emf induced in the loop.

**16.** Figure (38-E5) shows a square loop of side 5 cm being moved towards right at a constant speed of 1 cm/s. The front edge enters the 20 cm wide magnetic field at $t = 0$. Find the emf induced in the loop at (a) $t = 2$ s, (b) $t = 10$ s, (c) $t = 22$ s and (d) $t = 30$ s.

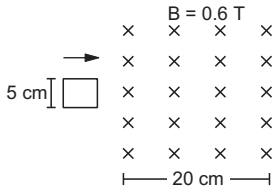



Figure 38-E5

**17.** Find the total heat produced in the loop of the previous problem during the interval 0 to 30 s if the resistance of the loop is 4·5 mΩ.

**18.** A uniform magnetic field $B$ exists in a cylindrical region of radius 10 cm as shown in figure (38-E6). A uniform wire of length 80 cm and resistance 4·0 Ω is bent into a square frame and is placed with one side along a diameter of the cylindrical region. If the magnetic field increases at a constant rate of 0·010 T/s, find the current induced in the frame.

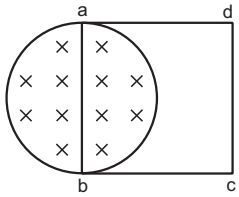



Figure 38-E6

**19.** The magnetic field in the cylindrical region shown in figure (38-E7) increases at a constant rate of 20·0 mT/s. Each side of the square loop *abcd* and *defa* has a length of 1·00 cm and a resistance of 4·00 Ω. Find the current (magnitude and sense) in the wire *ad* if (a) the switch $S_1$ is closed but $S_2$ is open, (b) $S_1$ is open but $S_2$ is closed, (c) both $S_1$ and $S_2$ are open and (d) both $S_1$ and $S_2$ are closed.

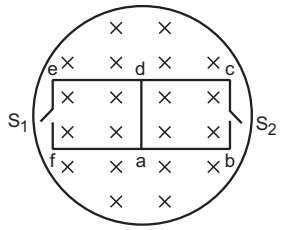



Figure 38-E7

**20.** Figure (38-E8) shows a circular coil of $N$ turns and radius $a$, connected to a battery of emf $\mathcal{E}$ through a rheostat. The rheostat has a total length $L$ and resistance $R$. The resistance of the coil is $r$. A small circular loop of radius $a'$ and resistance $r'$ is placed coaxially with the coil. The centre of the loop is at a distance $x$ from the centre of the coil. In the beginning, the sliding contact of the rheostat is at the left end and then onwards it is moved towards right at a constant speed $v$. Find the emf induced in the small circular loop at the instant (a) the contact begins to slide and (b) it has slid through half the length of the rheostat.

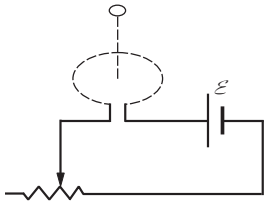



Figure 38-E8

**21.** A circular coil of radius 2·00 cm has 50 turns. A uniform magnetic field $B = 0{\cdot}200$ T exists in the space in a direction parallel to the axis of the loop. The coil is now rotated about a diameter through an angle of 60·0°. The operation takes 0·100 s. (a) Find the average emf induced in the coil. (b) If the coil is a closed one (with the two ends joined together) and has a resistance of 4·00 Ω, calculate the net charge crossing a cross-section of the wire of the coil.

**22.** A closed coil having 100 turns is rotated in a uniform magnetic field $B = 4{\cdot}0 \times 10^{-4}$ T about a diameter which is perpendicular to the field. The angular velocity of rotation is 300 revolutions per minute. The area of the coil is 25 $cm^2$ and its resistance is 4·0 Ω. Find (a) the average emf developed in half a turn from a position where the coil is perpendicular to the magnetic field, (b) the average emf in a full turn and (c) the net charge displaced in part (a).

**23.** A coil of radius 10 cm and resistance 40 Ω has 1000 turns. It is placed with its plane vertical and its axis parallel to the magnetic meridian. The coil is connected to a galvanometer and is rotated about the vertical diameter through an angle of 180°. Find the charge which flows through the galvanometer if the horizontal component of the earth's magnetic field is $B_H = 3{\cdot}0 \times 10^{-5}$ T.

**24.** A circular coil of one turn of radius 5·0 cm is rotated about a diameter with a constant angular speed of 80 revolutions per minute. A uniform magnetic field $B = 0{\cdot}010$ T exists in a direction perpendicular to the axis of rotation. Find (a) the maximum emf induced, (b)

the average emf induced in the coil over a long period and (c) the average of the squares of emf induced over a long period.

**25.** Suppose the ends of the coil in the previous problem are connected to a resistance of 100 Ω. Neglecting the resistance of the coil, find the heat produced in the circuit in one minute.

**26.** Figure (38-E9) shows a circular wheel of radius 10·0 cm whose upper half, shown dark in the figure, is made of iron and the lower half of wood. The two junctions are joined by an iron rod. A uniform magnetic field $B$ of magnitude $2{\cdot}00 \times 10^{-4}$ T exists in the space above the central line as suggested by the figure. The wheel is set into pure rolling on the horizontal surface. If it takes 2·00 seconds for the iron part to come down and the wooden part to go up, find the average emf induced during this period.

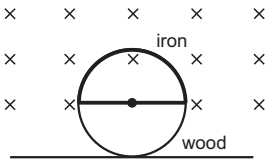



Figure 38-E9

**27.** A 20 cm long conducting rod is set into pure translation with a uniform velocity of 10 cm $s^{-1}$ perpendicular to its length. A uniform magnetic field of magnitude 0·10 T exists in a direction perpendicular to the plane of motion. (a) Find the average magnetic force on the free electrons of the rod. (b) For what electric field inside the rod, the electric force on a free electron will balance the magnetic force ? How is this electric field created ? (c) Find the motional emf between the ends of the rod.

**28.** A metallic metre stick moves with a velocity of 2 m $s^{-1}$ in a direction perpendicular to its length and perpendicular to a uniform magnetic field of magnitude 0·2 T. Find the emf induced between the ends of the stick.

**29.** A 10 m wide spacecraft moves through the interstellar space at a speed $3 \times 10^{7}$ m $s^{-1}$. A magnetic field $B = 3 \times 10^{-10}$ T exists in the space in a direction perpendicular to the plane of motion. Treating the spacecraft as a conductor, calculate the emf induced across its width.

**30.** The two rails of a railway track, insulated from each other and from the ground, are connected to a millivoltmeter. What will be the reading of the millivoltmeter when a train travels on the track at a speed of 180 km $h^{-1}$ ? The vertical component of earth's magnetic field is $0{\cdot}2 \times 10^{-4}$ T and the rails are separated by 1 m.

**31.** A right-angled triangle $abc$, made from a metallic wire, moves at a uniform speed $v$ in its plane as shown in figure (38-E10). A uniform magnetic field $B$ exists in the perpendicular direction. Find the emf induced (a) in the loop $abc$, (b) in the segment $bc$, (c) in the segment $ac$ and (d) in the segment $ab$.

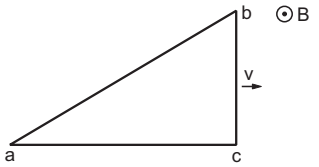



Figure 38-E10

**32.** A copper wire bent in the shape of a semicircle of radius $r$ translates in its plane with a constant velocity $v$. A uniform magnetic field $B$ exists in the direction perpendicular to the plane of the wire. Find the emf induced between the ends of the wire if (a) the velocity is perpendicular to the diameter joining free ends, (b) the velocity is parallel to this diameter.

**33.** A wire of length 10 cm translates in a direction making an angle of 60° with its length. The plane of motion is perpendicular to a uniform magnetic field of 1·0 T that exists in the space. Find the emf induced between the ends of the rod if the speed of translation is 20 cm $s^{-1}$.

**34.** A circular copper-ring of radius $r$ translates in its plane with a constant velocity $v$. A uniform magnetic field $B$ exists in the space in a direction perpendicular to the plane of the ring. Consider different pairs of diametrically opposite points on the ring. (a) Between which pair of points is the emf maximum ? What is the value of this maximum emf ? (b) Between which pair of points is the emf minimum ? What is the value of this minimum emf ?

**35.** Figure (38-E11) shows a wire sliding on two parallel, conducting rails placed at a separation $l$. A magnetic field $B$ exists in a direction perpendicular to the plane of the rails. What force is necessary to keep the wire moving at a constant velocity $v$ ?

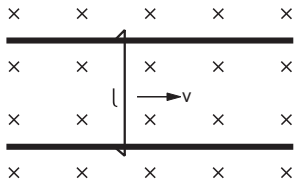



Figure 38-E11

**36.** Figure (38-E12) shows a long U-shaped wire of width $l$ placed in a perpendicular magnetic field $B$. A wire of length $l$ is slid on the U-shaped wire with a constant velocity $v$ towards right. The resistance of all the wires is $r$ per unit length. At $t = 0$, the sliding wire is close to the left edge of the U-shaped wire. Draw an equivalent circuit diagram, showing the induced emf as a battery. Calculate the current in the circuit.

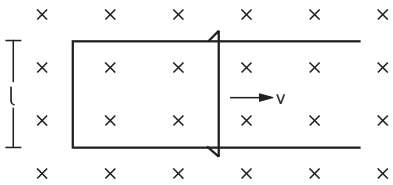



Figure 38-E12

**37.** Consider the situation of the previous problem. (a) Calculate the force needed to keep the sliding wire moving with a constant velocity $v$. (b) If the force needed just after $t = 0$ is $F_0$, find the time at which the force needed will be $F_0/2$.

**38.** Consider the situation shown in figure (38-E13). The wire $PQ$ has mass $m$, resistance $r$ and can slide on the smooth, horizontal parallel rails separated by a distance $l$. The resistance of the rails is negligible. A uniform magnetic field $B$ exists in the rectangular region and a resistance $R$ connects the rails outside the field region. At $t = 0$, the wire $PQ$ is pushed towards right with a speed $v_0$. Find (a) the current in the loop at an instant when the speed of the wire $PQ$ is $v$, (b) the acceleration of the wire at this instant, (c) the velocity $v$ as a function of $x$ and (d) the maximum distance the wire will move.

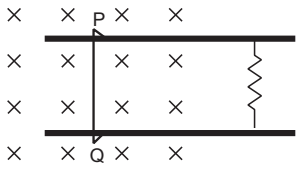



Figure 38-E13

**39.** A rectangular frame of wire $abcd$ has dimensions 32 cm × 8·0 cm and a total resistance of 2·0 Ω. It is pulled out of a magnetic field $B = 0{\cdot}020$ T by applying a force of $3{\cdot}2 \times 10^{-5}$ N (figure 38-E14). It is found that the frame moves with constant speed. Find (a) this constant speed, (b) the emf induced in the loop, (c) the potential difference between the points $a$ and $b$ and (d) the potential difference between the points $c$ and $d$.

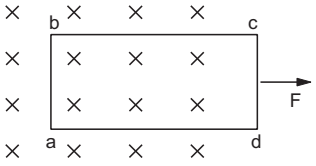



Figure 38-E14

**40.** Figure (38-E15) shows a metallic wire of resistance 0·20 Ω sliding on a horizontal, U-shaped metallic rail. The separation between the parallel arms is 20 cm. An electric current of 2·0 μA passes through the wire when it is slid at a rate of 20 cm $s^{-1}$. If the horizontal component of the earth's magnetic field is $3{\cdot}0 \times 10^{-5}$ T, calculate the dip at the place.

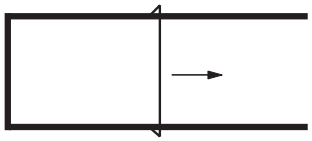


Figure 38-E15

**41.** A wire $ab$ of length $l$, mass $m$ and resistance $R$ slides on a smooth, thick pair of metallic rails joined at the bottom as shown in figure (38-E16). The plane of the rails makes an angle θ with the horizontal. A vertical magnetic field $B$ exists in the region. If the wire slides on the rails at a constant speed $v$, show that

$$B = \sqrt{\frac{mgR\sin\theta}{vl^2\cos^2\theta}}.$$

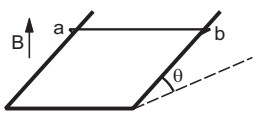



Figure 38-E16

**42.** Consider the situation shown in figure (38-E17). The wires $P_1Q_1$ and $P_2Q_2$ are made to slide on the rails with the same speed 5 cm $s^{-1}$. Find the electric current in the 19 Ω resistor if (a) both the wires move towards right and (b) if $P_1Q_1$ moves towards left but $P_2Q_2$ moves towards right.

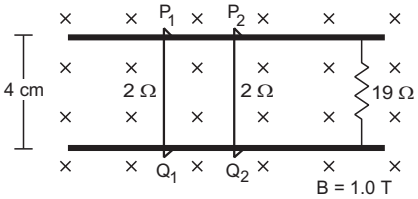



Figure 38-E17

**43.** Suppose the 19 Ω resistor of the previous problem is disconnected. Find the current through $P_2Q_2$ in the two situations (a) and (b) of that problem.

**44.** Consider the situation shown in figure (38-E18). The wire $PQ$ has a negligible resistance and is made to slide on the three rails with a constant speed of 5 cm $s^{-1}$. Find the current in the 10 Ω resistor when the switch $S$ is thrown to (a) the middle rail (b) the bottom rail.

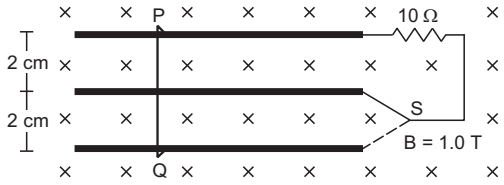



Figure 38-E18

**45.** The current generator $I_g$, shown in figure (38-E19), sends a constant current $i$ through the circuit. The wire $cd$ is fixed and $ab$ is made to slide on the smooth, thick rails with a constant velocity $v$ towards right. Each of these wires has resistance $r$. Find the current through the wire $cd$.

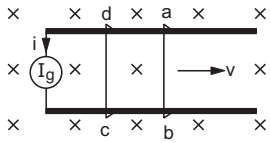



Figure 38-E19

**46.** The current generator $I_g$, shown in figure (38-E20), sends a constant current $i$ through the circuit. The wire $ab$ has a length $l$ and mass $m$ and can slide on the smooth, horizontal rails connected to $I_g$. The entire system lies in a vertical magnetic field $B$. Find the velocity of the wire as a function of time.

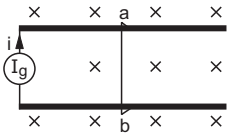



Figure 38-E20

**47.** The system containing the rails and the wire of the previous problem is kept vertically in a uniform horizontal magnetic field $B$ that is perpendicular to the plane of the rails (figure 38-E21). It is found that the wire stays in equilibrium. If the wire $ab$ is replaced by

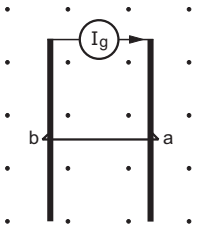



Figure 38-E21

another wire of double its mass, how long will it take in falling through a distance equal to its length ?

**48.** The rectangular wire-frame, shown in figure (38-E22), has a width $d$, mass $m$, resistance $R$ and a large length. A uniform magnetic field $B$ exists to the left of the frame. A constant force $F$ starts pushing the frame into the magnetic field at $t = 0$. (a) Find the acceleration of the frame when its speed has increased to $v$. (b) Show that after some time the frame will move with a constant velocity till the whole frame enters into the magnetic field. Find this velocity $v_0$. (c) Show that the velocity at time $t$ is given by

$$v = v_0(1 - e^{-Ft/mv_0}).$$

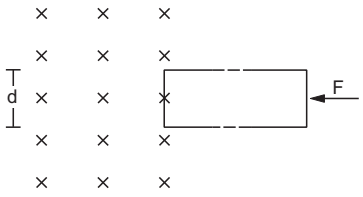



Figure 38-E22

**49.** Figure (38-E23) shows a smooth pair of thick metallic rails connected across a battery of emf $\mathcal{E}$ having a negligible internal resistance. A wire $ab$ of length $l$ and resistance $r$ can slide smoothly on the rails. The entire system lies in a horizontal plane and is immersed in a uniform vertical magnetic field $B$. At an instant $t$, the wire is given a small velocity $v$ towards right. (a) Find the current in it at this instant. What is the direction of the current ? (b) What is the force acting on the wire at this instant ? (c) Show that after some time the wire $ab$ will slide with a constant velocity. Find this velocity.

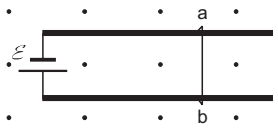



Figure 38-E23

**50.** A conducting wire $ab$ of length $l$, resistance $r$ and mass $m$ starts sliding at $t = 0$ down a smooth, vertical, thick pair of connected rails as shown in figure (38-E24). A uniform magnetic field $B$ exists in the space in a direction perpendicular to the plane of the rails. (a) Write the induced emf in the loop at an instant $t$ when the speed of the wire is $v$. (b) What would be the magnitude and direction of the induced current in the wire ? (c) Find the downward acceleration of the wire at this instant. (d) After sufficient time, the wire starts moving with a constant velocity. Find this velocity $v_m$. (e) Find the velocity of the wire as a function of time. (f) Find the displacement of the wire as a function of time. (g) Show that the rate of heat developed in the wire is equal to the rate at which the gravitational potential energy is decreased after steady state is reached.

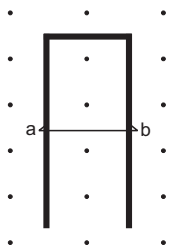



Figure 38-E24

**51.** A bicycle is resting on its stand in the east–west direction and the rear wheel is rotated at an angular speed of 100 revolutions per minute. If the length of each spoke is 30·0 cm and the horizontal component of the earth's magnetic field is $2{\cdot}0 \times 10^{-5}$ T, find the emf induced between the axis and the outer end of a spoke. Neglect centripetal force acting on the free electrons of the spoke.

**52.** A conducting disc of radius $r$ rotates with a small but constant angular velocity $\omega$ about its axis. A uniform magnetic field $B$ exists parallel to the axis of rotation. Find the motional emf between the centre and the periphery of the disc.

**53.** Figure (38-E25) shows a conducting disc rotating about its axis in a perpendicular magnetic field $B$. A resistor of resistance $R$ is connected between the centre and the rim. Calculate the current in the resistor. Does it enter the disc or leave it at the centre ? The radius of the disc is 5·0 cm, angular speed $\omega = 10$ rad/s, $B = 0{\cdot}40$ T and $R = 10\ \Omega$.

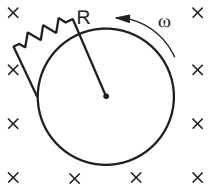



Figure 38-E25

**54.** The magnetic field in a region is given by $\vec{B} = \vec{k}\frac{B_0}{L}y$ where $L$ is a fixed length. A conducting rod of length $L$ lies along the $Y$-axis between the origin and the point (0, $L$, 0). If the rod moves with a velocity $v = v_0\vec{i}$, find the emf induced between the ends of the rod.

**55.** Figure (38-E26) shows a straight, long wire carrying a current $i$ and a rod of length $l$ coplanar with the wire and perpendicular to it. The rod moves with a constant velocity $v$ in a direction parallel to the wire. The distance of the wire from the centre of the rod is $x$. Find the motional emf induced in the rod.

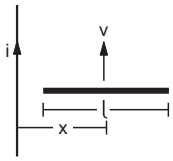



Figure 38-E26

**56.** Consider a situation similar to that of the previous problem except that the ends of the rod slide on a pair of thick metallic rails laid parallel to the wire. At one end the rails are connected by resistor of resistance $R$. (a) What force is needed to keep the rod sliding at a constant speed $v$? (b) In this situation what is the current in the resistance $R$? (c) Find the rate of heat developed in the resistor. (d) Find the power delivered by the external agent exerting the force on the rod.

**57.** Figure (38-E27) shows a square frame of wire having a total resistance $r$ placed coplanarly with a long, straight wire. The wire carries a current $i$ given by $i = i_0 \sin \omega t$. Find (a) the flux of the magnetic field through the square frame, (b) the emf induced in the frame and (c) the heat developed in the frame in the time interval 0 to $\frac{20\pi}{\omega}$.

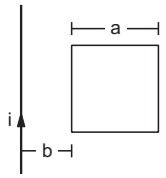



Figure 38-E27

**58.** A rectangular metallic loop of length $l$ and width $b$ is placed coplanarly with a long wire carrying a current $i$ (figure 38-E28). The loop is moved perpendicular to the wire with a speed $v$ in the plane containing the wire and the loop. Calculate the emf induced in the loop when the rear end of the loop is at a distance $a$ from the wire. Solve by using Faraday's law for the flux through the loop and also by replacing different segments with equivalent batteries.

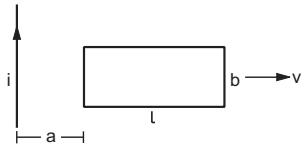



Figure 38-E28

**59.** Figure (38-E29) shows a conducting circular loop of radius $a$ placed in a uniform, perpendicular magnetic field $B$. A thick metal rod $OA$ is pivoted at the centre $O$. The other end of the rod touches the loop at $A$. The centre $O$ and a fixed point $C$ on the loop are connected by a wire $OC$ of resistance $R$. A force is applied at the middle point of the rod $OA$ perpendicularly, so that the rod rotates clockwise at a uniform angular velocity $\omega$. Find the force.

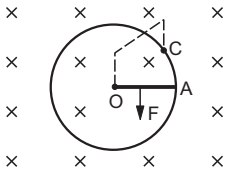



Figure 38-E29

**60.** Consider the situation shown in the figure of the previous problem. Suppose the wire connecting $O$ and $C$ has zero resistance but the circular loop has a resistance $R$ uniformly distributed along its length. The rod $OA$ is made to rotate with a uniform angular speed $\omega$ as shown in the figure. Find the current in the rod when $\angle AOC = 90°$.

**61.** Consider a variation of the previous problem (figure 38-E29). Suppose the circular loop lies in a vertical plane. The rod has a mass $m$. The rod and the loop have negligible resistances but the wire connecting $O$ and $C$ has a resistance $R$. The rod is made to rotate with a uniform angular velocity $\omega$ in the clockwise direction by applying a force at the midpoint of $OA$ in a direction perpendicular to it. Find the magnitude of this force when the rod makes an angle $\theta$ with the vertical.

**62.** Figure (38-E30) shows a situation similar to the previous problem. All parameters are the same except that a battery of emf $\mathcal{E}$ and a variable resistance $R$ are connected between $O$ and $C$. Neglect the resistance of the connecting wires. Let $\theta$ be the angle made by the rod from the horizontal position (shown in the figure), measured in the clockwise direction. During the part of the motion $0 < \theta < \pi/4$ the only forces acting on the rod are gravity and the forces exerted by the magnetic field and the pivot. However, during the part of the motion, the resistance $R$ is varied in such a way that the rod continues to rotate with a constant angular velocity $\omega$. Find the value of $R$ in terms of the given quantities.

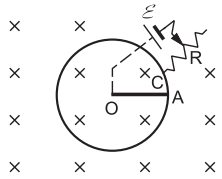



Figure 38-E30

**63.** A wire of mass $m$ and length $l$ can slide freely on a pair of smooth, vertical rails (figure 38-E31). A magnetic field $B$ exists in the region in the direction perpendicular to the plane of the rails. The rails are connected at the top end by a capacitor of capacitance $C$. Find the acceleration of the wire neglecting any electric resistance.

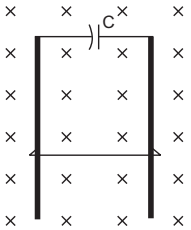



Figure 38-E31

**64.** A uniform magnetic field $B$ exists in a cylindrical region, shown dotted in figure (38-E32). The magnetic field increases at a constant rate $\frac{dB}{dt}$. Consider a circle of

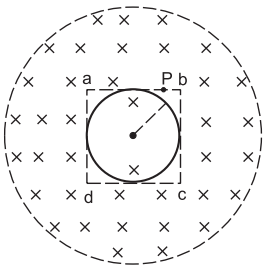



Figure 38-E32

radius $r$ coaxial with the cylindrical region. (a) Find the magnitude of the electric field $E$ at a point on the circumference of the circle. (b) Consider a point $P$ on the side of the square circumscribing the circle. Show that the component of the induced electric field at $P$ along $ba$ is the same as the magnitude found in part (a).

**65.** The current in an ideal, long solenoid is varied at a uniform rate of $0{\cdot}01\ \text{As}^{-1}$. The solenoid has 2000 turns/m and its radius is 6·0 cm. (a) Consider a circle of radius 1·0 cm inside the solenoid with its axis coinciding with the axis of the solenoid. Write the change in the magnetic flux through this circle in 2·0 seconds. (b) Find the electric field induced at a point on the circumference of the circle. (c) Find the electric field induced at a point outside the solenoid at a distance 8·0 cm from its axis.

**66.** An average emf of 20 V is induced in an inductor when the current in it is changed from 2·5 A in one direction to the same value in the opposite direction in 0·1 s. Find the self-inductance of the inductor.

**67.** A magnetic flux of $8 \times 10^{-4}$ weber is linked with each turn of a 200-turn coil when there is an electric current of 4 A in it. Calculate the self-inductance of the coil.

**68.** The current in a solenoid of 240 turns, having a length of 12 cm and a radius of 2 cm, changes at a rate of $0{\cdot}8\ \text{A s}^{-1}$. Find the emf induced in it.

**69.** Find the value of $t/\tau$ for which the current in an $LR$ circuit builds up to (a) 90%, (b) 99% and (c) 99·9% of the steady-state value.

**70.** An inductor-coil carries a steady-state current of 2·0 A when connected across an ideal battery of emf 4·0 V. If its inductance is 1·0 H, find the time constant of the circuit.

**71.** A coil having inductance 2·0 H and resistance 20 Ω is connected to a battery of emf 4·0 V. Find (a) the current at the instant 0·20 s after the connection is made and (b) the magnetic field energy at this instant.

**72.** A coil of resistance 40 Ω is connected across a 4·0 V battery. 0·10 s after the battery is connected, the current in the coil is 63 mA. Find the inductance of the coil.

**73.** An inductor of inductance 5·0 H, having a negligible resistance, is connected in series with a 100 Ω resistor and a battery of emf 2·0 V. Find the potential difference across the resistor 20 ms after the circuit is switched on.

**74.** The time constant of an $LR$ circuit is 40 ms. The circuit is connected at $t = 0$ and the steady-state current is found to be 2·0 A. Find the current at (a) $t = 10$ ms (b) $t = 20$ ms, (c) $t = 100$ ms and (d) $t = 1$ s.

**75.** An $LR$ circuit has $L = 1{\cdot}0$ H and $R = 20\ \Omega$. It is connected across an emf of 2·0 V at $t = 0$. Find $di/dt$ at (a) $t = 100$ ms, (b) $t = 200$ ms and (c) $t = 1{\cdot}0$ s.

**76.** What are the values of the self-induced emf in the circuit of the previous problem at the times indicated therein ?

**77.** An inductor-coil of inductance 20 mH having resistance 10 Ω is joined to an ideal battery of emf 5·0 V. Find the rate of change of the induced emf at $t = 0$, (b) $t = 10$ ms and (c) $t = 1{\cdot}0$ s.

**78.** An $LR$ circuit contains an inductor of 500 mH, a resistor of 25·0 Ω and an emf of 5·00 V in series. Find the potential difference across the resistor at $t =$ (a) 20·0 ms, (b) 100 ms and (c) 1·00 s.

**79.** An inductor-coil of resistance 10 Ω and inductance 120 mH is connected across a battery of emf 6 V and internal resistance 2 Ω. Find the charge which flows through the inductor in (a) 10 ms, (b) 20 ms and (c) 100 ms after the connections are made.

**80.** An inductor-coil of inductance 17 mH is constructed from a copper wire of length 100 m and cross-sectional area $1\ \text{mm}^2$. Calculate the time constant of the circuit if this inductor is joined across an ideal battery. The resistivity of copper $= 1{\cdot}7 \times 10^{-8}\ \Omega\text{m}$.

**81.** An $LR$ circuit having a time constant of 50 ms is connected with an ideal battery of emf $\mathcal{E}$. Find the time elapsed before (a) the current reaches half its maximum value, (b) the power dissipated in heat reaches half its maximum value and (c) the magnetic field energy stored in the circuit reaches half its maximum value.

**82.** A coil having an inductance $L$ and a resistance $R$ is connected to a battery of emf $\mathcal{E}$. Find the time taken for the magnetic energy stored in the circuit to change from one fourth of the steady-state value to half of the steady-state value.

**83.** A solenoid having inductance 4·0 H and resistance 10 Ω is connected to a 4·0 V battery at $t = 0$. Find (a) the time constant, (b) the time elapsed before the current reaches 0·63 of its steady-state value, (c) the power delivered by the battery at this instant and (d) the power dissipated in Joule heating at this instant.

**84.** The magnetic field at a point inside a 2·0 mH inductor-coil becomes 0·80 of its maximum value in 20 μs when the inductor is joined to a battery. Find the resistance of the circuit.

**85.** An $LR$ circuit with emf $\mathcal{E}$ is connected at $t = 0$. (a) Find the charge $Q$ which flows through the battery during 0 to $t$. (b) Calculate the work done by the battery during this period. (c) Find the heat developed during this period. (d) Find the magnetic field energy stored in the circuit at time $t$. (e) Verify that the results in the three parts above are consistent with energy conservation.

**86.** An inductor of inductance 2·00 H is joined in series with a resistor of resistance 200 Ω and a battery of emf 2·00 V. At $t = 10$ ms, find (a) the current in the circuit, (b) the power delivered by the battery, (c) the power dissipated in heating the resistor and (d) the rate at which energy is being stored in magnetic field.

**87.** Two coils $A$ and $B$ have inductances 1·0 H and 2·0 H respectively. The resistance of each coil is 10 Ω. Each coil is connected to an ideal battery of emf 2·0 V at $t = 0$. Let $i_A$ and $i_B$ be the currents in the two circuit at time $t$. Find the ratio $i_A / i_B$ at (a) $t = 100$ ms, (b) $t = 200$ ms and (c) $t = 1$ s.

**88.** The current in a discharging $LR$ circuit without the battery drops from 2·0 A to 1·0 A in 0·10 s. (a) Find the time constant of the circuit. (b) If the inductance of the circuit is 4·0 H, what is its resistance ?

**89.** A constant current exists in an inductor-coil connected to a battery. The coil is short-circuited and the battery is removed. Show that the charge flown through the coil after the short-circuiting is the same as that which flows in one time constant before the short-circuiting.

**90.** Consider the circuit shown in figure (38-E33). (a) Find the current through the battery a long time after the switch $S$ is closed. (b) Suppose the switch is again opened at $t = 0$. What is the time constant of the discharging circuit ? (c) Find the current through the inductor after one time constant.

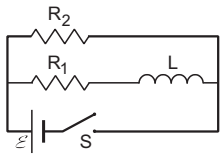



Figure 38-E33

**91.** A current of 1·0 A is established in a tightly wound solenoid of radius 2 cm having 1000 turns/metre. Find the magnetic energy stored in each metre of the solenoid.

**92.** Consider a small cube of volume 1 $mm^3$ at the centre of a circular loop of radius 10 cm carrying a current of 4 A. Find the magnetic energy stored inside the cube.

**93.** A long wire carries a current of 4·00 A. Find the energy stored in the magnetic field inside a volume of 1·00 $mm^3$ at a distance of 10·0 cm from the wire.

**94.** The mutual inductance between two coils is 2·5 H. If the current in one coil is changed at the rate of 1 $As^{-1}$, what will be the emf induced in the other coil ?

**95.** Find the mutual inductance between the straight wire and the square loop of figure (38-E27).

**96.** Find the mutual inductance between the circular coil and the loop shown in figure (38-E8).

**97.** A solenoid of length 20 cm, area of cross-section 4·0 $cm^2$ and having 4000 turns is placed inside another solenoid of 2000 turns having a cross-sectional area 8·0 $cm^2$ and length 10 cm. Find the mutual inductance between the solenoids.

**98.** The current in a long solenoid of radius $R$ and having $n$ turns per unit length is given by $i = i_0 \sin \omega t$. A coil having $N$ turns is wound around it near the centre. Find (a) the induced emf in the coil and (b) the mutual inductance between the solenoid and the coil.

□

## ANSWERS

### OBJECTIVE I

| | | | | | |
|---|---|---|---|---|---|
| 1. (b) | 2. (a) | 3. (d) | 4. (c) | 5. (b) | 6. (c) |
| 7. (a) | 8. (b) | 9. (b) | 10. (a) | 11. (c) | 12. (d) |

### OBJECTIVE II

1. (b), (c)
2. (d)
3. (c), (d)
4. (a), (c), (d)
5. (a), (b), (c)
6. (b), (d)
7. (d)
8. (b)
9. (a), (b), (c)
10. (b), (c)

### EXERCISE

1. $ML^2I^{-1}T^{-3}$ in each case
2. (a) volt/sec, volt, volt-sec (or weber) (b) 1·2 volt
3. (a) – 2·0 mV, – 4·0 mV, 4·0 mV, 2·0 mV
   (b)10 ms to 20 ms and 20 ms to 30 ms
4. $7{\cdot}8 \times 10^{-3}$ V
5. $1 \times 10^{-10}$ V
6. (a) 50 V (b) 50 V (c) zero
7. (a) 25 J (b) 25 J (c) 50 J
8. (a) 0·015 V (b) $7{\cdot}5 \times 10^{-3}$ V (c) zero
9. 10 μV
10. 5·0 T
11. $BA/R$
12. $2 \times 10^{-4}$ C
13. (a) $2Bav$ (b) $2Bav/R$ (c) $a^2B/R$
14. $\mathcal{E} = 1{\cdot}0$ V, anticlockwise
15. zero
16. (a) $3 \times 10^{-4}$ V, (b) zero, (c) $3 \times 10^{-4}$ V and (d) zero
17. $2 \times 10^{-4}$ J
18. $3{\cdot}9 \times 10^{-5}$ A
19. (a) $1{\cdot}25 \times 10^{-7}$ A, $a$ to $d$ (b) $1{\cdot}25 \times 10^{-7}$ A, $d$ to $a$, (c) zero (d) zero
20. $\dfrac{\pi\mu_0 N a^2 a'^2 \mathcal{E} R v}{2L(a^2 + x^2)^{3/2}(R' + r)^2}$ where $R' = R$ for part (a) and $R/2$ for part (b)
21. (a) $6{\cdot}28 \times 10^{-2}$ V (b) $1{\cdot}57 \times 10^{-3}$ C
22. (a) $2{\cdot}0 \times 10^{-3}$ V (b) zero (c) $5{\cdot}0 \times 10^{-5}$ C
23. $4{\cdot}7 \times 10^{-5}$ C
24. (a) $6{\cdot}6 \times 10^{-4}$ V (b) zero (c) $2{\cdot}2 \times 10^{-7}$ $V^2$
25. $1{\cdot}3 \times 10^{-7}$ J

26. $1{\cdot}57 \times 10^{-6}$ V

27. (a) $1{\cdot}6 \times 10^{-21}$ N (b) $1{\cdot}0 \times 10^{-2}$ Vm$^{-1}$ (c) $2{\cdot}0 \times 10^{-3}$ V

28. 0·4 V

29. 0·09 V

30. 1 mV

31. (a) zero (b) $vB(bc)$, positive at $c$ (c) zero
(d) $vB(bc)$, positive at $a$

32. (a) $2rvB$ (b) zero

33. $17 \times 10^{-3}$ V

34. (a) at the ends of the diameter perpendicular to the velocity, $2\,rvB$ (b) at the ends of the diameter parallel to the velocity, zero

35. zero

36. $\dfrac{Blv}{2r(l+vt)}$

37. (a) $\dfrac{B^2l^2v}{2r(l+vt)}$ (b) $l/v$

38. (a) $\dfrac{Blv}{R+r}$ (b) $\dfrac{B^2l^2v}{m(R+r)}$ towards left (c) $v = v_0 - \dfrac{B^2l^2x}{m(R+r)}$
(d) $\dfrac{mv_0(R+r)}{B^2l^2}$

39. (a) 25 m s$^{-1}$ (b) $4{\cdot}0 \times 10^{-2}$ V (c) $3{\cdot}6 \times 10^{-2}$ V
(d) $4{\cdot}0 \times 10^{-3}$ V

40. $\tan^{-1}(1/3)$

42. (a) 0·1 mA (b) zero

43. (a) zero (b) 1 mA

44. (a) 0·1 mA (b) 0·2 mA

45. $\dfrac{ir - Blv}{2r}$

46. $ilBt/m$, away from the generator

47. $2\sqrt{l/g}$

48. (a) $\dfrac{RF - vl^2B^2}{mR}$ (b) $\dfrac{RF}{l^2B^2}$

49. (a) $\dfrac{1}{r}(E - vBl)$ from $b$ to $a$ (b) $\dfrac{lB}{r}(E - vBl)$ towards right (c) $\dfrac{E}{Bl}$

50. (a) $vBl$ (b) $\dfrac{vBl}{r}$, $b$ to $a$ (c) $g - \dfrac{B^2l^2}{mr}v$ (d) $\dfrac{mgr}{B^2l^2}$
(e) $v_m(1 - e^{-gt/v_m})$ (f) $v_mt - \dfrac{v_m^2}{g}(1 - e^{-gt/v_m})$

51. $9{\cdot}4 \times 10^{-6}$ V

52. $\dfrac{1}{2}\omega r^2 B$

53. 0·5 mA, leaves

54. $\dfrac{B_0v_0l}{2}$

55. $\dfrac{\mu_0 iv}{2\pi}\ln\left(\dfrac{2x+l}{2x-l}\right)$

56. (a) $\dfrac{v}{R}\left\{\dfrac{\mu_0 i}{2\pi}\ln\dfrac{2x+l}{2x-l}\right\}^2$ (b) $\dfrac{\mu_o iv}{2\pi R}\ln\dfrac{2x+l}{2x-l}$
(c) $\dfrac{1}{R}\left\{\dfrac{\mu_0 iv}{2\pi}\ln\dfrac{2x+l}{2x-l}\right\}^2$ (d) same as (c)

57. (a) $\dfrac{\mu_0 ia}{2\pi}\ln\left(1+\dfrac{a}{b}\right)$ (b) $\dfrac{\mu_0 ai_0\omega\cos\omega t}{2\pi}\ln\left(1+\dfrac{a}{b}\right)$
(c) $\dfrac{5\mu_0^2a^2i_0^2\omega}{2\pi r}\ln^2\left(1+\dfrac{a}{b}\right)$

58. $\dfrac{\mu ilvb}{2\pi a(a+l)}$

59. $\dfrac{\omega a^3B^2}{2R}$ to the right of $OA$ in the figure

60. $\dfrac{8}{3}\dfrac{\omega a^2B}{R}$

61. $\dfrac{\omega a^3B^2}{2R} - mg\sin\theta$

62. $\dfrac{aB(2\mathcal{E} + \omega a^2B)}{2\,mg\cos\theta}$

63. $\dfrac{mg}{m + CB^2l^2}$

64. (a) $\dfrac{r}{2}\dfrac{dB}{dt}$

65. (a) $1{\cdot}6 \times 10^{-8}$ weber (b) $1{\cdot}2 \times 10^{-7}$ V m$^{-1}$
(c) $5{\cdot}6 \times 10^{-7}$ V m$^{-1}$

66. 0·4 H

67. $4 \times 10^{-2}$ H

68. $6 \times 10^{-4}$ V

69. 2·3, 4·6, 6·9

70. 0·50 s

71. (a) 0·17 A (b) 0·03 J

72. 4·0 H

73. 0·66 V

74. (a) 0·44 A (b) 0·79 A (c) 1·8 A and (d) 2·0 A

75. (a) 0·27 A s$^{-1}$ (b) 0·036 A s$^{-1}$ and (c) $4{\cdot}1 \times 10^{-9}$ A s$^{-1}$

76. (a) 0·27 V (b) 0·036 V (c) $4{\cdot}1 \times 10^{-9}$ V

77. (a) $2{\cdot}5 \times 10^{3}$ V s$^{-1}$ (b) 17 V s$^{-1}$ and (c) 0·00 V s$^{-1}$

78. (a) 3·16 V (b) 4·97 V and (c) 5·00 V

79. (a) 1·8 mC (b) 5·7 mC and (c) 45 mC

80. 10 ms

81. (a) 35 ms (b) 61 ms (c) 61 ms

82. $\tau\ln\dfrac{1}{2-\sqrt{2}}$

83. (a) 0·40 s (b) 0·40 s (c) 1·0 W and (d) 0·64 W

84. 160 Ω

85. (a) $\frac{\mathcal{E}}{R}\{t-\frac{L}{R}(1-x)\}$

(b) $\frac{\mathcal{E}^2}{R}\{t-\frac{L}{R}(1-x)\}$

(c) $\frac{\mathcal{E}^2}{R}\{t-\frac{L}{2R}(3-4x+x^2)\}$

(d) $\frac{L\mathcal{E}^2}{2R^2}(1-x)^2$, where $x=e^{-Rt/L}$

86. (a) 6·3 mA (b) 12·6 mW (c) 8·0 mW and (d) 4·6 mW

87. (a) 1·6 (b) 1·4 (c) 1·0

88. (a) 0·14 s (b) 28 Ω

90. (a) $\frac{\mathcal{E}(R_1+R_2)}{R_1R_2}$ (b) $\frac{L}{R_1+R_2}$ (c) $\frac{\mathcal{E}}{R_1e}$

91. $7{\cdot}9\times10^{-4}$ J

92. $8\pi\times10^{-14}$ J

93. $2{\cdot}55\times10^{-14}$ J

94. 2·5 V

95. $\frac{\mu_0 a}{2\pi}\ln\left(1+\frac{a}{b}\right)$

96. $N\frac{\mu_0\pi a^2 a'^{1/2}}{2(a^2+x^2)^{3/2}}$

97. $2{\cdot}0\times10^{-2}$ H

98. (a) $\pi\mu_0 i_0 nN\omega R^2\cos\omega t$ (b) $\pi\mu_0 nNR^2$

□

## CHAPTER 39

# ALTERNATING CURRENT

### 39.1 ALTERNATING CURRENT

When a resistor is connected across the terminals of a battery, a current is established in the circuit. The current has a unique direction, it goes from the positive terminal to the negative terminal via the external resistor. The magnitude of the current also remains almost constant. If the direction of the current in a resistor or in any other element changes alternately, the current is called an *alternating current* (AC). In this chapter, we shall study the alternating current that varies sinusoidally with time. Such a current is given by

$$i = i_0 \sin(\omega t + \varphi). \quad \ldots \quad (39.1)$$

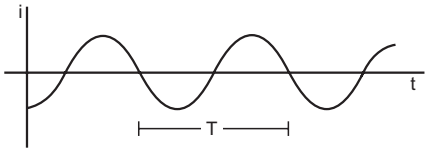



Figure 39.1

The current repeats its value after each time interval $T = 2\pi/\omega$. This time interval is called the *time period.* The current is positive for half the time period and is negative for the remaining half period. This means, its direction reverses after each half time period. The maximum value of the current is $i_0$ which is called the *peak current* or the *current amplitude.* To get sinusoidally varying alternating current, we need a source which can generate sinusoidally varying emf. An *AC generator,* also called an *AC dynamo*, can be used as such a source. It converts mechanical energy into electrical energy, producing an alternating emf.

### 39.2 AC GENERATOR, OR AC DYNAMO

#### Construction

A schematic design of an AC dynamo is shown in figure (39.2a). A simplified diagram of the same is shown in figure (39.2b). It consists of three main parts: a magnet, an armature with slip rings and brushes.

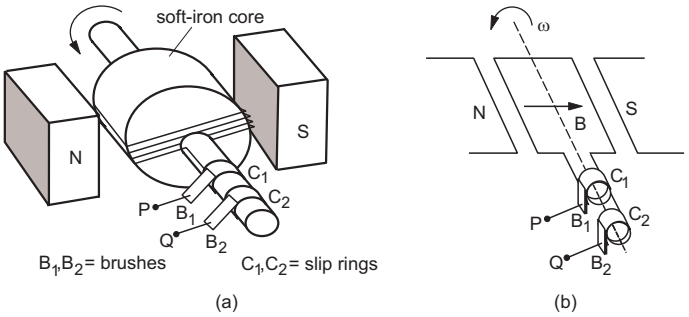



Figure 39.2

*Magnet*: It may be a permanent magnet or an electromagnet. The poles of the magnet face each other so that a strong uniform magnetic field $\vec{B}$ is produced between the poles.

*Armature*: It is a coil generally wound over a soft-iron core. The core increases the magnetic field due to its magnetization. The two ends of the coil are connected to two slip rings $C_1$ and $C_2$. The coil together with the rings can rotate in the magnetic field. The axis of rotation is in the plane of the coil but perpendicular to the magnetic field.

*Brushes*: Two graphite brushes $B_1$ and $B_2$ permanently touch the slip rings. As the armature rotates, the slip rings $C_1$ and $C_2$ slip against the brushes so that the contact is maintained all the time. These brushes are connected to two terminals $P$ and $Q$. The external circuit is connected to these terminals.

#### emf Induced as the Coil Rotates

Suppose the area of the coil is $A$, it contains $N$ turns and it is rotated at a constant angular velocity $\omega$. Suppose, the plane of the coil is perpendicular to the magnetic field at $t = 0$. The total magnetic flux through each turn of the coil is $BA$ in this position. In time $t$, the coil rotates through an angle $\theta = \omega t$. The flux through each turn of the coil at this time $t$ is

$$\Phi = BA \cos \omega t.$$

Using Faraday's law, the emf induced in each turn of the coil is

$$-\frac{d\Phi}{dt} = BA\omega \sin \omega t.$$

The total emf induced in the coil is,

$$\mathcal{E} = NBA\ \omega \sin \omega t$$

$$= \mathcal{E}_0 \sin \omega t. \qquad \ldots \quad (39.2)$$

We see that the emf varies sinusoidally with time with an angular frequency ω and hence with a time period $T = 2\pi/\omega$. The maximum magnitude of the emf, known as *peak emf*, is $\mathcal{E}_0$.

If the terminals $P$ and $Q$ are connected to an external circuit, this emf drives a current in the circuit which also varies sinusoidally with time as shown in figure (39.1).

**Household Power Generation**

The electricity that we use in our houses is generally AC electricity and is produced in power plants using the same principle as described above. The armature is connected to a *turbine*. The turbine has a rotor with blades. Steam at high pressure, water from a height or air at high speed strikes the blades. This rotates the rotor of the turbine. As the armature is connected to the turbine, the armature also rotates and alternating emf is produced. Gensets, which are used in houses at the time of power failure, at marriage functions, at public meetings, in fields where regular electric power is not available, etc., also work on the same principle. Here a diesel or a petrol engine drives the armature.

### 39.3 INSTANTANEOUS AND RMS CURRENT

An alternating current is given by

$$i = i_0 \sin(\omega t + \varphi). \qquad \ldots \quad (i)$$

This equation gives the instantaneous current at any instant $t$. The current changes with time, sometimes it is positive and sometimes negative. We define the *average current* or *mean current* over a time interval 0 to $t$ as

$$\bar{i} = \frac{\int_0^t i\,dt}{\int_0^t dt} = \frac{1}{T}\int_0^t i dt\ .$$

Using (i),

$$\bar{i} = \frac{i_0}{t}\int_0^t \sin(\omega t + \varphi)\,dt = -\frac{i_0}{t}\left[\frac{\cos(\omega t + \varphi)}{\omega}\right]_0^t.$$

or, $$\bar{i} = -\frac{i_0}{t}\left[\frac{\cos(\omega t + \varphi) - \cos\varphi}{\omega}\right]. \qquad \ldots \quad (ii)$$

For a time period, $t = T$ and $\omega T = 2\pi$ so that,

$$\bar{i} = -\frac{i_0}{T\omega}\left[\cos(2\pi + \varphi) - \cos\varphi\right] = 0.$$

If we take the average over a long time, the value of $i$ will be the same as for one time period. This can be easily seen from equation (ii). As cosine of an angle must remain between ± 1, the numerator has a finite value. If $t$ is large, the denominator is large and the average current $\bar{i}$ tends to zero.

The instantaneous current $i$ could be positive or negative at a given instant but the quantity $i^2$ always remains positive and hence its average is also positive. The average of $i^2$ over a time period is

$$\overline{i^2} = \frac{\int_0^T i^2\,dt}{\int_0^T dt}$$

$$= \frac{1}{T}\int_0^T i_0^2 \sin^2(\omega t + \varphi)\,dt$$

$$= \frac{i_0^2}{2T}\int_0^T [1 - \cos 2(\omega t + \varphi)]dt$$

$$= \frac{i_0^2}{2T}\left[t - \frac{\sin 2(\omega t + \varphi)}{2\omega}\right]_0^T$$

$$= \frac{i_0^2}{2T}\left[T - \frac{\sin(4\pi + 2\varphi) - \sin 2\varphi}{2\omega}\right] = \frac{i_0^2}{2}\ .$$

This is known as the *mean square current*. The square root of mean square current is called *root-mean-square current* or rms current. This is also known as the *virtual current*. Thus, the rms current or the virtual current corresponding to the current $i = i_0 \sin(\omega t + \varphi)$ is

$$i_{rms} = \sqrt{\overline{i^2}} = \frac{i_0}{\sqrt{2}}\ . \qquad \ldots \quad (39.3)$$

The equations for mean square current and root-mean-square current are derived for one time period. They are also valid if the average is calculated over a long period of time.

An alternating voltage (potential difference) may be written as

$$V = V_0 \sin(\omega t + \varphi).$$

This gives the instantaneous voltage. The mean voltage $\bar{V}$ over a complete cycle is zero, the mean square voltage over a cycle is $V_0^2/2$ and the root-mean-square voltage (rms voltage or virtual voltage) is $V_0/\sqrt{2}$. The significance of rms current and rms voltage may be shown by considering a resistor of resistance $R$ carrying a current

$$i = i_0 \sin(\omega t + \varphi). \qquad \ldots \quad (i)$$

The voltage across the resistor is

$$V = Ri = (i_0 R) \sin(\omega t + \varphi). \quad \ldots \text{(ii)}$$

The thermal energy developed in the resistor during the time $t$ to $t + dt$ is

$$i^2 R\, dt = i_0^2 R \sin^2(\omega t + \varphi) dt.$$

The thermal energy developed in one time period is

$$U = \int_0^T i^2 R dt$$

$$= R \int_0^T i_0^2 \sin^2(\omega t + \varphi) dt$$

$$= RT \left[ \frac{1}{T} \int_0^T i_0^2 \sin^2(\omega t + \varphi) dt \right]$$

$$= i_{rms}^2 RT.$$

Thus, if we pass a constant current $i_{rms}$ through the resistor, it will produce the same thermal energy in a time period as that produced when the alternating current $i$ passes through it. Similarly, a constant voltage $V_{rms}$ applied across a resistor produces the same thermal energy as that produced by the voltage $V = V_0 \sin(\omega t + \varphi)$. These statements are also valid if we consider a long period of time. The alternating voltage and the alternating current are generally measured and mentioned in terms of their rms values. When we say that the household supply is 220 V AC we mean that the rms value is 220 V. The peak value would be (220 V) $\sqrt{2}$ = 311 V.

***Example 39.1***

*The peak value of an alternating current is* 5 A *and its frequency is* 60 Hz. *Find its rms value. How long will the current take to reach the peak value starting from zero* ?

***Solution*** :

The rms current is

$$i_{rms} = \frac{i_0}{\sqrt{2}} = \frac{5 \text{ A}}{\sqrt{2}} = 3{\cdot}5 \text{ A}.$$

The time period is

$$T = \frac{1}{\nu} = \frac{1}{60} \text{ s}.$$

The current takes one fourth of the time period to reach the peak value starting from zero. Thus, the time required is

$$t = \frac{T}{4} = \frac{1}{240} \text{ s}.$$

## 39.4 SIMPLE AC CIRCUITS

### AC Circuit Containing only a Resistor

Figure (39.3) shows a circuit containing an AC source $\mathcal{E} = \mathcal{E}_0 \sin \omega t$ and a resistor of resistance $R$.

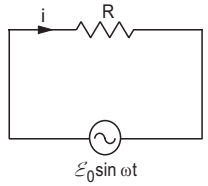



Figure 39.3

Such a circuit is also known as a *purely resistive circuit*. Notice the symbol for an AC source.

If the current at time $t$ is $i$, Kirchhoff's loop law gives

$$\mathcal{E}_0 \sin \omega t = Ri$$

or, $$i = \frac{\mathcal{E}_0}{R} \sin \omega t$$

$$= i_0 \sin \omega t \quad \ldots \text{(39.4)}$$

where $$i_0 = \frac{\mathcal{E}_0}{R}. \quad \ldots \text{(39.5)}$$

### AC Circuit Containing only a Capacitor

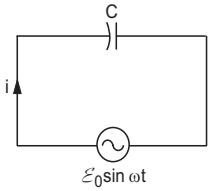



Figure 39.4

Figure (39.4) shows an AC source connected across a capacitor. The resistance of the circuit is assumed to be zero. Such a circuit is also known as a *purely capacitive circuit*. Suppose the charge on the capacitor is $q$ and the current is $i$ at time $t$. Any charge that goes through a wire accumulates on the capacitor, so that

$$idt = dq$$

or, $$i = \frac{dq}{dt}.$$

Using Kirchhoff's loop law,

$$\mathcal{E}_0 \sin \omega t = \frac{q}{C}$$

or, $$q = C \mathcal{E}_0 \sin \omega t$$

or, $$i = \frac{dq}{dt} = C \mathcal{E}_0 \omega \cos \omega t$$

or, $$i = i_0 \cos \omega t \quad \ldots \text{(39.6)}$$

where $$i_0 = C \mathcal{E}_0 \omega = \frac{\mathcal{E}_0}{1/\omega C}. \quad \ldots \text{(39.7)}$$

There are several points to be discussed. If a battery is connected across a capacitor, there is a current only for a short time in which the capacitor gets charged. After this the current becomes negligible. In case of an AC source, the current exists as along as the source is connected. We say that *a capacitor*

*stops direct current but allows alternating current.* The physical reason behind it is obvious. The charge on a capacitor is determined by the emf of the source. In case of an AC source, the emf keeps on changing. Accordingly, the charge $q$ keeps on changing and we get continuous current through the connecting wires and the source.

Another important point to note is the relation between the peak emf and the peak current. We have

$$i_0 = \frac{\mathcal{E}_0}{1/\omega C}$$

or, $$i_0 = \frac{\mathcal{E}_0}{X_c} \text{ where } X_c = \frac{1}{\omega C}.$$

We see that $X_c = 1/\omega C$ plays the role of effective resistance. It is called the *reactance* of the capacitor and its unit is ohm. It depends on the capacitance of the capacitor as well as on the frequency of the AC source. For a source of high frequency, the reactance $X_c = 1/\omega C$ is small and the peak current $i_0$ is large. For a small frequency, the reactance $1/\omega C$ is large and consequently the peak current is small.

If the frequency is zero, we get a direct-current (DC) source producing a constant emf. In this case, the reactance $1/\omega C$ is infinity and $i_0 = 0$. So the response of a capacitor to an alternating-current source depends on the frequency of the source.

The third important point concerns the phase difference between the emf and the current. We have,

$$\mathcal{E} = \mathcal{E}_0 \sin \omega t$$

and $$i = i_0 \cos \omega t = i_0 \sin(\omega t + \pi/2).$$

Thus, the current leads the emf by $\pi/2$. When the emf $\mathcal{E}$ is zero, the current has maximum magnitude. When the emf has maximum magnitude, the current is zero. Figure (39.5) shows variations in the current through the capacitor and in the emf as time passes.

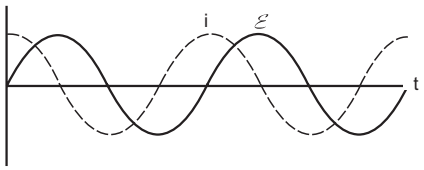



Figure 39.5

***Example 39.1***

*Find the reactance of a capacitor* ($C = 200\ \mu F$) *when it is connected to (a) a* 10 Hz *AC source, (b) a* 50 Hz *AC source and (c) a* 500 Hz *AC source.*

***Solution :***

The reactance is $X_c = \frac{1}{\omega C} = \frac{1}{2\pi \nu C}$.

(a) $$X_c = \frac{1}{2\pi(10\ \text{Hz})\ (200 \times 10^{-6}\ \text{F})}$$

$= 80\ \Omega$.

Similarly, the reactance is $16\ \Omega$ for 50 Hz and $1{\cdot}6\ \Omega$ for 500 Hz.

**AC Circuit Containing only an Inductor**

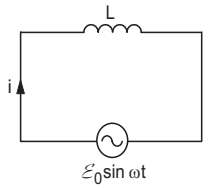



Figure 39.6

Figure (39.6) shows an inductor connected to an AC source. Such a circuit is also known as a *purely inductive circuit.*

The induced emf across the inductor is $-L\frac{di}{dt}$ so that from Kirchhoff's loop law,

$$\mathcal{E}_0 \sin \omega t - L\frac{di}{dt} = 0$$

or, $$\frac{di}{dt} = \frac{\mathcal{E}_0}{L} \sin \omega t$$

or, $$i = -\frac{\mathcal{E}_0}{\omega L} \cos \omega t + c \quad \ldots \text{(i)}$$

where $c$ is a constant. Now, average of $\cos \omega t$ over one time period is zero. Also, in the circuit we are discussing, the emf is sinusoidal and we expect the current to be sinusoidal too. Thus, average of $i$ must be zero over one time period. Hence, from (i), $c = 0$ and

$$i = -\frac{\mathcal{E}_0}{\omega L} \cos \omega t$$

or, $$i = \frac{\mathcal{E}_0}{\omega L} \sin(\omega t - \pi/2) \quad \ldots (39.8)$$

or, $$i = i_0 \sin(\omega t - \pi/2)$$

where $i_0 = \frac{\mathcal{E}_0}{\omega L}$. $\quad \ldots (39.9)$

The constant $X_L = \omega L$ plays the role of effective resistance in this circuit. It is called the *reactance* of the inductor. It is zero for direct current ($\omega = 0$) and increases as the frequency is increased. We see from equation (39.8) that the phase of the current is $\pi/2$ less than that of the emf. The current lags behind the emf. Figure (39.7) shows plots of the current through an inductor and of the emf as time passes.

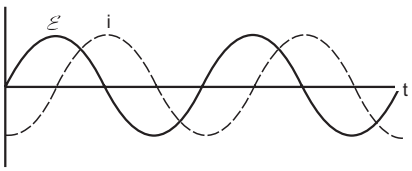



Figure 39.7

***Example 39.2***

*An inductor (L = 200 mH) is connected to an AC source of peak emf 210 V and frequency 50 Hz. Calculate the peak current. What is the instantaneous voltage of the source when the current is at its peak value?*

***Solution*** : The reactance of the inductor is

$$X_L = \omega L = (2\pi \times 50\ \text{s}^{-1}) \times (200 \times 10^{-3}\ \text{H})$$

$$= 62{\cdot}8\ \Omega.$$

The peak current is

$$i_0 = \frac{\mathcal{E}_0}{X_L} = \frac{210\ \text{V}}{62{\cdot}8\ \Omega} = 3{\cdot}3\ \text{A}.$$

As the current lags behind the voltage by π/2, the voltage is zero when the current has its peak value.

**Impedance**

The peak current and the peak emf in all the three circuits discussed above may be written as

$$i_0 = \frac{\mathcal{E}_0}{Z} \qquad \ldots \quad (39.10)$$

where $Z = R$ for a purely resistive circuit

$Z = \dfrac{1}{\omega C}$ for a purely capacitive circuit

and $Z = \omega L$ for a purely inductive circuit.

The peak current and the peak emf are related by equation (39.10) for any series circuit (one-loop circuit) having an AC source. The general name for $Z$ is *impedance*. Thus, the impedance of a purely resistive circuit is $R$, that of a purely capacitive circuit is $1/\omega C$ and that of a purely inductive circuit is $\omega L$.

**Phase factor**

We have seen that the current and the emf are, in general, not in phase in an AC circuit. If the emf is

$$\mathcal{E} = \mathcal{E}_0 \sin \omega t,$$

the current may be written as

$$i = i_0 \sin(\omega t + \varphi).$$

For a purely resistive circuit, $\varphi = 0$; for a purely capacitive circuit, $\varphi = \pi/2$ and for a purely inductive circuit, $\varphi = -\pi/2$. We shall call the constant $\varphi$ the phase factor.

## 39.5 VECTOR METHOD TO FIND THE CURRENT IN AN AC CIRCUIT

Let us now describe a simple method by which we can calculate the current in an AC circuit. We shall confine the discussion to series circuits only. Suppose an emf

$$\mathcal{E} = \mathcal{E}_0 \sin \omega t$$

is applied in a series AC circuit which may contain a resistance, a capacitor, an inductor or any combination of these. Let us represent the resistance of a resistor by a vector of magnitude $R$, the reactance of a capacitance by a vector of magnitude $X_c = 1/\omega C$ and the reactance of an inductor by a vector of magnitude $X_L = \omega L$. The vector corresponding to the resistance is drawn along the $X$-axis, the vector for the capacitive reactance is drawn π/2 ahead of the resistance, that is, along the positive $Y$-axis and the vector for the inductive reactance is drawn π/2 behind the resistance, that is, along the negative $Y$-axis.

The impedance of the circuit, $Z$, and the phase factor φ are obtained by the vector sum of these three vectors. The magnitude of the vector sum gives the impedance $Z$, and its angle with the $X$-axis gives the phase factor.

Thus, if the resistance of the circuit is $R$ and the net reactance is $X$, the impedance is $Z = \sqrt{R^2 + X^2}$ and $\tan\varphi = \dfrac{X}{R}$.

Once $Z$ and φ are obtained, the current in the circuit can be easily written as

$$i = \frac{\mathcal{E}_0}{Z} \sin(\omega t + \varphi). \qquad \ldots \quad (39.11)$$

*It should be clearly understood that the resistance, capacitance, inductance, etc., are not vector quantities. The above description is only a method to derive easily the equations for the current in an AC circuit.* Figure (39.8) shows the construction of vector diagrams for the three circuits discussed above.

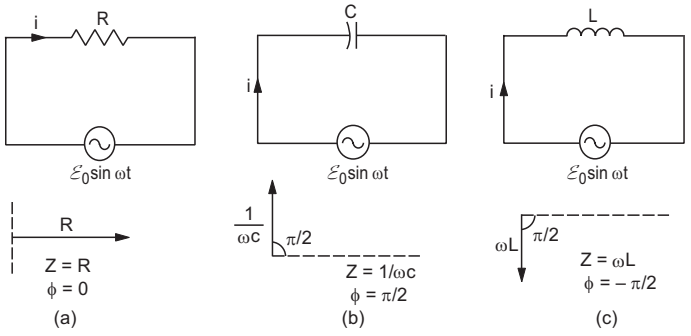



Figure 39.8

## 39.6 MORE AC CIRCUITS

When an AC source is connected in a circuit with a resistance and a reactance, the current varies initially in a complex way. After sufficient time, a sinusoidally varying current persists in the circuit. This steady-state current has a frequency equal to that of the source and may have a phase difference with the source. This steady-state current may be obtained by the vector method described above.

***CR* Circuit**

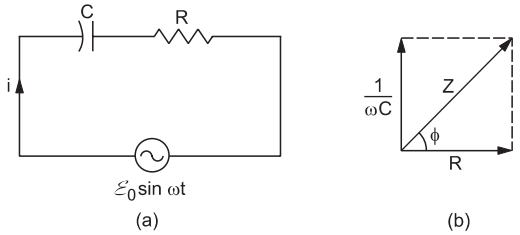



Figure 39.9

Let us find the current in a *CR* circuit using the vector method. The circuit and the corresponding vector diagram are drawn in figure (39.9). The resistance is represented by a vector of magnitude $R$ along the $x$-axis and the capacitive reactance by a vector of magnitude $1/\omega C$ along the positive $y$-axis. The impedance of the circuit is given by the magnitude of the resultant of these two. It is

$$Z = \sqrt{R^2 + (1/\omega C)^2} \qquad \ldots \text{ (i)}$$

and hence the peak current is

$$i_0 = \frac{\mathcal{E}_0}{Z} = \frac{\mathcal{E}_0}{\sqrt{R^2 + (1/\omega C)^2}}.$$

Also, the direction of the resultant makes an angle φ with the $x$-axis where

$$\tan\varphi = \frac{1}{\omega CR}. \qquad \ldots \text{ (ii)}$$

The steady-state current in the circuit is

$$i = \frac{\mathcal{E}_0}{Z} \sin(\omega t + \varphi)$$

where $Z$ and φ are given by equations (i) and (ii).

The reactance of the circuit is $1/\omega C$. We see that the current *leads* the emf.

***LR* Circuit**

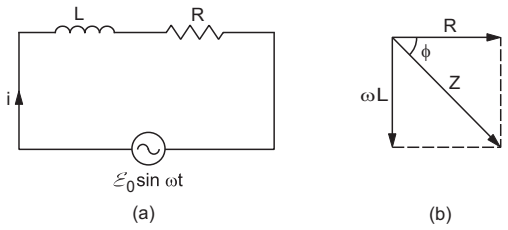



Figure 39.10

Figure (39.10) shows an inductor, a resistor and an AC source connected in series together with its vector diagram. The resistance is represented by a vector of magnitude $R$ along the $x$-axis and the inductive reactance by a vector of magnitude $\omega L$ along the negative $y$-axis. The impedance of the circuit is equal to the magnitude of the resultant of these two. Its value is

$$Z = \sqrt{R^2 + \omega^2 L^2}. \qquad \ldots \text{ (i)}$$

The resultant is at an angle φ below the $x$-axis where

$$\tan\varphi = \frac{\omega L}{R}. \qquad \ldots \text{ (ii)}$$

The current in steady state is, therefore, given by

$$i = \frac{\mathcal{E}_0}{\sqrt{R^2 + \omega^2 L^2}} \sin(\omega t - \varphi)$$

where φ is given by equation (ii). The reactance of the circuit is $\omega L$. We see that the current *lags* behind the emf.

***LCR* Circuit**

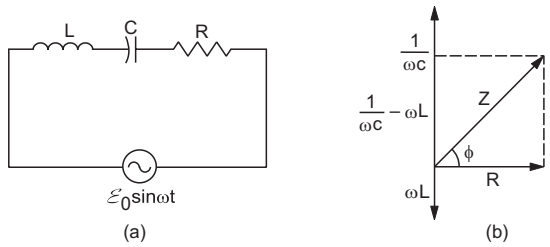



Figure 39.11

Figure (39.11) shows an inductor, a capacitor and a resistor connected in series with an AC source and the vector diagram to find the steady-state current.

The resultant of $1/\omega C$ and $\omega L$ is

$$X = X_c - X_L = \left(\frac{1}{\omega C} - \omega L\right)$$

in the direction of the positive $y$-axis. This is the net reactance of the circuit. The resultant of the vector for $R$ and that for the reactance $\left(\frac{1}{C\omega} - L\omega\right)$ has a magnitude

$$Z = \sqrt{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2} \qquad \ldots \text{ (39.12)}$$

which is the impedance of the circuit. This resultant makes an angle φ with the $x$-axis where

$$\tan\varphi = \frac{\frac{1}{\omega C} - \omega L}{R}. \qquad \ldots \text{ (39.13)}$$

The steady-state current in the circuit is given by

$$i = \frac{\mathcal{E}_0}{\sqrt{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}} \sin(\omega t + \varphi)$$

where φ is given by equation (39.13).

If $X_c = 1/\omega C$ is greater than $X_L = \omega L$, the vector for the net reactance $X_c - X_L$ is along the positive $Y$-axis. From equation (39.13), the phase factor φ is positive. Thus, the current *leads* the emf. If $X_c < X_L$, the vector for the net reactance is along the negative $Y$-axis and φ is negative. In this case, the current *lags* behind the

emf. If $X_L = X_c$, the net reactance is zero. It behaves as purely resistive circuit and the vector for $Z$ is along the $X$-axis. The current is in phase with the emf in this case.

If we vary the angular frequency ω of the AC source, the peak current

$$i_0 = \frac{\mathcal{E}_0}{\sqrt{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}}$$

also varies. It is maximum when

$$\frac{1}{\omega C} - \omega L = 0$$

or, $$\omega = \sqrt{\frac{1}{LC}} \cdot$$

The corresponding frequency is

$$\nu = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{LC}} \cdot \quad \ldots (39.14)$$

This frequency is known as the *resonant frequency* of the given circuit. The peak current in this case is $i_0 = \mathcal{E}_0/R$ and the reactance is zero.

Figure (39.12) shows the variation in the peak current $i_0$ with the applied frequency ν of the AC source in two different circuits. The values of $L$ as well as the values of $C$ are the same for the two circuits. We see that if $R$ is small, the resonance is sharp. This means, if the applied frequency is close to the resonant frequency $\nu_0$, the current is high, otherwise it is small. An *LCR* circuit used at a frequency close to the resonance frequency is called *resonant circuit.*

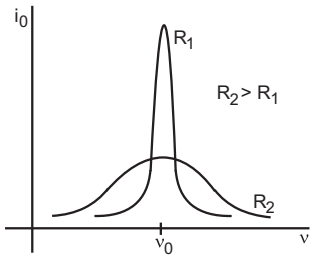



Figure 39.12

The tuning circuit of a radio or a television is an example of *LCR* resonant circuit. Signals are transmitted by different stations at different frequencies. The antenna receives these signals and drives a current in the tuning circuit. Only the signal corresponding to the resonant frequency is able to drive appreciable current and is further processed. When we 'tune' a radio, we change the capacitance of the tuning circuit and hence the resonant frequency changes. When this frequency matches with the frequency of the signal from the desired station, the tuning is complete.

***LC* oscillations**

If the resistance $R$ in an *LCR* circuit is zero, the peak current at resonance is

$$i = \frac{\mathcal{E}_0}{zero} \cdot$$

This means, there can be a finite current in the pure *LC* circuit even without any applied emf. This is the case when a charged capacitor is connected to a pure inductor. There is a current in the circuit at frequency $\nu = \frac{1}{2\pi}\sqrt{\frac{1}{LC}} \cdot$ The capacitor gets discharged sending a current in the inductor and induced emf in the inductor charges the capacitor again. Thus, the energy oscillates between electric field energy in the capacitor and magnetic field energy in the inductor. This phenomenon is called *LC oscillation*.

***Example 39.3***

*An LCR series circuit with $L = 100$ mH, $C = 100$ μF, $R = 120$ Ω is connected to an AC source of emf $\mathcal{E} = (30\text{ V})\sin(100\text{ s}^{-1})t$. Find the impedance, the peak current and the resonant frequency of the circuit.*

***Solution* :**

The reactance of the circuit is

$$X = \frac{1}{\omega C} - \omega L$$

$$= \frac{1}{(100\text{ s}^{-1})(100 \times 10^{-6}\text{ F})} - (100\text{ s}^{-1}) \times (100 \times 10^{-3}\text{ H})$$

$$= 100\ \Omega - 10\ \Omega = 90\ \Omega.$$

The resistance is $R = 120\ \Omega$.

The impedance is

$$Z = \sqrt{R^2 + X^2}$$

$$= \sqrt{(120\ \Omega)^2 + (90\ \Omega)^2} = 150\ \Omega.$$

The peak current is

$$i_0 = \frac{\mathcal{E}_0}{Z} = \frac{30\text{ V}}{150\ \Omega} = 0{\cdot}2\text{ A}.$$

The resonant frequency of the circuit is

$$\nu = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{1}{(100 \times 10^{-3}\text{ H})(100 \times 10^{-6}\text{ F})}}$$

$$\approx 50\text{ Hz}.$$

## 39.7 POWER IN AC CIRCUITS

Suppose an emf $\mathcal{E} = \mathcal{E}_0 \sin\omega t$ is applied in a circuit and a current $i = i_0 \sin(\omega t + \varphi)$ results. The work done by the source during the time interval $t$ to $t + dt$ is

$$dW = \mathcal{E}\, i\, dt$$

$$= \mathcal{E}_0 \, i_0 \sin \omega t \sin(\omega t + \varphi) \, dt$$

$$= \mathcal{E}_0 \, i_0 (\sin^2 \omega t \cos \varphi + \sin \omega t \cos \omega t \sin \varphi) dt.$$

The total work done in a complete cycle is

$$W = \mathcal{E}_0 \, i_0 \cos\varphi \int_0^T \sin^2 \omega t \, dt + \mathcal{E}_0 \, i_0 \sin \varphi \int_0^T \sin \omega t \cos \omega t \, dt$$

$$= \frac{1}{2} \mathcal{E}_0 \, i_0 \cos \varphi \int_0^T (1 - \cos 2\omega t) dt + \frac{1}{2} \mathcal{E}_0 \, i_0 \sin \varphi \int_0^T \sin 2\omega t \, dt$$

$$= \frac{1}{2} \mathcal{E}_0 \, i_0 \, T \cos \varphi.$$

The average power delivered by the source is, therefore,

$$P = \frac{W}{T} = \frac{1}{2} \mathcal{E}_0 \, i_0 \cos\varphi = \left(\frac{\mathcal{E}_0}{\sqrt{2}}\right)\left(\frac{i_0}{\sqrt{2}}\right)(\cos\varphi)$$

$$= \mathcal{E}_{rms} \, i_{rms} \cos\varphi. \quad \ldots \quad (39.15)$$

This equation is derived for the average power in a complete cycle. It also represents the average power delivered in a long time.

The term $\cos\varphi$ is called the *power factor* of the circuit. For a purely resistive circuit, $\varphi = 0$ so that $\cos \varphi = 1$ and $P = \mathcal{E}_{rms} \, i_{rms}$. For purely reactive circuits (no resistance, only capacitance and/or inductance), $\varphi = \pi/2$ or $-\pi/2$. In these cases, $\cos\varphi = 0$ and hence no power is absorbed in such circuits.

## 39.8 CHOKE COIL

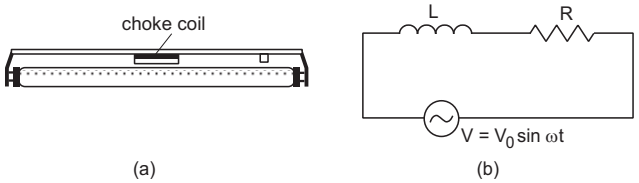



Figure 39.13

Choke coil is simply a coil having a large inductance but a small resistance. Choke coils are used with fluorescent mercury-tube fittings in houses (figure 39.13a).

At most places, the household electric power is supplied at 220 V, 50 Hz. If such a source is directly connected to a mercury tube, the tube will be damaged. To reduce the current, a choke coil is connected in series with the tube. Representing the tube by a resistor and the choke coil by an ideal inductor, the equivalent circuit is drawn in figure (39.13b). This is a simple *LR* circuit with impedance $Z = \sqrt{R^2 + \omega^2 L^2}$.

If the voltage applied is $V = V_0 \sin \omega t$, the peak current through the circuit is

$$i_0 = \frac{V_0}{\sqrt{R^2 + \omega^2 L^2}}.$$

The rms current is

$$i_{rms} = \frac{i_0}{\sqrt{2}} = \frac{V_0/\sqrt{2}}{\sqrt{R^2 + \omega^2 L^2}} = \frac{V_{rms}}{\sqrt{R^2 + \omega^2 L^2}}.$$

The rms voltage appearing across the resistor is

$$V_{R,\,rms} = R \, i_{rms} = \frac{R}{\sqrt{R^2 + \omega^2 L^2}} V_{rms}.$$

If the choke coil were not used, the voltage across the resistor would be the same as the applied voltage. Thus, by using the choke coil, the voltage across the resistor is reduced by a factor

$$\frac{R}{\sqrt{R^2 + \omega^2 L^2}}.$$

The advantage of using a choke coil to reduce the voltage is that an inductor does not consume power. Hence, we do not lose electric energy in the form of heat. If we connect an additional resistor in series with the tube to reduce the voltage, power will be lost in heating this additional resistor.

## 39.9 HOT-WIRE INSTRUMENTS

In an ordinary ammeter or voltmeter, a coil is free to rotate in the magnetic field of a fixed magnet. To measure a current or a voltage, current is passed through the coil and the coil deflects due to the torque acting on it. If an alternating current is passed through such a coil, the torque will reverse its direction each time the current changes direction and the average value of the torque will be zero. Because of friction, etc., the coil does not quickly respond to the changing torque and remains undeflected. To measure alternating currents or voltages, one would have to use a property so that the deflection of the moving part depends on $i^2$ and not on $i$. This ensures that the deflection remains independent of the direction of the current. The average of $i^2$ is not zero and hence a steady deflection may be obtained. Hot-wire instruments are designed to work on this principle.

### Hot-wire Ammeter

The construction of a hot-wire ammeter is shown in figure (39.14). A platinum–iridium wire *AB* is fixed tightly between two fixed ends *A* and *B*. A spring is fixed at one end *C* and is permanently connected to a thin wire at the other end. The thin wire is wound several times over a cylinder *D* and the end is connected to the middle point of *AB*. The cylinder can rotate about its axis. A pointer connected to the

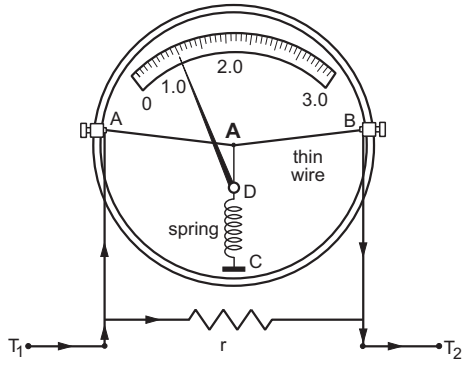



Figure 39.14

cylinder moves along a graduated scale when the cylinder rotates. A small resistance $r$ is connected in parallel to the wire $AB$ as a shunt. This makes the total resistance of the ammeter small so that it does not appreciably alter the current in the circuit. The points $A$ and $B$ are connected to the outer terminals $T_1$ and $T_2$.

The current to be measured is passed through the instrument via $T_1$, $T_2$. The wire $AB$ gets heated due to the current, the rise in temperature being proportional to $i_{rms}^2$. The length of the wire increases and consequently its tension decreases. Because of the tension in the spring on the other side, the cylinder rotates a little and the pointer deflects along the scale. The deflection is proportional to $i_{rms}^2$ but the scale is graduated in such a way that the reading gives directly the rms current.

**Hot-wire Voltmeter**

The construction of a hot-wire voltmeter is almost identical to a hot-wire ammeter except that a high resistance $R$ is connected in series with the wire $AB$ in place of the shunt $r$ (figure 39.15). The alternating voltage to be measured is applied across $T_1$ and $T_2$. A current passes through $AB$ and the pointer attached to the cylinder deflects. The deflection is proportional to $V_{rms}^2$. The scale is graduated in such a way that it reads directly the rms voltage.

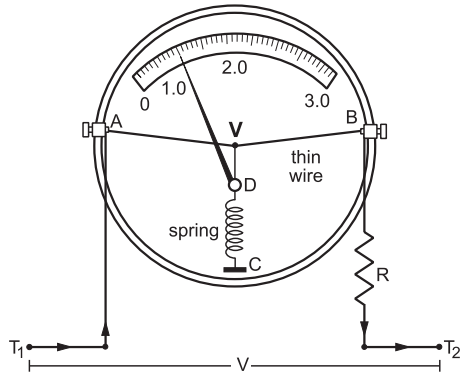



Figure 39.15

## 38.10 DC DYNAMO

An AC dynamo converts mechanical energy into electrical energy and it supplies alternating current in the circuit connected to it. A DC dynamo also converts mechanical energy into electrical energy but it supplies current in one direction only in the circuit connected to it.

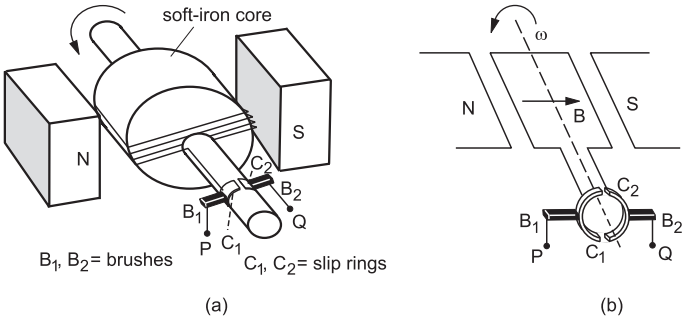



Figure 39.16

The basic design of a DC dynamo (figure 39.16a) is the same as that of an AC dynamo except for the slip rings. Figure (39.16b) shows a simplified diagram of the same. The slip rings are in the form of a split cylinder (figure 39.16). The ends of the armature (coil) are connected separately to the two halves $C_1$ and $C_2$ of the cylinders. The armature is rotated by some external agency. The split cylinder rotates with the armature. Two carbon brushes $B_1$ and $B_2$ press against the rotating halves $C_1$ and $C_2$. As the gaps pass under the brushes, the contacts to the external circuit are reversed. For half of a period of rotation, the terminal $P$ is connected to $C_1$ and the terminal $Q$ to $C_2$. For the other half of the period, $P$ is connected to $C_2$ and $Q$ to $C_1$. It is arranged in such a way that the gaps pass under the brushes at the time the emf becomes zero. Thus, although emf becomes negative, the current in the external circuit continues in the same direction (figure 39.17a). The system consisting of the split cylinders with brushes is also called a *slip-ring commutator*.

Although the current is unidirectional, its magnitude oscillates in time. To reduce the variation in the current, another coil perpendicular to the first

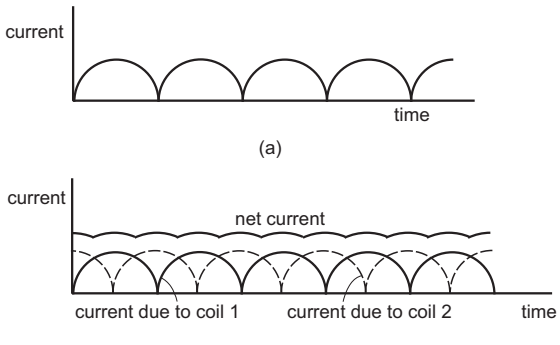



Figure 39.17

one is added in the system. The emf from this coil, again arranged properly with slip-ring commutator, is fed to the external circuit. The emf from this coil is maximum when the emf from the first coil is zero and vice versa. The sum of the two contains less oscillations and the current is more nearly constant (figure 39.17b). One can increase the number of coils to reduce the variation further.

### 39.11 DC MOTOR

A motor is used to convert electrical energy into mechanical energy and rotate a mechanical load. The principle of a DC motor is the same as that of a moving-coil galvanometer. The arrangement is basically the same as that of a DC dynamo. We can refer to figure (39.16) for its description. It has the field magnets, the armature, the slip rings and the brushes. A battery or the output of a DC generator is connected to the brushes through the outer terminals $P$ and $Q$. The battery drives a current in the coil and because of the magnetic field, a torque acts on it. This torque rotates the coil which is on a shaft to which the mechanical load is attached. This way the load is rotated. The torque depends on the orientation of the coil besides the strength of the current in it. It is zero when the coil is perpendicular to the field and is maximum in magnitude when it is parallel to the field. As the coil rotates, an induced emf $e$ is produced opposite to the applied emf $\mathcal{E}$. If the resistance of the circuit is $R$, the current at any instant is given by $i = (\mathcal{E} - e)/R$.

### 39.12 TRANSFORMER

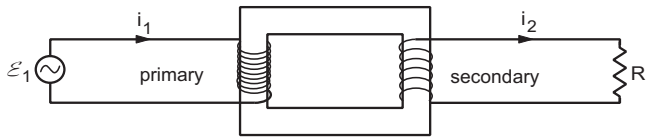



Figure 39.18

A transformer is used either to obtain a high AC voltage from a low-voltage AC source or to obtain a low AC voltage from a high-voltage AC source. The design of a simple transformer is shown in figure (39.18). Two coils are wound separately on a laminated soft-iron core. One of the coils is called the *primary* and the other is called the *secondary*. The original source of alternating voltage is connected across the primary. An induced emf appears across the ends of the secondary which is used to drive current in the desired circuit.

Suppose there are $N_1$ turns in the primary and $N_2$ turns in the secondary. An alternating emf $\mathcal{E}_1$ is applied across the primary which produces a current $i_1$ in the primary circuit and a current $i_2$ in the secondary circuit. The currents in the coils produce a magnetization in the soft-iron core and there is a corresponding magnetic field $B$ inside the core. The field due to magnetization of the core is large as compared to the field due to the currents in the coils. We assume that the field is constant in magnitude everywhere in the core and hence its flux ($BA$) through each turn is the same for the primary as well as for the secondary coil. Let the flux through each turn be $\Phi$. The emf induced in the primary is $-N_1 \frac{d\Phi}{dt}$ and that induced in the secondary is $-N_2 \frac{d\Phi}{dt} = \mathcal{E}_2$. If we neglect the resistance in the primary circuit, Kirchhoff's loop law applied to the primary circuit gives

$$\mathcal{E}_1 - N_1 \frac{d\Phi}{dt} = 0$$

or, $$\mathcal{E}_1 = N_1 \frac{d\Phi}{dt} . \quad \ldots \text{(i)}$$

Also, $$\mathcal{E}_2 = -N_2 \frac{d\Phi}{dt} . \quad \ldots \text{(ii)}$$

From (i) and (ii),

$$\mathcal{E}_2 = -\frac{N_2}{N_1} \mathcal{E}_1 . \quad \ldots \text{(39.16)}$$

The minus sign shows that $\mathcal{E}_2$ is 180° out of phase with $\mathcal{E}_1$. Equations (i), (ii) and (39.16) are valid for all values of currents in the primary and the secondary circuits.

#### Power Transfer

Let us first consider the case when the terminals of the secondary are not connected to any external circuit. The secondary circuit is incomplete and the current through it is zero. Suppose, the current in the primary is $i_s$ in this case (the subscript $s$ stands for the source and not for the secondary). As we have neglected the resistance in the primary circuit, it is a purely inductive circuit. The current has a phase difference of 90° with the applied emf $\mathcal{E}_1$ and hence the power delivered by the AC source is zero. The power in the secondary circuit is anyway zero as there is no current in this circuit.

Now suppose, the terminals of the secondary are joined to a resistance $R$. There will be an alternating current $i_2$ through $R$. There will be additional emf's induced in the primary as well as in the secondary due to $i_2$. But the net induced emf in the primary should remain equal and opposite to the source-emf $\mathcal{E}_1$ by (i). So, there will be an additional current $i_1$ in the primary circuit which will cancel the emf induced due to $i_2$. Thus the current in the primary will be $i_s + i_1$ and in

the secondary $i_2$. The emf in the secondary will remain $\mathcal{E}_2$ as given by (ii).

As the induced emf's due to $i_1$ and $i_2$ always cancel each other, the two alternating currents should be 180° out of phase. Also, $i_2$ is in phase with $\mathcal{E}_2$ (purely resistive circuit), and $\mathcal{E}_1$ is 180° out of phase with $\mathcal{E}_2$ (equation 39.16). This shows that $i_1$ is in phase with $\mathcal{E}_1$ (figure 39.19).

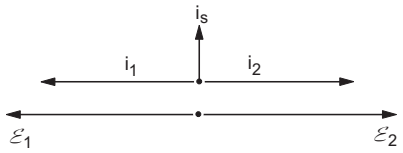



Figure 39.19

The primary current (current in the primary circuit) $i_s + i_1$, therefore, has a component $i_s$ which is at a phase difference of 90° from the applied emf $\mathcal{E}_1$ and a component $i_1$ which is in phase with this emf. The power delivered by the AC source is, therefore, $\mathcal{E}_1 i_1$. The power consumed by the resistance in the secondary circuit is $\mathcal{E}_2 i_2$. Neglecting any loss of energy elsewhere,

$$\mathcal{E}_1 i_1 = \mathcal{E}_2 i_2. \quad \ldots \text{(i)}$$

Using equation (39.16),

$$i_2 = -\frac{N_1}{N_2} i_1. \quad \ldots (39.17)$$

The minus sign shows that $i_2$ is 180° out of phase with $i_1$.

Quite often, the additional current $i_1$ in the primary is much larger than the original current $i_s$. This can be easily shown by connecting an electric bulb in series with the primary. The bulb glows much brighter when the secondary circuit is completed than when it is open. If $i_s$ is negligible as compared to $i_1$, equation (39.17) gives the relation between the net currents.

**Step-up and Step-down Transformers**

If $N_2 > N_1$, the secondary emf $\mathcal{E}_2$ is larger in magnitude than the primary emf $\mathcal{E}_1$. This type of transformer is called a *step-up* transformer. The secondary current is less than the primary current. The primary coil is made from a thick wire so that it can sustain the high current.

If $N_2 < N_1$, the emf in the secondary circuit is smaller in magnitude than the primary emf. This type of transformer is called a *step-down* transformer. The secondary current is more than the primary current and the wire used to make the secondary coil should be sufficiently thick to carry the high current.

**Efficiency of a Transformer**

In an ordinary transformer, there is some loss of energy due to primary resistance, hysteresis in the core, eddy currents in the core, etc. The efficiency of a transformer is defined as

$$\eta = \frac{\text{output power}}{\text{input power}}.$$

Efficiencies of the order of 99% can be easily achieved.

***Example 39.5***

*A radio set operates at* 6 V *DC. A transformer with* 18 *turns in the secondary coil is used to step down the input* 220 V *AC emf to* 6 V *AC emf. This AC emf is then rectified by another circuit to give* 6 V *DC which is fed to the radio. Find the number of turns in the primary.*

***Solution*** :

We have,

$$\left|\frac{\mathcal{E}_2}{\mathcal{E}_1}\right| = \frac{N_2}{N_1}$$

or, $$N_1 = \left|\frac{\mathcal{E}_1}{\mathcal{E}_2}\right| N_2 = \frac{220}{6} \times 18 = 660.$$

**Transmission of Power**

The fact that an AC voltage can be stepped up or stepped down, has application in transmission of power from the electricity generation plants to the users. Generally, these plants are quite far away from the actual areas where the power is used. Power is transmitted through several hundred kilometres of wires before it is used. Because of the resistance of these wires, some energy is lost in Joule heating in the form of $i^2Rt$. The plant can supply a fixed power depending on its capacity. If this power is supplied at a high voltage, the current is small. Correspondingly, the loss of power in transmission is small. So, the voltage at the electricity generation plant is stepped up to, say, 66 kV and fed to the transmission lines. In a town or city, the voltage is stepped down to the required value such as 220 V.

## *Worked Out Examples*

**1.** *A resistance of* 20 Ω *is connected to a source of alternating current rated* 110 V, 50 Hz. *Find (a) the rms current, (b) the maximum instantaneous current in the resistor and (c) the time taken by the current to change from its maximum value to the rms value.*

***Solution :***

(a) The rms potential difference = 110 V and so

the rms current $= \frac{110\ \text{V}}{20\ \Omega} = 5{\cdot}5\ \text{A}.$

(b) The maximum instantaneous current

$= \sqrt{2}$ (rms current)

$= \sqrt{2} \times 5{\cdot}5\ \text{A} = 7{\cdot}8\ \text{A}.$

(c) Let the current be $i = i_0 \sin \omega t$.

If $t_1$ and $t_2$ be the time instants for consecutive appearances of the maximum value and the rms value of the current,

$$i_0 = i_0 \sin \omega t_1$$

and $$\frac{i_0}{\sqrt{2}} = i_0 \sin \omega t_2.$$

If $\omega t_1 = \frac{\pi}{2}$, $\omega t_2 = \frac{\pi}{2} + \frac{\pi}{4}$.

Hence, $t_2 - t_1 = \frac{\pi}{4\omega}$

$$= \frac{\pi}{4 \times 2\pi\nu} = \frac{1}{8 \times 50}\ \text{s} = 2{\cdot}5\ \text{ms}.$$

**2.** *The electric current in a circuit is given by* $i = i_0(t/\tau)$ *for some time. Calculate the rms current for the period* $t = 0$ to $t = \tau$.

***Solution :***

The mean square current is

$$\overline{i^2} = \frac{1}{\tau}\int_0^{\tau} i_0^2\,(t/\tau)^2\,dt = \frac{i_0^2}{\tau^3}\int_0^{\tau} t^2\,dt = \frac{i_0^2}{3}.$$

Thus, the rms current is

$$i_{rms} = \sqrt{\overline{i^2}} = \frac{i_0}{\sqrt{3}}.$$

**3.** *A coil having a resistance of* 50·0 Ω *and an inductance of* 0·500 *henry is connected to an AC source of* 110 *volts*, 50·0 cycle/s. *Find the rms value of the current in the circuit.*

***Solution :***

The angular frequency $\omega = 2\pi\nu = 100\pi\ \text{s}^{-1}$.

The impedance of the coil

$$= \sqrt{R^2 + L^2\omega^2}$$

$$= \sqrt{(50\ \Omega)^2 + (0{\cdot}50\ \text{H} \times 100\pi\ \text{s}^{-1})^2}$$

$$= \sqrt{2500\ \Omega^2 + 2500\,\pi^2\Omega^2} = 164{\cdot}8\ \Omega.$$

The rms current is $\frac{\mathcal{E}_{rms}}{Z} = \frac{110\ \text{V}}{164{\cdot}8\ \Omega} \approx 0{\cdot}667\ \text{A}.$

The peak current $= \sqrt{2}$ (rms current) $\approx 0{\cdot}943\ \text{A}.$

**4.** *A capacitor of capacitance* 100 μF *and a coil of resistance* 50 Ω *and inductance* 0·5 H *are connected in series with a* 110 V, 50 Hz *AC source. Find the rms value of the current.*

***Solution :***

The resistance of the circuit is $R = 50\ \Omega$.

The reactance of the capacitor $= \frac{1}{\omega C}$

$$= \frac{1}{(2\pi \times 50\ \text{s}^{-1})\,(100 \times 10^{-6}\ \text{F})} = 31{\cdot}8\ \Omega.$$

The reactance of the inductor $= \omega L$

$$= (2\pi \times 50\ \text{s}^{-1})\,(0{\cdot}5\ \text{henry}) = 157\ \Omega.$$

The reactance of the circuit $= X = \frac{1}{\omega C} - L\omega$

$$= 31{\cdot}8\ \Omega - 157\ \Omega = -125{\cdot}2\ \Omega.$$

Hence, the impedance $Z = \sqrt{R^2 + X^2}$

$$= \sqrt{(50\ \Omega)^2 + (125{\cdot}2\ \Omega)^2} \approx 134{\cdot}6\ \Omega.$$

The rms current $= \frac{E_{rms}}{Z} = \frac{110\ \text{V}}{134{\cdot}6\ \Omega} = 0{\cdot}82\ \text{A}.$

**5.** *A capacitor of capacitance* 12·0 μF *is joined to an AC source of frequency* 200 Hz. *The rms current in the circuit is* 2·00 A. *(a) Find the rms voltage across the capacitor. (b) Find the average energy stored in the electric field between the plates of the capacitor.*

***Solution :***

(a) The impedance of the capacitor $= \frac{1}{\omega C}$

$$= \frac{1}{(2\pi \times 200\ \text{s}^{-1})\,(12\ \mu\text{F})} = 66{\cdot}3\ \Omega.$$

The rms voltage across the capacitor

$$= i_{rms}\,Z = 2{\cdot}0\ \text{A} \times 66{\cdot}3\ \Omega \approx 133\ \text{V}.$$

(b) The energy stored in the electric field $= \frac{1}{2}CV^2$.

Hence the average energy stored $= \frac{1}{2}C\overline{V^2}$.

But $\overline{V^2} = (V_{rms})^2$.

Thus, the average energy stored

$$= \frac{1}{2} \times (12\ \mu\text{F}) \times (133\ \text{V})^2 \approx 0{\cdot}106\ \text{J}.$$

**6.** *A series AC circuit contains an inductor* (20 mH), *a capacitor* (100 μF), *a resistor* (50 Ω) *and an AC source of* 12 V, 50 Hz. *Find the energy dissipated in the circuit in* 1000 s.

***Solution*** :

The time period of the source is

$$T = 1/\nu = 20 \text{ ms}.$$

The given time 1000 s is much larger than the time period. Hence we can write the average power dissipated as

$$P_{av} = V_{rms}\, i_{rms} \cos\varphi$$

where $\cos\varphi = R/Z$ is the power factor. Thus,

$$P_{av} = V_{rms} \frac{V_{rms}}{Z}\frac{R}{Z} = \frac{R\,V_{rms}^2}{Z^2}$$

$$= \frac{(50\ \Omega)\ (12\ \text{V})^2}{Z^2}$$

$$= \frac{7200}{Z^2}\ \Omega\text{V}^2. \qquad \ldots \text{(i)}$$

The capacitive reactance $\frac{1}{\omega C} = \frac{1}{2\pi \times 50 \times 100 \times 10^{-6}}\ \Omega$

$$= \frac{100}{\pi}\ \Omega.$$

The inductive reactance $= \omega L$

$$= 2\pi \times 50 \times 20 \times 10^{-3}\ \Omega = 2\pi\ \Omega.$$

The net reactance is $X = \frac{1}{\omega C} - \omega L$

$$= \frac{100}{\pi}\ \Omega - 2\pi\ \Omega \approx 25{\cdot}5\ \Omega.$$

Thus,

$$Z^2 = (50\ \Omega)^2 + (25{\cdot}5\ \Omega)^2 = 3150\ \Omega^2.$$

From (i), average power $P_{av} = \frac{7200\ \Omega\text{–V}^2}{3150\ \Omega^2} = 2{\cdot}286$ W.

The energy dissipated in 1000 s $= P_{av} \times 1000$ s

$$\approx 2{\cdot}3 \times 10^3 \text{ J}.$$

**7.** *An inductor of inductance* 100 mH *is connected in series with a resistance, a variable capacitance and an AC source of frequency* 2·0 kHz. *What should be the value of the capacitance so that maximum current may be drawn into the circuit* ?

***Solution*** :

This is an *LCR* series circuit. The current will be maximum when the net reactance is zero. For this,

$$\frac{1}{\omega C} = \omega L$$

or, $$C = \frac{1}{\omega^2 L} = \frac{1}{4\pi^2 \times (2{\cdot}0 \times 10^3\ \text{s}^{-1})^2\ (0{\cdot}1\ \text{H})}$$

$$= 63 \text{ nF}.$$

**8.** *An inductor coil joined to a* 6 V *battery draws a steady current of* 12 A. *This coil is connected to a capacitor and an AC source of rms voltage* 6 V *in series. If the current in the circuit is in phase with the emf, find the rms current.*

***Solution*** :

The resistance of the coil is $R = \frac{6\ \text{V}}{12\ \text{A}} = 0{\cdot}5\ \Omega.$

In the AC circuit, the current is in phase with the emf. This means that the net reactance of the circuit is zero. The impedance is equal to the resistance, i.e., $Z = 0{\cdot}5\ \Omega$.

The rms current $= \frac{\text{rms voltage}}{Z} = \frac{6\ \text{V}}{0{\cdot}5\ \Omega} = 12$ A.

□

## QUESTIONS FOR SHORT ANSWER

**1.** What is the reactance of a capacitor connected to a constant DC source ?

**2.** The voltage and current in a series AC circuit are given by

$$V = V_0 \cos \omega t \text{ and } i = i_0 \sin \omega t.$$

What is the power dissipated in the circuit ?

**3.** Two alternating currents are given by

$$i_1 = i_0 \sin \omega t \text{ and } i_2 = i_0 \sin\left(\omega t + \frac{\pi}{3}\right).$$

Will the rms values of the currents be equal or different ?

**4.** Can the peak voltage across the inductor be greater than the peak voltage of the source in an *LCR* circuit ?

**5.** In a circuit containing a capacitor and an AC source, the current is zero at the instant the source voltage is maximum. Is it consistent with Ohm's law ?

**6.** An AC source is connected to a capacitor. Will the rms current increase, decrease or remain constant if a dielectric slab is inserted into the capacitor ?

**7.** When the frequency of the AC source in an *LCR* circuit equals the resonant frequency, the reactance of the circuit is zero. Does it mean that there is no current through the inductor or the capacitor ?

**8.** When an AC source is connected to a capacitor there is a steady-state current in the circuit. Does it mean that

the charges jump from one plate to the other to complete the circuit ?

**9.** A current $i_1 = i_0 \sin \omega t$ passes through a resistor of resistance $R$. How much thermal energy is produced in one time period ? A current $i_2 = -i_0 \sin \omega t$ passes through the resistor. How much thermal energy is produced in one time period ? If $i_1$ and $i_2$ both pass through the resistor simultaneously, how much thermal energy is produced ? Is the principle of superposition obeyed in this case ?

**10.** Is energy produced when a transformer steps up the voltage ?

**11.** A transformer is designed to convert an AC voltage of 220 V to an AC voltage of 12 V. If the input terminals are connected to a DC voltage of 220 V, the transformer usually burns. Explain.

**12.** Can you have an AC series circuit in which there is a phase difference of (a) 180° (b) 120° between the emf and the current ?

**13.** A resistance is connected to an AC source. If a capacitor is included in the series circuit, will the average power absorbed by the resistance increase or decrease ? If an inductor of small inductance is also included in the series circuit, will the average power absorbed increase or decrease further ?

**14.** Can a hot-wire ammeter be used to measure a direct current having a constant value ? Do we have to change the graduations ?

## OBJECTIVE I

**1.** A capacitor acts as an infinite resistance for
(a) DC (b) AC
(c) DC as well as AC (d) neither AC nor DC.

**2.** An AC source producing emf
$$\mathcal{E} = \mathcal{E}_0 \left[\cos(100\ \pi\ \mathrm{s}^{-1})t + \cos(500\ \pi\ \mathrm{s}^{-1})t\right]$$
is connected in series with a capacitor and a resistor. The steady-state current in the circuit is found to be
$$i = i_1 \cos\left[(100\ \pi\ \mathrm{s}^{-1})t + \varphi_1\right] + i_2 \cos\left[(500\ \pi\ \mathrm{s}^{-1})t + \phi_2\right].$$
(a) $i_1 > i_2$ (b) $i_1 = i_2$ (c) $i_1 < i_2$
(d) The information is insufficient to find the relation between $i_1$ and $i_2$.

**3.** The peak voltage in a 220 V AC source is
(a) 220 V (b) about 160 V
(c) about 310 V (d) 440 V.

**4.** An AC source is rated 220 V, 50 Hz. The average voltage is calculated in a time interval of 0·01 s. It
(a) must be zero (b) may be zero
(c) is never zero (d) is $(220/\sqrt{2})$V.

**5.** The magnetic field energy in an inductor changes from maximum value to minimum value in 5·0 ms when connected to an AC source. The frequency of the source is
(a) 20 Hz (b) 50 Hz (c) 200 Hz (d) 500 Hz.

**6.** Which of the following plots may represent the reactance of a series $LC$ combination ?

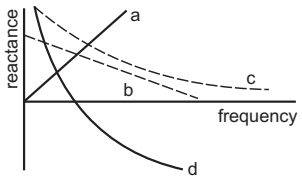



Figure 39-Q1

**7.** A series AC circuit has a resistance of 4 Ω and a reactance of 3 Ω. The impedance of the circuit is
(a) 5 Ω (b) 7 Ω (c) 12/7 Ω (d) 7/12 Ω.

**8.** Transformers are used
(a) in DC circuits only (b) in AC circuits only
(c) in both DC and AC circuits
(d) neither in DC nor in AC circuits.

**9.** An alternating current is given by
$$i = i_1 \cos \omega t + i_2 \sin \omega t.$$
The rms current is given by
(a) $\dfrac{i_1 + i_2}{\sqrt{2}}$ (b) $\dfrac{|i_1 + i_2|}{\sqrt{2}}$ (c) $\sqrt{\dfrac{i_1^2 + i_2^2}{2}}$ (d) $\sqrt{\dfrac{i_1^2 + i_2^2}{\sqrt{2}}}$.

**10.** An alternating current having peak value 14 A is used to heat a metal wire. To produce the same heating effect, a constant current $i$ can be used where $i$ is
(a) 14 A (b) about 20 A (c) 7 A (d) about 10 A.

**11.** A constant current of 2·8 A exists in a resistor. The rms current is
(a) 2·8 A (b) about 2 A
(c) 1·4 A (d) undefined for a direct current.

## OBJECTIVE II

**1.** An inductor, a resistor and a capacitor are joined in series with an AC source. As the frequency of the source is slightly increased from a very low value, the reactance
(a) of the inductor increases
(b) of the resistor increases
(c) of the capacitor increases
(d) of the circuit increases.

**2.** The reactance of a circuit is zero. It is possible that the circuit contains
(a) an inductor and a capacitor
(b) an inductor but no capacitor

(c) a capacitor but no inductor
(d) neither an inductor nor a capacitor.

**3.** In an AC series circuit, the instantaneous current is zero when the instantaneous voltage is maximum. Connected to the source may be a
(a) pure inductor (b) pure capacitor
(c) pure resistor
(d) combination of an inductor and a capacitor.

**4.** An inductor-coil having some resistance is connected to an AC source. Which of the following quantities have zero average value over a cycle ?
(a) Current (b) Induced emf in the inductor
(c) Joule heat
(d) Magnetic energy stored in the inductor

**5.** The AC voltage across a resistance can be measured using
(a) a potentiometer (b) a hot-wire voltmeter
(c) a moving-coil galvanometer
(d) a moving-magnet galvanometer.

**6.** To convert mechanical energy into electrical energy, one can use
(a) DC dynamo (b) AC dynamo
(c) motor (d) transformer.

**7.** An AC source rated 100 V (rms) supplies a current of 10 A (rms) to a circuit. The average power delivered by the source
(a) must be 1000 W (b) may be 1000 W
(c) may be greater than 1000 W
(d) may be less than 1000 W.

## EXERCISES

**1.** Find the time required for a 50 Hz alternating current to change its value from zero to the rms value.

**2.** The household supply of electricity is at 220 V (rms value) and 50 Hz. Find the peak voltage and the least possible time in which the voltage can change from the rms value to zero.

**3.** A bulb rated 60 W at 220 V is connected across a household supply of alternating voltage of 220 V. Calculate the maximum instantaneous current through the filament.

**4.** An electric bulb is designed to operate at 12 volts DC. If this bulb is connected to an AC source and gives normal brightness, what would be the peak voltage of the source ?

**5.** The peak power consumed by a resistive coil when connected to an AC source is 80 W. Find the energy consumed by the coil in 100 seconds which is many times larger than the time period of the source.

**6.** The dielectric strength of air is $3{\cdot}0 \times 10^6$ V/m. A parallel-plate air-capacitor has area 20 $cm^2$ and plate separation 0·10 mm. Find the maximum rms voltage of an AC source which can be safely connected to this capacitor.

**7.** The current in a discharging *LR* circuit is given by $i = i_0 e^{-t/\tau}$ where $\tau$ is the time constant of the circuit. Calculate the rms current for the period $t = 0$ to $t = \tau$.

**8.** A capacitor of capacitance 10 μF is connected to an oscillator giving an output voltage $\mathcal{E} = (10\text{ V})\sin \omega t$. Find the peak currents in the circuit for $\omega = 10\text{ s}^{-1}$, $100\text{ s}^{-1}$, $500\text{ s}^{-1}$, $1000\text{ s}^{-1}$.

**9.** A coil of inductance 5·0 mH and negligible resistance is connected to the oscillator of the previous problem. Find the peak currents in the circuit for $\omega = 100\text{ s}^{-1}$, $500\text{ s}^{-1}$, $1000\text{ s}^{-1}$.

**10.** A coil has a resistance of 10 Ω and an inductance of 0·4 henry. It is connected to an AC source of 6·5 V, $\frac{30}{\pi}$ Hz. Find the average power consumed in the circuit.

**11.** A resistor of resistance 100 Ω is connected to an AC source $\mathcal{E} = (12\text{ V})\sin(250\pi\text{ s}^{-1})t$. Find the energy dissipated as heat during $t = 0$ to $t = 1{\cdot}0$ ms.

**12.** In a series *RC* circuit with an AC source, $R$ = 300 Ω, $C$ = 25 μF, $\mathcal{E}_0 = 50$ V and $\nu = 50/\pi$ Hz. Find the peak current and the average power dissipated in the circuit.

**13.** An electric bulb is designed to consume 55 W when operated at 110 volts. It is connected to a 220 V, 50 Hz line through a choke coil in series. What should be the inductance of the coil for which the bulb gets correct voltage ?

**14.** In a series *LCR* circuit with an AC source, $R = 300\ \Omega$, $C = 20\ \mu$F, $L = 1{\cdot}0$ henry, $\mathcal{E}_{rms} = 50$ V and $\nu = 50/\pi$ Hz. Find (a) the rms current in the circuit and (b) the rms potential differences across the capacitor, the resistor and the inductor. Note that the sum of the rms potential differences across the three elements is greater than the rms voltage of the source.

**15.** Consider the situation of the previous problem. Find the average electric field energy stored in the capacitor and the average magnetic field energy stored in the coil.

**16.** An inductance of 2·0 H, a capacitance of 18 μF and a resistance of 10 kΩ are connected to an AC source of 20 V with adjustable frequency. (a) What frequency should be chosen to maximise the current in the circuit ? (b) What is the value of this maximum current ?

**17.** An inductor-coil, a capacitor and an AC source of rms voltage 24 V are connected in series. When the frequency of the source is varied, a maximum rms current of 6·0 A is observed. If this inductor coil is connected to a battery of emf 12 V and internal resistance 4·0 Ω, what will be the current ?

**18.** Figure (39-E1) shows a typical circuit for low-pass filter. An AC input $V_i = 10$ mV is applied at the left end and

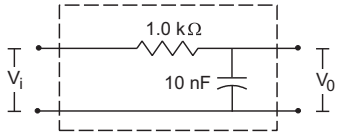


Figure 39-E1

the output $V_0$ is received at the right end. Find the output voltages for $\nu = 10\,\text{kHz}, 100\,\text{kHz}$, 1·0 MHz and 10·0 MHz. Note that as the frequency is increased the output decreases and hence the name low-pass filter.

**19.** A transformer has 50 turns in the primary and 100 in the secondary. If the primary is connected to a 220 V DC supply, what will be the voltage across the secondary ?

□

## ANSWERS

### OBJECTIVE I

1. (a) 2. (c) 3. (c) 4. (b) 5. (b) 6. (d)
7. (a) 8. (b) 9. (c) 10. (d) 11. (a)

### OBJECTIVE II

1. (a) 2. (a), (d) 3. (a), (b), (d)
4. (a), (b) 5. (b) 6. (a), (b)
7. (b), (d)

### EXERCISES

1. 2·5 ms
2. 311 V, 2·5 ms
3. 0·39 A
4. 17 volts
5. 4·0 kJ
6. 210 V
7. $\frac{i_0}{e}\sqrt{(e^2-1)/2}$
8. $1{\cdot}0\times10^{-3}$ A, 0·01 A, 0·05 A, 0·1 A
9. 20 A, 4·0 A, 0·20 A
10. 5/8 W
11. $2{\cdot}61\times10^{-4}$ J
12. 0·10 A, 1·5 W
13. 1·2 H
14. (a) 0·10 A (b) 50 V, 30 V, 10 V
15. 25 mJ, 5 mJ
16. (a) 27 Hz (b) 2 mA
17. 1·5 A
18. 8·5 mV, 1·6 mV, 0·16 mV, 16 μV
19. zero.

□

## CHAPTER 40

# ELECTROMAGNETIC WAVES

### 40.1 INTRODUCTION

We have seen that in certain situations light may be described as a wave. The wave equation for light propagating in $x$-direction in vacuum may be written as

$$E = E_0 \sin \omega(t - x/c)$$

where $E$ is the sinusoidally varying electric field at the position $x$ at time $t$. The constant $c$ is the speed of light in vacuum. The electric field $E$ is in the $Y$–$Z$ plane, that is, perpendicular to the direction of propagation.

There is also a sinusoidally varying magnetic field associated with the electric field when light propagates. This magnetic field is perpendicular to the direction of propagation as well as to the electric field $E$. It is given by

$$B = B_0 \sin \omega(t - x/c).$$

Such a combination of mutually perpendicular electric and magnetic fields is referred to as an *electromagnetic wave* in vacuum. The theory of electromagnetic wave was mainly developed by Maxwell around 1864. We give a brief discussion of this theory.

### 40.2 MAXWELL'S DISPLACEMENT CURRENT

We have stated Ampere's law as

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \, i \qquad \ldots \quad (40.1)$$

where $i$ is the electric current crossing a surface bounded by a closed curve and the line integral of $\vec{B}$ (circulation) is calculated along that closed curve. This equation is valid only when the electric field at the surface does not change with time. This law tells us that an electric current produces magnetic field and gives a method to calculate the field.

Ampere's law in this form is not valid if the electric field at the surface varies with time. As an example, consider a parallel-plate capacitor with circular plates, being charged by a battery (figure 40.1). If we place a compass needle in the space between the plates, the needle, in general, deflects. This shows that there is a magnetic field in this region. Figure (40.1) also shows a closed curve $\gamma$ which lies completely in the region between the plates. The plane surface $S$ bounded by this curve is also parallel to the plates and lies completely inside the region between the plates.

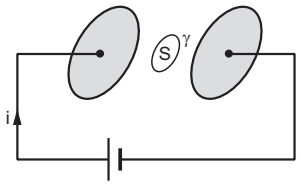



Figure 40.1

During the charging process, there is an electric current through the connecting wires. Charge is accumulated on the plates and the electric field at the points on the surface $S$ changes. It is found that there is a magnetic field at the points on the curve $\gamma$ and the circulation

$$\oint \vec{B} \cdot d\vec{l}$$

has a nonzero value. As no charge crosses the surface $S$, the electric current $i$ through the surface is zero. Hence,

$$\oint \vec{B} \cdot d\vec{l} \neq \mu_0 \, i. \qquad \ldots \quad \text{(i)}$$

Now, Ampere's law (40.1) can be deduced from Biot–Savart law. We can calculate the magnetic field due to each current element from Biot–Savart law and then its circulation along the closed curve $\gamma$. The circulation of the magnetic field due to these current elements must satisfy equation (40.1). If we denote this magnetic field by $\vec{B}'$,

$$\oint \vec{B}' \cdot d\vec{l} = 0. \qquad \ldots \quad \text{(ii)}$$

This shows that the actual magnetic field $\vec{B}$ is different from the field $\vec{B}'$ produced by the electric currents only. So, there must be some other source of magnetic field. This other source is nothing but the

changing electric field. As the capacitor gets charged, the electric field between the plates changes and this changing electric field produces magnetic field.

We know that a changing magnetic field produces an electric field. The relation between the two is given by Faraday's law

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt}.$$

Here, $\Phi_B = \int \vec{B}\cdot d\vec{S}$ is the flux of the magnetic field through the area bounded by the closed curve along which the circulation of $\vec{E}$ is calculated. Now we find that a changing electric field produces a magnetic field. The relation between the changing electric field and the magnetic field resulting from it is given by

$$\oint \vec{B}\cdot d\vec{l} = \mu_0\,\varepsilon_0\frac{d\Phi_E}{dt}. \qquad \ldots \quad (40.2)$$

Here, $\Phi_E$ is the flux of the electric field through the area bounded by the closed curve along which the circulation of $\vec{B}$ is calculated. Equation (40.1) gives the magnetic field resulting from an electric current due to flow of charges and equation (40.2) gives the magnetic field due to the changing electric field. If there exists an electric current as well as a changing electric field, the resultant magnetic field is given by

$$\oint \vec{B}\cdot d\vec{l} = \mu_0\, i + \mu_0\,\varepsilon_0\left(\frac{d\Phi_E}{dt}\right)$$

or,
$$\oint \vec{B}\cdot d\vec{l} = \mu_0(i + i_d) \qquad \ldots \quad (40.3)$$

where
$$i_d = \varepsilon_0\frac{d\Phi_E}{dt}.$$

It was James Clerk Maxwell who generalised Ampere's law from equation (40.1) to equation (40.3). Maxwell named the term $i_d = \varepsilon_0\frac{d\Phi_E}{dt}$ as *displacement current*. The current due to flow of charges is often called *conduction current* and is denoted by $i_c$.

***Example 40.1***

*A parallel-plate capacitor is being charged. Show that the displacement current across an area in the region between the plates and parallel to it (figure 40.1) is equal to the conduction current in the connecting wires.*

***Solution*** :

The electric field between the plates is

$$E = \frac{Q}{\varepsilon_0 A}$$

where $Q$ is the charge accumulated at the positive plate. The flux of this field through the given area is

$$\Phi_E = \frac{Q}{\varepsilon_0 A}\times A = \frac{Q}{\varepsilon_0}.$$

The displacement current is

$$i_d = \varepsilon_0\frac{d\Phi_E}{dt} = \varepsilon_0\frac{d}{dt}\left(\frac{Q}{\varepsilon_0}\right) = \frac{dQ}{dt}.$$

But $\frac{dQ}{dt}$ is the rate at which the charge is carried to the positive plate through the connecting wire. Thus, $i_d = i_c$.

## 40.3 CONTINUITY OF ELECTRIC CURRENT

Consider a closed surface enclosing a volume (figure 40.2). Suppose charges are entering into the volume and are also leaving it. If no charge is accumulated inside the volume, the total charge going into the volume in any time is equal to the total charge leaving it during the same time. The conduction current is then continuous.

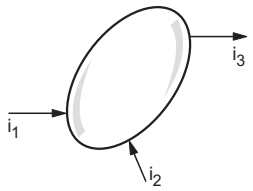



Figure 40.2

If charge is accumulated inside the volume, this continuity breaks. However, if we consider the conduction current plus the displacement current, the total current is still continuous. Any loss of conduction current $i_c$ appears as displacement current $i_d$. This can be shown as follows.

Suppose a total conduction current $i_1$ goes into the volume and a total conduction current $i_2$ goes out of it. The charge going into the volume in a time $dt$ is $i_1\,dt$ and that coming out is $i_2\,dt$. The charge accumulated inside the volume is

$$d(q_{inside}) = i_1\,dt - i_2\,dt$$

or,
$$\frac{d}{dt}(q_{inside}) = i_1 - i_2. \qquad \ldots \quad (i)$$

From Gauss's law,

$$\Phi_E = \oint \vec{E}\cdot d\vec{S} = \frac{q_{inside}}{\varepsilon_o}$$

or,
$$\varepsilon_0\frac{d\Phi_E}{dt} = \frac{d}{dt}(q_{inside})$$

or,
$$i_d = \frac{d}{dt}(q_{inside}).$$

Comparing with (i),

$$i_1 - i_2 = i_d$$

or,
$$i_1 = i_2 + i_d.$$

Thus, the total current (conduction + displacement) going into the volume is equal to the total current coming out of it.

## 40.4 MAXWELL'S EQUATIONS AND PLANE ELECTROMAGNETIC WAVES

The whole subject of electricity and magnetism may be described mathematically with the help of four fundamental equations:

Gauss's law for electricity $\oint \vec{E}\cdot d\vec{S} = \frac{q}{\varepsilon_0}$

Gauss's law for magnetism $\oint \vec{B}\cdot d\vec{S} = 0$

Faraday's law $\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt}$

Ampere's law $\oint \vec{B}\cdot d\vec{l} = \mu_0 i + \varepsilon_0\mu_0 \frac{d\Phi_E}{dt}$.

These equations are collectively known as Maxwell's equations.

In vacuum, there are no charges and hence no conduction currents. Faraday's law and Ampere's law take the form

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt} \quad \ldots \text{(i)}$$

and

$$\oint \vec{B}\cdot d\vec{l} = \mu_0\,\varepsilon_0 \frac{d\Phi_E}{dt} \quad \ldots \text{(ii)}$$

respectively.

Let us check if these equations are satisfied by a plane electromagnetic wave given by

$$\left.\begin{aligned} E = E_y = E_0 \sin \omega(t - x/c) \\ \text{and} \quad B = B_z = B_0 \sin \omega(t - x/c). \end{aligned}\right] \quad \ldots (40.4)$$

The wave described above propagates along the positive $x$-direction, the electric field remains along the $y$-direction and the magnetic field along the $z$-direction. The magnitudes of the fields oscillate between $\pm E_0$ and $\pm B_0$ respectively. It is a linearly polarized light, polarized along the $y$-axis.

### Faraday's Law

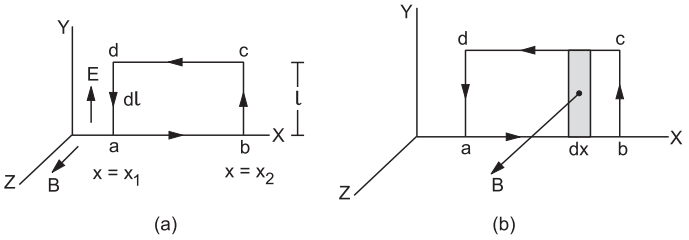



Figure 40.3

Let us consider the rectangular path $abcd$ in the $x$–$y$ plane as shown in figure (40.3a). Let us evaluate the terms in the Faraday's law on this path. The electric field is parallel to the $y$-axis. The circulation of $E$ is

$$\oint \vec{E}\cdot d\vec{l} = \int_a^b \vec{E}\cdot d\vec{l} + \int_b^c \vec{E}\cdot d\vec{l} + \int_c^d \vec{E}\cdot d\vec{l} + \int_d^a \vec{E}\cdot d\vec{l}$$

$$= 0 + E(x_2)\, l + 0 + E(x_1)\,(-l)$$

$$= E_0\, l\,[\sin \omega(t - x_2 / c) - \sin \omega(t - x_1/c)]. \quad \ldots \text{(i)}$$

Next, let us calculate the flux of the magnetic field $\Phi_B$, through the same rectangle $abcd$ (figure 40.3b). The flux through a strip of width $dx$ at $x$ is

$$B(x)\, l\, dx = B_0\,[\sin \omega(t - x/c)]\, l\, dx.$$

The flux through the rectangle $abcd$ is

$$\Phi_B = \int_{x_1}^{x_2} B_0\, l \sin \omega(t - x/c)\, dx$$

$$= -\frac{c}{\omega} B_0\, l[-\cos \omega(t - x_2/c) + \cos \omega(t - x_1/c)].$$

Thus,

$$\frac{d\Phi_B}{dt} = -cB_0\, l[\sin \omega(t - x_2/c) - \sin \omega(t - x_1/c)].$$

$$\ldots \text{(ii)}$$

The Faraday's law for vacuum is

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt}.$$

Putting from (i) and (ii) in this equation, we see that Faraday's law is satisfied by the wave given by equation (40.4a) if

$$E_0 = cB_0. \quad \ldots (40.5)$$

### Ampere's Law

Let us consider the rectangular path $efgh$ in the $x$–$z$ plane as shown in figure (40.4a).

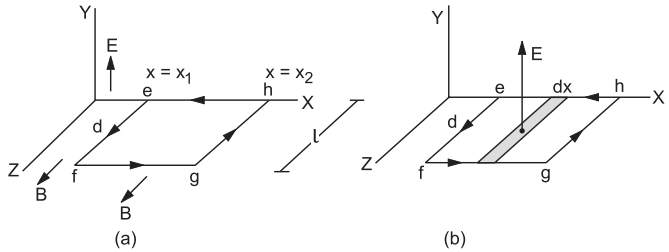



Figure 40.4

The circulation of $\vec{B}$ is

$$\oint \vec{B}\cdot d\vec{l} = \int_e^f \vec{B}\cdot d\vec{l} + \int_f^g \vec{B}\cdot d\vec{l} + \int_g^h \vec{B}\cdot d\vec{l} + \int_h^e \vec{B}\cdot d\vec{l}$$

$$= B(x_1)\, l + 0 - B(x_2)\, l + 0$$

$$= B_0\, l[\sin \omega(t - x_1/c) - \sin \omega(t - x_2/c)].$$

$$\ldots \text{(i)}$$

The flux of the electric field through the same rectangle $efgh$ (figure 40.4b) is

$$\Phi_E = \int \vec{E}\cdot d\vec{S}$$

$$= \int_{x_1}^{x_2} E(x)\, l\, dx$$

$$= E_0 \, l \int_{x_1}^{x_2} \sin \omega(t - x/c) \, dx$$

$$= -\frac{c}{\omega} E_0 \, l[-\cos \omega(t - x_2/c) + \cos \omega(t - x_1/c)]$$

or, $$\frac{d\Phi_E}{dt} = -cE_0 \, l[\sin \omega(t - x_2/c) - \sin \omega(t - x_1/c)]. \quad \ldots \text{(ii)}$$

The Ampere's law for vacuum is

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}.$$

Putting from (i) and (ii) in this equation, we see that Ampere's law is satisfied if

$$B_0 = \mu_0 \varepsilon_0 \, c \, E_0$$

or, $$\mu_0 \varepsilon_0 = \frac{B_0}{E_0 c}.$$

Using equation (40.5),

$$\mu_0 \varepsilon_0 = \frac{1}{c^2}$$

or, $$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}. \quad \ldots \text{(40.6)}$$

Thus, Maxwell's equations have a solution giving a plane electromagnetic wave of the form (40.4) with $E_0 = cB_0$ and the speed of this wave is $\frac{1}{\sqrt{\mu_0 \varepsilon_0}}$.

In older days, $\mu_0$ and $\varepsilon_0$ were defined in terms of electric and magnetic measurements. Putting these values of $\mu_0$ and $\varepsilon_0$, the speed of electromagnetic waves came out to be $c = 2{\cdot}99793 \times 10^8$ m/s which was the same as the measured speed of light in vacuum. This provided a confirmatory proof that light is an electromagnetic wave.

It may be recalled that the speed of electromagnetic waves, which is the same as the speed of light, is now an exactly defined constant. Similarly, the constant $\mu_0 = 4\pi \times 10^{-7}$ T–m/A is an exactly defined constant. The quantity $\varepsilon_0$ is defined by the equation (40.6).

***Example 40.2***

*The maximum electric field in a plane electromagnetic wave is* 600 N C$^{-1}$. *The wave is going in the x-direction and the electric field is in the y-direction. Find the maximum magnetic field in the wave and its direction.*

***Solution :***

We have $B_0 = \frac{E_0}{c} = \frac{600 \text{ N C}^{-1}}{3 \times 10^8 \text{ m s}^{-1}} = 2 \times 10^{-6}$ T.

As $\vec{E}, \vec{B}$ and the direction of propagation are mutually perpendicular, $\vec{B}$ should be along the *z*-direction.

## 40.5 ENERGY DENSITY AND INTENSITY

The electric and magnetic field in a plane electromagnetic wave are given by

$$E = E_0 \sin \omega(t - x/c)$$

and $$B = B_0 \sin \omega(t - x/c).$$

In any small volume $dV$, the energy of the electric field is

$$U_E = \frac{1}{2} \varepsilon_0 E^2 dV$$

and the energy of the magnetic field is

$$U_B = \frac{1}{2\mu_0} B^2 dV.$$

Thus, the total energy is

$$U = \frac{1}{2} \varepsilon_0 E^2 \, dV + \frac{1}{2\mu_0} B^2 dV.$$

The energy density is $u = \frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2$

$$= \frac{1}{2} \varepsilon_0 E_0^2 \sin^2 \omega(t - x/c) + \frac{1}{2\mu_0} B_0^2 \sin^2 \omega(t - x/c).$$

If we take the average over a long time, the $\sin^2$ terms have an average value of 1/2. Thus,

$$u_{au} = \frac{1}{4} \varepsilon_0 E_0^2 + \frac{1}{4\mu_0} B_0^2.$$

From equations (40.5) and (40.6),

$$E_0 = cB_0 \text{ and } \mu_0 \varepsilon_0 = \frac{1}{c^2} \text{ so that,}$$

$$\frac{1}{4\mu_0} B_0^2 = \frac{\varepsilon_0 c^2}{4} \left(\frac{E_0}{c}\right)^2 = \frac{1}{4} \varepsilon_0 E_0^2.$$

Thus, the electric energy density is equal to the magnetic energy density in average.

or, $$u_{av} = \frac{1}{4} \varepsilon_0 E_0^2 + \frac{1}{4} \varepsilon_0 E_0^2 = \frac{1}{2} \varepsilon_0 E_0^2. \quad \ldots \text{(40.7)}$$

Also, $$u_{av} = \frac{1}{4\mu_0} B_0^2 + \frac{1}{4\mu_0} B_0^2 = \frac{1}{2\mu_0} B_0^2. \quad \ldots \text{(40.8)}$$

***Example 40.3***

*The electric field in an electromagnetic wave is given by*

$$E = (50 \text{ N C}^{-1}) \sin \omega(t - x/c).$$

*Find the energy contained in a cylinder of cross-section* 10 cm$^2$ *and length* 50 cm *along the x-axis.*

***Solution :***

The energy density is

$$u_{av} = \frac{1}{2} \varepsilon_0 E_0^2 = \frac{1}{2} \times (8{\cdot}85 \times 10^{-12} \text{ C}^2 \text{N}^{-1} \text{m}^{-2}) \times (50 \text{ N C}^{-1})^2$$

$$= 1{\cdot}1 \times 10^{-8} \text{ J m}^{-3}.$$

The volume of the cylinder is

$$V = 10 \text{ cm}^2 \times 50 \text{ cm} = 5 \times 10^{-4} \text{ m}^3.$$

The energy contained in this volume is

$$U = (1{\cdot}1 \times 10^{-8}\ \text{J m}^{-3}) \times (5 \times 10^{-4}\ \text{m}^{3})$$
$$= 5{\cdot}5 \times 10^{-12}\ \text{J}.$$

**Intensity**

The energy crossing per unit area per unit time perpendicular to the direction of propagation is called the intensity of a wave.

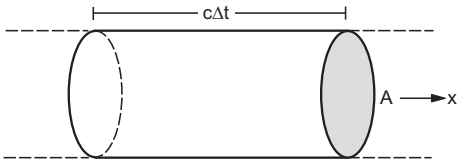



Figure 40.5

Consider a cylindrical volume with area of cross-section $A$ and length $c\,\Delta t$ along the $X$-axis (figure 40.5). The energy contained in this cylinder crosses the area $A$ in time $\Delta t$ as the wave propagates at speed $c$. The energy contained is

$$U = u_{av}(c\,\Delta t)A.$$

The intensity is $I = \dfrac{U}{A\Delta t} = u_{av}\,c.$

In terms of maximum electric field,

$$I = \frac{1}{2}\,\varepsilon_0 E_0^2 c. \qquad \ldots\ (40.9)$$

***Example 40.4***

*Find the intensity of the wave discussed in example (40.3).*

***Solution :***

The intensity is

$$I = \frac{1}{2}\,\varepsilon_0 E_0^2\, c = (1{\cdot}1 \times 10^{-8}\ \text{J m}^{-3}) \times (3 \times 10^{8}\ \text{m s}^{-1})$$
$$= 3{\cdot}3\ \text{W m}^{-2}.$$

## 40.6 MOMENTUM

The electromagnetic wave also carries linear momentum with it. The linear momentum carried by the portion of wave having energy $U$ is given by

$$p = \frac{U}{c}. \qquad \ldots\ (40.10)$$

Thus, if the wave incident on a material surface is completely absorbed, it delivers energy $U$ and momentum $p = U/c$ to the surface. If the wave is totally reflected, the momentum delivered is $2U/c$ because the momentum of the wave changes from $p$ to $-p$. It follows that electromagnetic waves incident on a surface exert a force on the surface.

## 40.7 ELECTROMAGNETIC SPECTRUM AND RADIATION IN ATMOSPHERE

Maxwell's equations are applicable for electromagnetic waves of all wavelengths. Visible light has wavelengths roughly in the range 380 nm to 780 nm. Today we are familiar with electromagnetic waves having wavelengths as small as 30 fm ($1\ \text{fm} = 10^{-15}\ \text{m}$) to as large as 30 km. Figure (40.6) shows the electromagnetic spectrum we are familiar with. The boundaries separating different regions of spectrum are not sharply defined. The gamma ray region and the X-ray region overlap considerably. We can only say that on the average, wavelengths of gamma rays are shorter than those of X-rays.

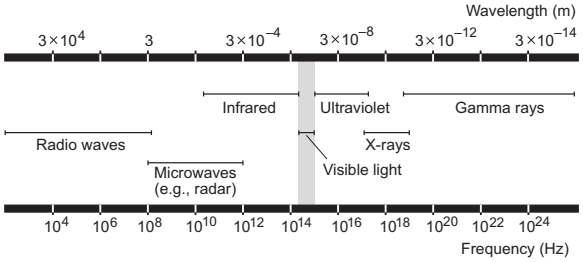



Figure 40.6

The basic source of electromagnetic waves is an accelerated charge. This produces changing electric field and changing magnetic field which constitute the wave. Radio waves may be produced by charges accelerating in AC circuits having an inductor and a capacitor. These waves are used in radio and TV communication. Microwaves are also produced by such electric circuits with oscillating current. They are used for radar systems among other applications. Microwave ovens are used for cooking. Infrared waves are emitted by the atoms and molecules of hot bodies. These waves are used in physical therapy. Among the electromagnetic waves, visible light is most familiar to us. This is emitted by atoms under suitable conditions. An atom contains electrons and the light emission is related to the acceleration of an electron inside the atom. The mechanism of emission of ultraviolet radiation is similar to that for visible light. The sun emits large amount of ultraviolet radiation. This radiation is harmful to us if absorbed in large amount. X-rays are produced most commonly when fast-moving electrons decelerate inside a metal target. X-rays are widely used in medical diagnosis. They are harmful to living tissues. Gamma rays are emitted by the nuclei and have the shortest wavelengths among the electromagnetic waves we generally deal with.

**Radiation in Atmosphere**

The earth is surrounded by atmosphere up to a height of about 300 km. The composition of atmosphere

differs widely as one moves up. Most of the water droplets, vapour and ice particles forming clouds, are contained in a layer starting from the earth's surface up to height of about 12 km. This part is called *troposphere*. The density of air at the top of the troposphere is about one tenth of the density near the earth's surface. The atmosphere between the heights of 12 km and 50 km is called *stratosphere*. In the upper part of the stratosphere, we have a layer of ozone. The density of air at the top of the stratosphere is about $10^{-3}$ times the density at the surface of the earth. Then we have *mesosphere* between a height of 50 km and 80 km. The atmosphere above that is called *ionosphere*. There are no sharp boundaries between the above divisions and the numbers given are only a rough guide.

The main source of electromagnetic radiation in the atmosphere is the sun. The sun sends electromagnetic waves of different wavelengths towards the earth. A major part of it is absorbed by the atmosphere. Visible light is only weakly absorbed. Most of the infrared radiation is absorbed by the atmosphere and used to heat it. The radiation from the sun has a lot of ultraviolet radiation. The ozone layer absorbs most of this radiation and other radiations of lower wavelengths and thus protects us from their harmful effects. The ozone layer converts the ultraviolet radiation to infrared which is used to heat the atmosphere and the earth's surface. It is suspected that ozone layer is slowly being depleted and this is causing great concern to scientists and environmentalists.

## *Worked Out Examples*

**1.** *A parallel-plate capacitor with plate area A and separation between the plates d, is charged by a constant current i. Consider a plane surface of area A/2 parallel to the plates and drawn symmetrically between the plates. Find the displacement current through this area.*

***Solution*** :

Suppose the charge on the capacitor at time $t$ is $Q$. The electric field between the plates of the capacitor is $E = \frac{Q}{\varepsilon_0 A}$. The flux through the area considered is

$$\Phi_E = \frac{Q}{\varepsilon_0 A} \cdot \frac{A}{2} = \frac{Q}{2\varepsilon_0}.$$

The displacement current is

$$i_d = \varepsilon_0 \frac{d\Phi_E}{dt} = \varepsilon_0 \left(\frac{1}{2\varepsilon_0}\right)\frac{dQ}{dt} = \frac{i}{2}.$$

**2.** *A plane electromagnetic wave propagating in the x-direction has a wavelength of* 5·0 mm. *The electric field is in the y-direction and its maximum magnitude is* 30 V $m^{-1}$. *Write suitable equations for the electric and magnetic fields as a function of x and t.*

***Solution*** :

The equation for the electric and the magnetic fields in the wave may be written as

$$E = E_0 \sin \omega\left(t - \frac{x}{c}\right)$$

$$B = B_0 \sin \omega\left(t - \frac{x}{c}\right).$$

We have,

$$\omega = 2\pi\nu = \frac{2\pi}{\lambda} c.$$

Thus, $$E = E_0 \sin\left[\frac{2\pi}{\lambda}(ct - x)\right]$$

$$= (30 \text{ V m}^{-1}) \sin\left[\frac{2\pi}{5{\cdot}0 \text{ mm}}(ct - x)\right].$$

The maximum magnetic field is

$$B_0 = \frac{E_0}{c} = \frac{30 \text{ V m}^{-1}}{3 \times 10^8 \text{ m s}^{-1}} = 10^{-7} \text{ T}.$$

So, $$B = B_0 \sin\left[\frac{2\pi}{\lambda}(ct - x)\right]$$

$$= (10^{-7} \text{ T}) \sin\left[\frac{2\pi}{5{\cdot}0 \text{ mm}}(ct - x)\right].$$

The magnetic field is along the $z$-axis.

**3.** *A light beam travelling in the x-direction is described by the electric field* $E_y = (300 \text{ V m}^{-1}) \sin \omega(t - x/c)$. *An electron is constrained to move along the y-direction with a speed of* $2{\cdot}0 \times 10^7$ m $s^{-1}$. *Find the maximum electric force and the maximum magnetic force on the electron.*

***Solution*** :

The maximum electric field is $E_0 = 300 \text{ V m}^{-1}$. The maximum magnetic field is

$$B_0 = \frac{E_0}{c} = \frac{300 \text{ V m}^{-1}}{3 \times 10^8 \text{ m s}^{-1}} = 10^{-6} \text{ T}$$

along the $z$-direction.

The maximum electric force on the electron is

$$F_e = qE_0 = (1{\cdot}6 \times 10^{-19} \text{ C}) \times (300 \text{ V m}^{-1})$$

$$= 4{\cdot}8 \times 10^{-17} \text{ N}.$$

The maximum magnetic force on the electron is

$$F_b = |q\vec{v} \times \vec{B}|_{\max} = qvB_0$$

$$= (1{\cdot}6 \times 10^{-19}\ \text{C}) \times (2{\cdot}0 \times 10^{7}\ \text{m s}^{-1}) \times (10^{-6}\ \text{T})$$
$$= 3{\cdot}2 \times 10^{-18}\ \text{N}.$$

**4.** *Find the energy stored in a* 60 cm *length of a laser beam operating at* 4 mW.

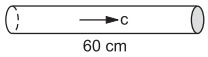



Figure 40-W1

***Solution*** **:**

The time taken by the electromagnetic wave to move through a distance of 60 cm is $t = \frac{60\ \text{cm}}{c} = 2 \times 10^{-9}$ s. The energy contained in the 60 cm length passes through a cross-section of the beam in $2 \times 10^{-9}$ s (figure 40-W1). But the energy passing through any cross section in $2 \times 10^{-9}$ s is

$$U = (4\ \text{mW}) \times (2 \times 10^{-9}\ \text{s})$$
$$= (4 \times 10^{-3}\ \text{Js}^{-1}) \times (2 \times 10^{-9}\ \text{s})$$
$$= 8 \times 10^{-12}\ \text{J}.$$

This is the energy contained in 60 cm length.

**5.** *Find the amplitude of the electric field in a parallel beam of light of intensity* $2{\cdot}0\ \text{W m}^{-2}$.

***Solution*** **:**

The intensity of a plane electromagnetic wave is

$$I = u_{av}\, c = \frac{1}{2}\varepsilon_0 E_0^2\, c$$

or, $$E_0 = \sqrt{\frac{2I}{\varepsilon_0 c}}$$

$$= \sqrt{\frac{2 \times (2{\cdot}0\ \text{W m}^{-2})}{(8{\cdot}85 \times 10^{-12}\ \text{C}^2\text{N}^{-1}\text{m}^{-2}) \times (3 \times 10^{8}\ \text{m s}^{-1})}}$$

$$= 38{\cdot}8\ \text{N C}^{-1}.$$

□

## QUESTIONS FOR SHORT ANSWER

**1.** In a microwave oven, the food is kept in a plastic container and the microwave is directed towards the food. The food is cooked without melting or igniting the plastic container. Explain.

**2.** A metal rod is placed along the axis of a solenoid carrying a high-frequency alternating current. It is found that the rod gets heated. Explain why the rod gets heated.

**3.** Can an electromagnetic wave be deflected by an electric field ? By a magnetic field ?

**4.** A wire carries an alternating current $i = i_0 \sin \omega t$. Is there an electric field in the vicinity of the wire ?

**5.** A capacitor is connected to an alternating-current source. Is there a magnetic field between the plates ?

**6.** Can an electromagnetic wave be polarized ?

**7.** A plane electromagnetic wave is passing through a region. Consider the quantities (a) electric field, (b) magnetic field, (c) electrical energy in a small volume and (d) magnetic energy in a small volume. Construct pairs of the quantities that oscillate with equal frequencies.

## OBJECTIVE I

**1.** A magnetic field can be produced by
(a) a moving charge (b) a changing electric field
(c) none of them (d) both of them.

**2.** A compass needle is placed in the gap of a parallel plate capacitor. The capacitor is connected to a battery through a resistance. The compass needle
(a) does not deflect
(b) deflects for a very short time and then comes back to the original position
(c) deflects and remains deflected as long as the battery is connected
(d) deflects and gradually comes to the original position in a time which is large compared to the time constant.

**3.** Dimensions of $1/(\mu_0\varepsilon_0)$ is
(a) L/T (b) T/L (c) $L^2/T^2$ (d) $T^2/L^2$.

**4.** Electromagnetic waves are produced by
(a) a static charge (b) a moving charge
(c) an accelerating charge (d) chargeless particles.

**5.** An electromagnetic wave going through vacuum is described by
$$E = E_0 \sin(kx - \omega t);\ B = B_0 \sin(kx - \omega t).$$
Then
(a) $E_0 k = B_0 \omega$ (b) $E_0 B_0 = \omega k$
(c) $E_0 \omega = B_0 k$ (d) none of these.

**6.** An electric field $\vec{E}$ and a magnetic field $\vec{B}$ exist in a region. The fields are not perpendicular to each other.
(a) This is not possible.
(b) No electromagnetic wave is passing through the region.
(c) An electromagnetic wave may be passing through the region.
(d) An electromagnetic wave is certainly passing through the region.

**7.** Consider the following two statements regarding a linearly polarized, plane electromagnetic wave:
(A) The electric field and the magnetic field have equal average values.
(B) The electric energy and the magnetic energy have equal average values.
(a) Both A and B are true. (b) A is false but B is true.
(c) B is false but A is true. (d) Both A and B are false.

**8.** A free electron is placed in the path of a plane electromagnetic wave. The electron will start moving
(a) along the electric field
(b) along the magnetic field
(c) along the direction of propagation of the wave
(d) in a plane containing the magnetic field and the direction of propagation.

**9.** A plane electromagnetic wave is incident on a material surface. The wave delivers momentum $p$ and energy $E$.
(a) $p = 0,\ E \neq 0$. (b) $p \neq 0,\ E = 0$.
(c) $p \neq 0,\ E \neq 0$. (d) $p = 0,\ E = 0$.

## OBJECTIVE II

**1.** An electromagnetic wave going through vacuum is described by
$$E = E_0 \sin(kx - \omega t).$$
Which of the following is/are independent of the wavelength ?
(a) $k$ (b) $\omega$ (c) $k/\omega$ (d) $k\omega$.

**2.** Displacement current goes through the gap between the plates of a capacitor when the charge of the capacitor
(a) increases (b) decreases
(c) does not change (d) is zero.

**3.** Speed of electromagnetic waves is the same
(a) for all wavelengths (b) in all media
(c) for all intensities (d) for all frequencies.

**4.** Which of the following have zero average value in a plane electromagnetic wave ?
(a) electric field (b) magnetic field
(c) electric energy (d) magnetic energy.

**5.** The energy contained in a small volume through which an electromagnetic wave is passing oscillates with
(a) zero frequency (b) the frequency of the wave
(c) half the frequency of the wave
(d) double the frequency of the wave.

## EXERCISES

**1.** Show that the dimensions of the displacement current $\varepsilon_0 \frac{d\varphi_E}{dt}$ are that of an electric current.

**2.** A point charge is moving along a straight line with a constant velocity $v$. Consider a small area $A$ perpendicular to the direction of motion of the charge (figure 40-E1). Calculate the displacement current through the area when its distance from the charge is $x$. The value of $x$ is not large so that the electric field at any instant is essentially given by Coulomb's law.

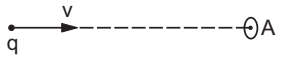



Figure 40-E1

**3.** A parallel-plate capacitor having plate-area $A$ and plate separation $d$ is joined to a battery of emf $\mathscr{E}$ and internal resistance $R$ at $t = 0$. Consider a plane surface of area $A/2$, parallel to the plates and situated symmetrically between them. Find the displacement current through this surface as a function of time.

**4.** Consider the situation of the previous problem. Define displacement resistance $R_d = V/i_d$ of the space between the plates where $V$ is the potential difference between the plates and $i_d$ is the displacement current. Show that $R_d$ varies with time as
$$R_d = R(e^{t/\tau} - 1).$$

**5.** Using $B = \mu_0 H$ find the ratio $E_0/H_0$ for a plane electromagnetic wave propagating through vacuum. Show that it has the dimensions of electric resistance. This ratio is a universal constant called the *impedance of free space.*

**6.** The sunlight reaching the earth has maximum electric field of 810 V m$^{-1}$. What is the maximum magnetic field in this light ?

**7.** The magnetic field in a plane electromagnetic wave is given by
$$B = (200\ \mu\text{T}) \sin[(4{\cdot}0 \times 10^{15}\ \text{s}^{-1})(t - x/c)].$$
Find the maximum electric field and the average energy density corresponding to the electric field.

**8.** A laser beam has intensity $2{\cdot}5 \times 10^{14}$ W m$^{-2}$. Find the amplitudes of electric and magnetic fields in the beam.

**9.** The intensity of the sunlight reaching the earth is 1380 W m$^{-2}$. Assume this light to be a plane, monochromatic wave. Find the amplitudes of electric and magnetic fields in this wave.

□

## ANSWERS

### OBJECTIVE I

1. (d) 2. (d) 3. (c) 4. (c) 5. (a) 6. (c)
7. (a) 8. (a) 9. (c)

### OBJECTIVE II

1. (c) 2. (a), (b) 3. (c), 4. (a), (b)
5. (d)

### EXERCISES

2. $\frac{q\,Av}{2\pi x^{3}}$
3. $\frac{\mathscr{E}}{2R}\,e^{-\frac{td}{\varepsilon\,AR}}$
5. 377 Ω
6. 2·7 μT
7. $6\times10^{4}$ N C$^{-1}$, 0·016 J m$^{-3}$
8. $4{\cdot}3\times10^{8}$ N C$^{-1}$, 1·44 T
9. $1{\cdot}02\times10^{3}$ N C$^{-1}$, $3{\cdot}40\times10^{-6}$ T

□

CHAPTER 41

# ELECTRIC CURRENT THROUGH GASES

Gases are, in general, poor conductors of electricity. This is because they do not have free charged particles in large numbers which may respond to an applied electric field. There may be some ionization due to cosmic rays and other factors, but regular recombination of ions of opposite polarity also takes place and the number of charged particles does not increase much. Electric current may be passed through a gas if we ensure that by some mechanism charged particles are continuously produced in the gas. This can be done in many ways. One such method is to apply a large potential difference across a gas column at very low pressure. Another method is to heat a metal kept in an evacuated chamber to high temperatures at which electrons are ejected from the metal. Yet another method is to pass X-rays through the gas. There are several other methods.

## 41.1 DISCHARGE THROUGH GASES AT LOW PRESSURE

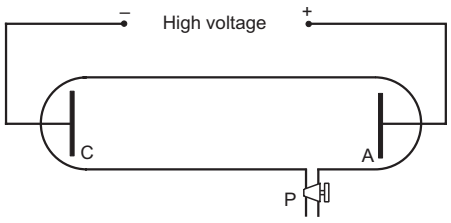



Figure 41.1

To study electric currents through gases at low pressures, one uses a glass tube known as *discharge tube*. Normally, it is a closed tube of length of about 30 cm and diameter of about 4 cm. It is fitted with two metal electrodes $C$ and $A$ (figure 41.1). A side tube $P$ is used to pump out the enclosed gas so as to obtain the desired low pressure. The electrodes are connected to the secondary of an induction coil so that a high potential difference may be applied across the gas. The electrode $C$ connected to the negative terminal is called the cathode and the electrode $A$ connected to the positive terminal is called the anode.

**Sparking Potential**

If the potential difference between the electrodes is gradually increased, sparking occurs in the gas at a certain stage. The minimum potential difference which can cause sparks in a gas is called the *sparking potential*. Sparking potential depends on the pressure of the gas as well as on the separation between the electrodes. After careful studies, Paschen found that the sparking potential of a gas in a discharge tube is a function of the product of the pressure of the gas and the separation between the electrodes:

$$V = f(pd). \qquad \ldots \quad (41.1)$$

This is called the *Paschen's law*.

**Low-Pressure Phenomena**

In general, when a high potential difference is applied across a gas, sparking occurs in the form of irregular streaks of light. Suppose, in a typical case, the pressure of the gas is about 10 cm of mercury and sparking occurs. The sparking is accompanied by crackling noise. If the pressure of the gas is gradually decreased by pumping out the gas, a series of phenomena take place. At a pressure of about 10 mm of mercury, the irregular streaks broaden out in a luminous column extending from the anode almost up to the cathode (figure 41.2a). The crackling sound is replaced by a continuous buzzing sound. This column is known as *positive column*. The colour of the positive column depends on the nature of the enclosed gas. It

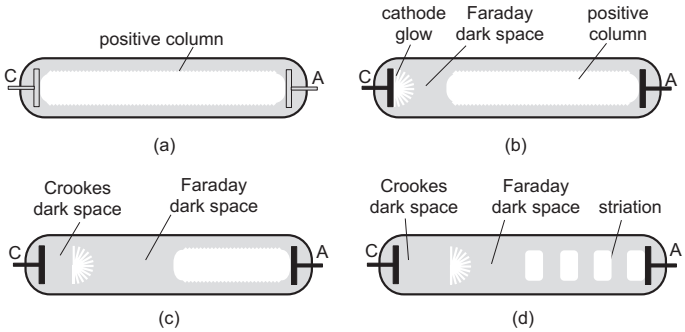



Figure 41.2

is reddish for air, bright red for neon, bluish for $CO_2$, etc.

As the pressure is further decreased to about 3–4 mm of mercury, the positive column decreases in length. It starts from the anode but ends well before the cathode. A bluish glow is seen around the cathode and there is a dark space between this glow and the positive column (figure 41.2b). The glow around the cathode is called *cathode glow* or *negative glow*. The dark space between the cathode glow and the positive column is called *Faraday dark space*.

When the pressure is reduced to about 1 mm of mercury, the positive column is further shortened and the length of the Faraday dark space increases. The cathode glow is detached from the cathode and a new dark space, called *Crookes dark space* (figure 41.2c), appears between the cathode and the cathode glow.

As the pressure is decreased further, the Crookes dark space and the cathode glow expand. At about 0·1 mm of mercury, the positive column is split into alternate bright and dark bands called *striations* (figure 41.2d).

With further reduction in pressure, the striations move towards the anode and finally vanish. The cathode glow also vanishes at around 0·01 mm of mercury and the Crookes dark space fills the entire tube. A new phenomenon starts at this stage. The walls of the tube begin to glow. This is called *fluorescence*. The colour of the glow depends on the nature of the glass. It is yellowish green for soda glass.

If the pressure is still decreased, the current through the gas gradually decreases and finally the tube stops conducting.

The values of pressure mentioned in the above discussion represent only typical values. The actual pressures at which these phenomena start, depend on the geometry of the discharge tube, the potential difference applied, the gas contained in the tube, etc.

**Explanation of Discharge Phenomena**

Due to cosmic rays and other factor, some ions are always present in a gas. When a potential difference is applied across a discharge tube, the ions are accelerated due to the electric field. They soon collide with other molecules of the gas and share the excess energy acquired. If the potential difference is sufficiently high, the ions get enough energy to ionize the molecules on collision. This way, ions are produced in large number and conductions starts. Generally, an electron is detached from a molecule to make the molecule a positive ion. At low pressures, this electron can move through a considerably large distance before attaching to another molecule forming a negative ion. These free electrons and the positive ions play important roles in discharge-tube phenomena.

Let us consider the case when the discharge tube looks the most beautiful—there is Crookes dark space, cathode glow, Faraday dark space and then alternate dark and bright bands.

The positive ions produced near the surface of the cathode are attracted towards it. Hence, the ions are accelerated as they move towards the cathode. In the process they gain kinetic energy. The ions strike the cathode with sufficient kinetic energy to liberate more electrons from its surface. These electrons are accelerated away from the cathode and they ionize the gas further by collision. Thus, ionization takes place much more rapidly near the cathode.

When a molecule gets ionized, the electron moves towards the anode and the positive ion towards the cathode. The electron being light, is swept away very fast by the electric field as compared to the slow-moving, heavy, positive ions. Thus, a positive charge density builds up near the cathode and an intense electric field is produced between the cathode and this region. The electrons emitted by the cathode acquire sufficient kinetic energy while passing through this intense electric field to ionize the neutral molecules with which they collide. These ionized molecules emit light which appears as the cathode glow. The electrons emitted from the cathode travel on the average a distance equal to the mean free path before they collide and cause cathode glow. Thus cathode glow appears some distance away from the cathode and this explains the Crookes dark space.

The slow-moving, positive ions create a positive charge density in the region of cathode glow. As the fast-moving electrons, coming from the cathode, pass through this region they are slowed down and hence lose their ionizing capacity. Thus, there is no emission of light for some distance beyond the cathode glow and this makes the Faraday dark space. After coming out of the cathode glow, the electrons again accelerate due to the electric field and acquire sufficient energy to ionize the molecules. This starts the positive column. The Faraday dark space is several times longer than the Crookes dark space because the electric field here is not as intense as it is near the cathode.

The successive process of ionizing, losing ionizing power, accelerating for some distance and again ionizing, is repeated till the electrons reach the anode. Thus, we have alternate dark and bright bands, i.e., the striations.

As the pressure is still lowered, the mean free path increases and hence the length of the Crookes dark space increases. At very low pressure, when the mean free path becomes larger than the length of the tube, the Crookes dark space fills the entire tube and no cathode glow or positive column is observed. The

electrons coming out of the cathode strike the walls of the tube and this impact causes fluorescence.

## 41.2 CATHODE RAYS

When the pressure of the gas in a discharge tube is lowered, at a certain stage the Crookes dark space fills the whole length of the tube. The fact that the walls of the tube glows (fluorescence) shows that something is coming out from the cathode, travelling through the length of the tube and falling on the walls. As we have discussed above, this something is a stream of fast-moving electrons. This fact was recognised after a series of experiments carried out by Crookes, Thomson and others. They named these invisible streams coming from the cathode *cathode rays* and established the following main properties.

(a) *Cathode rays are emitted normally from the cathode surface. Their direction is independent of the position of the anode.*

Figure 41.3

This can be shown by taking the cathode in the shape of a concave surface. If a fluorescent material is placed at the centre of curvature of the cathode surface, the material glows with maximum intensity. Any lateral shift reduces the glow.

(b) *Cathode rays travel in straight lines.*

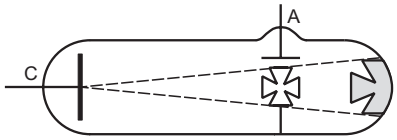



Figure 41.4

This can be shown by placing a metal cross in the path of the cathode rays. A shadow appears on the wall on the opposite side as the cathode rays do not reach there (there is no fluorescence in the shadow region). If the cross is lowered, the shadow disappears.

(c) *Cathode rays exert mechanical force on the object they strike.*

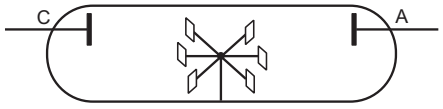



Figure 41.5

To show this, one can put a light wheel of mica in the path of the cathode rays. With proper arrangement, the wheel starts rotating as the cathode rays fall on it.

(d) *Cathode rays produce heat when they strike a material surface.*

If a blackened platinum strip is placed at the centre of curvature of a concave-shaped cathode, the strip becomes red-hot after some time.

(e) *Cathode rays produce fluorescence when they strike a number of crystals, minerals and salts.*

(f) *When cathode rays strike a solid object, specially a metal,* X-*rays are emitted from the object.*

(g) *Cathode rays can be deflected by an electric field and also by a magnetic field. The direction of deflection is the same as that of a stream of negatively charged particles. The deflection in such a condition is independent of the gas present, the material of the cathode, the position of the anode, etc.*

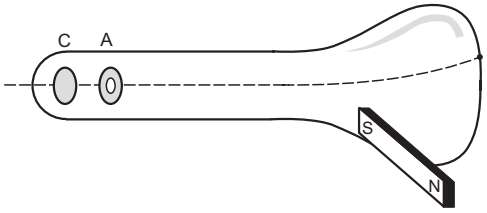



Figure 41.6

Such a deflection was studied by Thomson using an apparatus of the design similar to that shown in figure (41.6). Cathode rays start from the cathode *C* and pass undeflected into the larger bulb where it causes fluorescence on the opposite surface. The particular design ensures that the glow is in the shape of a small dot. If a magnet is now brought closer to the larger bulb, the dot moves on the wall showing that the rays have been deflected. The direction of the deflection confirms that cathode rays contain negatively charged particles. Similar deflection can be studied in electric field by bringing a charged rod near the larger bulb.

(h) *Cathode rays ionize the gas through which they are passed.*

(i) *Cathode rays can penetrate thin foils of metal.*

(j) *Cathode rays affect photographic plates.*

All the above properties can be easily understood once we recognise that cathode rays are nothing but a stream of fast-moving electrons.

## 41.3 CANAL RAYS OR POSITIVE RAYS

If the cathode of a discharge tube has holes in it and the pressure of the gas is around 1 mm of mercury, streams of faint luminous glow come out from each hole on the back side of the cathode. This shows that something is coming out of the holes. These are called *canal rays* or *positive rays.* The origin of the canal rays can be easily understood. When the molecules near the cathode are ionized, the positive ions move slowly towards the cathode. The positive ions passing through the holes constitute the positive or canal rays. The positive rays are deflected by electric and magnetic

field. The direction of the deflection is the same as that of a stream of positively charged particles. They also cause fluorescence when incident on certain materials.

### 41.4 DISCOVERY AND PROPERTIES OF ELECTRON

The experiments on discharge tube towards the end of the nineteenth century played an important role in the discovery of electron. The fact that cathode rays are deflected in electric and magnetic field led scientists to believe that they are made of tiny negatively charged particles. These particles are the electrons as we know them today. J J Thomson (1856–1940) is often credited with the discovery of electron. This is not because he was the first to perform the discharge-tube experiment or was the first to explain cathode rays in terms of negatively charged particles. The credit goes to him for studying the properties of electrons and suggesting that electrons are necessary constituents of all atoms. The basic physics behind his famous experiment to measure the "charge by mass" ($e/m$) ratio of electrons is described below. This ratio is also called the *specific charge* of the electron.

**Determination of *e/m***

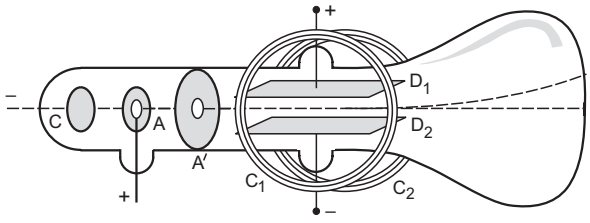



Figure 41.7

Figure (41.7) shows the basic design of Thomson's experiment to measure $e/m$. A large potential difference $V$ is applied between the cathode $C$ and the anode $A$ sealed in a highly evacuated tube. A narrow beam of electrons, ejected from the cathode, passes through the holes in the anode $A$ and in a parallel metal electrode $A'$. The beam passes through the region between the two metal plates $D_1$, $D_2$ and then strikes the end of the tube. A fluorescent material is coated on the inner surface of the tube at this end so that a visible glow is produced when electrons strike the end. The plate $D_1$ is connected to the positive terminal of a power supply and $D_2$ to the negative. Thus, a constant potential difference is maintained between them. Knowing the separation between the plates, the electric field $E$ in the region between the plates can be computed. This field $E$ is in the downward direction in figure (41.7) and exerts a force in the upward direction on the electrons. When the potential difference is applied, the glow at the end of the tube shifts in the upward direction.

A magnetic field $B$ can also be applied in the region between the plates by passing electric currents in circular coils $C_1$, $C_2$. This field is perpendicular to the electric field as well as to the undeviated path of the cathode rays. If this field alone is present, the electrons move in a circular arc in the field region and hence are deflected from their straight path. The direction of the current in $C_1$, $C_2$ is so chosen that the magnetic force on the electrons is in the downward direction in the figure. Thus, the electron beam is deflected downwards due to the magnetic force.

If both the electric and the magnetic field are switched on and the values are so chosen that

$$v = E/B, \qquad \ldots \text{ (i)}$$

the magnetic force $evB$ will exactly cancel the electric force $eE$ and the beam will pass undeflected. If the potential difference between the anode $A$ and the cathode $C$ is $V$, the speed of the electrons coming out of $A$ is given by

$$\frac{1}{2} m v^2 = eV. \qquad \ldots \text{ (ii)}$$

Putting the value of $v$ from (i) into (ii),

$$\frac{1}{2} m \left(\frac{E}{B}\right)^2 = eV$$

or,
$$\frac{e}{m} = \frac{E^2}{2 B^2 V}. \qquad \ldots (41.2)$$

In an experiment, the position of the glow at the end of the tube is noted without applying any electric or magnetic field. The fields are now applied and the potential difference between $D_1$ and $D_2$ is adjusted till the glow returns to its original position. The value of $e/m$ is calculated by equation (41.2) in this situation.

**Millikan Oil-drop Experiment : Determination of *e***

Thomson's experiment described above, could determine the value of $e/m$. The value of electronic charge $e$ was measured by Robert Andrews Millikan about fifteen years after Thomson's experiment. Millikan was awarded the Nobel Prize in physics for 1923 for this

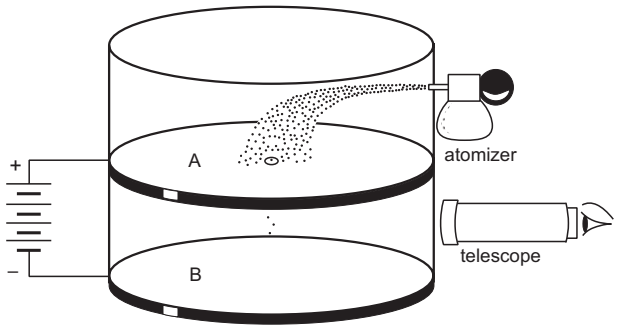



Figure 41.8

classic experiment. The basic design of the Millikan oil-drop experiment is shown in figure (41.8).

Two accurately-aligned, parallel metal plates $A$ and $B$ are separated by a small distance of the order of few millimetres. The plates are enclosed in a chamber with glass walls. An electric potential difference is applied between the plates to produce an electric field $E$ in the vertically downward direction. Small droplets of oil are sprayed in the region between the plates through a hole in the upper plate $A$. Some of the droplets acquire electric charge due to friction with air (this process may be aided by passing X-rays through the air or by putting some radioactive material in the chamber). The chamber is illuminated by sending light horizontally through it. The drops can be seen by using a telescope placed perpendicular to the light beam. A drop looks like a bright star moving either downwards or upwards. If a droplet has a mass $m$ and charge $q$, the forces on it are (a) its weight $mg$, (b) electric force $qE$, (c) buoyancy $B$ and (d) viscous force $F$.

Most of the drops acquire a negative charge so that the electric force is upwards. A particular drop is chosen in the field of view of the telescope and the magnitude of the field $E$ is adjusted to make the drop stationary. In this case, there is no force of viscosity and the other three forces add to zero. Thus,

$$qE + B = mg$$

or, $$qE = mg - B$$

$$= \frac{4}{3}\pi r^3(\rho - \sigma)g$$

or, $$q = \frac{4\pi r^3(\rho - \sigma)g}{3E}. \quad \ldots \text{(i)}$$

Here $r$ is the radius of the drop, and $\rho$ and $\sigma$ are the densities of the oil and the air respectively.

To determine the radius of the drop, the electric field is switched off. The drop accelerates downwards. After a while, the speed of the drop becomes constant. This speed $v$ is measured by noting the time taken by the drop to fall through a predetermined distance. In this case, the viscous force $6\pi\eta r v$ and the buoyancy $B$ taken together balance the weight $mg$.

$$6\pi\eta r v + \frac{4}{3}\pi r^3\sigma g = \frac{4}{3}\pi r^3\rho g$$

or, $$6\pi\eta v = \frac{4}{3}\pi r^2(\rho - \sigma)g$$

or, $$r = \left[\frac{9\eta v}{2(\rho - \sigma)g}\right]^{1/2}.$$

Putting this value of $r$ in (i)

$$q = \frac{18\pi}{E}\sqrt{\frac{\eta^3 v^3}{2(\rho - \sigma)g}}.$$

Millikan and his coworkers repeated the experiement thousands of times and measured the charges on so many drops. It was found that each drop had a charge that was a small integral multiple of a basic value $e = 1{\cdot}6 \times 10^{-19}$ C within the accuracy of the experiment. From this observation, Millikan concluded that $e = 1{\cdot}6 \times 10^{-19}$ C is the charge on an electron. If one electron is attached to a drop, the charge on the drop is $e$. If two electrons are attached to a drop, the charge on the drop is $2e$ and so on.

## 41.5 THERMIONIC EMISSION

When a metal is heated to a high temperature, electrons escape from its surface. This phenomenon is called *thermionic emission* and the electrons coming out are called *thermions*. When an electron attempts to come out, the remaining metal, which becomes positively charged, pulls it back. The electron is thus slowed down and is able to come out only if it has got some extra kinetic energy to overcome the pull. The minimum energy that must be given to an electron to take it out of the metal is called the *work function* of the metal. Work function is denoted by the symbol $\varphi$ and is different for different metals.

At ordinary temperatures, the free electrons of a metal do not have sufficient energy to come out. As the temperature is increased, the average kinetic energy of the electrons increases. Some of the electrons having sufficient kinetic energy are able to come out from the metal.

If $n$ thermions are ejected per unit time by a metal surface, the thermionic current is $i = ne$. This current is given by the *Richardson–Dushman equation*

$$i = ne = AST^2 e^{-\varphi/kT}. \quad \ldots \text{(41.3)}$$

Here $S$ is the surface area, $T$ is the absolute temperature of the surface, $\varphi$ is the work function of the metal, $k$ is the Boltzmann constant and $A$ is a constant which depends only on the nature of the metal.

***Example 41.1***

*The work function of a thermionic emitter is* 4·5 eV. *By what factor does the thermionic current increase if its temperature is raised from* 1500 K *to* 2000 K ?

***Solution*** :

If $i_1$ and $i_2$ represent the thermionic currents at temperatures $T_1 = 1500$ K and $T_2 = 2000$ K respectively,

$$i_1 = AST_1^2 e^{-\varphi/kT_1}$$

and $$i_2 = AST_2^2 e^{-\varphi/kT_2}.$$

Thus, $$\frac{i_2}{i_1} = \left(\frac{T_2}{T_1}\right)^2 e^{-\frac{\varphi}{k}\left(\frac{1}{T_2} - \frac{1}{T_1}\right)}.$$

Putting the values of $T_1$, $T_2$, φ and $k$

$$\frac{i_2}{i_1} = 10625.$$

## 41.6 DIODE VALVE

A diode valve consists of a cathode which can be heated to a high temperature to emit electrons and an anode which can collect these electrons. The cathode and the anode are sealed in an evacuated glass bulb. The cathode is also called *filament* and the anode is also called *plate*. In one design, a pure tungsten wire or a thorium-coated tungsten wire is used as the cathode. This wire can be directly connected to an external circuit driving current through it and hence increasing its temperature. In another design, a hollow nickel tube coated with barium oxide is used as the cathode. The tube surrounds a heater coil. When the external circuit drives a current through the heater coil, the nickel tube gets heated by radiation. The anode is in the form of a hollow nickel cylinder which surrounds the cathode. Potential difference may be applied between the cathode and the anode by connecting a battery through the leads coming out of the sealed valve. The positive terminal of the battery is connected to the anode and the negative terminal to the cathode.

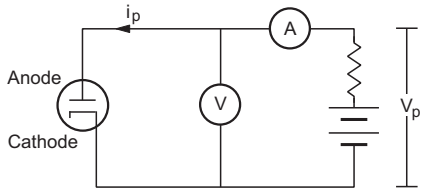



Figure 41.9

Figure (41.9) shows the symbolic representation of a diode valve and the circuit to study the current through the valve. When the cathode is heated, it emits electrons. These electrons are attracted towards the anode when a positive potential $V_p$ (with respect to the cathode) is applied to it. This potential $V_p$ is called the *plate voltage.* The electrons pass through the battery and then return to the cathode. Thus, an electric current $i_p$ is established in the circuit which can be measured by the ammeter connected in the circuit. This current is called the *plate current.* If the plate voltage $V_p$ is changed, the plate current $i_p$ is also changed. A set of plots between the plate current and the plate voltage is called *diode characteristics.*

### Diode Charactersitics

Figure (41.10) shows the nature of diode characteristics at different temperatures of the cathode. The space between the cathode and the anode contains electrons and hence, is negatively charged.

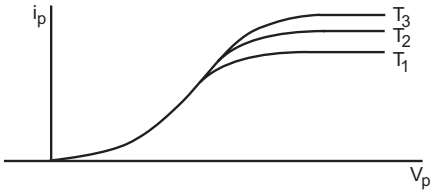



Figure 41.10

This negative charge is called the *space charge* and it repels the newly ejected electrons back to the cathode. The plate current $i_p$ is thus reduced due to the space charge. As the plate voltage is increased, the electrons are pulled with a greater force and the effect of space charge decreases. Thus, the plate current $i_p$ increases as the plate voltage $V_p$ increases. When $V_p$ is sufficiently large, all the electrons emitted by the cathode are collected by the anode and the current becomes saturated. Further increase in $V_p$ does not increase $i_p$.

If the anode is given a negative potential with respect to the cathode, the electrons are pushed back towards the cathode and no current flows in the external circuit. We say that the diode *does not conduct* in this case. Thus, the diode allows current only in one direction. The electrons can leave the diode at the anode and enter into it at the cathode. Correspondingly, positive current can enter the diode at the anode and may come out at the cathode.

If the temperature of the filament is increased, greater number of electrons are ejected from the cathode and the saturation current increases.

Far from the saturation, the current is roughly proportional to $V_p^{3/2}$. Thus,

$$i_p = k\, V_p^{3/2}$$

where $k$ is a constant for a given diode. This law is called *Langmuir–Child law.*

### Dynamic Plate Resistance

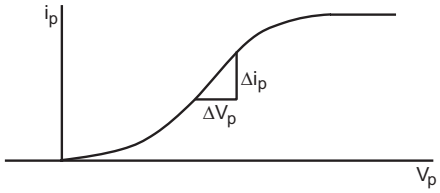



Figure 41.11

The resistance of a metallic conductor is defined as $R = V/i$. One can define the resistance of a diode using this equation. However, a more useful quantity, called the *dynamic plate resistance* of the diode, is defined as follows. Suppose the diode is operated at a plate voltage $V_p$ and the plate current is $i_p$ (figure 41.11). Now the plate voltage $V_p$ is changed by a small amount to $V_p + \Delta V_p$. Consequently, the current will

change by a small amount from $i_p$ to $i_p + \Delta i_p$. The dynamic plate resistance of the diode is defined as

$$r_p = \frac{\Delta V_p}{\Delta i_p} . \qquad \ldots \quad (41.4)$$

It is clear that $r_p$ depends on the operating conditions ($V_p$, $i_p$). Referring to figure (41.11), the dynamic plate resistance is the inverse of the slope of the $i_p$–$V_p$ characteristic at the operating point.

***Example 41.2***

*When the plate voltage applied to a diode valve is changed from* 40 V *to* 42 V*, the plate current increases from* 50 mA *to* 60 mA*. Calculate the dynamic plate resistance at the operating condition.*

***Solution*** : Here $\Delta V_p = 42\ \text{V} - 40\ \text{V} = 2\ \text{V}$

and $\Delta i_p = 60\ \text{mA} - 50\ \text{mA} = 10\ \text{mA}.$

Thus, the dynamic plate resistance is

$$r_p = \frac{\Delta V_p}{\Delta i_p} = \frac{2\ \text{V}}{10\ \text{mA}} = 200\ \Omega.$$

**Rectification**

If a source of alternating current is conneted to a resistor, the direction of the current in the resistor changes alternately. By using a diode in the circuit, one can have current in the resistor in a fixed direction. *Obtaining a unidirectional current from an AC source is called rectification.*

**Half-Wave Rectification**

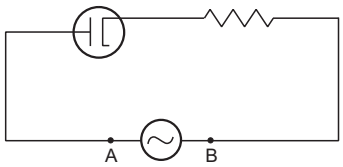



Figure 41.12

Figure (41.12) shows a circuit containing an AC source, a diode and a resistor. When the point $A$ is at a potential higher than that of $B$, the anode is also at a potential higher than that of the cathode. The diode allows current to pass through. Thus, the AC source sends a current which goes through the diode, through the resistor and then back to the source. The current in the resistor is from left to right.

When the potential at $A$ becomes less than that at $B$, the diode does not allow a current. There is no current in the resistance in this case.

Figure (41.13) shows the variations in the applied AC voltage (potential of $A$ with respect to $B$) and the current in the resistor as time passes. The current is allowed only in the positive half-cycles and is stopped in the negative half-cycles.

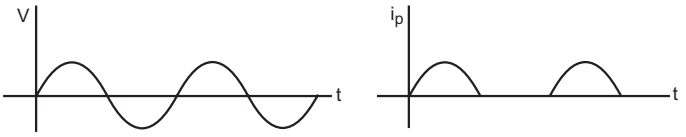
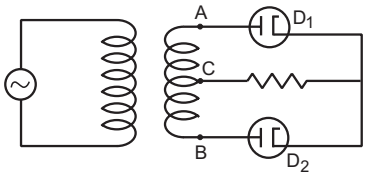



Figure 41.13

We see that the current in the resistor is always from left to right. The diode has *rectified* the AC voltage. As the current is allowed only for half the time, the rectification is called *half-wave rectificition.*

**Full-Wave Rectification**




Figurre 41.14

Figure (41.14) shows the circuit for full-wave rectification. The AC source is connected to the primary of a transformer. The terminals of the secondary are connected to the anodes of two diodes $D_1$ and $D_2$. The two cathodes are connected to one end of a resistor. The other end of the resistor is connected to the centre $C$ of the secondary of the transformer.

Let us take the potential at $C$ to be zero. Consider a half-cycle in which the potential of $A$ is positive and that of $B$ is negative. The diode $D_1$ conducts but $D_2$ does not. So a current passes through $D_1$ and then through the resistor back to the secondary. The current in the resistor is from right to left. In the next half-cycle, the potential at $B$ is positive and that at $A$ is negative. The diode $D_2$ conducts, but $D_1$ does not. The current passes through $D_2$ and then through the resistor back to the secondary. Again, the current in the resistor is from right to left. So, the current in the resistor is from right to left in both the half-cycles. Figure (41.15) shows the variations in the applied AC voltage and the current in the resistor as time passes.

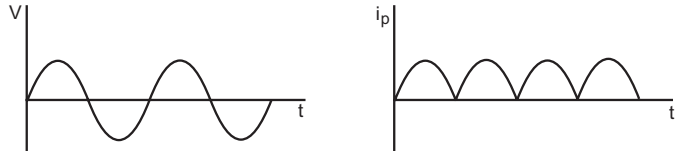



Figure 41.15

**41.7 TRIODE VALVE**

A triode valve is similar in construction to a diode valve except that a wire grid is inserted between the cathode and the anode. Thus, there are three external terminals in a triode which are connected to the cathode, the grid and the anode. Figure (41.16) shows the symbol of a triode valve.

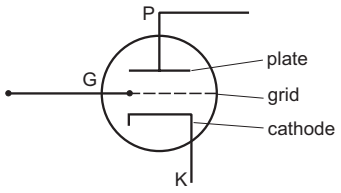



Figure 41.16

When the cathode is heated, it emits thermions. These thermions pass through the holes in the grid and reach the anode. If the anode is given a positive potential and the grid is kept at zero potential (both with respect to the cathode), the anode attracts the electrons and the triode works like a diode. The current through the anode is called the *plate current* or *anode current* and is denoted by $i_p$. This current is affected by the space charge near the cathode. The space charge is controlled both by the *grid voltage* $V_g$ (with respect to the cathode) and the plate voltage $V_p$. If the *grid voltage* $V_g$ is made negative, the grid repels the electrons coming from the cathode and the current decreases. If the grid voltage is made positive, it will help the electrons to go towards the anode, increasing the current. As the grid is closer to the cathode, changing grid voltage is more effective than changing plate voltage in bringing about a change in the plate current. In absence of the grid, the current is roughly given by $i_p = k\, V_p^{3/2}$. When the grid is added, the current is given by

$$i_p = k(V_p + \mu\, V_g)^{3/2}. \qquad \ldots \quad (41.4)$$

Thus, a grid voltage $V_g$ has the same effect as a plate voltage $\mu V_g$. The constant $\mu$, which may be typically of the order of 10, is called *amplification factor*.

**Triode Characteristics**

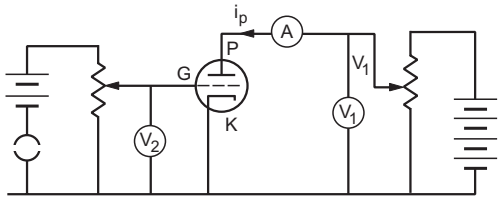



Figure 41.17

Figure (41.17) shows the circuit used to study the current in a triode. The plate current is measured by the ammeter $A$, the plate voltage $V_p$ by the voltmeter $V_1$ and the grid voltage $V_g$ by the voltmeter $V_2$. These voltages may be changed by changing the rheostat settings. The grid potential may be made positive or negative.

In a triode, there are several currents and voltages and one can draw a number of characterstic curves. More important are $i_p$–$V_p$ curves at constant $V_g$ and $i_p$–$V_g$ curves at constant $V_p$. The curves in the first set are called the *anode characteristics* and those in the second set are called the *mutual characteristics*. Figure (41.18a) and (41.18b) show the two characteristics qualitatively.

The operating point of a triode is determined by specifying $V_p$, $V_g$ and $i_p$. Generally, the triode is operated in the linear portions of its characteristics (figure 41.18).

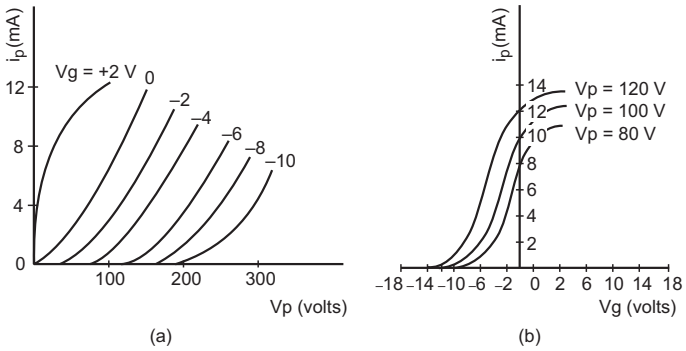



Figure 41.18

**Dynamic plate resistance**

If the grid voltage is kept constant and the plate voltage is changed, the plate current changes. We define the *dynamic plate resistance* $r_p$ as

$$r_p = \left(\frac{\Delta V_p}{\Delta i_p}\right)_{\Delta V_g = 0}. \qquad \ldots \quad (41.5)$$

**Mutual conductance**

If the plate voltage $V_p$ is kept constant and the grid voltage $V_g$ is changed, the plate current changes. The *mutual conductance* $g_m$ is defined as

$$g_m = \left(\frac{\Delta i_p}{\Delta V_g}\right)_{\Delta V_p = 0}. \qquad \ldots \quad (41.6)$$

Mutual conductance is also called *transconductance*.

If both $V_g$ and $V_p$ change, the plate current changes by

$$\Delta i_p = \left(\frac{\Delta i_p}{\Delta V_p}\right)_{\Delta V_g = 0} \Delta V_p + \left(\frac{\Delta i_p}{\Delta V_g}\right)_{\Delta V_p = 0} \Delta V_g$$

$$= \frac{1}{r_p}\, \Delta V_p + g_m\, \Delta V_g. \qquad \ldots \quad (41.7)$$

**Amplification factor**

If the grid voltage is increased, the plate current $i_p$ also increases. One can bring the current back to $i_p$ by decreasing the plate voltage $V_p$. The *amplification factor* $\mu$ is defined as

$$\mu = -\left(\frac{\Delta V_p}{\Delta V_g}\right)_{\Delta i_p = 0}. \qquad \ldots \quad (41.8)$$

The negative sign is put in the definition because if the grid voltage is increased, one should decrease the plate voltage so as to maintain the same current. Thus, $\Delta V_p$ and $\Delta V_g$ are of opposite signs so that the definition makes μ positive. This same μ is used in equation (41.4).

From equation (41.7), if $\Delta i_p = 0$,

$$-\frac{\Delta V_p}{\Delta V_g} = \frac{g_m}{1/r_p}$$

or, $$\mu = r_p \times g_m. \quad \ldots (41.9)$$

## 41.8 TRIODE AS AN AMPLIFIER

We have seen that a small change in the grid voltage leads to a large change in the plate current. This fact is used for amplification of small signals. Figure (41.19) shows the basic circuit for amplification. A small fluctuating voltage $v_s$ (from a microphone for example) is included in the grid circuit. This voltage is called the *signal*. The value $v_s$ also gives the amount by which the grid voltage changes. Thus,

$$v_s = \Delta V_g.$$

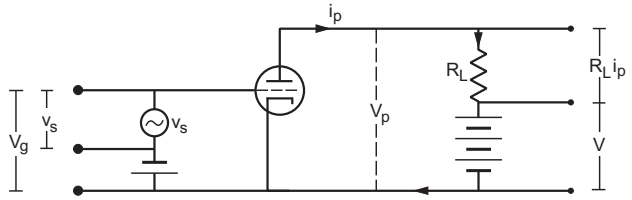



Figure 41.19

A load resistance $R_L$ is included in the plate circuit. The value of $V_g$ (with zero signal), and $V_p$ are so adjusted that the operating point lies on the linear portion of the mutual characteristics (figure 41.18). As the signal $v_s$ changes, the grid potential changes and consequently the plate current $i_p$ changes. The voltage appearing across the load resistance also changes accordingly. Suppose that the signal is small enough so that the operating point always remains on the linear portion. The change in the voltage across the load resistance, then, follows the pattern of the signal but the amplitudes are much larger. The change in the voltage across the load resistance is called the *output voltage* $v_0$.

Thus,

$$v_0 = R_L \Delta i_p. \quad \ldots (i)$$

The ratio $A = \frac{v_0}{v_s}$ is called the *voltage-gain* or *gain factor*.

If $V$ is the potential difference across the battery in the plate circuit, that across the triode is

$$V_p = V - R_L\, i_p$$

or, $$\Delta V_p = -\, R_L\, \Delta i_p = -\, v_0.$$

From equation (41.7),

$$\Delta i_p = \frac{\Delta V_p}{r_p} + g_m\, \Delta V_g$$

$$= -\,\frac{v_0}{r_p} + g_m\, v_s.$$

From (i), the output voltage is,

$$v_0 = R_L\, \Delta i_p$$

$$= -\,\frac{R_L v_0}{r_p} + R_L\, g_m\, v_s$$

or, $$v_0\left[1 + \frac{R_L}{r_p}\right] = R_L\, g_m v_s$$

or, $$\frac{v_0}{v_s} = \frac{R_L\, g_m}{1 + \frac{R_L}{r_p}} = \frac{R_L\, g_m r_p}{r_p + R_L}$$

or, $$A = \frac{R_L\, \mu}{r_p + R_L} = \frac{\mu}{1 + \frac{r_p}{R_L}} \quad \ldots (41.10)$$

where $\mu = g_m r_p$ is the amplification factor.

### *Worked Out Examples*

**1.** *The mean free path of the electrons in a discharge tube is* 20 cm. *The tube itself is* 10 cm *long. What is the length of the Crookes dark space* ?

***Solution*** **:** The mean free path of the electrons is much longer than the length of the tube. Thus, the electrons, in general, do not collide in between, no ionization takes place and hence no light is emitted. The Crookes dark space fills the entire tube and hence is 10 cm long.

**2.** *Consider a cylindrical tube closed at one end and fitted with a conducting, movable piston at the other end. A cathode is fixed in the tube near the closed end and an anode is fixed with the piston. A gas is filled in the tube at pressure p. Using Paschen equation V = f(pd), show that the sparking potential does not change as the piston is slowly moved in or out. Assume that the temperature does not change in the process.*

***Solution*** : As the piston is moved, the volume of the gas and hence its pressure changes. As the tube is cylindrical, the volume is proportional to the length of the tube. From Boyle's law, $pd$ = constant and hence the sparking potential does not change as the piston is moved.

**3.** *The number of thermions emitted in a given time increases* 100 times *as the temperature of the emitting surface is increased from* 600 K *to* 800 K. *Find the work function of the emitter. Boltzmann constant* $k = 8{\cdot}62 \times 10^{-5}$ eV $K^{-1}$.

***Solution*** : The number of thermions $n$, emitted by a surface, in a given time is given by (Richardson–Dushman equation)

$$n = A'ST^2 e^{-\varphi/kT}$$

where $A'$ is a constant and other symbols have their usual meanings. Let $n_1$ and $n_2$ be the number of electrons emitted at temperatures $T_1$ and $T_2$. Then

$$\frac{n_1}{n_2} = \frac{T_1^2}{T_2^2}\frac{e^{-\varphi/kT_1}}{e^{-\varphi/kT_2}}$$

or, $$\frac{n_1 T_2^2}{n_2 T_1^2} = e^{-\frac{\varphi}{k}\left(\frac{1}{T_1}-\frac{1}{T_2}\right)}$$

or, $$-\frac{\varphi}{k}\left(\frac{1}{T_1}-\frac{1}{T_2}\right) = \ln\frac{n_1 T_2^2}{n_2 T_1^2}$$

or, $$\frac{\varphi(T_2 - T_1)}{kT_1T_2} = \ln\frac{n_2 T_1^2}{n_1 T_2^2}$$

or, $$\varphi = \frac{kT_1 T_2}{T_2 - T_1}\ln\frac{n_2 T_1^2}{n_1 T_2^2}$$

$$= \frac{(8{\cdot}62 \times 10^{-5}\text{ eV K}^{-1})\,(600\text{ K})\,(800\text{ K})}{200\text{ K}}\ln\left(100\times\frac{36}{64}\right)$$

$$= 0{\cdot}83\text{ eV}.$$

**4.** *The constant A in the Richardson–Dushman equation is* $60 \times 10^4$ A $m^{-2}K^{-2}$ *for tungsten. A tungsten cathode has a total surface area of* $2{\cdot}0 \times 10^{-5}$ $m^2$ *and operates at* 2000 K. *The work function of tungsten is* 4·55 eV. *Calculate the electric current due to thermionic emission.*

***Solution*** :

The Richardson–Dushman equation is

$$i = AST^2 e^{-\varphi/kT}.$$

We have, $$\frac{\varphi}{kT} = \frac{4{\cdot}55\text{ eV}}{(8{\cdot}62 \times 10^{-5}\text{ eV K}^{-1})\,(2000\text{ K})}$$

$$= 26{\cdot}4.$$

Thus, the thermionic current $i$ is

$$= 60 \times 10^4\text{ A m}^{-2}\text{K}^{-2} \times (2{\cdot}0 \times 10^{-5}\text{ m}^2) \times (2000\text{ K})^2 \times e^{-26{\cdot}4}$$

$$= (4{\cdot}8 \times 10^5\text{ A})\, e^{-26{\cdot}4} = 0{\cdot}16\text{ mA}.$$

**5.** *Calculate the saturation thermionic current if* 120 W *is applied to a thoriated-tungsten filament of surface area* 1·0 $cm^2$. *Assume that the surface radiates like a blackbody. The required constants are*

$A = 3 \times 10^4$ A $m^{-2}$–$K^2$, $\varphi = 2{\cdot}6$ eV, $k = 8{\cdot}62 \times 10^{-5}$ eV $K^{-1}$ *and* $\sigma = 6 \times 10^{-8}$ W $m^{-2}K^{-4}$.

***Solution*** : The thermionic current is given by the Richardson–Dushman equation

$$i = AST^2 e^{-\varphi/kT}. \qquad \ldots \text{(i)}$$

When the power input to the filament equals the power radiated, the temperature becomes constant. The thermionic current then becomes saturated. The power radiated is given by the Stefan's law

$$P = S\sigma T^4$$

or, $120\text{ W} = (1{\cdot}0 \times 10^{-4}\text{ m}^2) \times (6 \times 10^{-8}\text{ W m}^{-2}\text{K}^{-4}) \times T^4$

or, $T = 2114$ K.

Now $$\frac{\varphi}{kT} = \frac{2{\cdot}6\text{ eV}}{(8{\cdot}62 \times 10^{-5}\text{ eV K}^{-1})\,(2114\text{ K})} = 14{\cdot}26.$$

Putting in (i),

$$i = 3 \times 10^4\text{ A m}^{-2}\text{K}^{-2} \times (1{\cdot}0 \times 10^{-4}\text{ m}^2)\,(2114\text{ K})^2\, e^{-14{\cdot}26}$$

$$= (1{\cdot}34 \times 10^7\text{ A})\, e^{-14{\cdot}26} = 8{\cdot}6\text{ A}.$$

**6.** *In a Millikan-type oil-drop experiment, the plates are* 8 mm *apart. An oil drop is found to remain at rest when the upper plate is at a potential* 136 V *higher than that of the lower one. When the electric field is switched off, the drop is found to fall a distance of* 2·0 mm *in* 36 seconds *with a uniform speed. Find (a) the charge on the drop and (b) the number of electrons attached to this drop. Density of oil* = 880 kg $m^{-3}$ *and coefficient of viscosity of air* = 180 μpoise.

***Solution*** : (a) The charge on the drop is

$$q = \frac{18\pi}{E}\sqrt{\frac{\eta^3 v^3}{2(\rho - \sigma)g}}. \qquad \ldots \text{(i)}$$

Here $$E = \frac{136\text{ V}}{8 \times 10^{-3}\text{ m}} = 1{\cdot}7 \times 10^4\text{ V m}^{-1}$$

$$\eta = 180\ \mu\text{poise} = 1{\cdot}8 \times 10^{-5}\text{ N sm}^{-2}$$

$$v = \frac{2{\cdot}0\text{ mm}}{36\text{ s}} = \frac{1}{18} \times 10^{-3}\text{ m s}^{-1}$$

and $\rho = 880$ kg $m^{-3}$.

The density of air $\sigma$ (1·29 kg $m^{-3}$) may be neglected in comparison to that of the oil. Putting values in (i),

$$q = 7{\cdot}93 \times 10^{-19}\text{ C}.$$

(b) The number of electrons attached to the drop is,

$$n = \frac{7{\cdot}93 \times 10^{-19}\ \text{C}}{1{\cdot}6 \times 10^{-19}\ \text{C}} = 4{\cdot}96.$$

It is clear that 5 electrons are attached to the drop.

**7.** *Show that the dynamic plate resistance of a diode is* $\frac{2V}{3i}$ *where V and i are the plate voltage and the plate current respectively. Assume Langmuir–Child equation to hold.*

***Solution*** :

The dynamic plate resistance of the diode is $R = \frac{dV}{di}$.

The Langmuir–Child equation is

$$i = cV^{3/2} \quad \ldots \text{(i)}$$

where $c$ is a constant for a given diode. This gives

$$\frac{di}{dV} = \frac{3}{2} cV^{1/2}. \quad \ldots \text{(ii)}$$

Dividing (ii) by (i), $\frac{1}{i}\frac{di}{dV} = \frac{3}{2V}$

or, $$\frac{dV}{di} = \frac{2V}{3i}.$$

**8.** *The mutual conductance of a triode valve is* 2·5 millimho. *Find the change in the plate current if the grid voltage is changed from* –2·0 V *to* –4·5 V.

***Solution*** :

The mutual conductance of a triode valve is

$$g_m = \left[\frac{\Delta i_p}{\Delta V_g}\right]_{\Delta V_p = 0}$$

or, $$\Delta i_p = g_m \Delta V_g$$

$$= (2{\cdot}5 \times 10^{-3}\ \Omega^{-1}) \times (-4{\cdot}5\ \text{V} + 2{\cdot}0\ \text{V})$$

$$= -6{\cdot}25 \times 10^{-3}\ \text{A}.$$

**9.** *A triode valve has amplification factor* 21 *and dynamic plate resistance* 10 kΩ. *This is used as an amplifier with a load of* 20 kΩ. *Find the gain factor of the amplifier.*

***Solution*** :

The gain factor of a triode valve amplifier is

$$A = \frac{\mu}{1 + \frac{r_p}{R_L}}$$

where μ is the amplification factor, $r_p$ is the plate resistance and $R_L$ is the load resistance. Thus,

$$A = \frac{21}{1 + \frac{10\ \text{k}\Omega}{20\ \text{k}\Omega}} = 14.$$

□

## QUESTIONS FOR SHORT ANSWER

**1.** Why is conduction easier in gases if the pressure is low ? Will the conduction continue to improve if the pressure is made as low as nearly zero ?

**2.** An AC source is connected to a diode and a resistor in series. Is the current through the resistor AC or DC ?

**3.** How will the thermionic current vary if the filament current is increased ?

**4.** Would you prefer a material having a high melting point or a low melting point to be used as a cathode in a diode ?

**5.** Would you prefer a material having a high work function or a low work function to be used as a cathode in a diode ?

**6.** An isolated metal sphere is heated to a high temperature. Will it become positively charged due to thermionic emission ?

**7.** A diode valve is connected to a battery and a load resistance. The filament is heated so that a constant current is obtained in the circuit. As the cathode continuously emits electrons, does it get more and more positively charged ?

**8.** Why does thermionic emission not take place in nonconductors ?

**9.** The cathode of a diode valve is replaced by another cathode of double the surface area. Keeping the voltage and temperature conditions the same, will the plate current decrease, increase or remain the same ?

**10.** Why is the linear portion of the triode characteristic chosen to operate the triode as an amplifier ?

## OBJECTIVE I

**1.** Cathode rays constitute a stream of
(a) electrons (b) protons
(c) positive ions (d) negative ions.

**2.** Cathode rays are passing through a discharge tube. In the tube, there is
(a) an electric field but no magnetic field

(b) a magnetic field but no electric field
(c) an electric as well as a magnetic field
(d) neither an electric nor a magnetic field.

**3.** Let $i_0$ be the thermionic current from a metal surface when the absolute temperature of the surface is $T_0$. The temperature is slowly increased and the thermionic current is measured as a function of temperature. Which of the following plots may represent the variation in $(i/i_0)$ against $(T/T_0)$ ?

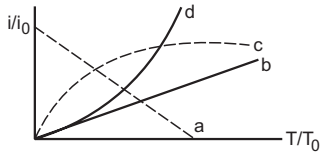



Figure 41-Q1

**4.** When the diode shows saturated current, dynamic plate resistance is
(a) zero (b) infinity
(c) indeterminate (d) different for different diodes.

**5.** The anode of a thermionic diode is connected to the negative terminal of a battery and the cathode to its positive terminal.
(a) No appreciable current will pass through the diode.
(b) A large current will pass through the diode from the anode to the cathode.
(c) A large current will pass through the diode from the cathode to the anode.
(d) The diode will be damaged.

**6.** A diode, a resistor and a 50 Hz AC source are connected in series. The number of current pulses per second through the resistor is
(a) 25 (b) 50 (c) 100 (d) 200.

**7.** A triode is operated in the linear region of its characteristics. If the plate voltage is slightly increased, the dynamic plate resistance will
(a) increase (b) decrease
(c) remain almost the same
(d) become zero.

**8.** The plate current in a triode valve is maximum when the potential of the grid is
(a) positive (b) zero (c) negative (d) nonpositive.

**9.** The amplification factor of a triode operating in the linear region depends strongly on
(a) the temperature of the cathode
(b) the plate potential (c) the grid potential
(d) the separations of the grid from the cathode and the anode.

## OBJECTIVE II

**1.** Electric conduction takes place in a discharge tube due to the movement of
(a) positive ions (b) negative ions
(c) electrons (d) protons.

**2.** Which of the following are true for cathode ray ?
(a) It travels along straight lines.
(b) It emits X-ray when strikes a metal.
(c) It is an electromagnetic wave.
(d) It is not deflected by magnetic field.

**3.** Because of the space charge in a diode valve,
(a) the plate current decreases
(b) the plate voltage increases
(c) the rate of emission of thermions increases
(d) the saturation current increases.

**4.** The saturation current in a triode valve can be changed by changing
(a) the grid voltage (b) the plate voltage
(c) the separation between the grid and the cathode
(d) the temperature of the cathode.

**5.** Mark the correct options.
(a) A diode valve can be used as a rectifier.
(b) A triode valve can be used as a rectifier.
(c) A diode valve can be used as an amplifier.
(d) A triode valve can be used as an amplifier.

**6.** The plate current in a diode is zero. It is possible that
(a) the plate voltage is zero
(b) the plate voltage is slightly negative
(c) the plate voltage is slightly positive
(d) the temperature of the filament is low.

**7.** The plate current in a triode valve is zero. The temperature of the filament is high. It is possible that
(a) $V_g > 0,\ V_p > 0$ (b) $V_g > 0,\ V_p < 0$
(c) $V_g < 0,\ V_p > 0$ (d) $V_g < 0,\ V_p < 0$.

## EXERCISES

**1.** A discharge tube contains helium at a low pressure. A large potential difference is applied across the tube. Consider a helium atom that has just been ionized due to the detachment of an atomic electron. Find the ratio of the distance travelled by the free electron to that by the positive ion in a short time $dt$ after the ionization.

**2.** A molecule of a gas, filled in a discharge tube, gets ionized when an electron is detached from it. An electric field of 5·0 kV m$^{-1}$ exists in the vicinity of the event. (a) Find the distance travelled by the free electron in 1 μs assuming no collision. (b) If the mean free path of the electron is 1·0 mm, estimate the time of transit of the free electron between successive collisions.

**3.** The mean free path of electrons in the gas in a discharge tube is inversely proportional to the pressure inside it. The Crookes dark space occupies half the length of the discharge tube when the pressure is 0·02 mm of mercury. Estimate the pressure at which the dark space will fill the whole tube.

**4.** Two discharge tubes have identical material structure and the same gas is filled in them. The length of one tube is 10 cm and that of the other tube is 20 cm. Sparking starts in both the tubes when the potential difference between the cathode and the anode is 100 V. If the pressure in the shorter tube is 1·0 mm of mercury, what is the pressure in the longer tube ?

**5.** Calculate $n(T)/n(1000\text{ K})$ for tungsten emitter at $T$ = 300 K, 2000 K and 3000 K where $n(T)$ represents the number of thermions emitted per second by the surface at temperature $T$. Work function of tungsten is 4·52 eV.

**6.** The saturation current from a thoriated-tungsten cathode at 2000 K is 100 mA. What will be the saturation current for a pure-tungsten cathode of the same surface area operating at the same temperature ? The constant $A$ in the Richardson–Dushman equation is $60 \times 10^4$ A m$^{-2}$K$^{-2}$ for pure tungsten and $3{\cdot}0 \times 10^4$ A m$^{-2}$K$^{-2}$ for thoriated tungsten. The work function of pure tungsten is 4·5 eV and that of thoriated tungsten is 2·6 eV.

**7.** A tungsten cathode and a thoriated-tungsten cathode have the same geometrical dimensions and are operated at the same temperature. The thoriated-tungsten cathode gives 5000 times more current than the other one. Find the operating temperature. Take relevant data from the previous problem.

**8.** If the temperature of a tungsten filament is raised from 2000 K to 2010 K, by what factor does the emission current change ? Work function of tungsten is 4·5 eV.

**9.** The constant $A$ in the Richardson–Dushman equation for tungsten is $60 \times 10^4$ A m$^{-2}$ K$^{-2}$. The work function of tungsten is 4·5 eV. A tungsten cathode having a surface area $2{\cdot}0 \times 10^{-5}$ m$^2$ is heated by a 24 W electric heater. In steady state, the heat radiated by the cathode equals the energy input by the heater and the temperature becomes constant. Assuming that the cathode radiates like a blackbody, calculate the saturation current due to thermions. Take Stefan constant = $6 \times 10^{-8}$ W m$^{-2}$ K$^{-4}$. Assume that the thermions take only a small fraction of the heat supplied.

**10.** A plate current of 10 mA is obtained when 60 volts are applied across a diode tube. Assuming the Langmuir–Child equation $i_p \propto V_p^{3/2}$ to hold, find the dynamic resistance $r_p$ in this operating condition.

**11.** The plate current in a diode is 20 mA when the plate voltage is 50 V or 60 V. What will be the current if the plate voltage is 70 V ?

**12.** The power delivered in the plate circuit of a diode is 1·0 W when the plate voltage is 36 V. Find the power delivered if the plate voltage is increased to 49 V. Assume Langmuir–Child equation to hold.

**13.** A triode valve operates at $V_p$ = 225 V and $V_g$ = – 0·5 V. The plate current remains unchanged if the plate voltage is increased to 250 V and the grid voltage is decreased to – 2·5 V. Calculate the amplification factor.

**14.** Calculate the amplification factor of a triode valve which has plate resistance of 2 kΩ and transconductance of 2 millimho.

**15.** The dynamic plate resistance of a triode valve is 10 kΩ. Find the change in the plate current if the plate voltage is changed from 200 V to 220 V.

**16.** Find the values of $r_p$, μ and $g_m$ of a triode operating at plate voltage 200 V and grid voltage –6 V. The plate characteristics are shown in figure (41-E1).

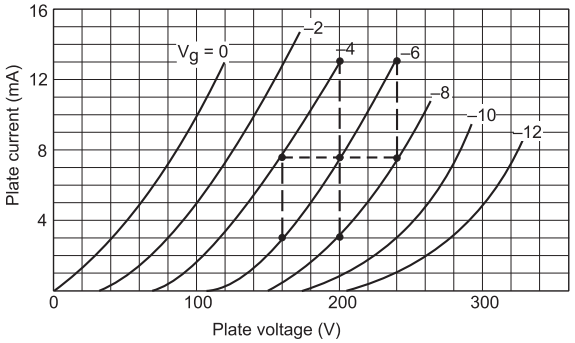



Figure 41-E1

**17.** The plate resistance of a triode is 8 kΩ and the transconductance is 2·5 millimho. (a) If the plate voltage is increased by 48 V, and the grid voltage is kept constant, what will be the increase in the plate current ? (b) With plate voltage kept constant at this increased value, how much should the grid voltage be decreased in order to bring the plate current back to its initial value ?

**18.** The plate resistance and the amplification factor of a triode are 10 kΩ and 20. The tube is operated at plate voltage 250 V and grid voltage –7·5 V. The plate current is 10 mA. (a) To what value should the grid voltage be changed so as to increase the plate current to 15 mA ? (b) To what value should the plate voltage be changed to take the plate current back to 10 mA ?

**19.** The plate current, plate voltage and grid voltage of a 6F6 triode tube are related as

$$i_p = 41(V_p + 7\,V_g)^{1{\cdot}41}$$

where $V_p$ and $V_g$ are in volts and $i_p$ in microamperes. The tube is operated at $V_p$ = 250 V, $V_g$ = –20 V. Calculate (a) the tube current, (b) the plate resistance, (c) the mutual conductance and (d) the amplification factor.

**20.** The plate current in a triode can be written as

$$i_p = k\left(V_g + \frac{V_p}{\mu}\right)^{3/2}.$$

Show that the mutual conductance is proportional to the cube root of the plate current.

**21.** A triode has mutual conductance = 2·0 millimho and plate resistance = 20 kΩ. It is desired to amplify a signal by a factor of 30. What load resistance should be added in the circuit ?

**22.** The gain factor of an amplifier is increased from 10 to 12 as the load resistance is changed from 4 kΩ to 8 kΩ. Calculate (a) the amplification factor and (b) the plate resistance.

**23.** Figure (41-E2) shows two identical triode tubes connected in parallel. The anodes are connected together, the grids are connected together and the cathodes are connected together. Show that the equivalent plate resistance is half of the individual plate resistance, the equivalent mutual conductance is double the individual mutual conductance and the equivalent amplification factor is the same as the individual amplification factor.

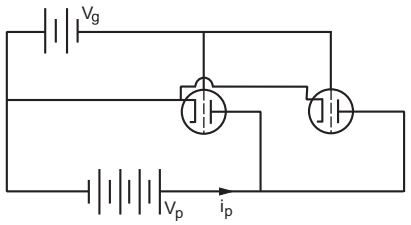



Figure 41-E2

□

## ANSWERS

### OBJECTIVE I

1. (a) 2. (c) 3. (d) 4. (b) 5. (a) 6. (b)
7. (c) 8. (a) 9. (d)

### OBJECTIVE II

1. (a), (b), (c) 2. (a), (b) 3. (a)
4. (d) 5. (a), (b), (d) 6. all
7. (b), (c), (d)

### EXERCISES

1. 7340
2. (a) 440 m (b) 1·5 ns
3. 0·01 mm of mercury
4. 0·5 mm of mercury
5. $6{\cdot}57 \times 10^{-55}$, $9{\cdot}73 \times 10^{11}$, $1{\cdot}37 \times 10^{16}$
6. 33 μA
7. 1914 K
8. 1·14
9. 1·0 mA
10. 4 kΩ
11. 20 mA
12. 2·2 W
13. 12·5
14. 4
15. 2 mA
16. 8·0 kΩ, 20 and 2·5 millimho
17. (a) 6 mA (b) 2·4 V
18. (a) –5·0 V (b) 200 V
19. (a) 30 mA (b) 2·53 kΩ (c) 2·77 millimho (d) 7
21. 60 kΩ
22. (a) 15 (b) 2 kΩ

□

CHAPTER 42

# PHOTOELECTRIC EFFECT AND WAVE–PARTICLE DUALITY

## 42.1 PHOTON THEORY OF LIGHT

We have learnt that light has wave character as well as particle character. Depending on the situation, one of the two characters dominates. When light is passed through a double slit, it shows interference. This observation can only be understood in terms of wave theory which was discussed in detail in an earlier chapter. There are some phenomena which can only be understood in terms of the particle theory of light. When light of sufficiently low wavelength falls on a metal surface, electrons are ejected. This phenomenon is called the *photoelectric effect* and can be understood only in terms of the particle nature of light.

The particles of light have several properties in common with the material particles and several other properties which are different from the material particles. The particles of light are called *photons*. We list some of the important properties of photons.

(a) A photon always travels at a speed $c = 299,\ 792,\ 458\ \text{ms}^{-1} \approx 3{\cdot}0 \times 10^{8}\ \text{m s}^{-1}$ in vacuum. This is true for any frame of reference used to observe the photon.

(b) The mass of a photon is not defined in the sense of Newtonian mechanics. We shall ignore this concept. We simply state that the *rest mass* of a photon is zero.

(c) Each photon has a definite energy and a definite linear momentum.

(d) Let $E$ and $p$ be the energy and linear momentum of a photon of light, and $\nu$ and $\lambda$ be the frequency and wavelength of the same light when it behaves as a wave. Then,

$$\text{and} \qquad \begin{aligned} E &= h\nu = hc/\lambda \\ p &= h/\lambda = E/c \end{aligned} \qquad \ldots\ (42.1)$$

where $h$ is a universal constant known as the *Planck constant* and has a value $6{\cdot}626 \times 10^{-34}$ Js $= 4{\cdot}136 \times 10^{-15}$ eVs.

Thus, all photons of light of a particular wavelength $\lambda$ have the same energy $E = hc/\lambda$ and the same momentum $p = h/\lambda$.

(e) A photon may collide with a material particle. The total energy and the total momentum remain conserved in such a collision. The photon may get absorbed and/or a new photon may be emitted. Thus, the number of photons may not be conserved.

(f) If the intensity of light of a given wavelength is increased, there is an increase in the number of photons crossing a given area in a given time. The energy of each photon remains the same.

***Example 42.1***

*Consider a parallel beam of light of wavelengh* 600 nm *and intensity* 100 W m$^{-2}$. *(a) Find the energy and linear momentum of each photon. (b) How many photons cross* 1 cm$^2$ *area perpendicular to the beam in one second ?*

***Solution :***

(a) The energy of each photon $E = hc/\lambda$

$$= \frac{(4{\cdot}14 \times 10^{-15}\ \text{eVs}) \times (3 \times 10^{8}\ \text{m s}^{-1})}{600 \times 10^{-9}\ \text{m}} = 2{\cdot}07\ \text{eV}.$$

The linear momentum is

$$p = \frac{E}{c} = \frac{2{\cdot}07\ \text{eV}}{3 \times 10^{8}\ \text{m s}^{-1}} = 0{\cdot}69 \times 10^{-8}\ \text{eVs m}^{-1}.$$

(b) The energy crossing 1 cm$^2$ in one second

$$= (100\ \text{W m}^{-2}) \times (1\ \text{cm}^{2}) \times (1\ \text{s}) = 1{\cdot}0 \times 10^{-2}\ \text{J}.$$

The number of photons making up this amount of energy is

$$n = \frac{1{\cdot}0 \times 10^{-2}\ \text{J}}{2{\cdot}07\ \text{eV}} = \frac{1{\cdot}0 \times 10^{-2}}{2{\cdot}07 \times 1{\cdot}6 \times 10^{-19}} = 3{\cdot}0 \times 10^{16}.$$

For a given wavelength λ, the energy of light is an integer times $hc/\lambda$. Thus, the energy of light can be varied only in quantums (steps) of $\frac{hc}{\lambda}$. The photon theory is, therefore, also called the quantum theory of light.

## 42.2 PHOTOELECTRIC EFFECT

When light of sufficiently small wavelength is incident on a metal surface, electrons are ejected from the metal. This phenomenon is called the *photoelectric effect*. The electrons ejected from the metal are called *photoelectrons*. Let us try to understand photoelectric effect on the basis of the photon theory of light.

We know that there are large number of free electrons in a metal which wander throughout the body of the metal. However, these electrons are not free to leave the surface of the metal. As they try to come out of the metal, the metal attracts them back. A minimum energy, equal to the work function φ, must be given to an electron so as to bring it out of the metal.

When light is incident on a metal surface, the photons collide with the free electrons. In a particular collision, the photon may give all of its energy to the free electron. If this energy is more than the work function φ, the electron may come out of the metal. It is not necessary that if the energy supplied to an electron is more than φ, it will come out. The electron after receiving the energy, may lose energy to the metal in course of collisions with the atoms of the metal. Only if an electron near the surface gets the extra energy and heads towards the outside, it is able to come out. If it is given an energy $E$ which is greater than φ, and it makes the most economical use of it, it will have a kinetic energy $(E - \varphi)$ after coming out. If it makes some collisions before coming out, the kinetic energy will be less than $(E - \varphi)$. The actual kinetic energy of such an electron will depend on the total energy lost in collisions. It is also possible that the electron makes several collisions inside the metal and loses so much energy that it fails to come out. So, the kinetic energy of the photoelectron coming out may be anything between zero and $(E - \varphi)$ where $E$ is the energy supplied to the individual electrons. We can, therefore, write

$$K_{max} = E - \varphi.$$

**Table 42.1 : *Work functions of some photosensitive metals***

| Metal | Work function (eV) | Metal | Work function (eV) |
|---|---|---|---|
| Cesium | 1·9 | Calcium | 3·2 |
| Potassium | 2·2 | Copper | 4·5 |
| Sodium | 2·3 | Silver | 4·7 |
| Lithium | 2·5 | Platinum | 5·6 |

Let monochromatic light of wavelength λ be incident on the metal surface. In the particle picture, photons of energy $hc/\lambda$ fall on the surface. Suppose, a particular photon collides with a free electron and supplies all its energy to the electron. The electron gets an extra energy $E = hc/\lambda$ and may come out of metal. The maximum kinetic energy of this electron is, therefore,

$$K_{max} = \frac{hc}{\lambda} - \varphi = h\nu - \varphi. \quad \ldots (42.2)$$

As all the photons have the same energy $hc/\lambda$, equation (42.2) gives the maximum kinetic energy of any of the ejected electrons.

Equation (42.2) is called ***Einstein's photoelectric equation.*** Einstein, after an average academic career, put forward this theory in 1905 while working as a grade III technical officer in a patent office. He was awarded the Nobel Prize in physics for 1921 for this work.

### Threshold Wavelength

Equation (42.2) tells that if the wavelength λ is equal to

$$\lambda_0 = hc/\varphi,$$

the maximum kinetic energy is zero. An electron may just come out in this case. If $\lambda > \lambda_0$, the energy $hc/\lambda$ supplied to the electron is smaller than the work function φ and no electron will come out. Thus, photoelectric effect takes place only if $\lambda \le \lambda_0$. This wavelength $\lambda_0$ is called the *threshold wavelength* for the metal. The corresponding frequency

$$\nu_0 = c/\lambda_0 = \varphi/h$$

is called the *threshold frequency* for the metal. Threshold wavelength and threshold frequency depend on the metal used.

Writing $\varphi = h\nu_0$, equation (42.2) becomes

$$K_{max} = h(\nu - \nu_0). \quad \ldots (42.3)$$

***Example 42.2***

*Find the maximum wavelength of light that can cause photoelectric effect in lithium.*

***Solution :*** From table (42.1), the work function of lithium is 2·5 eV. The threshold wavelength is

$$\lambda = hc/\varphi.$$

$$= \frac{(4{\cdot}14 \times 10^{-15}\ \text{eVs}) \times (3 \times 10^{8}\ \text{m s}^{-1})}{2{\cdot}5\ \text{eV}}$$

$$= \frac{1242\ \text{eVnm}}{2{\cdot}5\ \text{eV}} = 497\ \text{nm}.$$

This is the required maximum wavelength.

**Experimental Arrangement**

A systematic study of photoelectric effect can be made in the laboratory with the apparatus shown in figure (42.1). Two metal plates $C$ and $A$ are sealed in a vacuum chamber. Light of reasonably short wavelength passes through a transparent window in the wall of the chamber and falls on the plate $C$ which is called the *cathode* or the *emitter*. The electrons are emitted by $C$ and collected by the plate $A$ called the *anode* or the *collector*. The potential difference between the cathode and the anode can be changed with the help of the batteries, rheostat and the commutator. The anode potential can be made positive or negative with respect to the cathode. The electrons collected by the anode $A$ flow through the ammeter, batteries, etc., and are back to the cathode $C$ and hence an electric current is established in the circuit. Such a current is called a *photocurrent*.

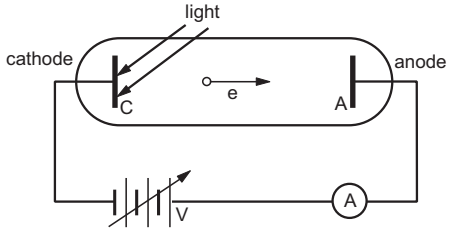



Figure 42.1

As photoelectrons are emitted from the cathode $C$, they move towards the anode $A$. At any time, the space between the cathode and the anode contains a number of electrons making up the *space charge*. This negative charge repels the fresh electrons coming from the cathode. However, some electrons are able to reach the anode and there is a photocurrent. When the anode is given a positive potential with respect to the cathode, electrons are attracted towards the anode and the photocurrent increases. The current thus depends on the potential applied to the anode. Figure (42.2) shows the variation in current with potential. If the potential

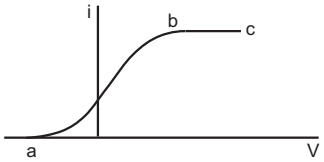



Figure 42.2

of the anode is increased gradually, a situation arrives when the effect of the space charge becomes negligible and any electron that is emitted from the cathode is able to reach the anode. The current then becomes constant and is known as the *saturation current*. This is shown by the part *bc* in figure (42.2). Further increase in the anode potential does not change the magnitude of the photocurrent.

If the potential of the anode is made negative with respect to the cathode, the electrons are repelled by the anode. Some electrons go back to the cathode so that the current decreases. At a certain value of this negative potential, the current is completely stopped. The smallest magnitude of the anode potential which just stops the photocurrent, is called the *stopping potential.*

The stopping potential is related to the maximum kinetic energy of the ejected electrons. To stop the current, we must ensure that even the fastest electron fails to reach the anode. Suppose, the anode is kept at a negative potential of magnitude $V_0$ with respect to the cathode. As a photoelectron travels from the cathode to the anode, the potential energy increases by $eV_0$. This is equal to the decrease in the kinetic energy of the photoelectron. The kinetic energy of the fastest photoelectron, as it reaches the anode, is $K_{max} - eV_0$. If the fastest electron just fails to reach the anode, we should have

$$eV_0 = K_{max} = \frac{hc}{\lambda} - \varphi$$

or,
$$V_0 = \frac{hc}{e}\left(\frac{1}{\lambda}\right) - \frac{\varphi}{e}. \qquad \ldots\ (42.3)$$

We see that the stopping potential $V_0$ depends on the wavelength of the light and the work function of the metal. It does not depend on the intensity of light. Thus, if an anode potential of $-2{\cdot}0$ V stops the photocurrent from a metal when a 1 W source of light is used, the same potential of $-2{\cdot}0$ V will stop the photocurrent when a 100 W source of light of the same wavelength is used.

The saturation current increases as the intensity of light increases. This is because, a larger number of photons now fall on the metal surface and hence a larger number of electrons interact with photons. The number of electrons emitted increases and hence the current increases.

Figure (42.3a) shows plots of photocurrent versus anode potential for three different intensities of light.

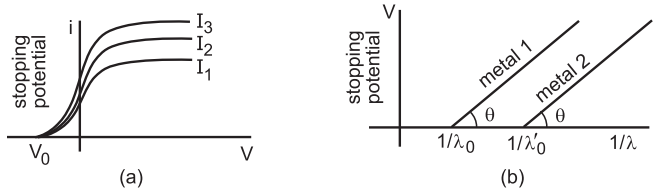



Figure 42.3

Note that the stopping potential $V_0$ is indepedent of the intensity of light.

The variation in stopping potential $V_0$ with $1/\lambda$ is shown in figure (42.3b) for cathodes of two different metals. From equation (42.3), the slope of each curve is

$$\tan\theta = \frac{hc}{e}$$

which is the same for all metals. Also, the curves intersect the $1/\lambda$ axis where $V_0$ is zero. Using equation (42.3), this corresponds to

$$\frac{hc}{\lambda_0} = \varphi$$

or,
$$\frac{1}{\lambda_0} = \frac{\varphi}{hc}$$

which is inverse of the threshold wavelength.

Let us summarise the results obtained from the experiments described above.

1. When light of sufficiently small wavelength falls on a metal surface, the metal emits electrons. The emission is almost instantaneous.

2. There is a threshold wavelength $\lambda_0$ for a given metal such that if the wavelength of light is more than $\lambda_0$, no photoelectric effect takes place.

3. The kinetic energies of the photoelectrons vary from zero to a maximum of $K_{max}$ where

$$K_{max} = \frac{hc}{\lambda} - \varphi$$

with usual meanings of the symbols.

4. The photocurrent may be stopped by applying a negative potential to the anode with respect to the cathode. The minimum magnitude of the potential needed to stop the photocurrent is called the stopping potential. It is proportional to the maximum kinetic energy of the photoelectrons.

5. The stopping potential does not depend on the intensity of the incident light. This means that the kinetic energy of the photoelectrons is independent of intensity of light.

6. The stopping potential depends on the wavelength of the incident light.

7. The photocurrent increases if the intensity of the incident light is increased.

**Photoelectric Effect and Wave Theory of Light**

According to wave thoery, when light falls on a metal surface, energy is continuously distributed over the surface. All the free electrons at the surface receive light energy. An electron may be ejected only when it acquires energy more than the work function. If we use a low-intensity source, it may take hours before an electron acquires this much energy from the light. In this period, there will be many collisions and any extra energy accumulated so far will be shared with the remaining metal. This will result in no photoelectron. This is contrary to experimental observations. No matter how small is the intensity, photoelectrons are ejected and that too without any appreciable time delay. In the photon theory, low intensity means less number of photons and hence less number of electrons get a chance to absorb energy. But any fortunate electron on which a photon falls, gets the full energy of the photon and may come out immediately.

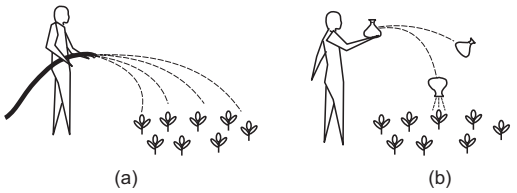



Figure 42.4

In figure (42.4), we illustrate an analogy to the wave and particle behaviour of light. In part (a), water is sprayed from a distance on an area containing several plants. Each plant receives water at nearly the same rate. It takes time for a particular plant to receive a certain amount of water. In part (b) of the figure, water is filled in identical, loosely-tied water bags and a particle physicist throws the bags randomly at the plants. When a bag collides with a plant, it sprays all its water on that plant in a very short time. In the same way, whole of the energy associated with a photon is absorbed by a free electron when the photon hits it.

The maximum kinetic energy of a photoelectron does not depend on the intensity of the incident light. This fact is also not understood by the wave theory. According to this theory, more intensity means more energy and the maximum kinetic energy must increase with the increase in intensity which is not true. The dependence of maximum kinetic energy on wavelength is also against the wave theory. There should not be any threshold wavelength according to the wave thoery. According to this theory, by using sufficiently intense light of any wavelength, an electron may be given the required amount of energy to come out. Experiments, however, show the existence of threshold wavelength.

***Example 42.3***

*A point source of monochromatic light of* 1·0 mW *is placed at a distance of* 5·0 m *from a metal surface. Light falls perpendicularly on the surface. Assume wave theory of light to hold and also that all the light falling on the circular area with radius* $= 1{\cdot}0 \times 10^{-9}$ m *(which is few times the diameter of an atom) is absorbed by a single electron on the surface. Calculate the time required by the electron to receive sufficient energy to come out of the metal if the work function of the metal is* 2·0 eV.

***Solution*** : The energy radiated by the light source per second is 1·0 mJ. This energy is spread over the total solid angle $4\pi$. The solid angle subtended at the source by the circular area mentioned is

$$d\Omega = \frac{dA}{r^2} = \frac{\pi \times (1{\cdot}0 \times 10^{-9}\ \text{m})^2}{(5{\cdot}0\ \text{m})^2} = \frac{\pi}{25} \times 10^{-18}\ \text{sr}.$$

Hence the energy heading towards the circular area per second is

$$\frac{d\Omega}{4\pi}(1{\cdot}0\ \text{mJ}) = 10^{-20}\ \text{mJ}.$$

The time required for accumulation of 2·0 eV of energy on this circular area is

$$t = \frac{2{\cdot}0 \times 1{\cdot}6 \times 10^{-19}\ \text{J}}{10^{-20}\ \text{mJ s}^{-1}} = 3{\cdot}2 \times 10^4\ \text{s} = 8{\cdot}8\ \text{hours}.$$

The assumption of continuous absorption of energy is based on the wave theory. The above calculation shows that if this theory were correct, the first electron would be ejected not before 8·8 hours of continuous irradiation. However, in actual case, photoelectrons come out almost without any time delay after light falls on the metal.

## 42.3 MATTER WAVES

We have seen that light behaves in certain situations as waves and in certain other situations as particles. We know that electrons behave as particles in many of the situations. Can electrons also show wave nature in some suitable situations ? The answer is *yes*. A large number of experiments are now available in which electrons interfere like waves and produce fringes. Electron microscope is built on the basis of the wave properties of electrons. Protons, neutrons or even bigger particles have intrinsic wave properties. It is only a question of putting them under proper experimental situations to bring out their wave character.

If an electron behaves as waves, what is its wavelength ? The relation was proposed by Prince Louis Victor de Broglie in his PhD thesis for which he was awarded the Nobel Prize in physics for 1929. The wavelength is given by

$$\lambda = \frac{h}{p} \qquad \ldots \quad (42.4)$$

where $p$ is the momentum of the electron and $h$ is the Planck constant. This wavelength is known as the *de Broglie wavelength* of the electron. Same is the case with other particles such as a neutron, a proton, a molecule, etc. In fact, the equation also applies to light. When light shows its photon character, each photon has a momentum $p = h/\lambda$ (equation 42.1).

Can we apply Newton's laws to find the motion of an electron if the electron has both particle and wave characters. Indeed we should not rely upon Newton's laws to discuss the behaviour of electrons in all situations. While discussing the "scope of classical physics" it was mentioned that the classical mechanics of Newton fails for particles of very small size. A rough estimate was given that the classical mechanics works well for particles of linear size greater than $10^{-4}$ cm. For smaller particles, we should use *quantum mechanics* which takes into account the dual nature (wave nature and particle nature) of electrons, protons and other subatomic particles.

### *Worked Out Examples*

Use $h = 6{\cdot}63 \times 10^{-34}$ Js $= 4{\cdot}14 \times 10^{-15}$ eVs, $c = 3 \times 10^8$ m s$^{-1}$ and $m_e = 9{\cdot}1 \times 10^{-31}$ kg wherever required.

---

**1.** *How many photons are emitted per second by a* 5 mW *laser source operating at* 632·8 nm ?

***Solution*** : The energy of each photon is

$$E = \frac{hc}{\lambda} = \frac{(6{\cdot}63 \times 10^{-34}\ \text{J s}) \times (3 \times 10^8\ \text{m s}^{-1})}{(632{\cdot}8 \times 10^{-9}\ \text{m})} = 3{\cdot}14 \times 10^{-19}\ \text{J}.$$

The energy of the laser emitted per second is $5 \times 10^{-3}$ J. Thus the number of photons emitted per second

$$= \frac{5 \times 10^{-3}\ \text{J}}{3{\cdot}14 \times 10^{-19}\ \text{J}} = 1{\cdot}6 \times 10^{16}.$$

**2.** *A monochromatic source of light operating at* 200 W *emits* $4 \times 10^{20}$ *photons per second. Find the wavelength of the light.*

***Solution*** : The energy of each photon $= \frac{200\ \text{J s}^{-1}}{4 \times 10^{20}\ \text{s}^{-1}} = 5 \times 10^{-19}$ J.

$$\text{Wavelength} = \lambda = \frac{hc}{E} = \frac{(6{\cdot}63 \times 10^{-34}\ \text{J s}) \times (3 \times 10^8\ \text{m s}^{-1})}{(5 \times 10^{-19}\ \text{J})} = 4{\cdot}0 \times 10^{-7}\ \text{m} = 400\ \text{nm}.$$

**3.** *A hydrogen atom moving at a speed v absorbs a photon of wavelength* 122 nm *and stops. Find the value of v. Mass of a hydrogen atom* $= 1{\cdot}67 \times 10^{-27}$ kg.

***Solution*** : The linear momentum of the photon

$$= \frac{h}{\lambda} = \frac{6{\cdot}63 \times 10^{-34}\ \text{Js}}{122 \times 10^{-9}\ \text{m}} = 5{\cdot}43 \times 10^{-27}\ \text{kg m s}^{-1}.$$

As the photon is absorbed and the atom stops, the total final momentum is zero. From conservation of linear momentum, the initial momentum must be zero. The atom should move opposite to the direction of motion of the photon and they should have the same magnitudes of linear momentum. Thus,

$$(1{\cdot}67 \times 10^{-27}\ \text{kg})\ v = 5{\cdot}43 \times 10^{-27}\ \text{kg m s}^{-1}$$

or, $$v = \frac{5{\cdot}43 \times 10^{-27}}{1{\cdot}67 \times 10^{-27}}\ \text{m s}^{-1} = 3{\cdot}25\ \text{m s}^{-1}.$$

**4.** *A parallel beam of monochromatic light of wavelength* 500 nm *is incident normally on a perfectly absorbing surface. The power through any cross-section of the beam is* 10 W. *Find (a) the number of photons absorbed per second by the surface and (b) the force exerted by the light beam on the surface.*

***Solution*** :

(a) The energy of each photon is

$$E = \frac{hc}{\lambda} = \frac{(4{\cdot}14 \times 10^{-15}\ \text{eVs}) \times (3 \times 10^{8}\ \text{m s}^{-1})}{500\ \text{nm}}$$

$$= \frac{1242\ \text{eVn m}}{500\ \text{nm}} = 2{\cdot}48\ \text{eV}.$$

In one second, 10 J of energy passes through any cross section of the beam. Thus, the number of photons crossing a cross section is

$$n = \frac{10\ \text{J}}{2{\cdot}48\ \text{eV}} = 2{\cdot}52 \times 10^{19}.$$

This is also the number of photons falling on the surface per second and being absorbed.

(b) The linear momentum of each photon is

$$p = \frac{h}{\lambda} = \frac{h\nu}{c}.$$

The total momentum of all the photons falling per second on the surface is

$$= \frac{nh\nu}{c} = \frac{10\ \text{J}}{c} = \frac{10\ \text{J}}{3 \times 10^{8}\ \text{m s}^{-1}} = 3{\cdot}33 \times 10^{-8}\ \text{Ns}.$$

As the photons are completely absorbed by the surface, this much momentum is transferred to the surface per second. The rate of change of the momentum of the surface, i.e., the force on it is

$$F = \frac{dp}{dt} = \frac{3{\cdot}33 \times 10^{-8}\ \text{Ns}}{1\ \text{s}} = 3{\cdot}33 \times 10^{-8}\ \text{N}.$$

**5.** *Figure (42-W1) shows a small, plane strip suspended from a fixed support through a string of length l. A continuous beam of monochromatic light is incident horizontally on the strip and is completely absorbed. The energy falling on the strip per unit time is W. (a) Find the deflection of the string from the vertical if the mirror stays in equilibrium. (b) If the strip is deflected slightly from its equilibrium position in the plane of the figure, what will be the time period of the resulting oscillations* ?

***Solution*** :

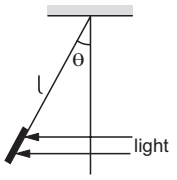



Figure 42-W1

(a) The linear momentum of the light falling per unit time on the strip is *W*/*c*. As the light is incident on the strip, its momentum is absorbed by the mirror. The change in momentum imparted to the strip per unit time is thus *W*/*c*. This is equal to the force on the strip by the light beam. In equilibrium, the force by the light beam, the weight of the strip and the force due to tension add to zero. If the string makes an angle θ with the vertical,

$$T\cos\theta = mg$$

and $$T\sin\theta = W/c.$$

Thus, $$\tan\theta = \frac{W}{mgc}.$$

(b) In equilibrium, the tension is

$$T = \left[ (mg)^2 + \left(\frac{W}{c}\right)^2 \right]^{1/2}$$

or, $$\frac{T}{m} = \left[ g^2 + \left(\frac{W}{mc}\right)^2 \right]^{1/2}.$$

This plays the role of effective *g*. The time period of small oscillations is

$$t = 2\pi\sqrt{\frac{l}{T/m}} = 2\pi \frac{\sqrt{l}}{\left[ g^2 + \left(\frac{W}{mc}\right)^2 \right]^{\frac{1}{4}}}.$$

**6.** *A point source of light is placed at the centre of curvature of a hemispherical surface. The radius of curvature is r and the inner surface is completely reflecting. Find the force on the hemisphere due to the light falling on it if the source emits a power W.*

***Solution*** :

The energy emitted by the source per unit time, i.e., *W* falls on an area $4\pi r^2$ at a distance *r* in unit time. Thus, the energy falling per unit area per unit time is $W/(4\pi r^2)$. Consider a small area *dA* at the point *P* of the hemisphere (figure 42-W2). The energy falling per

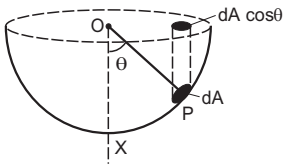



Figure 42-W2

unit time on it is $\frac{W\,dA}{4\pi r^2}$. The corresponding momentum incident on this area per unit time is $\frac{W\,dA}{4\pi r^2 c}$. As the light is reflected back, the change in momentum per unit time, i.e., the force on $dA$ is

$$dF = \frac{2W\,dA}{4\pi r^2 c}.$$

Suppose the radius $OP$ through the area $dA$ makes an angle $\theta$ with the symmetry axis $OX$. The force on $dA$ is along this radius. By symmetry, the resultant force on the hemisphere is along $OX$. The component of $dF$ along $OX$ is

$$dF\cos\theta = \frac{2W\,dA}{4\pi r^2 c}\cos\theta.$$

If we project the area $dA$ on the plane containing the rim, the projection is $dA\cos\theta$. Thus, the component of $dF$ along $OX$ is,

$$dF\cos\theta = \frac{2\ W}{4\pi r^2 c}(\text{projection of } dA).$$

The net force along $OX$ is

$$F = \frac{2\ W}{4\pi r^2 c}\left(\sum \text{ projection of } dA\right).$$

When all the small areas $dA$ are projected, we get the area enclosed by the rim which is $\pi r^2$. Thus,

$$F = \frac{2\ W}{4\pi r^2 c}\times \pi r^2 = \frac{W}{2c}.$$

**7.** *A perfectly reflecting solid sphere of radius r is kept in the path of a parallel beam of light of large aperture. If the beam carries an intensity I, find the force exerted by the beam on the sphere.*

***Solution*** **:**

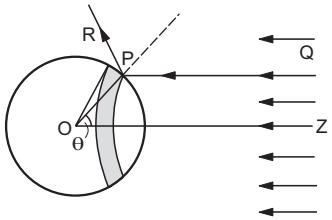



Figure 42-W3

Let $O$ be the centre of the sphere and $OZ$ be the line opposite to the incident beam (figure 42-W3). Consider a radius $OP$ of the sphere making an angle $\theta$ with $OZ$. Rotate this radius about $OZ$ to get a circle on the sphere. Change $\theta$ to $\theta + d\theta$ and rotate the radius about $OZ$ to get another circle on the sphere. The part of the sphere between these circles is a ring of area $2\pi r^2 \sin\theta\, d\theta$. Consider a small part $\Delta A$ of this ring at $P$. Energy of the light falling on this part in time $\Delta t$ is

$$\Delta U = I\Delta t(\Delta A\cos\theta).$$

The momentum of this light falling on $\Delta A$ is $\Delta U/c$ along $QP$. The light is reflected by the sphere along $PR$. The change in momentum is

$$\Delta p = 2\,\frac{\Delta U}{c}\cos\theta = \frac{2}{c}\,I\,\Delta t\,(\Delta A\cos^2\theta)$$

along the inward normal. The force on $\Delta A$ due to the light falling on it, is

$$\frac{\Delta p}{\Delta t} = \frac{2}{c}\,I\,\Delta A\cos^2\theta.$$

This force is along $PO$. The resultant force on the ring as well as on the sphere is along $ZO$ by symmetry. The component of the force on $\Delta A$, along $ZO$ is

$$\frac{\Delta p}{\Delta t}\cos\theta = \frac{2}{c}\,I\,\Delta A\cos^3\theta.$$

The force acting on the ring is

$$dF = \frac{2}{c}\,I\,(2\pi r^2\sin\theta\, d\theta)\cos^3\theta.$$

The force on the entire sphere is

$$F = \int_0^{\pi/2} \frac{4\pi r^2 I}{c}\cos^3\theta\sin\theta\, d\theta$$

$$= -\int_{\theta=0}^{\pi/2} \frac{4\pi r^2 I}{c}\cos^3\theta\, d(\cos\theta)$$

$$= -\frac{4\pi r^2 I}{c}\left[\frac{\cos^4\theta}{4}\right]_0^{\pi/2} = \frac{\pi r^2 I}{c}.$$

Note that integration is done only for the hemisphere that faces the incident beam.

**8.** *Find the threshold wavelengths for photoelectric effect from a copper surface, a sodium surface and a cesium surface. The work functions of these metals are* 4·5 eV, 2·3 eV *and* 1·9 eV *respectively.*

***Solution*** **:** If $\lambda_0$ be the threshold wavelength and $\varphi$ be the work function,

$$\lambda_0 = \frac{hc}{\varphi} = \frac{1242\text{ eV nm}}{\varphi}.$$

For copper, $\lambda_0 = \frac{1242\text{ eV nm}}{4{\cdot}5\text{ eV}} = 276\text{ nm}.$

For sodium, $\lambda_0 = \frac{1242\text{ eV nm}}{2{\cdot}3\text{ eV}} = 540\text{ nm}.$

For cesium, $\lambda_0 = \dfrac{1242 \text{ eV nm}}{1{\cdot}9 \text{ eV}} = 654 \text{ nm}$.

**9.** *Ultraviolet light of wavelength* 280 nm *is used in an experiment on photoelectric effect with lithium* ($\varphi = 2{\cdot}5$ eV) *cathode. Find (a) the maximum kinetic energy of the photoelectrons and (b) the stopping potential.*

***Solution*** **:**

(a) The maximum kinetic energy is

$$K_{max} = \frac{hc}{\lambda} - \varphi$$

$$= \frac{1242 \text{ eV nm}}{280 \text{ nm}} - 2{\cdot}5 \text{ eV}$$

$$= 4{\cdot}4 \text{ eV} - 2{\cdot}5 \text{ eV} = 1{\cdot}9 \text{ eV}.$$

(b) Stopping potential $V$ is given by

$$eV = K_{max}$$

or, $$V = \frac{K_{max}}{e} = \frac{1{\cdot}9 \text{ eV}}{e} = 1{\cdot}9 \text{ V}.$$

**10.** *In a photoelectric experiment, it was found that the stopping potential decreases from* 1·85 V *to* 0·82 V *as the wavelength of the incident light is varied from* 300 nm *to* 400 nm. *Calculate the value of the Planck constant from these data.*

***Solution*** **:**

The maximum kinetic energy of a photoelectron is

$$K_{max} = \frac{hc}{\lambda} - \varphi$$

and the stopping potential is

$$V = \frac{K_{max}}{e} = \frac{hc}{\lambda e} - \frac{\varphi}{e}.$$

If $V_1, V_2$ are the stopping potentials at wavelengths $\lambda_1$ and $\lambda_2$ respectively,

$$V_1 = \frac{hc}{\lambda_1 e} - \frac{\varphi}{e}$$

and $$V_2 = \frac{hc}{\lambda_2 e} - \frac{\varphi}{e}.$$

This gives, $$V_1 - V_2 = \frac{hc}{e}\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)$$

or, $$h = \frac{e(V_1 - V_2)}{c\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)}$$

$$= \frac{e(1{\cdot}85 \text{ V} - 0{\cdot}82 \text{ V})}{c\left(\frac{1}{300 \times 10^{-9} \text{ m}} - \frac{1}{400 \times 10^{-9} \text{ m}}\right)}$$

$$= \frac{1{\cdot}03 \text{ eV}}{(3 \times 10^8 \text{ m s}^{-1})\left(\frac{1}{12} \times 10^7 \text{ m}^{-1}\right)}$$

$$= 4{\cdot}12 \times 10^{-15} \text{ eVs}.$$

**11.** *A beam of* 450 nm *light is incident on a metal having work function* 2·0 eV *and placed in a magnetic field B. The most energetic electrons emitted perpendicular to the field are bent in circular arcs of radius* 20 cm. *Find the value of B.*

***Solution*** **:** The kinetic energy of the most energetic electrons is

$$K = \frac{hc}{\lambda} - \varphi$$

$$= \frac{1242 \text{ eV nm}}{450 \text{ nm}} - 2{\cdot}0 \text{ eV}$$

$$= 0{\cdot}76 \text{ eV} = 1{\cdot}2 \times 10^{-19} \text{ J}.$$

The linear momentum $= mv = \sqrt{2mK}$

$$= \sqrt{2 \times (9{\cdot}1 \times 10^{-31} \text{ kg}) \times (1{\cdot}2 \times 10^{-19} \text{ J})}$$

$$= 4{\cdot}67 \times 10^{-25} \text{ kgms}^{-1}.$$

When a charged particle is sent perpendicular to a magnetic field, it goes along a circle of radius

$$r = \frac{mv}{qB}.$$

Thus, $$0{\cdot}20 \text{ m} = \frac{4{\cdot}67 \times 10^{-25} \text{ kg m s}^{-1}}{(1{\cdot}6 \times 10^{-19} \text{ C}) \times B}$$

or, $$B = \frac{4{\cdot}67 \times 10^{-25} \text{ kg m s}^{-1}}{(1{\cdot}6 \times 10^{-19} \text{ C}) \times (0{\cdot}20 \text{ m})} = 1{\cdot}46 \times 10^{-5} \text{ T}.$$

**12.** *A monochromatic light of wavelength* $\lambda$ *is incident on an isolated metallic sphere of radius a. The threshold wavelength is* $\lambda_0$ *which is larger than* $\lambda$. *Find the number of photoelectrons emitted before the emission of photoelectrons will stop.*

***Solution*** **:** As the metallic sphere is isolated, it becomes positively charged when electrons are ejected from it. There is an extra attractive force on the photoelectrons. If the potential of the sphere is raised to $V$, the electron should have a minimum energy $\varphi + eV$ to be able to come out. Thus, emission of photoelectrons will stop when

$$\frac{hc}{\lambda} = \varphi + eV$$

$$= \frac{hc}{\lambda_0} + eV$$

or, $$V = \frac{hc}{e}\left(\frac{1}{\lambda} - \frac{1}{\lambda_0}\right).$$

The charge on the sphere needed to take its potential to $V$ is

$$Q = (4\pi\varepsilon_0 a)V.$$

The number of electrons emitted is, therefore,

$$n = \frac{Q}{e} = \frac{4\pi\varepsilon_0 aV}{e}$$

$$= \frac{4\pi\varepsilon_0 ahc}{e^2}\left(\frac{1}{\lambda} - \frac{1}{\lambda_0}\right).$$

**13.** *Light described at a place by the equation* $E = (100\ \text{V m}^{-1})\ [\sin (5 \times 10^{15}\ \text{s}^{-1})\ t + \sin (8 \times 10^{15}\ \text{s}^{-1})t]$ *falls on a metal surface having work function* 2·0 eV. *Calculate the maximum kinetic energy of the photoelectrons.*

***Solution*** : The light contains two different frequencies. The one with larger frequency will cause photoelectrons with largest kinetic energy. This larger frequency is

$$\nu = \frac{\omega}{2\pi} = \frac{8 \times 10^{15}\ \text{s}^{-1}}{2\pi}.$$

The maximum kinetic energy of the photoelectrons is

$$K_{\max} = h\nu - \varphi$$

$$= (4{\cdot}14 \times 10^{-15}\ \text{eVs}) \times \left(\frac{8 \times 10^{15}}{2\pi}\ \text{s}^{-1}\right) - 2{\cdot}0\ \text{eV}$$

$$= 5{\cdot}27\ \text{eV} - 2{\cdot}0\ \text{eV} = 3{\cdot}27\ \text{eV}.$$

□

## QUESTIONS FOR SHORT ANSWER

**1.** Can we find the mass of a photon by the definition $p = mv$ ?

**2.** Is it always true that for two sources of equal intensity, the number of photons emitted in a given time are equal ?

**3.** What is the speed of a photon with respect to another photon if (a) the two photons are going in the same direction and (b) they are going in opposite directions ?

**4.** Can a photon be deflected by an electric field ? By a magnetic field ?

**5.** A hot body is placed in a closed room maintained at a lower temperature. Is the number of photons in the room increasing ?

**6.** Should the energy of a photon be called its kinetic energy or its internal energy ?

**7.** In an experiment on photoelectric effect, a photon is incident on an electron from one direction and the photoelectron is emitted almost in the opposite direction. Does this violate conservation of momentum ?

**8.** It is found that yellow light does not eject photoelectrons from a metal. Is it advisable to try with orange light ? With green light ?

**9.** It is found that photosynthesis starts in certain plants when exposed to the sunlight but it does not start if the plant is exposed only to infrared light. Explain.

**10.** The threshold wavelength of a metal is $\lambda_0$. Light of wavelength slightly less than $\lambda_0$ is incident on an insulated plate made of this metal. It is found that photoelectrons are emitted for sometime and after that the emission stops. Explain.

**11.** Is $p = E/c$ valid for electrons ?

**12.** Consider the de Broglie wavelength of an electron and a proton. Which wavelength is smaller if the two particles have (a) the same speed (b) the same momentum (c) the same energy ?

**13.** If an electron has a wavelength, does it also have a colour ?

## OBJECTIVE I

**1.** Planck constant has the same dimensions as
(a) force × time (b) force × distance
(c) force × speed (d) force × distance × time.

**2.** Two photons having
(a) equal wavelengths have equal linear momenta
(b) equal energies have equal linear momenta
(c) equal frequencies have equal linear momenta
(d) equal linear momenta have equal wavelengths.

**3.** Let $p$ and $E$ denote the linear momentum and energy of a photon. If the wavelength is decreased,
(a) both $p$ and $E$ increase
(b) $p$ increases and $E$ decreases
(c) $p$ decreases and $E$ increases
(d) both $p$ and $E$ decrease.

**4.** Let $n_r$ and $n_b$ be respectively the number of photons emitted by a red bulb and a blue bulb of equal power in a given time.
(a) $n_r = n_b$ (b) $n_r < n_b$ (c) $n_r > n_b$
(d) The information is insufficient to get a relation between $n_r$ and $n_b$.

**5.** The equation $E = pc$ is valid
(a) for an electron as well as for a photon
(b) for an electron but not for a photon
(c) for a photon but not for an electron
(d) neither for an electron nor for a photon.

**6.** The work function of a metal is $h\nu_0$. Light of frequency $\nu$ falls on this metal. The photoelectric effect will take place only if
(a) $\nu \geq \nu_0$ (b) $\nu > 2\nu_0$ (c) $\nu < \nu_0$ (d) $\nu < \nu_0/2$.

**7.** Light of wavelength $\lambda$ falls on a metal having work function $hc/\lambda_0$. Photoelectric effect will take place only if
(a) $\lambda \geq \lambda_0$ (b) $\lambda \geq 2\lambda_0$ (c) $\lambda \leq \lambda_0$ (d) $\lambda < \lambda_0/2$.

**8.** When stopping potential is applied in an experiment on photoelectric effect, no photocurrent is observed. This means that
(a) the emission of photoelectrons is stopped
(b) the photoelectrons are emitted but are re-absorbed by the emitter metal
(c) the photoelectrons are accumulated near the collector plate
(d) the photoelectrons are dispersed from the sides of the apparatus.

**9.** If the frequency of light in a photoelectric experiment is doubled, the stopping potential will
(a) be doubled (b) be halved
(c) become more than double
(d) become less than double.

**10.** The frequency and intensity of a light source are both doubled. Consider the following statements.
(A) The saturation photocurrent remains almost the same.
(B) The maximum kinetic energy of the photoelectrons is doubled.
(a) Both A and B are true. (b) A is true but B is false.
(c) A is false but B is true. (d) Both A and B are false.

**11.** A point source of light is used in a photoelectric effect. If the source is removed farther from the emitting metal, the stopping potential
(a) will increase (b) will decrease
(c) will remain constant
(d) will either increase or decrease.

**12.** A point source causes photoelectric effect from a small metal plate. Which of the following curves may represent the saturation photocurrent as a function of the distance between the source and the metal ?

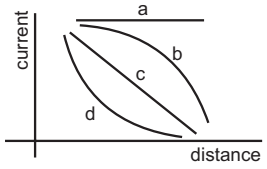



Figure 42-Q1

**13.** A nonmonochromatic light is used in an experiment on photoelectric effect. The stopping potential
(a) is related to the mean wavelength
(b) is related to the longest wavelength
(c) is related to the shortest wavelength
(d) is not related to the wavelength.

**14.** A proton and an electron are accelerated by the same potential difference. Let $\lambda_e$ and $\lambda_p$ denote the de Broglie wavelengths of the electron and the proton respectively.
(a) $\lambda_e = \lambda_p$ (b) $\lambda_e < \lambda_p$ (c) $\lambda_e > \lambda_p$
(d) The relation between $\lambda_e$ and $\lambda_p$ depends on the accelerating potential difference.

## OBJECTIVE II

**1.** When the intensity of a light source is increased,
(a) the number of photons emitted by the source in unit time increases
(b) the total energy of the photons emitted per unit time increases
(c) more energetic photons are emitted
(d) faster photons are emitted.

**2.** Photoelectric effect supports quantum nature of light because
(a) there is a minimum frequency below which no photoelectrons are emitted
(b) the maximum kinetic energy of photoelectrons depends only on the frequency of light and not on its intensity
(c) even when the metal surface is faintly illuminated the photoelectrons leave the surface immediately
(d) electric charge of the photoelectrons is quantized.

**3.** A photon of energy $h\nu$ is absorbed by a free electron of a metal having work function $\varphi < h\nu$.
(a) The electron is sure to come out.
(b) The electron is sure to come out with a kinetic energy $h\nu - \varphi$.
(c) Either the electron does not come out or it comes out with a kinetic energy $h\nu - \varphi$.
(d) It may come out with a kinetic energy less than $h\nu - \varphi$.

**4.** If the wavelength of light in an experiment on photoelectric effect is doubled,
(a) the photoelectric emission will not take place
(b) the photoelectric emission may or may not take place
(c) the stopping potential will increase
(d) the stopping potential will decrease.

**5.** The photocurrent in an experiment on photoelectric effect increases if
(a) the intensity of the source is increased
(b) the exposure time is increased
(c) the intensity of the source is decreased
(d) the exposure time is decreased.

**6.** The collector plate in an experiment on photoelectric effect is kept vertically above the emitter plate. Light source is put on and a saturation photocurrent is recorded. An electric field is switched on which has a vertically downward direction.
(a) The photocurrent will increase.
(b) The kinetic energy of the electrons will increase.
(c) The stopping potential will decrease.
(d) The threshold wavelength will increase.

**7.** In which of the following situations the heavier of the two particles has smaller de Broglie wavelength ? The two particles
(a) move with the same speed
(b) move with the same linear momentum
(c) move with the same kinetic energy
(d) have fallen through the same height.

## EXERCISES

Use $h = 6{\cdot}63 \times 10^{-34}$ J s $= 4{\cdot}14 \times 10^{-15}$ eVs, $c = 3 \times 10^{8}$ m s$^{-1}$ and $m_e = 9{\cdot}1 \times 10^{-31}$ kg wherever needed.

**1.** Visible light has wavelengths in the range of 400 nm to 780 nm. Calculate the range of energy of the photons of visible light.

**2.** Calculate the momentum of a photon of light of wavelength 500 nm.

**3.** An atom absorbs a photon of wavelength 500 nm and emits another photon of wavelength 700 nm. Find the net energy absorbed by the atom in the process.

**4.** Calculate the number of photons emitted per second by a 10 W sodium vapour lamp. Assume that 60% of the consumed energy is converted into light. Wavelength of sodium light = 590 nm.

**5.** When the sun is directly overhead, the surface of the earth receives $1{\cdot}4 \times 10^{3}$ W m$^{-2}$ of sunlight. Assume that the light is monochromatic with average wavelength 500 nm and that no light is absorbed in between the sun and the earth's surface. The distance between the sun and the earth is $1{\cdot}5 \times 10^{11}$ m. (a) Calculate the number of photons falling per second on each square metre of earth's surface directly below the sun. (b) How many photons are there in each cubic metre near the earth's surface at any instant ? (c) How many photons does the sun emit per second ?

**6.** A parallel beam of monochromatic light of wavelength 663 nm is incident on a totally reflecting plane mirror. The angle of incidence is 60° and the number of photons striking the mirror per second is $1{\cdot}0 \times 10^{19}$. Calculate the force exerted by the light beam on the mirror.

**7.** A beam of white light is incident normally on a plane surface absorbing 70% of the light and reflecting the rest. If the incident beam carries 10 W of power, find the force exerted by it on the surface.

**8.** A totally reflecting, small plane mirror placed horizontally faces a parallel beam of light as shown in figure (42-E1). The mass of the mirror is 20 g. Assume that there is no absorption in the lens and that 30% of the light emitted by the source goes through the lens. Find the power of the source needed to support the weight of the mirror. Take $g = 10$ m s$^{-2}$.

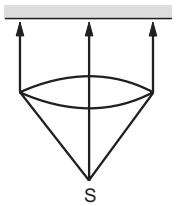



Figure 42-E1

**9.** A 100 W light bulb is placed at the centre of a spherical chamber of radius 20 cm. Assume that 60% of the energy supplied to the bulb is converted into light and that the surface of the chamber is perfectly absorbing. Find the pressure exerted by the light on the surface of the chamber.

**10.** A sphere of radius 1·00 cm is placed in the path of a parallel beam of light of large aperture. The intensity of the light is 0·50 W cm$^{-2}$. If the sphere completely absorbs the radiation falling on it, find the force exerted by the light beam on the sphere.

**11.** Consider the situation described in the previous problem. Show that the force on the sphere due to the light falling on it is the same even if the sphere is not perfectly absorbing.

**12.** Show that it is not possible for a photon to be completely absorbed by a free electron.

**13.** Two neutral particles are kept 1 m apart. Suppose by some mechanism some charge is transferred from one particle to the other and the electric potential energy lost is completely converted into a photon. Calculate the longest and the next smaller wavelength of the photon possible.

**14.** Find the maximum kinetic energy of the photoelectrons ejected when light of wavelength 350 nm is incident on a cesium surface. Work function of cesium = 1·9 eV.

**15.** The work function of a metal is $2{\cdot}5 \times 10^{-19}$ J. (a) Find the threshold frequency for photoelectric emission. (b) If the metal is exposed to a light beam of frequency $6{\cdot}0 \times 10^{14}$ Hz, what will be the stopping potential ?

**16.** The work function of a photoelectric material is 4·0 eV. (a) What is the threshold wavelength ? (b) Find the wavelength of light for which the stopping potential is 2·5 V.

**17.** Find the maximum magnitude of the linear momentum of a photoelectron emitted when light of wavelength 400 nm falls on a metal having work function 2·5 eV.

**18.** When a metal plate is exposed to a monochromatic beam of light of wavelength 400 nm, a negative potential of 1·1 V is needed to stop the photocurrent. Find the threshold wavelength for the metal.

**19.** In an experiment on photoelectric effect, the stopping potential is measured for monochromatic light beams corresponding to different wavelengths. The data collected are as follows:

| wavelength (nm): | 350 | 400 | 450 | 500 | 550 |
|---|---|---|---|---|---|
| stopping potential(V): | 1·45 | 1·00 | 0·66 | 0·38 | 0·16 |

Plot the stopping potential against inverse of wavelength (1/λ) on a graph paper and find (a) the Planck constant, (b) the work function of the emitter and (c) the threshold wavelength.

**20.** The electric field associated with a monochromatic beam becomes zero $1{\cdot}2 \times 10^{15}$ times per second. Find the maximum kinetic energy of the photoelectrons when this light falls on a metal surface whose work function is 2·0 eV.

**21.** The electric field associated with a light wave is given by

$$E = E_0 \sin [(1{\cdot}57 \times 10^{7} \text{ m}^{-1})(x - ct)].$$

Find the stopping potential when this light is used in

an experiment on photoelectric effect with the emitter having work function 1·9 eV.

**22.** The electric field at a point associated with a light wave is $E = (100\ \text{Vm}^{-1})\ \sin\ [(3{\cdot}0 \times 10^{15}\ \text{s}^{-1})t]\ \sin\ [(6{\cdot}0 \times 10^{15}\ \text{s}^{-1})t]$. If this light falls on a metal surface having a work function of 2·0 eV, what will be the maximum kinetic energy of the photoelectrons ?

**23.** A monochromatic light source of intensity 5 mW emits $8 \times 10^{15}$ photons per second. This light ejects photoelectrons from a metal surface. The stopping potential for this setup is 2·0 V. Calculate the work function of the metal.

**24.** Figure (42-E2) is the plot of the stopping potential versus the frequency of the light used in an experiment on photoelectric effect. Find (a) the ratio $h/e$ and (b) the work function.

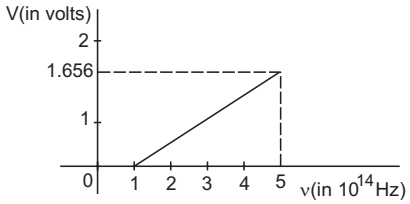



Figure 42-E2

**25.** A photographic film is coated with a silver bromide layer. When light falls on this film, silver bromide molecules dissociate and the film records the light there. A minimum of 0·6 eV is needed to dissociate a silver bromide molecule. Find the maximum wavelength of light that can be recorded by the film.

**26.** In an experiment on photoelectric effect, light of wavelength 400 nm is incident on a cesium plate at the rate of 5·0 W. The potential of the collector plate is made sufficiently positive with respect to the emitter so that the current reaches its saturation value. Assuming that on the average one out of every $10^6$ photons is able to eject a photoelectron, find the photocurrent in the circuit.

**27.** A silver ball of radius 4·8 cm is suspended by a thread in a vacuum chamber. Ultraviolet light of wavelength 200 nm is incident on the ball for some time during which a total light energy of $1{\cdot}0 \times 10^{-7}$ J falls on the surface. Assuming that on the average one photon out of every ten thousand is able to eject a photoelectron, find the electric potential at the surface of the ball assuming zero potential at infinity. What is the potential at the centre of the ball ?

**28.** In an experiment on photoelectric effect, the emitter and the collector plates are placed at a separation of 10 cm and are connected through an ammeter without any cell (figure 42-E3). A magnetic field $B$ exists parallel to the plates. The work function of the emitter is 2·39 eV and the light incident on it has wavelengths between 400 nm and 600 nm. Find the minimum value of $B$ for which the current registered by the ammeter is zero. Neglect any effect of space charge.

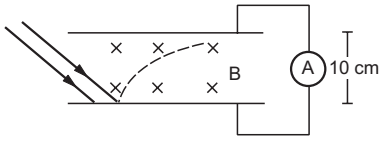



Figure 42-E3

**29.** In the arrangement shown in figure (42-E4), $y = 1{\cdot}0$ mm, $d = 0{\cdot}24$ mm and $D = 1{\cdot}2$ m. The work function of the material of the emitter is 2·2 eV. Find the stopping potential $V$ needed to stop the photocurrent.

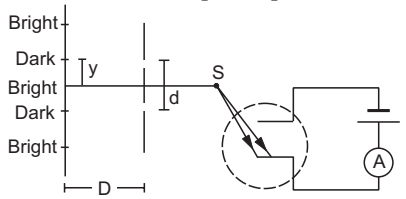



Figure 42-E4

**30.** In a photoelectric experiment, the collector plate is at 2·0 V with respect to the emitter plate made of copper ($\varphi = 4{\cdot}5$ eV). The emitter is illuminated by a source of monochromatic light of wavelength 200 nm. Find the minimum and maximum kinetic energy of the photoelectrons reaching the collector.

**31.** A small piece of cesium metal ($\varphi = 1{\cdot}9$ eV) is kept at a distance of 20 cm from a large metal plate having a charge density of $1{\cdot}0 \times 10^{-9}\ \text{C m}^{-2}$ on the surface facing the cesium piece. A monochromatic light of wavelength 400 nm is incident on the cesium piece. Find the minimum and the maximum kinetic energy of the photoelectrons reaching the large metal plate. Neglect any change in electric field due to the small piece of cesium present.

**32.** Consider the situation of the previous problem. Consider the fastest electron emitted parallel to the large metal plate. Find the displacement of this electron parallel to its initial velocity before it strikes the large metal plate.

**33.** A horizontal cesium plate ($\varphi = 1{\cdot}9$ eV) is moved vertically downward at a constant speed $v$ in a room full of radiation of wavelength 250 nm and above. What should be the minimum value of $v$ so that the vertically upward component of velocity is nonpositive for each photoelectron ?

**34.** A small metal plate (work function $\varphi$) is kept at a distance $d$ from a singly ionized, fixed ion. A monochromatic light beam is incident on the metal plate and photoelectrons are emitted. Find the maximum wavelength of the light beam so that some of the photoelectrons may go round the ion along a circle.

**35.** A light beam of wavelength 400 nm is incident on a metal plate of work function 2·2 eV. (a) A particular electron absorbs a photon and makes two collisions before coming out of the metal. Assuming that 10% of the extra energy is lost to the metal in each collision, find the kinetic energy of this electron as it comes out of the metal. (b) Under the same assumptions, find the maximum number of collisions the electron can suffer before it becomes unable to come out of the metal.

□

## ANSWERS

### OBJECTIVE I

1. (d) 2. (d) 3. (a) 4. (c) 5. (c) 6. (a)
7. (c) 8. (b) 9. (c) 10. (b) 11. (c) 12. (d)
13. (c) 14. (c)

### OBJECTIVE II

1. (a), (b) 2. (a), (b), (c) 3. (d)
4. (b), (d) 5. (a) 6. (b)
7. (a), (c), (d)

### EXERCISES

1. $2{\cdot}56 \times 10^{-19}$ J to $5{\cdot}00 \times 10^{-19}$ J
2. $1{\cdot}33 \times 10^{-27}$ kg m s$^{-1}$
3. $1{\cdot}1 \times 10^{-19}$ J
4. $1{\cdot}77 \times 10^{19}$
5. (a) $3{\cdot}5 \times 10^{21}$ (b) $1{\cdot}2 \times 10^{13}$ (c) $9{\cdot}9 \times 10^{44}$
6. $1{\cdot}0 \times 10^{-8}$ N
7. $4{\cdot}3 \times 10^{-8}$ N
8. 100 MW
9. $4{\cdot}0 \times 10^{-7}$ Pa
10. $5{\cdot}2 \times 10^{-9}$ N
13. 860 m, 215 m
14. 1·6 eV
15. (a) $3{\cdot}8 \times 10^{14}$ Hz (b) 0·91 V
16. (a) 310 nm (b) 190 nm
17. $4{\cdot}2 \times 10^{-25}$ kg m s$^{-1}$
18. 620 nm
19. (a) $4{\cdot}2 \times 10^{-15}$ eVs (b) 2·15 eV (c) 585 nm
20. 0·48 eV
21. 1·2 V
22. 3·93 eV
23. 1·9 eV
24. (a) $4{\cdot}14 \times 10^{-15}$ eVs (b) 0·414 eV
25. 2070 nm
26. 1·6 μA
27. 0·3 V in each case
28. $2{\cdot}85 \times 10^{-5}$ T
29. 0·9 V
30. 2·0 eV, 3·7 eV
31. 22·6 eV, 23·8 eV
32. 9·2 cm
33. $1{\cdot}04 \times 10^{6}$ m s$^{-1}$
34. $\dfrac{8\pi\varepsilon_0 dhc}{e^2 + 8\pi\varepsilon_0 \varphi d}$
35. (a) 0·31 eV (b) 4

□

CHAPTER 43

# BOHR'S MODEL AND PHYSICS OF THE ATOM

## 43.1 EARLY ATOMIC MODELS

The idea that all matter is made of very small indivisible particles is very old. It has taken a long time, intelligent reasoning and classic experiments to cover the journey from this idea to the present day atomic models.

We can start our discussion with the mention of English scientist Robert Boyle (1627–1691) who studied the expansion and compression of air. The fact that air can be compressed or expanded, tells that air is made of tiny particles with lot of empty space between the particles. When air is compressed, these particles get closer to each other, reducing the empty space. We mention Robert Boyle here, because, with him atomism entered a new phase, from mere reasoning to experimental observations. The smallest unit of an element, which carries all the properties of the element is called an atom. Experiments on discharge tube, measurement of $e/m$ by Thomson, etc., established the existence of negatively charged electrons in the atoms. And then started the search for the structure of the positive charge inside an atom because the matter as a whole is electrically neutral.

### Thomson's Model of the Atom

Thomson suggested in 1898 that the atom is a positively charged solid sphere and electrons are embedded in it in sufficient number so as to make the atom electrically neutral. One can compare Thomson's atom to a birthday cake in which cherries are embedded. This model was quite attractive as it could explain several observations available at that time. It could explain why only negatively charged particles are emitted when a metal is heated and never the positively charged particles. It could also explain the formation of ions and ionic compounds of chemistry.

### Lenard's Suggestion

Lenard had noted that cathode rays could pass through materials of small thickness almost undeviated. If the atoms were solid spheres, most of the electrons in the cathode rays would hit them and would not be able to go ahead in the forward direction. Lenard, therefore, suggested in 1903 that the atom must have a lot of empty space in it. He proposed that the atom is made of electrons and similar tiny particles carrying positive charge. But then, the question was, why on heating a metal, these tiny positively charged particles were not ejected ?

### Rutherford's Model of the Atom

Thomson's model and Lenard's model, both had certain advantages and disadvantages. Thomson's model made the positive charge immovable by assuming it to be spread over the total volume of the atom. On the other hand, electrons were tiny particles and could be ejected on heating a metal. But the almost free passage of cathode rays through an atom was not consistent with Thomson's model. For that, the atom should have a lot of empty space as suggested by Lenard. So, the positive charge should be in the form of tiny particles occupying a very small volume, yet these particles should not be able to come out on heating.

It was Ernest Rutherford who solved the problem by doing a series of experiments from 1906 to 1911 on alpha particle scattering.

In these experiments, a beam of alpha particles was bombarded on a thin gold foil and their deflections were studied (figure 43.1). Most of the alpha particles passed through the gold foil either undeviated or with a small deviation. This was expected because an alpha particle is a heavy particle and will brush aside any tiny particle coming in its way. However, some of the alpha particles were deflected by large angles.

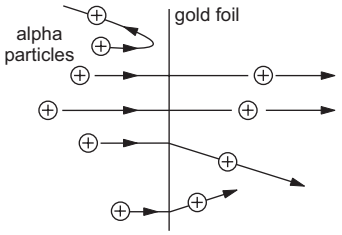



Figure 43.1

Rutherford found that some of the alpha particles, about one in 8000, were deflected by more than 90°, i.e., they were turned back by the foil.

This was interesting. When 8000 alpha particles could go through the gold atoms undeflected, why then one was forced to turn back. The alpha particle itself is about 7350 times heavier than the electron. So neither an electron, nor a similar positively charged particle could cause a large scale deflection of an alpha particle. The alpha particle must have encountered a very heavy particle in its path, a particle with mass of the order of the mass of the atom itself. Also, thousands of alpha particles go undeviated or almost undeviated. So this heavy mass in the atom should occupy a very small volume so that the atom may contain lot of empty space.

From the pattern of the scattering of alpha particles, Rutherford made quantitative analysis. He found that the heavy particle from which an alpha particle suffered large deflection, had a positive charge and virtually all the mass of the atom was concentrated in it. Its size was also estimated from the same experiment. The linear size was found to be about 10 fermi (1 fermi = 1 femtometre = $10^{-15}$ m) which was about $10^{-5}$ of the size of the linear atom. As the volume is proportional to the cube of the linear size, the volume of this positively charged particle was only about $10^{-15}$ of the volume of the atom.

Based on these observations, Rutherford proposed the model of *nuclear atom* which remains accepted to a large extent even today. According to this model, the atom contains a positively charged tiny particle at its centre called the *nucleus* of the atom. This nucleus contains almost all the mass of the atom. Outside this nucleus, there are electrons which move around it at some separation. The space between the nucleus and the electrons is empty and determines the size of the atom. The amount of the positive charge on the nucleus is exactly equal to the total amount of negative charges on all the electrons of the atom. Figure (43.2) shows schematic representations of an atom in Thomson's model and Rutherford's model.

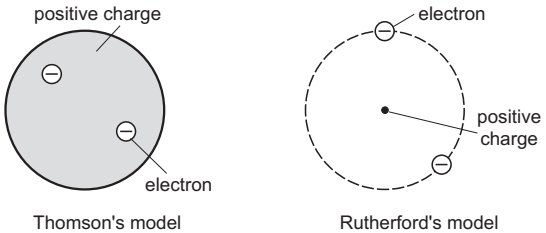



Figure 43.2

So, Rutherford's model explains the charge neutrality and the large empty space inside the atom as suggested by Lenard. It also explains why only negatively charged particles are ejected easily by an atom. It is so because the positively charged particle (nucleus) is so heavy that when an atom gets energy from heating or otherwise, this particle is hardly affected.

The movement of electrons around the nucleus was a necessary part of Rutherford's model. If the electrons were at rest, they would fall into the nucleus because of Coulomb attraction. If the electrons move in circular orbits, the Coulomb force will only change the direction of velocity providing the necessary centripetal force. This electronic motion, however, created difficulties for Rutherford's model as we shall now study.

## 43.2 HYDROGEN SPECTRA

When a material body is heated, it emits electromagnetic radiation. The radiation may consist of various components having different wavelengths. When the filament of an electric bulb is heated, it gives white light and all wavelengths in the visible range are present in the emitted radiation. If the emitted light is passed through a prism, components of different wavelengths deviate by different amounts and we get a continuous spectrum.

If hydrogen gas enclosed in a sealed tube is heated to high temperatures, it emits radiation. If this radiation is passed through a prism, components of different wavelengths are deviated by different amounts and thus we get the hydrogen spectrum. The most striking feature in this spectrum is that only some sharply defined, discrete wavelengths exist in the emitted radiation. For example, light of wavelength 656·3 nm is observed and then light of wavelength 486·1 nm is observed. Hydrogen atoms do not emit any radiation between 656·3 nm and 486·1 nm.

A hydrogen sample also emits radiation with wavelengths less than those in the visible range and also with wavelengths larger than those in the visible range. Figure (43.3) shows a schematic arrangement of the wavelengths present in a hydrogen spectrum. We see that the lines may be grouped in separate series. In each series, the separation between the consecutive wavelengths decreases as we go from higher wavelength to lower wavelength. In fact, the wavelengths in each series approach a limiting value known as the *series limit*. Thus, we have indicated the Lyman series (ultraviolet region), Balmer series

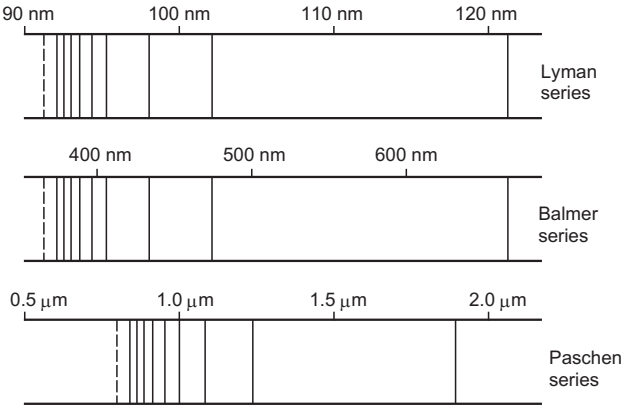



Figure 43.3

(visible region), Paschen series (infrared region), etc., in the figure.

The wavelengths nicely fit in the equation

$$\frac{1}{\lambda} = R\left(\frac{1}{n^2} - \frac{1}{m^2}\right) \quad \ldots \quad (43.1)$$

where $R \approx 1{\cdot}09737 \times 10^7\ \text{m}^{-1}$ and $n$ and $m$ are integers with $m > n$. Lyman series can be reproduced by setting $n = 1$ and varying $m$ from 2 onwards, Balmer series by setting $n = 2$ with $m > 2$, Paschen series by setting $n = 3$ with $m > 3$, etc.

It is said that John Jacob Balmer (1825–1898), a Swiss schoolteacher, was fond of playing with numbers. Once he complained to his physicist friend that he was getting bored as he had no numbers to play with. The friend gave him four wavelengths 656·3, 486·1, 434·1 and 420·2 nm of hydrogen spectrum and asked if Balmer could find a relation amongst them. And Balmer soon came out with his formula

$$\lambda = \frac{364{\cdot}56\, m^2}{m^2 - 4}, \text{ where } m = 3, 4, 5, 6$$

which was later put in more convenient form (equation 43.1) by Rydberg.

### 43.3 DIFFICULTIES WITH RUTHERFORD'S MODEL

The sharply defined, discrete wavelengths in hydrogen spectra posed a serious puzzle before physicists.

A hydrogen atom consists of an electron and a nucleus containing just a proton. The important question is why the electron does not hit the proton due to Coulomb attraction. In Rutherford's model, we assume that the electron revolves round the proton and the Coulomb force provides the necessary centripetal force to keep it moving in circular orbit. From the point of view of mechanics, a revolving electron in an atom is a satisfactory picture. But Maxwell's equations of electromagnetism show that any accelerated electron must continuously emit electromagnetic radiation. The revolving electron should, therefore, always emit radiation at all temperatures. The wavelength of the radiation should be related to the frequency of revolution. If the radiation is continuously emitted, the energy is spent and the radius of the circle should gradually decrease and the electron should finally fall into the proton. Also, the frequency of revolution changes continuously as the energy is spent, and so, the electron should emit radiation of continuously varying wavelength during the period of its motion.

The actual observations are quite different. At room temperature or below, hydrogen is very stable; it neither emits radiation nor does the electron collapse into the proton. When extra energy is supplied through heat or electric discharge, radiation is emitted, but the wavelengths are sharply defined as given by equation (43.1). These sharply defined wavelengths may be taken as the fingerprints of the element (hydrogen). Be it Calcutta, Delhi, Madras, Hyderabad, New York, London or Canberra, sun or upper atmosphere, hydrogen always emits only these fixed wavelengths. Such observations could not be explained by classical concepts and something new was about to take birth.

### 43.4 BOHR'S MODEL

In 1913, Niels Bohr, a great name in physics, suggested that the puzzle of hydrogen spectra may be solved if we make the following assumptions.

#### Bohr's Postulates

(a) The electron revolves round the nucleus in circular orbits.

(b) The orbit of the electron around the nucleus can take only some special values of radius. In these orbits of special radii, the electron does not radiate energy as expected from Maxwell's laws. These orbits are called *stationary orbits.*

(c) The energy of the atom has a definite value in a given stationary orbit. The electron can jump from one stationary orbit to other. If it jumps from an orbit of higher energy $E_2$ to an orbit of lower energy $E_1$, it emits a photon of radiation. The energy of the photon is $E_2 - E_1$. The wavelength of the emitted radiation is given by the Einstein–Planck equation

$$E_2 - E_1 = h\nu = \frac{hc}{\lambda}.$$

The electron can also absorb energy from some source and jump from a lower energy orbit to a higher energy orbit.

(d) In stationary orbits, the angular momentum $l$ of the electron about the nucleus is an integral multiple of the Planck constant $h$ divided by $2\pi$,

$$l = n\frac{h}{2\pi}.$$

This last assumption is called *Bohr's quantization rule* and the assumptions (a) to (d) are known as *Bohr's postulates*.

**Energy of a Hydrogen Atom**

Let us now use the above postulates to find the allowed energies of the atom for different allowed orbits of the electron. The theory developed is applicable to hydrogen atoms, and ions having just one electron. Thus, it is valid for $He^{+}$, $Li^{++}$, $Be^{+++}$, etc. These ions are often called *hydrogen-like ions*. Let us assume that the nucleus has a positive charge $Ze$ (i.e., there are $Z$ protons in it) and an electron moves with a constant speed $v$ along a circle of radius $r$ with the centre at the nucleus. The force acting on the electron is that due to Coulomb attraction and is equal to

$$F = \frac{Ze^2}{4\pi\varepsilon_0 r^2}.$$

The acceleration of the electron is towards the centre and has a magnitude $v^2/r$. If $m$ is the mass of the electron, from Newton's law,

$$\frac{Ze^2}{4\pi\varepsilon_0 r^2} = \frac{mv^2}{r}$$

or,
$$r = \frac{Ze^2}{4\pi\varepsilon_0 mv^2}. \quad \ldots \text{(i)}$$

Also, from Bohr's quantization rule, the angular momentum is

$$mvr = n\frac{h}{2\pi} \quad \ldots \text{(ii)}$$

where $n$ is a positive integer.

Eliminating $r$ from (i) and (ii), we get

$$v = \frac{Ze^2}{2\varepsilon_0 hn}. \quad \ldots (43.2)$$

Substituting this in (ii),

$$r = \frac{\varepsilon_0 h^2 n^2}{\pi m Ze^2}. \quad \ldots (43.3)$$

We see that the allowed radii are proportional to $n^2$. For each value of $n$, we have an allowed orbit. For $n = 1$, we have the first orbit (smallest radius), for $n = 2$, we have the second orbit and so on.

From equation (43.2), the kinetic energy of the electron in the $n$th orbit is

$$K = \frac{1}{2}mv^2 = \frac{mZ^2e^4}{8\varepsilon_0^2 h^2 n^2}. \quad \ldots (43.4)$$

The potential energy of the atom is

$$V = -\frac{Ze^2}{4\pi\varepsilon_0 r} = -\frac{mZ^2e^4}{4\varepsilon_0^2 h^2 n^2}. \quad \ldots (43.5)$$

We have taken the potential energy to be zero when the nucleus and the electron are widely separated.

The total energy of the atom is

$$E = K + V$$
$$= -\frac{mZ^2e^4}{8\varepsilon_0^2 h^2 n^2}. \quad \ldots (43.6)$$

Equations (43.2) through (43.6) give various parameters of the atom when the electron is in the $n$th orbit. The atom is also said to be in the *nth energy state* in this case. In deriving the expression for the total energy $E$, we have considered the kinetic energy of the electron and the potential energy of the electron–nucleus pair. It is assumed that the acceleration of the nucleus is negligible on account of its large mass.

**Radii of different orbits**

From equation (43.3), the radius of the smallest circle allowed to the electron is ($n = 1$)

$$r_1 = \frac{\varepsilon_0 h^2}{\pi m Ze^2}.$$

For hydrogen, $Z = 1$ and putting the values of other constants we get $r_1 = 53$ picometre (1 pm = $10^{-12}$ m) or 0·053 nm. This length is called the *Bohr radius* and is a convenient unit for measuring lengths in atomic physics. It is generally denoted by the symbol $a_0$.

The second allowed radius is $4a_0$, third is $9a_0$ and so on. In general, the radius of the $n$th orbit is

$$r_n = n^2 a_0.$$

For a hydrogen-like ion with $Z$ protons in the nucleus,

$$r_n = \frac{n^2 a_0}{Z}. \quad \ldots (43.7)$$

**Ground and excited states**

From equation (43.6), the total energy of the atom in the state $n = 1$ is

$$E_1 = -\frac{mZ^2e^4}{8\varepsilon_0^2 h^2}.$$

For hydrogen atom, $Z = 1$ and putting the values of the constants, $E_1 = -13{\cdot}6$ eV. This is the energy when the electron revolves in the smallest allowed orbit $r = a_0$, i.e., the one with radius 0·053 nm. We also see from equation (43.6) that the energy of the atom in the $n$th energy state is proportional to $\frac{1}{n^2}$. Thus,

$$E_n = \frac{E_1}{n^2} = -\frac{13{\cdot}6 \text{ eV}}{n^2}. \quad \ldots (43.8)$$

The energy in the state $n = 2$ is $E_2 = E_1/4 = -3{\cdot}4$ eV. In the state $n = 3$, it is $E_1/9 = -1{\cdot}5$ eV, etc. The lowest energy corresponds to the smallest circle. Note that the energy is negative and hence a larger magnitude means lower energy. The zero of energy corresponds to the state where the electron and the nucleus are widely separated. Figure (43.4) shows schematically the allowed orbits together with the energies of the atom. It also displays the allowed energies separately.

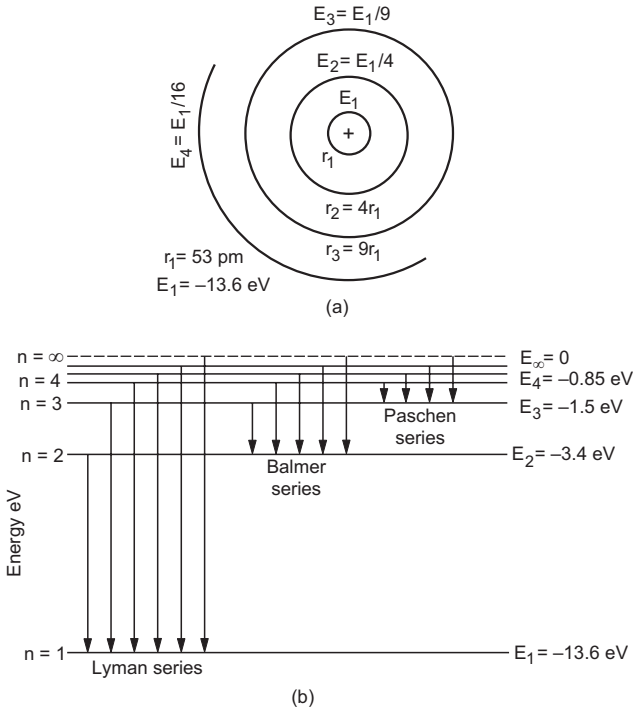



Figure 43.4

The state of an atom with the lowest energy is called its *ground state*. The states with higher energies are called *excited states*. Thus, the energy of a hydrogen atom in the ground state is –13·6 eV and in the first excited state –3·4 eV.

**Hydrogen Spectra**

We can now explain why hydrogen gas kept in a flask at room temperature does not emit radiation. This is because almost all the atoms are in the ground state and there are no orbits of lower energy to which an electron can jump. Hence, the atoms cannot emit any radiation. When energy is given in the form of heat or by electric discharge or by some other means, some of the electrons jump to the higher energy orbits $n = 2$, $n = 3$, etc. These electrons then jump back to lower energy orbits. The atoms radiate energy in the process. This explains why the atoms radiate only when they are heated or given energy in some other form.

If an electron makes a jump from the $m$th orbit to the $n$th orbit ($m > n$), the energy of the atom changes from $E_m$ to $E_n$. This extra energy $E_m - E_n$ is emitted as a photon of electromagnetic radiation. The corresponding wavelength is given by

$$\frac{1}{\lambda} = \frac{E_m - E_n}{hc}$$

$$= \frac{mZ^2e^4}{8\varepsilon_0^2 h^3 c}\left(\frac{1}{n^2} - \frac{1}{m^2}\right)$$

$$= RZ^2\left(\frac{1}{n^2} - \frac{1}{m^2}\right) \quad \ldots (43.9)$$

where $R = \frac{me^4}{8\varepsilon_0^2 h^3 c}$ is called the *Rydberg constant*. Putting the values of different constants, the Rydberg constant $R$ comes out to be $1{\cdot}0973 \times 10^7 \text{ m}^{-1}$ and equation (43.9) is in excellent agreement with the experimental formula (43.1). In terms of the Rydberg constant, the energy of the atom in the $n$th state is $E = \frac{-RhcZ^2}{n^2}$. Quite often, the energy of the atom is mentioned in unit of rydberg. An energy of 1 rydberg means –13·6 eV. It is useful to remember that $Rhc$ = 13·6 eV.

***Example 43.1***

*Calculate the energy of a* $He^+$ *ion in its first excited state.*

***Solution :***

The energy is $E_n = \frac{-RhcZ^2}{n^2} = -\frac{(13{\cdot}6 \text{ eV})Z^2}{n^2}$

For a $He^+$ ion, $Z = 2$ and for the first excited state, $n$ = 2 so that the energy of $He^+$ ion in the first excited state is –13·6 eV.

***Example 43.2***

*Calculate the wavelength of radiation emitted when* $He^+$ *makes a transition from the state* $n = 3$ *to the state* $n = 2$.

***Solution :***

The wavelength $\lambda$ is given by

$$\frac{1}{\lambda} = RZ^2\left(\frac{1}{n^2} - \frac{1}{m^2}\right)$$

$$= 4R\left(\frac{1}{4} - \frac{1}{9}\right) = \frac{5}{9}R$$

or, $$\lambda = \frac{9}{5R} = \frac{9}{5 \times 1{\cdot}09737 \times 10^7 \text{ m}^{-1}} = 164{\cdot}0 \text{ nm}.$$

**Series structure**

If a hydrogen atom makes transition from the state $n = 2$ to the state $n = 1$, the wavelength of the emitted radiation is given by

$$\frac{1}{\lambda} = R\left(1 - \frac{1}{4}\right) \text{ or, } \lambda = 121{\cdot}6 \text{ nm}.$$

If it makes transition from the state $n = \infty$ to the state $n = 1$, the wavelength emitted is given by

$$\frac{1}{\lambda} = R(1 - 0) \text{ or, } \lambda = 91{\cdot}2 \text{ nm}.$$

Thus, all the transitions ending at $n = 1$ correspond to wavelengths grouped between 121·6 nm and 91·2 nm. These lines constitute the *Lyman series*.

Similarly, transitions from higher states to $n = 2$ lead to emission of radiation with wavelengths between 656·3 nm and 365·0 nm. These wavelengths fall in the visible region and constitute the *Balmer series*. The transitions from higher states to $n = 3$ give rise to *Paschen series* with wavelengths between 1875 nm and 822 nm, and similarly for other series. This explains the grouping of wavelengths in different series as shown in figure (43.3).

**Ionization potential**

What happens if we supply more than 13·6 eV to a hydrogen atom in its ground state ? The total energy is then positive. The equations deduced above are not applicable in this case. In fact, a total energy of zero corresponds to electron and nucleus separated by an infinite distance. In this case, the electron is not bound to the nucleus and is free to move anywhere. The atom is said to be ionized, i.e., its electron has been detatched from the nucleus. Positive energy means that the atom is ionized and the electron is moving independently with some kinetic energy.

The minimum energy needed to ionize an atom is called *ionization energy*. The potential difference through which an electron should be accelerated to acquire this much energy is called *ionization potential*. Thus, ionization energy of hydrogen atom in ground state is 13·6 eV and ionization potential is 13·6 V.

**Binding energy**

*Binding energy* of a system is defined as the energy released when its constituents are brought from infinity to form the system. It may also be defined as the energy needed to separate its constituents to large distances. If an electron and a proton are initially at rest and brought from large distances to form a hydrogen atom, 13·6 eV energy will be released. The binding energy of a hydrogen atom is, therefore, 13·6 eV, same as its ionization energy.

**Excitation potential**

The energy needed to take the atom from its ground state to an excited state is called the *excitation energy* of that excited state. The hydrogen atom in ground state needs 10·2 eV to go into the first excited state. Thus, the excitation energy of hydrogen atom in the first excited state is 10·2 eV. The potential through which an electron should be accelerated to acquire this much of energy is called the *excitation potential*. Thus, the excitation potential of hydrogen atom in first excited state is 10·2 V.

***Example 43.3***

*The excitation energy of a hydrogen-like ion in its first excited state is* 40·8 eV. *Find the energy needed to remove the electron from the ion.*

***Solution*** :

The excitation energy in the first excited state is

$$E = RhcZ^2\left(\frac{1}{1^2} - \frac{1}{2^2}\right)$$
$$= (13{\cdot}6 \text{ eV}) \times Z^2 \times \frac{3}{4}.$$

Equating this to 40·8 eV, we get $Z = 2$. So, the ion in question is $He^+$. The energy of the ion in the ground state is

$$E = -\frac{RhcZ^2}{1^2} = -4 \times (13{\cdot}6 \text{ eV})$$
$$= -54{\cdot}4 \text{ eV}.$$

Thus 54·4 eV is required to remove the electron from the ion.

## 43.5 LIMITATIONS OF BOHR'S MODEL

Bohr's model was a great success at a time when the physicists were struggling hard to understand the discrete wavelengths in hydrogen spectra. Even today the model is very popular among beginners and nonphysicists, who can 'visualise' the inside of the atom as electrons going in circles around the nucleus. However, the model did not go too far. It could not be extended for atoms or ions having more than one electron. Even helium spectrum was beyond the scope of the Bohr's model. As technology improved and the wavelengths were measured with greater accuracy, deviations were observed even in the case of hydrogen spectral lines. Thus, at least seven components having slightly different wavelengths are revealed in what was previously known as the 656·3 nm line. On the theoretical side also, the model is not quite consistent with the physics in totality. Bohr's postulates look more like a patch on Maxwell's electromagnetism. Maxwell's theory is not replaced or refuted but it is arbitrarily assumed that in certain orbits, electrons get the licence

to disobey the laws of electromagnetism and are allowed not to radiate energy.

### 43.6 THE WAVE FUNCTION OF AN ELECTRON

Physicists now have a mathematically and logically sound theory in the name of *quantum mechanics* which describes the spectra in a much better way. A very brief introduction to this theory is given below.

We have seen in previous chapters that to understand the behaviour of light, we must use the wave picture (the electric field $\vec{E}$) as well as the particle picture (the photon). The energy of a particular 'photon' is related to the 'wavelength' of the $\vec{E}$ wave. Light going in $x$-direction is represented by the wave function

$$E(x, t) = E_0 \sin(kx - \omega t).$$

In general, if light can go in any direction, the wave function is

$$\vec{E}(\vec{r},t) = \vec{E}_0 \sin(\vec{k}\cdot\vec{r} - \omega t). \quad \ldots \text{(i)}$$

If $|\vec{E}|^2$ at a certain point $\vec{r}$ is large, the intensity of light is high and we say that the 'density of photons' at that position is high. Suppose the intensity is so low that we expect only a single photon in a large volume. Even this weak light is represented by a wave given by (i) with $|\vec{E}_0|^2$ having a small value. Where is this photon at time $t$ ? We can't assign a unique position to the photon because $\vec{E}$ is spread over a large space, and wherever $\vec{E} \neq 0$ there is light. But if we put an instrument to detect the photon, we shall not detect a part of photon here and a part there. The whole photon is detected at just one point. The probability of finding the photon is more where $|\vec{E}(\vec{r}, t)|^2$ is large. To know something about the 'photon', we have to get the wave function $\vec{E}(\vec{r}, t)$ of light and correlate the wave properties with the particle properties. The wave function $\vec{E}(\vec{r}, t)$ satisfies Maxwell's equations. Similar is the case with electrons. An electron also has a wave character as well as a particle character. Its wave function is $\psi(\vec{r}, t)$ which may be obtained by solving *Schrodinger's wave equation*. The particle properties of the electron must be understood through this wave function $\psi(\vec{r}, t)$. The wave function varies continuously in space and may be extended over a large part of space at a given instant. This does not mean that the electron is spread over that large part. If we put an instrument to detect the electron at a point, we shall either detect a whole electron or none. But where will this electron be found ? The answer is again hidden in $\psi(\vec{r}, t)$. Wherever $\psi \neq 0$, there is a chance to find the electron. Greater the value of $|\psi(\vec{r}, t)|^2$, greater is the probability of detecting the electron there. Not only the information about the electron's position but information about all the properties including energy is also contained in the wave function $\psi(\vec{r}, t)$.

### 43.7 QUANTUM MECHANICS OF THE HYDROGEN ATOM

The wave function $\psi(\vec{r}, t)$ of the electron and the possible energies $E$ of a hydrogen atom or a hydrogen-like ion are obtained from the Schrodinger's equation

$$\frac{-h^2}{8\pi^2 m}\left[\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right] - \frac{Ze^2\psi}{4\pi\varepsilon_0 r} = E\psi. \quad \ldots \text{(43.10)}$$

Here $(x,y,z)$ refers to a point with the nucleus as the origin and $r$ is the distance of this point from the nucleus. $E$ refers to energy. The constant $Z$ is the number of protons in the nucleus. For hydrogen, we have to put $Z = 1$. There are infinite number of functions $\psi(\vec{r})$ which satisfy equation (43.10). These functions, which are solutions of equation (43.10), may be characterised in terms of three parameters $n$, $l$ and $m_l$. With each solution $\psi_{nlm_l}$, there is associated a unique value of the energy $E$ of the atom or the ion. The energy $E$ corresponding to the wave function $\psi_{nlm_l}$ depends only on $n$ and may be written as

$$E_n = -\frac{mZ^2e^4}{8\varepsilon_0^2 h^2 n^2}. \quad \ldots \text{(43.11)}$$

These energies happen to be identical with the allowed energies calculated in Bohr's model. This explains the success of Bohr's model in quantitatively obtaining the wavelengths in a hydrogen spectrum. For each $n$ there are $n$ values of $l$, namely $l = 0, 1, 2, \ldots, n-1$ and for each $l$ there are $2l + 1$ values of $m_l$ namely $m_l = -l, -l+1, -l+2, \ldots, l-1, l$. The parameter $n$ is called the *principal quantum number, l* the *orbital angular momentum quantum number* and $m_l$ the *magnetic quantum number.*

The lowest possible energy for the hydrogen atom is $-Rhc = -13{\cdot}6$ eV and the wave function of the electron in this 'ground state' is

$$\psi(\vec{r}) = \psi_{100} = \sqrt{\frac{Z^3}{\pi a_0^2}}\, e^{-r/a_0}. \quad \ldots \text{(43.12)}$$

In Bohr's model, we say that the electron moves in a circular orbit of radius $a_0 = 0{\cdot}053$ nm in the ground state. In quantum mechanics, the very idea of orbit is invalid. In ground state, the wave function of the electron is given by equation (43.12). At any instant this wave function is spread over large distances in space, and wherever $\psi \neq 0$, the presence of electron may be felt. However, if the electron is detected by some experiment, it will be detected at one single

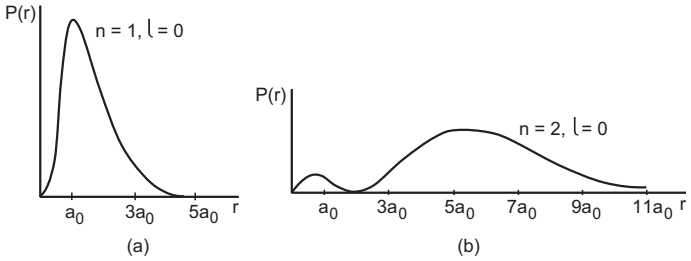



Figure 43.5

position only. The probability of finding the electron in a small volume $dV$ is $|\psi(\vec{r})|^2 dV$. One can calculate the probability $P(r)dr$ of finding the electron at a distance between $r$ and $r + dr$ from the nucleus. The function $P(r)$ is called linear probability density. In ground state, given by equation (43.12), $P(r)$ comes out to be

$$P(r) = \frac{4}{a_0^3} r^2 e^{-2r/a_0}.$$

A plot of $P(r)$ versus $r$ is shown in figure (43.5a). Note that $P(r)$ is maximum at $r = a_0$. This means that the electron is more likely to be found near $r = a_0$ than at any other distance.

It may be satisfying that at least the probability of finding the electron is maximum at the same radial distance from the nucleus where the Bohr's model assigns the electron to be. However, even this cannot be stretched too far. The linear probability density $P(r)$ for $n = 2$, $l = 0$, $m = 0$ is plotted in figure (43.5b) which has two maxima, one near $r = a_0$ and the other near $r = 5{\cdot}4\ a_0$. In Bohr's model, all $n = 2$ electrons should be at $r = 4\ a_0$.

### 43.8 NOMENCLATURE IN ATOMIC PHYSICS

We have neglected the *spin* of the electron in the discussion so far. A very interesting property of electrons is that each electron has a permanent angular momentum whose component along any given direction is $\frac{h}{4\pi}$ or $-\frac{h}{4\pi}$. This angular momentum is different from the angular momentum resulting from the motion of the electron and is known as the *spin angular momentum* of the electron. The complete wave function of an electron also has a part depending on the state of the spin. The spin part of the wave function is characterized by a spin quantum number $m_s$ which can take values $m_s = +1/2$ or $-1/2$. A wave function is thus characterised by $n$, $l$, $m_l$ and $m_s$. A particular wave function described by particular values of $n$, $l$, $m_l$, $m_s$ corresponds to a *quantum state*. For $n = 1$, we have $l = 0$ and $m_l = 0$. But $m_s$ can be $+1/2$ or $-1/2$. So there are two quantum states corresponding to $n = 1$. For $n = 2$ there are 8 quantum states, for $n = 3$ there are 18 quantum states and so on. In general, there are $2n^2$ quantum states corresponding to a particular $n$. The quantum states corresponding to a particular $n$ are together called a *major shell*. The major shell corresponding to $n = 1$ is called K shell, corresponding to $n = 2$ is called L shell, corresponding to $n = 3$ is called M shell, etc.

A very interesting and important law of nature is that *there cannot be more than one electron in any quantum state*. This is known as *Pauli exclusion principle*. Thus, a K shell can contain a maximum of 2 electrons, an L shell can contain a maximum of 8 electrons, an M shell can contain a maximum of 18 electrons and so on.

It is customary to use the symbols s, p, d, f, etc., to denote the values of the orbital angular momentum quantum number $l$. These symbols correspond to $l = 0$, 1, 2, 3, etc., respectively. The quantum states corresponding to a given principal quantum number $n$ and a given orbital angular momentum quantum number $l$ form what we call a *subshell*. Thus $n = 1$, $l = 0$ is called 1s subshell. Similarly $n = 2$, $l = 0$ is called 2s subshell, $n = 2$, $l = 1$ is called 2p subshell and so on. For atoms having more than one electron also, the concept of $n$, $l$, $m_l$, $m_s$ is valid. The energy then depends on $n$ as well as on $l$. Thus 1s, 2s, 2p, etc., also designate the *energy levels*. For an atom having many electrons, the quantum states are, in general, gradually filled from lower energy to higher energy to form the ground state of the atom.

We have seen that electrons obey Pauli exclusion principle. Apart from electrons, there are other particles which obey this principle. Protons and neutrons also obey this principle. Any particle that obeys Pauli exclusion principle, is called a *fermion*. Electrons, protons, neutrons are all fermions.

### 43.9 LASER

When an atom jumps from a higher energy state to a lower energy state, it emits a photon of light. In an ordinary source of light, atoms emit photons independently of each other. As a result, different photons have different phases and the light as a whole becomes incoherent. Also, the energy of transition differs slightly from photon to photon so that the wavelength is not uniquely defined. There is a spread $\Delta\lambda$ in the wavelength $\lambda$. The direction of light is also different for different transitions so that we do not get a strictly parallel beam of light.

LASER (Light Amplification by Stimulated Emission of Radiation) is a process by which we get a light beam which is coherent, highly monochromatic and almost perfectly parallel. The word 'laser' is also

used for the light beam obtained by this process. All the photons in the light beam, emitted by different atoms at different instants, are in phase. The spread $\Delta\lambda$ in wavelength is very small, of the order of $10^{-6}$ nm which is about 1000 times smaller than the spread in the usual $^{86}$Kr light. A beam of laser can go to the moon and return to the earth without much loss of intensity. This shows that laser may be obtained as an almost perfectly parallel beam.

To understand the process involved in laser, we have to first discuss *stimulated emission.*

**Stimulated Emission**

Consider an atom which has an allowed state at energy $E_1$ and another allowed state at a higher energy $E_2$. Suppose the atom is in the lower energy state $E_1$. If a photon of light having energy $E_2 - E_1$ is incident on this atom, the atom may absorb the photon and jump to the higher energy state $E_2$ (figure 43.6a). This process is called *stimulated absorption* of light photon. The incident photon has stimulated the atom to absorb the energy.

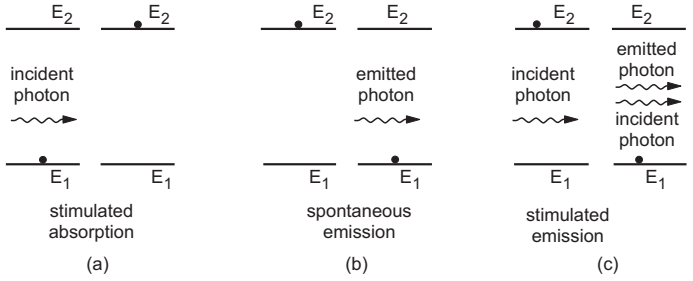



Figure 43.6

Now, suppose the atom is in the higher energy state $E_2$. If we just leave the atom there, it will eventually come down to the lower state by emitting a photon having energy $E_2 - E_1$ (figure 43.6b). This process is called *spontaneous emission.* Typically, an atom stays for about 10 ns in an excited state. The average time for which an atom stays in an excited state is called the *lifetime* of that state. There are atoms which have certain excited states having a lifetime of the order of a millisecond, i.e., about $10^5$ times longer than the usual lifetimes. Such states are called *metastable states.*

Finally, suppose the atom is in the higher energy state $E_2$ and a photon having energy $(E_2 - E_1)$ is incident on it (figure 43-6c). The incident photon interacts with the atom and may cause the atom to come down to the lower energy state. A fresh photon is emitted in the process. This process is different from spontaneous emission in which the atom jumps to the lower energy state on its own. In the present case, the incident photon has 'stimulated' the atom to make the jump.

*When an atom emits a photon due to its interaction with a photon incident on it, the process is called stimulated emission. The emitted photon has exactly the same energy, phase and direction as the incident photon.*

**Basic Process of Laser**

The basic scheme to get light amplification by stimulated emission is as follows.

A system is chosen which has a metastable state having an energy $E_2$ (figure 43.7). There is another allowed energy $E_1$ which is less than $E_2$. The system may be a gas or a liquid in a cylindrical tube or a solid in the shape of a cylindrical rod. Suppose, by some technique, the number of atoms in the metastable state $E_2$ is made much larger than that in $E_1$. Suppose a photon of light of energy $E_2 - E_1$ is incident on one of the atoms in the metastable state. This atom drops to the state $E_1$ emitting a photon in the same phase, energy and direction as the first one. These two photons interact with two more atoms in the state $E_2$ and so on. So the number of photons keeps on increasing. All these photons have the same phase, the same energy and the same direction. So the amplification of light is achieved.

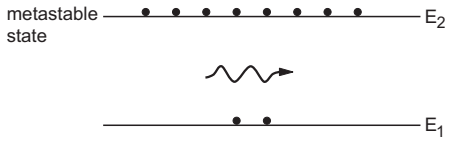



Figure 43.7

In this scheme, two arrangements are necessary. Firstly, the metastable state with energy $E_2$ must all the time have larger number of atoms than the number in the lower energy state. If the lower energy state has a larger number of atoms, these atoms will absorb a sizable number of photons to go up in energy. This way the stimulated emission will be weakened and the amplification will not be possible. When a higher energy state has more number of atoms than a lower energy state has, we say that *population inversion* has taken place. This is because, normally, the population in the lower energy state is higher. To sustain laser action, we need an arrangement which ensures population inversion between the states $E_1$ and $E_2$. The metastable state should continue to get atoms and the atoms should be continuously removed from the lower energy state $E_1$. This process is called *pumping.*

Secondly, the photons emitted due to stimulating action should stimulate other atoms to emit more photons. This means, the stimulated photons should spend enough time in the system, interacting with the atoms. To achieve this, two mirrors are fixed at the ends of the cylindrical region containing the lasing

material. The mirrors reflect the photons back and forth to keep them inside the region for a long time. One of the mirrors is made slightly transmitting so that a small fraction, say 1%, of the light comes out of the region. This is the laser which becomes available to us for use.

Any photon travelling in a direction not parallel to the axis of the cylindrical region is thrown out from the sides after few reflections. The photons moving parallel to the axis remain in the region for long time and hence only the light along the axis is amplified. This explains why the laser light is highly directional.

Let us now discuss a He–Ne laser which is most widely used in classroom demonstrations.

**He–Ne Laser**

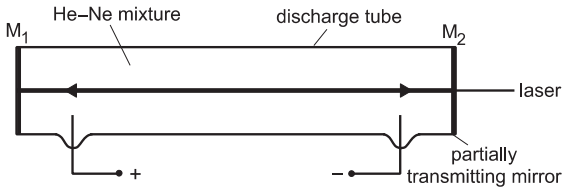



Figure 43.8

A schematic design of the system is shown in figure (43.8). A mixture of helium (about 90%) and neon (about 10%) at low pressures is taken in a cylindrical glass tube. Two parallel mirrors $M_1$ and $M_2$ are fixed at the ends. One of the two mirrors, $M_2$ in the figure, is slightly transmitting and laser light comes out of it. The tube contains two electrodes which are connected to a high-voltage power supply so that a large electric field is established in the tube.

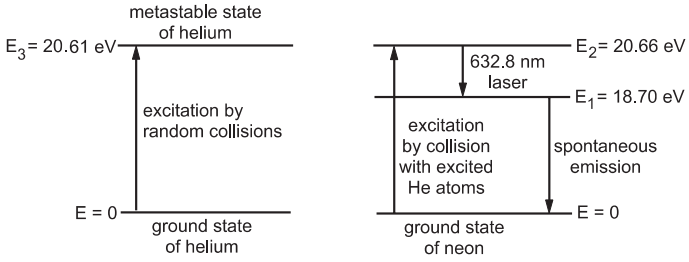



Figure 43.9

The relevant energy levels of helium and neon are shown in figure (43.9). Lasing action takes place between the state at energy $E_2 = 20{\cdot}66$ eV and the state at energy $E_1 = 18{\cdot}70$ eV of neon atoms. Helium has a metastable state at $E_3 = 20{\cdot}61$ eV which happens to be close to the level $E_2$ of neon. Helium is used to pump the neon atoms to the state $E_2$ from where they may come down to the state $E_1$ by stimulated emission. The energy difference is

$$E_2 - E_1 = 1{\cdot}96 \text{ eV}$$

so that the wavelength of He–Ne laser is

$$\lambda = \frac{hc}{E_2 - E_1} = 632{\cdot}8 \text{ nm}.$$

**Working**

When the power supply is switched on and the electric field is established, some of the atoms of the mixture get ionized. The electrons freed by these atoms are accelerated by the high electric field. These electrons collide with helium atoms to take them to the metastable state at energy $E_3$. Such an excited atom collides with a neon atom and transfers the extra energy to it. As a result, the helium atom comes back to its ground state and the neon atom is excited to the state at energy $E_2$. This process takes place continuously so that the neon atoms are continuously pumped to the state at energy $E_2$ to keep the population of this state large.

Stimulated emission takes place between the states at energies $E_2$ and $E_1$. As the state at energy $E_1$ has a small lifetime, of the order of 10 ns, these atoms readily jump to the still lower states. This way the population of the state at energy $E_1$ is always very small. Thus, population inversion between $E_2$ and $E_1$ is achieved and maintained.

Laser light comes out of the partially transmitting mirror.

Note that the higher energy state $E_2$ of neon is not itself metastable. But the metastable state of helium accumulates atoms at higher energy which take neon atoms to the level $E_2$ by means of collisions.

**Uses of Laser**

Laser was invented in 1960. Since then, laser technology has greatly advanced and now lasers have widespread use in industry, scientific research, surgery, etc.

Because of the near-perfect parallel and monochromatic character, a laser beam can be focused by a converging lens to a very small spot. This results in very high intensity over that tiny spot. It can, therefore, be used for very accurate microsurgery where a very small area is to be treated. Lasers in infrared region are used to burn away cervical tumours. These lasers are also used for cutting tissues. Lasers are used to spot-weld detached retina with great accuracy. Because of the high intensity, lasers are used to drill sharp holes in metals and diamond. In garment industry, lasers are used to cut many layers (say 50 layers at a time) of cloth without frayed edges.

Lasers are widely used to send telephone signals over long distances through optical fibres. They are also used in nuclear fusion research which is likely to be the ultimate source of energy for us.

Because of its directional properties, lasers are used in surveying. Another use of laser is to align tools and equipment in industry and scientific research. Laser light is sent to the moon from where it is reflected back to the earth without much loss of intensity. Thus, points on the moon's surface may be monitored from the earth using lasers. Laser has numerous military applications.

An interesting application of lasers is to produce holograms, which record a 3 D image of an object. When the hologram is viewed, again with a laser, the same 3 D perception is achieved as it is with the actual object.

In compact disc (CD) audio systems, a laser beam is used in place of the phonographic needle. Sound is recorded on the compact disc using digital electronic techniques. This results in great compression of the sound data and a very large number of songs, speeches etc. can be stored on a CD which is much smaller than a traditional record. Also, the playback of the music is more 'true' than traditional systems and almost without any distortion. Using lasers, video images can also be stored on discs which can be played back using a laser disc player and a TV. Since the combination of digital electronic techniques and CD allows us to store a large amount of data in a small volume, books of large volume like dictionaries and encyclopedias, are now available on CDs. This technique is now being used in computers for data retrieval and storage. Lasers are used in laser printers. The present book was also prepared with the help of a laser printer. Incredible new applications are being created everyday using lasers.

## *Worked Out Examples*

First Bohr radius $a_0 = 53$ pm, energy of hydrogen atom in ground state $= -13{\cdot}6$ eV, Planck constant $h = 4{\cdot}14 \times 10^{-15}$ eVs, speed of light $= 3 \times 10^8$ m s$^{-1}$.

**1.** *Find the radius of* $Li^{++}$ *ions in its ground state assuming Bohr's model to be valid.*

***Solution*** : For hydrogen-like ions, the radius of the $n$th orbit is

$$a_n = \frac{n^2 a_0}{Z}.$$

For $Li^{++}$, $Z = 3$ and in ground state $n = 1$. The radius is

$$a_1 = \frac{53 \text{ pm}}{3} \approx 18 \text{ pm}.$$

**2.** *A particular hydrogen-like ion emits radiation of frequency* $2{\cdot}467 \times 10^{15}$ Hz *when it makes transition from* $n = 2$ *to* $n = 1$. *What will be the frequency of the radiation emitted in a transition from* $n = 3$ *to* $n = 1$ ?

***Solution*** : The frequency of radiation emitted is given by

$$\nu = \frac{c}{\lambda} = K\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right).$$

Thus, $2{\cdot}467 \times 10^{15}$ Hz $= K\left(\frac{1}{1^2} - \frac{1}{2^2}\right)$

or, $$K = \frac{4}{3} \times 2{\cdot}467 \times 10^{15} \text{ Hz}.$$

The frequency of the radiation emitted in the transition $n = 3$ to $n = 1$ is

$$\nu' = K\left[\frac{1}{1^2} - \frac{1}{3^2}\right]$$

$$= \frac{8}{9}K = \frac{8}{9} \times \frac{4}{3} \times 2{\cdot}467 \times 10^{15} \text{ Hz}$$

$$= 2{\cdot}92 \times 10^{15} \text{ Hz}.$$

**3.** *Calculate the two highest wavelengths of the radiation emitted when hydrogen atoms make transitions from higher states to* $n = 2$ *states.*

***Solution*** : The highest wavelength corresponds to the lowest energy of transition. This will be the case for the transition $n = 3$ to $n = 2$. The second highest wavelength corresponds to the transition $n = 4$ to $n = 2$.

The energy of the state $n$ is $E_n = \frac{E_1}{n^2}$.

Thus, $$E_2 = -\frac{13{\cdot}6 \text{ eV}}{4} = -3{\cdot}4 \text{ eV}$$

$$E_3 = -\frac{13{\cdot}6 \text{ eV}}{9} = -1{\cdot}5 \text{ eV}$$

and $$E_4 = -\frac{13{\cdot}6 \text{ eV}}{16} = -0{\cdot}85 \text{ eV}.$$

The highest wavelength is $\lambda_1 = \frac{hc}{\Delta E}$

$$= \frac{1242 \text{ eVnm}}{(3{\cdot}4 \text{ eV} - 1{\cdot}5 \text{ eV})} = 654 \text{ nm}.$$

The second highest wavelength is

$$\lambda_2 = \frac{1242 \text{ eVnm}}{(3{\cdot}4 \text{ eV} - 0{\cdot}85 \text{ eV})} = 487 \text{ nm}.$$

**4.** *What is the wavelength of the radiation emitted when the electron in a hydrogen atom jumps from* $n = \infty$ *to* $n = 2$ ?

***Solution*** : The energy of $n = 2$ state is

$$E_2 = \frac{-13{\cdot}6 \text{ eV}}{4} = -3{\cdot}4 \text{ eV}.$$

The energy of $n = \infty$ state is zero.

The wavelength emitted in the given transition is

$$\lambda = \frac{hc}{\Delta E}$$

$$= \frac{1242 \text{ eV nm}}{3{\cdot}4 \text{ eV}} = 365 \text{ nm}.$$

**5.** *(a) Find the wavelength of the radiation required to excite the electron in* $Li^{++}$ *from the first to the third Bohr orbit. (b) How many spectral lines are observed in the emission spectrum of the above excited system ?*

***Solution*** **:** (a) The energy in the first orbit $= E_1 = Z^2 E_0$ where $E_0 = -13{\cdot}6$ eV is the energy of a hydrogen atom in ground state. Thus for $Li^{++}$,

$$E_1 = 9E_0 = 9 \times (-13{\cdot}6 \text{ eV}).$$

The energy in the third orbit is

$$E_3 = \frac{E_1}{n^2} = \frac{E_1}{9} = -13{\cdot}6 \text{ eV}.$$

Thus, $E_3 - E_1 = 8 \times 13{\cdot}6 \text{ eV} = 108{\cdot}8 \text{ eV}.$

The wavelength of radiation required to excite $Li^{++}$ from the first orbit to the third orbit is given by

$$\frac{hc}{\lambda} = E_3 - E_1$$

or,
$$\lambda = \frac{hc}{E_3 - E_1}$$

$$= \frac{1242 \text{ eV nm}}{108{\cdot}8 \text{ eV}} \approx 11{\cdot}4 \text{ nm}.$$

(b) The spectral lines emitted are due to the transitions $n = 3 \to n = 2$, $n = 3 \to n = 1$ and $n = 2 \to n = 1$. Thus, there will be three spectral lines in the spectrum.

**6.** *Find the wavelengths present in the radiation emitted when hydrogen atoms excited to* $n = 3$ *states return to their ground states.*

***Solution*** **:** A hydrogen atom may return directly to the ground state or it may go to $n = 2$ and from there to the ground state. Thus, wavelengths corresponding to $n = 3 \to n = 1$, $n = 3 \to n = 2$ and $n = 2 \to n = 1$ are present in the radiation.

The energies in $n = 1$, 2 and 3 states are

$$E_1 = -13{\cdot}6 \text{ eV}$$

$$E_2 = -\frac{13{\cdot}6}{4} \text{ eV} = -3{\cdot}4 \text{ eV}$$

and
$$E_3 = -\frac{13{\cdot}6}{9} \text{ eV} = -1{\cdot}5 \text{ eV}.$$

The wavelength emitted in the transition $n = 3$ to the ground state is

$$\lambda = \frac{hc}{\Delta E}$$

$$= \frac{1242 \text{ eV nm}}{13{\cdot}6 \text{ eV} - 1{\cdot}5 \text{ eV}} = 103 \text{ nm}.$$

Similarly, the wavelength emitted in the transition $n = 3$ to $n = 2$ is 654 nm and that emitted in the transition $n = 2$ to $n = 1$ is 122 nm. The wavelengths present in the radiation are, therefore, 103 nm, 122 nm and 654 nm.

**7.** *How many different wavelengths may be observed in the spectrum from a hydrogen sample if the atoms are excited to states with principal quantum number* $n$ *?*

***Solution*** **:** From the $n$th state, the atom may go to $(n-1)$th state, … , 2nd state or 1st state. So there are $(n-1)$ possible transitions starting from the $n$th state. The atoms reaching $(n-1)$th state may make $(n-2)$ different transitions. Similarly for other lower states. The total number of possible transitions is

$$(n-1) + (n-2) + (n-3) + \ldots 2 + 1$$

$$= \frac{n(n-1)}{2}.$$

**8.** *Monochromatic radiation of wavelength* $\lambda$ *is incident on a hydrogen sample in ground state. Hydrogen atoms absorb a fraction of light and subsequently emit radiation of six different wavelengths. Find the value of* $\lambda$.

***Solution*** **:** As the hydrogen atoms emit radiation of six different wavelengths, some of them must have been excited to $n = 4$. The energy in $n = 4$ state is

$$E_4 = \frac{E_1}{4^2} = -\frac{13{\cdot}6 \text{ eV}}{16} = -0{\cdot}85 \text{ eV}.$$

The energy needed to take a hydrogen atom from its ground state to $n = 4$ is

$$13{\cdot}6 \text{ eV} - 0{\cdot}85 \text{ eV} = 12{\cdot}75 \text{ eV}.$$

The photons of the incident radiation should have 12·75 eV of energy. So

$$\frac{hc}{\lambda} = 12{\cdot}75 \text{ eV}$$

or,
$$\lambda = \frac{hc}{12{\cdot}75 \text{ eV}}$$

$$= \frac{1242 \text{ eV nm}}{12{\cdot}75 \text{ eV}} = 97{\cdot}5 \text{ nm}.$$

**9.** *The energy needed to detach the electron of a hydrogen-like ion in ground state is* 4 rydberg. *(a) What is the wavelength of the radiation emitted when the electron jumps from the first excited state to the ground state ? (b) What is the radius of the first orbit for this atom ?*

***Solution*** **:** (a) In energy units, 1 rydberg = 13·6 eV. The energy needed to detach the electron is $4 \times 13{\cdot}6$ eV. The energy in the ground state is, therefore, $E_1 = -4 \times 13{\cdot}6$ eV. The energy of the first excited state $(n = 2)$ is $E_2 = \frac{E_1}{4} = -13{\cdot}6$ eV. The energy difference is $E_2 - E_1 = 3 \times 13{\cdot}6 \text{ eV} = 40{\cdot}8$ eV. The wavelength of the

radiation emitted is

$$\lambda = \frac{hc}{\Delta E}$$

$$= \frac{1242 \text{ eV nm}}{40{\cdot}8 \text{ eV}} = 30{\cdot}4 \text{ nm}.$$

(b) The energy of a hydrogen-like ion in ground state is $E = Z^2 E_0$ where $Z =$ atomic number and $E_0 = -13{\cdot}6$ eV. Thus, $Z = 2$. The radius of the first orbit is $\frac{a_0}{Z}$ where $a_0 = 53$ pm. Thus,

$$r = \frac{53 \text{ pm}}{2} = 26{\cdot}5 \text{ pm}.$$

**10.** *A hydrogen sample is prepared in a particular excited state A. Photons of energy* 2·55 eV *get absorbed into the sample to take some of the electrons to a further excited state B. Find the quantum numbers of the states A and B.*

***Solution*** : The allowed energies of hydrogen atoms are

$$E_1 = -13{\cdot}6 \text{ eV}$$

$$E_2 = -3{\cdot}4 \text{ eV}$$

$$E_3 = -1{\cdot}5 \text{ eV}$$

$$E_4 = -0{\cdot}85 \text{ eV}$$

$$E_5 = -0{\cdot}54 \text{ eV}.$$

We see that a difference of 2·55 eV can only be absorbed in transition $n = 2$ to $n = 4$. So the state $A$ has quantum number 2 and the state $B$ has quantum number 4.

**11.** *(a) Find the maximum wavelength* $\lambda_0$ *of light which can ionize a hydrogen atom in its ground state. (b) Light of wavelength* $\lambda_0$ *is incident on a hydrogen atom which is in its first excited state. Find the kinetic energy of the electron coming out.*

***Solution*** : (a) To ionize a hydrogen atom in ground state, a minimum of 13·6 eV energy should be given to it. A photon of light should have this much of energy in order to ionize a hydrogen atom. Thus,

$$\frac{hc}{\lambda_0} = 13{\cdot}6 \text{ eV}$$

or,
$$\lambda_0 = \frac{1242 \text{ eV nm}}{13{\cdot}6 \text{ eV}} = 91{\cdot}3 \text{ nm}.$$

(b) The energy of the hydrogen atom in its first excited state is $-\frac{13{\cdot}6 \text{ eV}}{4} = -3{\cdot}4$ eV. Thus, 3·4 eV of energy is needed to take the electron out of the atom. The energy of a photon of the light of wavelength $\lambda_0$ is 13·6 eV. Thus, the electron coming out will have a kinetic energy 13·6 eV – 3·4 eV = 10·2 eV.

**12.** *Derive an expression for the magnetic field at the site of the nucleus in a hydrogen atom due to the circular motion of the electron. Assume that the atom is in its ground state and give the answer in terms of fundamental constants.*

***Solution*** : We have

$$\frac{mv^2}{r} = \frac{e^2}{4\pi\varepsilon_0 r^2}$$

or,
$$v^2 r = \frac{e^2}{4\pi\varepsilon_0 m} \cdot \quad \ldots \text{(i)}$$

From Bohr's quantization rule, in ground state,

$$vr = \frac{h}{2\pi m} \cdot \quad \ldots \text{(ii)}$$

From (i) and (ii),

$$v = \frac{e^2}{2\varepsilon_0 h} \quad \ldots \text{(iii)}$$

and
$$r = \frac{\varepsilon_0 h^2}{\pi m e^2} \cdot \quad \ldots \text{(iv)}$$

As the electron moves along a circle, it crosses any point on the circle $\frac{v}{2\pi r}$ times per unit time. The charge crossing per unit time, that is the current, is $i = \frac{ev}{2\pi r}$. The magnetic field at the centre due to this circular current is

$$B = \frac{\mu_0 i}{2r} = \frac{\mu_0 ev}{4\pi r^2} \cdot$$

From (iii) and (iv),

$$B = \frac{\mu_0 e}{4\pi} \frac{e^2}{2\varepsilon_0 h} \times \frac{\pi^2 m^2 e^4}{\varepsilon_0^2 h^4}$$

$$= \frac{\mu_0 e^7 \pi m^2}{8\varepsilon_0^3 h^5} \cdot$$

**13.** *A lithium atom has three electrons. Assume the following simple picture of the atom. Two electrons move close to the nucleus making up a spherical cloud around it and the third moves outside this cloud in a circular orbit. Bohr's model can be used for the motion of this third electron but* $n = 1$ *states are not available to it. Calculate the ionization energy of lithium in ground state using the above picture.*

***Solution*** : In this picture, the third electron moves in the field of a total charge + 3$e$ – 2$e$ = + $e$. Thus, the energies are the same as that of hydrogen atoms. The lowest energy is

$$E_2 = \frac{E_1}{4} = \frac{-13{\cdot}6 \text{ eV}}{4} = -3{\cdot}4 \text{ eV}.$$

Thus, the ionization energy of the atom in this picture is 3·4 eV.

**14.** *A particle known as* μ*-meson, has a charge equal to that of an electron and mass* 208 *times the mass of the electron. It moves in a circular orbit around a nucleus of charge* +3 $e$. *Take the mass of the nucleus to be infinite. Assuming that the Bohr's model is applicable to this system, (a) derive an expression for the radius of the nth Bohr orbit, (b) find the value of n for which the radius of the orbit is approximately the same as that of*

*the first Bohr orbit for a hydrogen atom and (c) find the wavelength of the radiation emitted when the* μ*-meson jumps from the third orbit to the first orbit.*

***Solution*** : (a) We have,

$$\frac{mv^2}{r} = \frac{Ze^2}{4\pi\varepsilon_0 r^2}$$

or, $$v^2 r = \frac{Ze^2}{4\pi\varepsilon_0 m}. \quad \ldots \text{(i)}$$

The quantization rule is $vr = \frac{nh}{2\pi m}$.

The radius is $r = \frac{(vr)^2}{v^2 r} = \frac{n^2 h^2}{4\pi^2 m^2}\frac{4\pi\varepsilon_0 m}{Ze^2}$

$$= \frac{n^2 h^2 \varepsilon_0}{Z\pi m e^2}. \quad \ldots \text{(ii)}$$

For the given system, $Z = 3$ and $m = 208\ m_e$.

Thus $$r_\mu = \frac{n^2 h^2 \varepsilon_0}{624\pi m_e e^2}.$$

(b) From (ii), the radius of the first Bohr orbit for the hydrogen atom is

$$r_h = \frac{h^2 \varepsilon_0}{\pi m_e e^2}.$$

For $r_\mu = r_h$,

$$\frac{n^2 h^2 \varepsilon_0}{624\pi m_e e^2} = \frac{h^2 \varepsilon_0}{\pi m_e e^2}$$

or, $$n^2 = 624$$

or, $$n \approx 25.$$

(c) From (i), the kinetic energy of the atom is

$$\frac{mv^2}{2} = \frac{Ze^2}{8\pi\varepsilon_0 r}$$

and the potential energy is $-\frac{Ze^2}{4\pi\varepsilon_0 r}$.

The total energy is $E_n = -\frac{Ze^2}{8\pi\varepsilon_0 r}$.

Using (ii),

$$E_n = -\frac{Z^2 \pi m e^4}{8\pi\varepsilon_0^2 n^2 h^2} = -\frac{9 \times 208\ m_e e^4}{8\varepsilon_0^2 n^2 h^2}$$

$$= \frac{1872}{n^2}\left(-\frac{m_e e^4}{8\varepsilon_0^2 h^2}\right). \quad \ldots \text{(iii)}$$

But $\left(-\frac{m_e e^4}{8\varepsilon_0^2 h^2}\right)$ is the ground state energy of hydrogen atom and hence is equal to –13·6 eV.

From (iii), $E_n = -\frac{1872}{n^2} \times 13{\cdot}6 \text{ eV} = \frac{-25459{\cdot}2 \text{ eV}}{n^2}$.

Thus, $E_1 = -25459{\cdot}2$ eV and $E_3 = \frac{E_1}{9} = -2828{\cdot}8$ eV. The energy difference is $E_3 - E_1 = 22630{\cdot}4$ eV.

The wavelength emitted is

$$\lambda = \frac{hc}{\Delta E}$$

$$= \frac{1242 \text{ eV nm}}{22630{\cdot}4 \text{ eV}} = 55 \text{ pm}.$$

**15.** *Find the wavelengths in a hydrogen spectrum between the range* 500 nm to 700 nm.

***Solution*** : The energy of a photon of wavelength 500 nm is

$$\frac{hc}{\lambda} = \frac{1242 \text{ eV nm}}{500 \text{ nm}} = 2{\cdot}44 \text{ eV}.$$

The energy of a photon of wavelength 700 nm is

$$\frac{hc}{\lambda} = \frac{1242 \text{ eV nm}}{700 \text{ nm}} = 1{\cdot}77 \text{ eV}.$$

The energy difference between the states involved in the transition should, therefore, be between 1·77 eV and 2·44 eV.

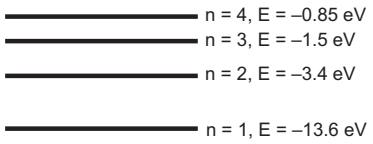



Figure 43-W1

Figure (43-W1) shows some of the energies of hydrogen states. It is clear that only those transitions which end at $n = 2$ may emit photons of energy between 1·77 eV and 2·44 eV. Out of these only $n = 3 \rightarrow n = 2$ falls in the proper range. The energy of the photon emitted in the transition $n = 3$ to $n = 2$ is $\Delta E = (3{\cdot}4 - 1{\cdot}5)$ eV = 1·9 eV. The wavelength is

$$\lambda = \frac{hc}{\Delta E}$$

$$= \frac{1242 \text{ eV nm}}{1{\cdot}9 \text{ eV}} = 654 \text{ nm}.$$

**16.** *A beam of ultraviolet radiation having wavelength between* 100 nm *and* 200 nm *is incident on a sample of atomic hydrogen gas. Assuming that the atoms are in ground state, which wavelengths will have low intensity in the transmitted beam ? If the energy of a photon is equal to the difference between the energies of an excited state and the ground state, it has large probability of being absorbed by an atom in the ground state.*

***Solution*** : The energy of a photon corresponding to λ = 100 nm is

$$\frac{1242 \text{ eV nm}}{100 \text{ nm}} = 12{\cdot}42 \text{ eV}$$

and that corresponding to λ = 200 nm is 6·21 eV.

The energy needed to take the atom from the ground state to the first excited state is

$$E_2 - E_1 = 13{\cdot}6 \text{ eV} - 3{\cdot}4 \text{ eV} = 10{\cdot}2 \text{ eV},$$

to the second excited state is

$$E_3 - E_1 = 13{\cdot}6 \text{ eV} - 1{\cdot}5 \text{ eV} = 12{\cdot}1 \text{ eV},$$

to the third excited state is

$$E_4 - E_1 = 13{\cdot}6 \text{ eV} - 0{\cdot}85 \text{ eV} = 12{\cdot}75 \text{ eV, etc.}$$

Thus, 10·2 eV photons and 12·1 eV photons have large probability of being absorbed from the given range 6·21 eV to 12·42 eV. The corresponding wavelengths are

$$\lambda_1 = \frac{1242 \text{ eV nm}}{10{\cdot}2 \text{ eV}} = 122 \text{ nm}$$

and $$\lambda_2 = \frac{1242 \text{ eV nm}}{12{\cdot}1 \text{ eV}} = 103 \text{ nm.}$$

These wavelengths will have low intensity in the transmitted beam.

**17.** *A neutron moving with speed $v$ makes a head-on collision with a hydrogen atom in ground state kept at rest. Find the minimum kinetic energy of the neutron for which inelastic (completely or partially) collision may take place. The mass of neutron ≈ mass of hydrogen* $= 1{\cdot}67 \times 10^{-27}$ kg.

***Solution*** : Suppose the neutron and the hydrogen atom move at speeds $v_1$ and $v_2$ after the collision. The collision will be inelastic if a part of the kinetic energy is used to excite the atom. Suppose an energy $\Delta E$ is used in this way. Using conservation of linear momentum and energy,

$$mv = mv_1 + mv_2 \quad \ldots \text{(i)}$$

and $$\frac{1}{2}mv^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 + \Delta E. \quad \ldots \text{(ii)}$$

From (i), $$v^2 = v_1^2 + v_2^2 + 2v_1v_2.$$

From (ii), $$v^2 = v_1^2 + v_2^2 + \frac{2\Delta E}{m}.$$

Thus, $$2v_1v_2 = \frac{2\Delta E}{m}.$$

Hence, $$(v_1 - v_2)^2 = (v_1 + v_2)^2 - 4v_1v_2 = v^2 - \frac{4\Delta E}{m}.$$

As $v_1 - v_2$ must be real,

$$v^2 - \frac{4\Delta E}{m} \geq 0$$

or, $$\frac{1}{2}mv^2 > 2\Delta E.$$

The minimum energy that can be absorbed by the hydrogen atom in ground state to go in an excited state is 10·2 eV. Thus, the minimum kinetic energy of the neutron needed for an inelastic collision is

$$\frac{1}{2}mv_{min}^2 = 2 \times 10{\cdot}2 \text{ eV} = 20{\cdot}4 \text{ eV.}$$

**18.** *Light corresponding to the transition $n = 4$ to $n = 2$ in hydrogen atoms falls on cesium metal (work function* = 1·9 eV*). Find the maximum kinetic energy of the photoelectrons emitted.*

***Solution*** : The energy of the photons emitted in transition $n = 4$ to $n = 2$ is

$$h\nu = 13{\cdot}6 \text{ eV}\left[\frac{1}{2^2} - \frac{1}{4^2}\right] = 2{\cdot}55 \text{ eV.}$$

The maximum kinetic energy of the photoelectrons is

$$= 2{\cdot}55 \text{ eV} - 1{\cdot}9 \text{ eV} = 0{\cdot}65 \text{ eV.}$$

**19.** *A small particle of mass m moves in such a way that the potential energy $U = \frac{1}{2}m^2\omega^2r^2$ where $\omega$ is a constant and $r$ is the distance of the particle from the origin. Assuming Bohr's model of quantization of angular momentum and circular orbits, show that radius of the nth allowed orbit is proportional to $\sqrt{n}$.*

***Solution*** : The force at a distance $r$ is

$$F = -\frac{dU}{dr} = -m\omega^2r. \quad \ldots \text{(i)}$$

Suppose the particle moves along a circle of radius $r$. The net force on it should be $mv^2/r$ along the radius. Comparing with (i),

$$\frac{mv^2}{r} = m\omega^2r$$

or, $$v = \omega r. \quad \ldots \text{(ii)}$$

The quantization of angular momentum gives

$$mvr = \frac{nh}{2\pi}$$

or, $$v = \frac{nh}{2\pi mr}. \quad \ldots \text{(iii)}$$

From (ii) and (iii),

$$r = \left(\frac{nh}{2\pi m\omega}\right)^{1/2}.$$

Thus, the radius of the $n$th orbit is proportional to $\sqrt{n}$.

□

## QUESTIONS FOR SHORT ANSWER

**1.** How many wavelengths are emitted by atomic hydrogen in visible range (380 nm–780 nm) ? In the range 50 nm to 100 nm ?

**2.** The first excited energy of a $He^+$ ion is the same as the ground state energy of hydrogen. Is it always true that one of the energies of any hydrogen-like ion will be the same as the ground state energy of a hydrogen atom ?

**3.** Which wavelengths will be emitted by a sample of atomic hydrogen gas (in ground state) if electrons of energy 12·2 eV collide with the atoms of the gas ?

**4.** When white radiation is passed through a sample of hydrogen gas at room temperature, absorption lines are observed in Lyman series only. Explain.

**5.** Balmer series was observed and analysed before the other series. Can you suggest a reason for such an order ?

**6.** What will be the energy corresponding to the first excited state of a hydrogen atom if the potential energy of the atom is taken to be 10 eV when the electron is widely separated from the proton ? Can we still write $E_n = E_1 / n^2$, or $r_n = a_0 n^2$ ?

**7.** The difference in the frequencies of series limit of Lyman series and Balmer series is equal to the frequency of the first line of the Lyman series. Explain.

**8.** The numerical value of ionization energy in eV equals the ionization potential in volts. Does the equality hold if these quantities are measured in some other units ?

**9.** We have stimulated emission and spontaneous emission. Do we also have stimulated absorption and spontaneous absorption ?

**10.** An atom is in its excited state. Does the probability of its coming to ground state depend on whether the radiation is already present or not ? If yes, does it also depend on the wavelength of the radiation present ?

## OBJECTIVE I

**1.** The minimum orbital angular momentum of the electron in a hydrogen atom is
(a) $h$ (b) $h/2$ (c) $h/2\pi$ (d) $h/\lambda$.

**2.** Three photons coming from excited atomic-hydrogen sample are picked up. Their energies are 12·1 eV, 10·2 eV and 1·9 eV. These photons must come from
(a) a single atom (b) two atoms
(c) three atoms (d) either two atoms or three atoms.

**3.** Suppose, the electron in a hydrogen atom makes transition from $n = 3$ to $n = 2$ in $10^{-8}$ s. The order of the torque acting on the electron in this period, using the relation between torque and angular momentum as discussed in the chapter on rotational mechanics is
(a) $10^{-34}$ N m (b) $10^{-24}$ N m
(c) $10^{-42}$ N m (d) $10^{-8}$ N m.

**4.** In which of the following transitions will the wavelength be minimum ?
(a) $n = 5$ to $n = 4$ (b) $n = 4$ to $n = 3$
(c) $n = 3$ to $n = 2$ (d) $n = 2$ to $n = 1$.

**5.** In which of the following systems will the radius of the first orbit ($n = 1$) be minimum ?
(a) Hydrogen atom (b) Deuterium atom
(c) Singly ionized helium (d) Doubly ionized lithium.

**6.** In which of the following systems will the wavelength corresponding to $n = 2$ to $n = 1$ be minimum ?
(a) Hydrogen atom (b) Deuterium atom
(c) Singly ionized helium (d) Doubly ionized lithium.

**7.** Which of the following curves may represent the speed of the electron in a hydrogen atom as a function of the principal quantum number $n$ ?

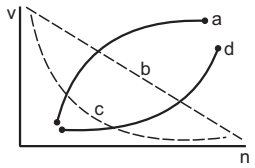



Figure 43-Q1

**8.** As one considers orbits with higher values of $n$ in a hydrogen atom, the electric potential energy of the atom
(a) decreases (b) increases
(c) remains the same (d) does not increase.

**9.** The energy of an atom (or ion) in its ground state is –54·4 eV. It may be
(a) hydrogen (b) deuterium (c) $He^+$ (d) $Li^{++}$.

**10.** The radius of the shortest orbit in a one-electron system is 18 pm. It may be
(a) hydrogen (b) deuterium (c) $He^+$ (d) $Li^{++}$.

**11.** A hydrogen atom in ground state absorbs 10·2 eV of energy. The orbital angular momentum of the electron is increased by
(a) $1{\cdot}05 \times 10^{-34}$ J s (b) $2{\cdot}11 \times 10^{-34}$ J s
(c) $3{\cdot}16 \times 10^{-34}$ J s (d) $4{\cdot}22 \times 10^{-34}$ J s.

**12.** Which of the following parameters are the same for all hydrogen-like atoms and ions in their ground states ?
(a) Radius of the orbit (b) Speed of the electron
(c) Energy of the atom
(d) Orbital angular momentum of the electron

**13.** In a laser tube, all the photons
(a) have same wavelength (b) have same energy
(c) move in same direction (d) move with same speed.

## OBJECTIVE II

**1.** In a laboratory experiment on emission from atomic hydrogen in a discharge tube, only a small number of lines are observed whereas a large number of lines are present in the hydrogen spectrum of a star. This is because in a laboratory
(a) the amount of hydrogen taken is much smaller than that present in the star
(b) the temperature of hydrogen is much smaller than that of the star
(c) the pressure of hydrogen is much smaller than that of the star
(d) the gravitational pull is much smaller than that in the star.

**2.** An electron with kinetic energy 5 eV is incident on a hydrogen atom in its ground state. The collision
(a) must be elastic (b) may be partially elastic
(c) must be completely inelastic
(d) may be completely inelastic.

**3.** Which of the following products in a hydrogen atom are independent of the principal quantum number $n$ ? The symbols have their usual meanings.
(a) $vn$ (b) $Er$ (c) $En$ (d) $vr$

**4.** Let $A_n$ be the area enclosed by the $n$th orbit in a hydrogen atom. The graph of $\ln (A_n / A_1)$ against $\ln(n)$
(a) will pass through the origin
(b) will be a straight line with slope 4
(c) will be a monotonically increasing nonlinear curve
(d) will be a circle.

**5.** Ionization energy of a hydrogen-like ion $A$ is greater than that of another hydrogen-like ion $B$. Let $r$, $u$, $E$ and $L$ represent the radius of the orbit, speed of the electron, energy of the atom and orbital angular momentum of the electron respectively. In ground state
(a) $r_A > r_B$ (b) $u_A > u_B$ (c) $E_A > E_B$ (d) $L_A > L_B$.

**6.** When a photon stimulates the emission of another photon, the two photons have
(a) same energy (b) same direction
(c) same phase (d) same wavelength.

## EXERCISES

Planck constant $h = 6{\cdot}63 \times 10^{-34}$ Js $= 4{\cdot}14 \times 10^{-15}$ eVs, first Bohr radius of hydrogen $a_0 = 53$ pm, energy of hydrogen atom in ground state $= -13{\cdot}6$ eV, Rydberg's constant $= 1{\cdot}097 \times 10^{7}\ \text{m}^{-1}$.

**1.** The Bohr radius is given by $a_0 = \dfrac{\varepsilon_0 h^2}{\pi m e^2}$. Verify that the RHS has dimensions of length.

**2.** Find the wavelength of the radiation emitted by hydrogen in the transitions (a) $n = 3$ to $n = 2$, (b) $n = 5$ to $n = 4$ and (c) $n = 10$ to $n = 9$.

**3.** Calculate the smallest wavelength of radiation that may be emitted by (a) hydrogen, (b) $He^+$ and (c) $Li^{++}$.

**4.** Evaluate Rydberg constant by putting the values of the fundamental constants in its expression.

**5.** Find the binding energy of a hydrogen atom in the state $n = 2$.

**6.** Find the radius and energy of a $He^+$ ion in the states (a) $n = 1$, (b) $n = 4$ and (c) $n = 10$.

**7.** A hydrogen atom emits ultraviolet radiation of wavelength 102·5 nm. What are the quantum numbers of the states involved in the transition ?

**8.** (a) Find the first excitation potential of $He^+$ ion. (b) Find the ionization potential of $Li^{++}$ ion.

**9.** A group of hydrogen atoms are prepared in $n = 4$ states. List the wavelengths that are emitted as the atoms make transitions and return to $n = 2$ states.

**10.** A positive ion having just one electron ejects it if a photon of wavelength 228 Å or less is absorbed by it. Identify the ion.

**11.** Find the maximum Coulomb force that can act on the electron due to the nucleus in a hydrogen atom.

**12.** A hydrogen atom in a state having a binding energy of 0·85 eV makes transition to a state with excitation energy 10·2 eV. (a) Identify the quantum numbers $n$ of the upper and the lower energy states involved in the transition. (b) Find the wavelength of the emitted radiation.

**13.** Whenever a photon is emitted by hydrogen in Balmer series, it is followed by another photon in Lyman series. What wavelength does this latter photon correspond to ?

**14.** A hydrogen atom in state $n = 6$ makes two successive transitions and reaches the ground state. In the first transition a photon of 1·13 eV is emitted. (a) Find the energy of the photon emitted in the second transition. (b) What is the value of $n$ in the intermediate state ?

**15.** What is the energy of a hydrogen atom in the first excited state if the potential energy is taken to be zero in the ground state ?

**16.** A hot gas emits radiation of wavelengths 46·0 nm, 82·8 nm and 103·5 nm only. Assume that the atoms have only two excited states and the difference between consecutive energy levels decreases as energy is increased. Taking the energy of the highest energy state to be zero, find the energies of the ground state and the first excited state.

**17.** A gas of hydrogen-like ions is prepared in a particular excited state $A$. It emits photons having wavelength equal to the wavelength of the first line of the Lyman series together with photons of five other wavelengths. Identify the gas and find the principal quantum number of the state $A$.

**18.** Find the maximum angular speed of the electron of a hydrogen atom in a stationary orbit.

**19.** A spectroscopic instrument can resolve two nearby wavelengths $\lambda$ and $\lambda + \Delta\lambda$ if $\lambda/\Delta\lambda$ is smaller than 8000. This is used to study the spectral lines of the Balmer series of hydrogen. Approximately how many lines will be resolved by the instrument ?

**20.** Suppose, in certain conditions only those transitions are allowed to hydrogen atoms in which the principal quantum number $n$ changes by 2. (a) Find the smallest wavelength emitted by hydrogen. (b) List the wavelengths emitted by hydrogen in the visible range (380 nm to 780 nm).

**21.** According to Maxwell's theory of electrodynamics, an electron going in a circle should emit radiation of frequency equal to its frequency of revolution. What

should be the wavelength of the radiation emitted by a hydrogen atom in ground state if this rule is followed ?

**22.** The average kinetic energy of molecules in a gas at temperature $T$ is 1·5 $kT$. Find the temperature at which the average kinetic energy of the molecules of hydrogen equals the binding energy of its atoms. Will hydrogen remain in molecular form at this temperature ? Take $k = 8{\cdot}62 \times 10^{-5}$ eV $K^{-1}$.

**23.** Find the temperature at which the average thermal kinetic energy is equal to the energy needed to take a hydrogen atom from its ground state to $n = 3$ state. Hydrogen can now emit red light of wavelength 653·1 nm. Because of Maxwellian distribution of speeds, a hydrogen sample emits red light at temperatures much lower than that obtained from this problem. Assume that hydrogen molecules dissociate into atoms.

**24.** Average lifetime of a hydrogen atom excited to $n = 2$ state is $10^{-8}$ s. Find the number of revolutions made by the electron on the average before it jumps to the ground state.

**25.** Calculate the magnetic dipole moment corresponding to the motion of the electron in the ground state of a hydrogen atom.

**26.** Show that the ratio of the magnetic dipole moment to the angular momentum ($l = mvr$) is a universal constant for hydrogen-like atoms and ions. Find its value.

**27.** A beam of light having wavelengths distributed uniformly between 450 nm to 550 nm passes through a sample of hydrogen gas. Which wavelength will have the least intensity in the transmitted beam ?

**28.** Radiation coming from transitions $n = 2$ to $n = 1$ of hydrogen atoms falls on helium ions in $n = 1$ and $n = 2$ states. What are the possible transitions of helium ions as they absorb energy from the radiation ?

**29.** A hydrogen atom in ground state absorbs a photon of ultraviolet radiation of wavelength 50 nm. Assuming that the entire photon energy is taken up by the electron, with what kinetic energy will the electron be ejected ?

**30.** A parallel beam of light of wavelength 100 nm passes through a sample of atomic hydrogen gas in ground state. (a) Assume that when a photon supplies some of its energy to a hydrogen atom, the rest of the energy appears as another photon moving in the same direction as the incident photon. Neglecting the light emitted by the excited hydrogen atoms in the direction of the incident beam, what wavelengths may be observed in the transmitted beam ? (b) A radiation detector is placed near the gas to detect radiation coming perpendicular to the incident beam. Find the wavelengths of radiation that may be detected by the detector.

**31.** A beam of monochromatic light of wavelength λ ejects photoelectrons from a cesium surface ($\Phi = 1{\cdot}9$ eV). These photoelectrons are made to collide with hydrogen atoms in ground state. Find the maximum value of λ for which (a) hydrogen atoms may be ionized, (b) hydrogen atoms may get excited from the ground state to the first excited state and (c) the excited hydrogen atoms may emit visible light.

**32.** Electrons are emitted from an electron gun at almost zero velocity and are accelerated by an electric field $E$ through a distance of 1·0 m. The electrons are now scattered by an atomic hydrogen sample in ground state. What should be the minimum value of $E$ so that red light of wavelength 656·3 nm may be emitted by the hydrogen ?

**33.** A neutron having kinetic energy 12·5 eV collides with a hydrogen atom at rest. Nelgect the difference in mass between the neutron and the hydrogen atom and assume that the neutron does not leave its line of motion. Find the possible kinetic energies of the neutron after the event.

**34.** A hydrogen atom moving at speed $v$ collides with another hydrogen atom kept at rest. Find the minimum value of $v$ for which one of the atoms may get ionized. The mass of a hydrogen atom = $1{\cdot}67 \times 10^{-27}$ kg.

**35.** A neutron moving with a speed $v$ strikes a hydrogen atom in ground state moving towards it with the same speed. Find the minimum speed of the neutron for which inelastic (completely or partially) collision may take place. The mass of neutron ≈ mass of hydrogen = $1{\cdot}67 \times 10^{-27}$ kg.

**36.** When a photon is emitted by a hydrogen atom, the photon carries a momentum with it. (a) Calculate the momentum carried by the photon when a hydrogen atom emits light of wavelength 656·3 nm. (b) With what speed does the atom recoil during this transition ? Take the mass of the hydrogen atom = $1{\cdot}67 \times 10^{-27}$ kg. (c) Find the kinetic energy of recoil of the atom.

**37.** When a photon is emitted from an atom, the atom recoils. The kinetic energy of recoil and the energy of the photon come from the difference in energies between the states involved in the transition. Suppose, a hydrogen atom changes its state from $n = 3$ to $n = 2$. Calculate the fractional change in the wavelength of light emitted, due to the recoil.

**38.** The light emitted in the transition $n = 3$ to $n = 2$ in hydrogen is called $H_\alpha$ light. Find the maximum work function a metal can have so that $H_\alpha$ light can emit photoelectrons from it.

**39.** Light from Balmer series of hydrogen is able to eject photoelectrons from a metal. What can be the maximum work function of the metal ?

**40.** Radiation from hydrogen discharge tube falls on a cesium plate. Find the maximum possible kinetic energy of the photoelectrons. Work function of cesium is 1·9 eV.

**41.** A filter transmits only the radiation of wavelength greater than 440 nm. Radiation from a hydrogen-discharge tube goes through such a filter and is incident on a metal of work function 2·0 eV. Find the stopping potential which can stop the photoelectrons.

**42.** The earth revolves round the sun due to gravitational attraction. Suppose that the sun and the earth are point particles with their existing masses and that Bohr's quantization rule for angular momentum is valid in the case of gravitation. (a) Calculate the minimum radius the earth can have for its orbit. (b) What is the value of the principal quantum number $n$ for the present

radius ? Mass of the earth $= 6{\cdot}0 \times 10^{24}$ kg, mass of the sun $= 2{\cdot}0 \times 10^{30}$ kg, earth–sun distance $= 1{\cdot}5 \times 10^{11}$ m.

**43.** Consider a neutron and an electron bound to each other due to gravitational force. Assuming Bohr's quantization rule for angular momentum to be valid in this case, derive an expression for the energy of the neutron–electron system.

**44.** A uniform magnetic field $B$ exists in a region. An electron projected perpendicular to the field goes in a circle. Assuming Bohr's quantization rule for angular momentum, calculate (a) the smallest possible radius of the electron (b) the radius of the $n$th orbit and (c) the minimum possible speed of the electron.

**45.** Suppose in an imaginary world the angular momentum is quantized to be even integral multiples of $h/2\pi$. What is the longest possible wavelength emitted by hydrogen atoms in visible range in such a world according to Bohr's model ?

**46.** Consider an excited hydrogen atom in state $n$ moving with a velocity $v (v << c)$. It emits a photon in the direction of its motion and changes its state to a lower state $m$. Apply momentum and energy conservation principles to calculate the frequency $\nu$ of the emitted radiation. Compare this with the frequency $\nu_0$ emitted if the atom were at rest.

□

## ANSWERS

### OBJECTIVE I

1. (c) 2. (d) 3. (b) 4. (d) 5. (d) 6. (d)
7. (c) 8. (b) 9. (c) 10. (d) 11. (a) 12. (d)
13. (d)

### OBJECTIVE II

1. (b) 2. (a) 3. (a), (b) 4. (a), (b)
5. (b) 6. all

### EXERCISES

2. (a) 654 nm (b) 4050 nm (c) 38860 nm
3. (a) 91 nm (b) 23 nm (c) 10 nm
4. $1{\cdot}097 \times 10^{7}\ m^{-1}$
5. 3·4 eV
6. (a) 0·265 A, –54·4 eV (b) 4·24 A, –3·4 eV
   (c) 26·5 A, –0·544 eV
7. 1 and 3
8. (a) 40·8 V (b) 122·4 V
9. 487 nm, 654 nm, 1910 nm
10. $He^{+}$
11. $8{\cdot}2 \times 10^{-8}$ N
12. (a) 4, 2 (b) 487 nm
13. 122 nm
14. 12·1 eV, 3
15. 23·8 eV
16. –27 eV, –12 eV
17. $He^{+}$, 4
18. $4{\cdot}1 \times 10^{16}\ rad\ s^{-1}$
19. 38
20. (a) 103 nm (b) 487 nm
21. 45·7 nm
22. $1{\cdot}05 \times 10^{5}$ K
23. $9{\cdot}4 \times 10^{4}$ K
24. $8{\cdot}2 \times 10^{6}$
25. $9{\cdot}2 \times 10^{-24}\ A\ m^{-2}$
26. $\frac{e}{2m} = 8{\cdot}8 \times 10^{10}\ C\ kg^{-1}$
27. 487 nm
28. $n = 2$ to $n = 3$ and $n = 2$ to $n = 4$
29. 11·24 eV
30. (a) 100 nm, 560 nm, 3880 nm
    (b) 103 nm, 121 nm, 654 nm
31. (a) 80 nm (b) 102 nm (c) 89 nm
32. $12{\cdot}1\ Vm^{-1}$
33. zero
34. $7{\cdot}2 \times 10^{4}\ m\ s^{-1}$
35. $3{\cdot}13 \times 10^{4}\ m\ s^{-1}$

36. (a) $1{\cdot}0 \times 10^{27}$ kg m $s^{-1}$ (b) 0·6 m $s^{-1}$
    (c) $1{\cdot}9 \times 10^{-9}$ eV
37. $10^{-9}$
38. 1·9 eV
39. 3·4 eV
40. 11·7 eV
41. 0·55 V
42. (a) $2{\cdot}3 \times 10^{-138}$ m (b) $2{\cdot}5 \times 10^{74}$
43. $-\dfrac{2\pi^2 G^2 m_n^2 m_e^3}{2h^2 n^2}$
44. (a) $\sqrt{\dfrac{h}{2\pi eB}}$ (b) $\sqrt{\dfrac{nh}{2\pi eB}}$ (c) $\sqrt{\dfrac{heB}{2\pi m^2}}$
45. 487 nm
46. $\nu = \nu_0\left(1 + \dfrac{v}{c}\right)$

□

## CHAPTER 44

# X-RAYS

### 44.1 PRODUCTION OF X-RAYS

When highly energetic electrons are made to strike a metal target, electromagnetic radiation comes out. A large part of this radiation has wavelength of the order of 0·1 nm (≈ 1 Å) and is known as *X-ray*.

X-ray was discovered by the German physicist W C Roentgen in 1895. He found that photographic film wrapped light-tight in black paper became exposed when placed near a cathode-ray tube. He concluded that some invisible radiation was coming from the cathode-ray tube which penetrated the black paper to affect the photographic plate. He named this radiation as X-ray because its nature and properties could not be known at that time. In mathematics, we generally use the symbol $x$ for unknown quantities. However, after some calculation we finally get the value of this unknown $x$. Similarly, we now know about the nature and properties of X-rays.

A device used to produce X-rays is generally called an *X-ray tube*. Figure (41.1) shows a schematic diagram of such a device. This was originally designed by Coolidge and is known as *Coolidge tube* to produce X-rays.

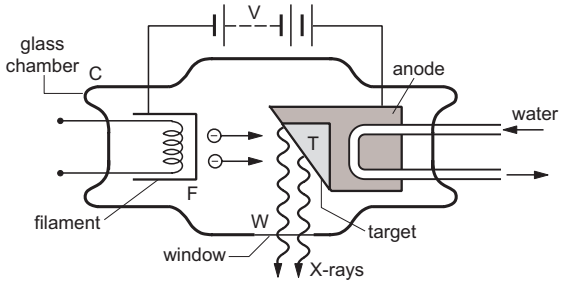



Figure 44.1

A filament $F$ and a metallic target $T$ are fixed in an evacuated glass chamber $C$. The filament is heated electrically and emits electrons by thermionic emission. A constant potential difference of several kilovolts is maintained between the filament and the target using a DC power supply so that the target is at a higher potential than the filament. The electrons emitted by the filament are, therefore, accelerated by the electric field set up between the filament and the target and hit the target with a very high speed. These electrons are stopped by the target and in the process X-rays are emitted. These X-rays are brought out of the tube through a window $W$ made of thin mica or mylar or some such material which does not absorb X-rays appreciably.

In the process, large amount of heat is developed, and thus an arrangement is provided to cool down the tube continuously by running water.

The exact design of the X-ray tube depends on the type of use for which these X-rays are required.

### 44.2 CONTINUOUS AND CHARACTERISTIC X-RAYS

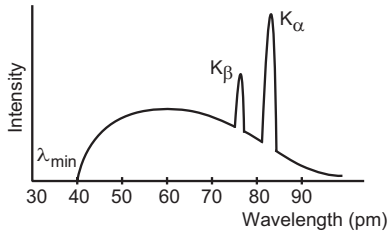



Figure 44.2

If the X-rays coming from a Coolidge tube are examined for the wavelengths present, and the intensity of different wavelength components are measured, we obtain a plot of the nature shown in figure (44.2). We see that there is a minimum wavelength below which no X-ray is emitted. This is called the *cutoff wavelength* or the *threshold wavelength*. The X-rays emitted can be clearly divided in two categories. At certain sharply defined wavelengths, the intensity of X-rays is very large as marked $K_\alpha$, $K_\beta$ in figure (44.2). These X-rays are known as *characteristic X-rays*. At other wavelengths the intensity varies gradually and these X-rays are called *continuous X-rays*. Let us examine the origin of these two types of X-rays.

Suppose, the potential difference applied between the target and the filament is $V$ and electrons are

emitted by the filament with negligible speed. The electrons are accelerated in their journey from the filament to the target. The kinetic energy of an electron when it hits the target is

$$K = eV. \quad \ldots \quad (44.1)$$

As the electron enters into the target material, it readily loses its kinetic energy and is brought to rest inside the metal. The electron before finally being stopped, makes several collisions with the atoms in the target. At each collision, one of the following two processes may occur:

(a) The kinetic energy of the electron is reduced. A part of this lost kinetic energy is converted into a photon of electromagnetic radiation and the remaining part increases the kinetic energy of the colliding particle of the target. The energy received by the colliding particle goes into heating the target. The electron makes another collision with its reduced energy.

(b) The electron knocks out an inner electron of the atom with which it collides.

The fraction of kinetic energy appearing as the energy of a photon varies from collision to collision. In a certain collision, the electron may lose its entire kinetic energy to bring out a photon or it may not create a photon at all. Thus, the energy of the photon created can be anything between 0 and $eV$ depending on how much energy has already been lost to the target and what fraction of the available energy is converted into the photon. The maximum energy of such a photon can be $E = eV$ when the electron converts all its kinetic energy into a photon in the first encounter itself.

The wavelength of the X-ray and the energy of the corresponding photon are related through the equation

$$\lambda = \frac{hc}{E}. \quad \ldots \quad (44.2)$$

As $E$ can take any value between zero and $eV$, the wavelength $\lambda$ can take any value between infinity and $hc/eV$. This explains the origin of continuous X-rays and the cutoff wavelength. We have,

$$\lambda_{min} = \frac{hc}{eV}. \quad \ldots \quad (44.3)$$

We see that the cutoff wavelength $\lambda_{min}$ depends only on the accelerating voltage $V$ applied between the target and the filament. It does not depend on the material of the target.

We shall now discuss what happens if the electron knocks out an inner electron from the atom with which it collides. The electrons in an atom occupy different quantum states characterized by the quantum numbers $n$, $l$, $m_l$, $m_s$. The energy primarily depends on the principal quantum number $n$. The two electrons corresponding to $n = 1$ are said to be in K shell, those corresponding to $n = 2$ are in L shell, etc. Suppose, the incident electron knocks out an electron from the K shell. This will create a vacancy in the K shell in the sense that now there is only one electron with $n$ = 1, whereas two could be accommodated by Pauli exclusion principle. An electron from a higher energy state may make a transition to this vacant state. When such a transition takes place, the difference of energy $\Delta E$ is converted into an X-ray photon of wavelength $\lambda = hc/\Delta E$. *X-rays emitted due to electronic transition from a higher energy state to a vacancy created in the K shell are called K X-rays.*

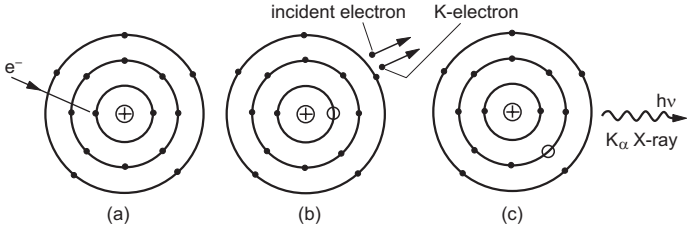



Figure 44.3

Figure (44.3) shows the process schematically. If an electron from the L shell (i.e., with $n = 2$) makes transition to the vacant state in the K shell, the X-ray emitted is called $K_\alpha$ *X-ray*. If an electron from the M shell makes transition to the K shell, a $K_\beta$ X-ray is emitted. Similarly one defines $K_\gamma$ X-ray. If a photon of $K_\alpha$ X-ray is emitted, the vacancy in the K shell is filled up but a vacancy is created in the L shell. This vacancy can be filled up by a transition of electron from higher shells giving L X-ray. If an electron jumps from the M shell to the vacant state in the L shell, we obtain $L_\alpha$ X-ray. If the vacancy in L shell is filled up by an electron of N shell ($n = 4$), $L_\beta$ X-ray is emitted, and so on.

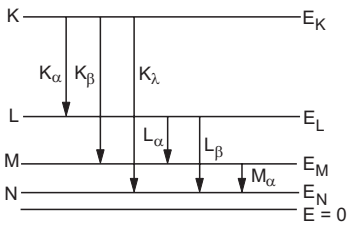



Figure 44.4

Figure (44.4) shows the energy levels of the atom when one electron is knocked out. The lowest line corresponds to the atom with all its electrons intact. This has been taken as zero energy. The energy level with label $E_K$ is the energy of the atom when an electron from the K shell is knocked out. Similar is the interpretation for $E_L$, $E_M$, $E_N$, etc. Note the convention of choosing $E = 0$ in the ground state. In hydrogen atom we had chosen $E = 0$ when the electron was knocked out. The ground state of hydrogen atom

then had an energy of –13·6 eV. Here the convention is opposite and the energy in the ground state is chosen to be zero. The energy in the ionized state is then positive. As the electrons in K shell are most tightly bound, maximum energy is to be given to the atom to knock out an electron from the K shell. That is why, in figure (44.4), the energy level of the atom with a vacancy in the K shell is shown highest.

The energies $E_K$, $E_L$, …, etc., are characteristic properties of the material. For different materials, the values of these energies will be different. The values of $E_K - E_L$, $E_L - E_M$, etc., also have definite values for a given material. The wavelengths of the X-rays emitted corresponding to these transitions are

$$\lambda = \frac{hc}{E_K - E_L} \text{ for } K_\alpha,$$

$$\lambda = \frac{hc}{E_K - E_M} \text{ for } K_\beta,$$

$$\lambda = \frac{hc}{E_L - E_M} \text{ for } L_\alpha,$$

etc. These wavelengths, therefore, have definite values for a particular material. The X-rays emitted in this way are the characteristic X-rays shown in figure (44.2). They are so named because their wavelengths may be used to identify the element from which they originate.

### 44.3 SOFT AND HARD X-RAYS

If the accelerating voltage applied between the filament and the target is increased, the cutoff wavelength $\lambda_{min}$ decreases further (equation 44.3). The X-rays of low wavelengths are called *hard X-rays* and those of large wavelength are called *soft X-rays*. Hard and soft are simply relative terms. In terms of energy, harder X-rays means more energy in each photon. So, if the voltage between the filament and the target is increased, we get harder X-rays. If the filament current in a Coolidge tube is increased by increasing the voltage in the filament circuit, more electrons are emitted per unit time. This results in an increase in the number of X-ray photons emitted per unit time and hence the intensity of X-rays is increased. The cutoff wavelength $\lambda_{min}$ remains unchanged as the maximum kinetic energy of the electrons reaching the target is not affected by the filament current.

### 44.4 MOSELEY'S LAW

Moseley's experiments (1913–1914) on characteristic X-rays played a very important role in developing the concept of atomic number. In those days, the elements were arranged in periodic table in the increasing order of atomic weight. The periodicity in chemical properties of elements was brought out from such arrangement, though some anomalies were present. Bohr had proposed his model in the same year and there was no concept of distribution of electrons in different energy levels. During those days, Moseley measured the frequencies of characteristic X-rays from a large number of elements and plotted the square root of the frequency against its position number in the periodic table. He discovered that the plot is very close to a straight line. A portion of Moseley's plot is shown in figure (44.5) where $\sqrt{\nu}$ of $K_\alpha$ X-rays is plotted against the position number. From this linear relation, Moseley concluded that there must be a fundamental property of the atom which increases by regular steps as one moves from one element to the other. This quantity was later identified to be the number of protons in the nucleus and was referred to as the *atomic number*.

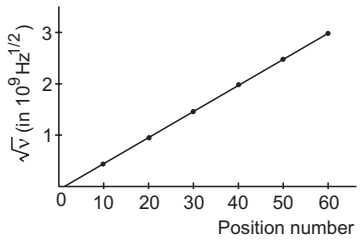



Figure 44.5

Thus, elements should be arranged in the ascending order of atomic number and not of atomic weight. This removed several discrepancies existing in the periodic table. For example, nickel has atomic weight 58·7 whereas the atomic weight of cobalt is 58·9. However, the frequency of $K_\alpha$ X-ray from cobalt is less than the frequency of $K_\alpha$ X-ray from nickel. Thus, Moseley rearranged the sequence as Co, Ni instead of Ni, Co. Similarly, several other rearrangements were made.

Moseley's observations can be mathematically expressed as

$$\sqrt{\nu} = a(Z - b) \quad \ldots \quad (44.4)$$

where $a$ and $b$ are constants. This relation is known as *Moseley's law*. We can understand Moseley's law qualitatively from Bohr's atomic model.

Consider an atom from which an electron from the K shell has been knocked out. Consider an electron from the L shell which is about to make a transition to the vacant site. It finds the nucleus of charge $Ze$ screened by the spherical cloud of the remaining one electron in the K shell (figure 44.6). If we neglect the effect of the outer electrons and the other $L$ electrons, the electron making the transition finds a charge $(Z - 1)e$ at the centre. One, therefore, may expect Bohr's model to give reasonable results if $Z$ is replaced by $Z - b$ with $b \approx 1$.

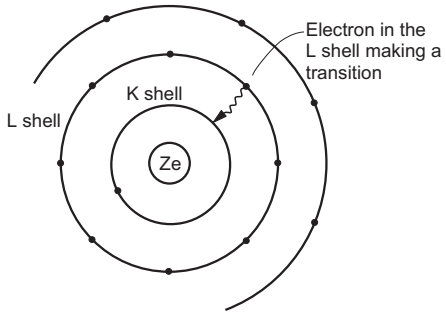



Figure 44.6

According to Bohr's model, the energy released during the transition from $n = 2$ to $n = 1$ is given by

$$\Delta E = h\nu = Rhc(Z - b)^2 \left(\frac{1}{1^2} - \frac{1}{2^2}\right)$$

so that $\sqrt{\nu} = \sqrt{\frac{3Rc}{4}}\ (Z - b)$

which is the same as equation (44.4) with $a = \sqrt{3Rc/4}$.

Moseley was killed in the First World War at an early age of 27 years.

### 44.5 BRAGG'S LAW

X-rays are electromagnetic waves of short wavelengths and may be diffracted by suitable diffracting centres. However, the diffraction effects are appreciable only when the diffracting apertures are of the order of the wavelength, i.e., of the order of 0·1 nm. This is almost the size of an atom and it is difficult to construct slits with such small gaps so that X-rays can be appreciably diffracted.

In solid crystals, atoms are arranged in fairly regular pattern with interatomic gaps of the order of 0·1 nm. Common salt is an example of such a crystalline solid. Almost all the metals at ordinary temperature are crystalline. These metals may act as natural three-dimensional gratings for the diffraction of X-rays.

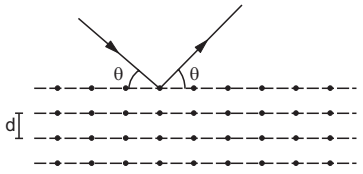



Figure 44.7

The structure of a solid can be viewed as a series of parallel planes of atoms separated by a distance $d$ (figure 44.7). Suppose, an X-ray beam is incident on a solid, making an angle θ with the planes of the atoms. These X-rays are diffracted by different atoms and the diffracted rays interfere. In certain directions, the interference is constructive and we obtain strong reflected X-rays. The analysis shows that there will be a strong reflected X-ray beam only if

$$2d \sin\theta = n\lambda \qquad \ldots \quad (44.5)$$

where $n$ is an integer. For monochromatic X-rays, λ is fixed and there are some specific angles $\theta_1$, $\theta_2$, $\theta_3$, …, etc., corresponding to $n = 1, 2, 3, \ldots$, etc., in equation (44.5). Thus, if the X-rays are incident at one of these angles, they are reflected; otherwise they are absorbed. When they are reflected, the laws of reflection are obeyed, i.e., (a) the angle of incidence is equal to the angle of reflection and (b) the incident ray, the reflected ray and the normal to the reflecting plane are coplanar.

Equation (44.5) is known as *Bragg's law*.

By using a monochromatic X-ray beam and noting the angles of strong reflection, the interplanar spacing $d$ and several informations about the structure of the solid can be obtained.

### 44.6 PROPERTIES AND USES OF X-RAYS

As discussed earlier, X-rays are electromagnetic waves of short wavelengths. Accordingly, it has many properties common with light. Here are some of the properties of X-rays.

(a) X-rays travel in straight lines in vacuum at a speed equal to that of light ($3 \times 10^8$ m/s).

(b) X-rays are diffracted by crystals in accordance with Bragg's law.

(c) X-rays are not deflected by electric or magnetic field as it contains no charged particles.

(d) X-rays affect a photographic plate more strongly than visible light.

(e) When incident on certain materials such as barium platinocyanide, X-rays cause fluorescence (light is emitted from the material).

(f) When passed through a gas, X-rays ionize the molecules of the gas.

(g) X-rays can penetrate into several metals and other materials. Thus, they can pass through small thicknesses of aluminium, woods, plastics, human flesh etc. They are stopped by materials of high density and high atomic number.

The penetrating power of X-ray has made it popular and familiar to the general public. It is used extensively to detect diseases inside the body. It passes quite freely through the flesh but is stopped by the bones. So it can photograph the bones inside the body on a photographic film. Such a photograph is called a *radiograph*. This is used to detect and study bone fractures due to an accident. Chest radiographs are used to study diseases in lungs. Dentists also use X-ray to study teeth-decay. X-ray is also used in cancer therapy.

X-ray is used in industry and material science research. It can detect structural defects, fault of joints, welding etc. X-ray machines are used to inspect suitcases, wooden boxes etc. without opening them and can be typically found at the custom, security counters at airports etc.

X-ray can be used to detect diseases and to cure them. At the same time, random and excess exposure to X-ray may induce diseases. X-ray has a damaging effect on the living cells of a body which may lead to cell-death. High exposure for a long period (say for years) may lead to cancer or genetic defects.

TV, computer terminals, oscilloscopes etc. use a cathode-ray tube in which a highly energetic electron beam strikes the screen. A fraction of kinetic energy is converted into X-rays which may come out. Screens of these equipments are generally designed to absorb these X-rays.

## *Worked Out Examples*

**1.** *Find the maximum frequency of the X-rays emitted by an X-ray tube operating at* 30 kV.

***Solution*** : For maximum frequency, the total kinetic energy ($eV$) should be converted into an X-ray photon. Thus,

$$h\nu = eV$$

or, $$\nu = \frac{e}{h} V$$

$$= \frac{e \times 30 \times 10^{3} \text{ V}}{4{\cdot}14 \times 10^{-15} \text{ eV–s}}$$

$$= \frac{30}{4{\cdot}14} \times 10^{18} \text{ Hz} \approx 7{\cdot}2 \times 10^{18} \text{ Hz}.$$

**2.** *An X-ray tube operates at* 20 kV. *A particular electron loses* 5% *of its kinetic energy to emit an X-ray photon at the first collision. Find the wavelength corresponding to this photon.*

***Solution*** : Kinetic energy acquired by the electron is

$$K = eV = 20 \times 10^{3} \text{ eV}.$$

The energy of the photon

$$= 0{\cdot}05 \times 20 \times 10^{3} \text{ eV} = 10^{3} \text{ eV}.$$

Thus, $$\frac{hc}{\lambda} = 10^{3} \text{ eV}$$

or, $$\lambda = \frac{hc}{10^{3} \text{ eV}}$$

$$= \frac{(4{\cdot}14 \times 10^{-15} \text{ eVs}) \times (3 \times 10^{8} \text{ m s}^{-1})}{10^{3} \text{ eV}}$$

$$= \frac{1242 \text{ eV nm}}{10^{3} \text{ eV}} = 1{\cdot}24 \text{ nm}.$$

**3.** *An X-ray tube is operated at* 20 kV *and the current through the tube is* 0·5 mA. *Find (a) the number of electrons hitting the target per second, (b) the energy falling on the target per second as the kinetic energy of the electrons and (c) the cutoff wavelength of the X-rays emitted.*

***Solution*** : (a) $$i = ne = 0{\cdot}5 \times 10^{-3} \text{ A}$$

or, $$n = \frac{0{\cdot}5 \times 10^{-3} \text{ A}}{1{\cdot}6 \times 10^{-19} \text{ C}} = 3{\cdot}1 \times 10^{15}.$$

(b) The kinetic energy of an electron reaching the target is $K = eV$. The energy falling on the target per second

$$= n\,eV = iV = (0{\cdot}5 \times 10^{-3} \text{ A}) \times (20 \times 10^{3} \text{ V})$$

$$= 10 \text{ J s}^{-1}.$$

(c) $$\frac{hc}{\lambda_{\min}} = eV$$

or, $$\lambda_{\min} = \frac{hc}{eV}$$

$$= \frac{1242 \text{ eV nm}}{e(20 \times 10^{3} \text{ V})} = 0{\cdot}062 \text{ nm}.$$

**4.** *Find the constants $a$ and $b$ in Moseley's equation $\sqrt{\nu} = a(Z - b)$ from the following data.*

| Element | $Z$ | Wavelength of $K_\alpha$ X-ray |
|---|---|---|
| Mo | 42 | 71 pm |
| Co | 27 | 178·5 pm |

***Solution*** : Moseley's equation is

$$\sqrt{\nu} = a(Z - b).$$

Thus, $$\sqrt{\frac{c}{\lambda_1}} = a(Z_1 - b) \quad \ldots \text{(i)}$$

and $$\sqrt{\frac{c}{\lambda_2}} = a(Z_2 - b). \quad \ldots \text{(ii)}$$

From (i) and (ii),

$$\sqrt{c}\left(\frac{1}{\sqrt{\lambda_1}} - \frac{1}{\sqrt{\lambda_2}}\right) = a(Z_1 - Z_2)$$

or, $$a = \frac{\sqrt{c}}{(Z_1 - Z_2)}\left(\frac{1}{\sqrt{\lambda_1}} - \frac{1}{\sqrt{\lambda_2}}\right)$$

$$= \frac{(3 \times 10^{8} \text{ m s}^{-1})^{1/2}}{42 - 27}\left[\frac{1}{(71 \times 10^{-12} \text{ m})^{1/2}} - \frac{1}{(178{\cdot}5 \times 10^{-12} \text{ m})^{1/2}}\right]$$

$$= 5{\cdot}0 \times 10^{7} \text{ (Hz)}^{1/2}.$$

Dividing (i) by (ii),

$$\sqrt{\frac{\lambda_2}{\lambda_1}} = \frac{Z_1 - b}{Z_2 - b}$$

or, $$\sqrt{\frac{178{\cdot}5}{71}} = \frac{42 - b}{27 - b}$$

or, $$b = 1{\cdot}37.$$

**5.** *The $K_\alpha$ X-ray of molybdenum has wavelength* 71 pm. *If the energy of a molybdenum atom with a K electron knocked out is* 23·32 keV, *what will be the energy of this atom when an L electron is knocked out* ?

***Solution*** : $K_\alpha$ X-ray results from the transition of an electron from L shell to K shell. If the energy of the atom with a vacancy in the K shell is $E_K$ and the energy with a vacancy in the L shell is $E_L$, the energy of the photon emitted is $E_K - E_L$. The energy of the 71 pm photon is

$$E = \frac{hc}{\lambda}$$

$$= \frac{1242 \text{ eV nm}}{71 \times 10^{-3} \text{ nm}} = 17{\cdot}5 \text{ keV}.$$

Thus, $E_K - E_L = 17{\cdot}5$ keV

or, $E_L = E_K - 17{\cdot}5$ keV

$= 23{\cdot}32$ keV $- 17{\cdot}5$ keV $= 5{\cdot}82$ keV.

**6.** *Show that the frequency of $K_\beta$ X-ray of a material equals the sum of the frequencies of $K_\alpha$ and $L_\alpha$ X-rays of the same material.*

***Solution*** :

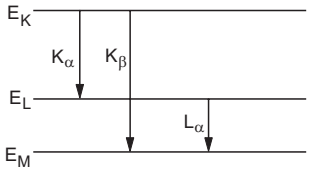



Figure 44-W1

The energy level diagram of an atom with one electron knocked out is shown in figure (44-W1).

Energy of $K_\alpha$ X-ray is $E_{K_\alpha} = E_K - E_L$

of $K_\beta$ X-ray is $E_{K_\beta} = E_K - E_M$,

and of $L_\alpha$ X-ray is $E_{L_\alpha} = E_L - E_M$.

Thus, $E_{K_\beta} = E_{K_\alpha} + E_{L_\alpha}$

or, $h\nu_{K_\beta} = h\nu_{K_\alpha} + h\nu_{L_\alpha}$

or, $\nu_{K_\beta} = \nu_{K_\alpha} + \nu_{L_\alpha}$.

□

## QUESTIONS FOR SHORT ANSWER

**1.** When a Coolidge tube is operated for some time it becomes hot. Where does the heat come from ?

**2.** In a Coolidge tube, electrons strike the target and stop inside it. Does the target get more and more negatively charged as time passes ?

**3.** Can X-rays be used for photoelectric effect ?

**4.** Can X-rays be polarized ?

**5.** X-ray and visible light travel at the same speed in vacuum. Do they travel at the same speed in glass ?

**6.** Characteristic X-rays may be used to identify the element from which they are coming. Can continuous X-rays be used for this purpose ?

**7.** Is it possible that in a Coolidge tube characterstic $L_\alpha$ X-rays are emitted but not $K_\alpha$ X-rays ?

**8.** Can $L_\alpha$ X-ray of one material have shorter wavelength than $K_\alpha$ X-ray of another ?

**9.** Can a hydrogen atom emit characteristic X-ray ?

**10.** Why is exposure to X-ray injurious to health but exposure to visible light is not, when both are electromagnetic waves ?

## OBJECTIVE I

**1.** X-ray beam can be deflected
(a) by an electric field (b) by a magnetic field
(c) by an electric field as well as by a magnetic field
(d) neither by an electric field nor by a magnetic field.

**2.** Consider a photon of continuous X-ray coming from a Coolidge tube. Its energy comes from
(a) the kinetic energy of the striking electron
(b) the kinetic energy of the free electrons of the target
(c) the kinetic energy of the ions of the target
(d) an atomic transition in the target.

**3.** The energy of a photon of characteristic X-ray from a Coolidge tube comes from
(a) the kinetic energy of the striking electron
(b) the kinetic energy of the free electrons of the target
(c) the kinetic energy of the ions of the target
(d) an atomic transition in the target.

**4.** If the potential difference applied to the tube is doubled and the separation between the filament and the target is also doubled, the cutoff wavelength
(a) will remain unchanged (b) will be doubled

(c) will be halved
(d) will become four times the original.

**5.** If the current in the circuit for heating the filament is increased, the cutoff wavelength
(a) will increase (b) will decrease
(c) will remain unchanged (d) will change.

**6.** Moseley's law for characteristic X-rays is $\sqrt{\nu} = a(Z - b)$. In this,
(a) both $a$ and $b$ are independent of the material
(b) $a$ is independent but $b$ depends on the material
(c) $b$ is independent but $a$ depends on the material
(d) both $a$ and $b$ depend on the material.

**7.** Frequencies of $K_\alpha$ X-rays of different materials are measured. Which one of the graphs in figure (44-Q1) may represent the relation between the frequency $\nu$ and the atomic number $Z$.

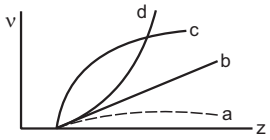



Figure 44-Q1

**8.** The X-ray beam coming from an X-ray tube
(a) is monochromatic
(b) has all wavelengths smaller than a certain maximum wavelength
(c) has all wavelengths greater than a certain minimum wavelength
(d) has all wavelengths lying between a minimum and a maximum wavelength.

**9.** One of the following wavelengths is absent and the rest are present in the X-rays coming from a Coolidge tube. Which one is the absent wavelength ?
(a) 25 pm (b) 50 pm (c) 75 pm (d) 100 pm.

**10.** Figure (44-Q2) shows the intensity–wavelength relations of X-rays coming from two different Coolidge tubes. The solid curve represents the relation for the tube $A$ in which the potential difference between the target and the filament is $V_A$ and the atomic number of the target material is $Z_A$. These quantities are $V_B$ and $Z_B$ for the other tube. Then,
(a) $V_A > V_B$, $Z_A > Z_B$ (b) $V_A > V_B$, $Z_A < Z_B$
(c) $V_A < V_B$, $Z_A > Z_B$ (d) $V_A < V_B$, $Z_A < Z_B$.

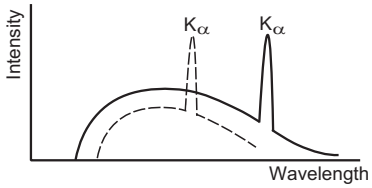



Figure 44-Q2

**11.** 50% of the X-ray coming from a Coolidge tube is able to pass through a 0·1 mm thick aluminium foil. If the potential difference between the target and the filament is increased, the fraction of the X-ray passing through the same foil will be
(a) 0% (b) < 50% (c) 50% (d) > 50%.

**12.** 50% of the X-ray coming from a Coolidge tube is able to pass through a 0·1 mm thick aluminium foil. The potential difference between the target and the filament is increased. The thickness of aluminium foil, which will allow 50% of the X-ray to pass through, will be
(a) zero (b) < 0·1 mm (c) 0·1 mm (d) > 0·1 mm.

**13.** X-ray from a Coolidge tube is incident on a thin aluminium foil. The intensity of the X-ray transmitted by the foil is found to be $I_0$. The heating current is increased so as to increase the temperature of the filament. The intensity of the X-ray transmitted by the foil will be
(a) zero (b) < $I_0$ (c) $I_0$ (d) > $I_0$.

**14.** Visible light passing through a circular hole forms a diffraction disc of radius 0·1 mm on a screen. If X-ray is passed through the same set-up, the radius of the diffraction disc will be
(a) zero (b) < 0·1 mm (c) 0·1 mm (d) > 0·1 mm.

## OBJECTIVE II

**1.** For harder X-rays,
(a) the wavelength is higher
(b) the intensity is higher
(c) the frequency is higher
(d) the photon energy is higher.

**2.** Cutoff wavelength of X-rays coming from a Coolidge tube depends on the
(a) target material (b) accelerating voltage
(c) separation between the target and the filament
(d) temperature of the filament.

**3.** Mark the correct options.
(a) An atom with a vacancy has smaller energy than a neutral atom.
(b) K X-ray is emitted when a hole makes a jump from the K shell to some other shell.
(c) The wavelength of K X-ray is smaller than the wavelength of L X-ray of the same material.
(d) The wavelength of $K_\alpha$ X-ray is smaller than the wavelength of $K_\beta$ X-ray of the same material.

**4.** For a given material, the energy and wavelength of characterstic X-rays satisfy
(a) $E(K_\alpha) > E(K_\beta) > E(K_\gamma)$ (b) $E(M_\alpha) > E(L_\alpha) > E(K_\alpha)$
(c) $\lambda(K_\alpha) > \lambda(K_\beta) > \lambda(K_\gamma)$ (d) $\lambda(M_\alpha) > \lambda(L_\alpha) > \lambda(K_\alpha)$.

**5.** The potential difference applied to an X-ray tube is increased. As a result, in the emitted radiation,
(a) the intensity increases
(b) the minimum wavelength increases
(c) the intensity remains unchanged
(d) the minimum wavelength decreases.

**6.** When an electron strikes the target in a Coolidge tube, its entire kinetic energy

(a) is converted into a photon
(b) may be converted into a photon
(c) is converted into heat
(d) may be converted into heat.

**7.** X-ray incident on a material
(a) exerts a force on it (b) transfers energy to it
(c) transfers momentum to it
(d) transfers impulse to it.

**8.** Consider a photon of continuous X-ray and a photon of characteristic X-ray of the same wavelength. Which of the following is/are different for the two photons ?
(a) Frequency (b) Energy
(c) Penetrating power (d) Method of creation

## EXERCISES

Planck constant $h = 4{\cdot}14 \times 10^{-15}$ eV s$^{-1}$, speed of light $c = 3 \times 10^{8}$ m s$^{-1}$.

**1.** Find the energy, the frequency and the momentum of an X-ray photon of wavelength 0·10 nm.

**2.** Iron emits $K_\alpha$ X-ray of energy 6·4 keV and calcium emits $K_\alpha$ X-ray of energy 3·69 keV. Calculate the times taken by an iron $K_\alpha$ photon and a calcium $K_\alpha$ photon to cross through a distance of 3 km.

**3.** Find the cutoff wavelength for the continuous X-rays coming from an X-ray tube operating at 30 kV.

**4.** What potential difference should be applied across an X-ray tube to get X-ray of wavelength not less than 0·10 nm ? What is the maximum energy of a photon of this X-ray in joule ?

**5.** The X-ray coming from a Coolidge tube has a cutoff wavelength of 80 pm. Find the kinetic energy of the electrons hitting the target.

**6.** If the operating potential in an X-ray tube is increased by 1%, by what percentage does the cutoff wavelength decrease ?

**7.** The distance between the cathode (filament) and the target in an X-ray tube is 1·5 m. If the cutoff wavelength is 30 pm, find the electric field between the cathode and the target.

**8.** The short-wavelength limit shifts by 26 pm when the operating voltage in an X-ray tube is increased to 1·5 times the original value. What was the original value of the operating voltage ?

**9.** The electron beam in a colour TV is accelerated through 32 kV and then strikes the screen. What is the wavelength of the most energetic X-ray photon ?

**10.** When 40 kV is applied across an X-ray tube, X-ray is obtained with a maximum frequency of $9{\cdot}7 \times 10^{18}$ Hz. Calculate the value of Planck constant from these data.

**11.** An X-ray tube operates at 40 kV. Suppose the electron converts 70% of its energy into a photon at each collision. Find the lowest three wavelengths emitted from the tube. Neglect the energy imparted to the atom with which the electron collides.

**12.** The wavelength of $K_\alpha$ X-ray of tungsten is 21·3 pm. It takes 11·3 keV to knock out an electron from the L shell of a tungsten atom. What should be the minimum accelerating voltage across an X-ray tube having tungsten target which allows production of $K_\alpha$ X-ray ?

**13.** The $K_\beta$ X-ray of argon has a wavelength of 0·36 nm. The minimum energy needed to ionize an argon atom is 16 eV. Find the energy needed to knock out an electron from the K shell of an argon atom.

**14.** The $K_\alpha$ X-rays of aluminium ($Z = 13$) and zinc ($Z = 30$) have wavelengths 887 pm and 146 pm respectively. Use Moseley's law $\sqrt{\nu} = a(Z - b)$ to find the wavelength of the $K_\alpha$ X-ray of iron ($Z = 26$).

**15.** A certain element emits $K_\alpha$ X-ray of energy 3·69 keV. Use the data from the previous problem to identify the element.

**16.** The $K_\beta$ X-rays from certain elements are given below. Draw a Moseley-type plot of $\sqrt{\nu}$ versus $Z$ for $K_\beta$ radiation.

| Element | Ne | P | Ca | Mn | Zn | Br |
|---|---|---|---|---|---|---|
| Energy (keV) | 0·858 | 2·14 | 4·02 | 6·51 | 9·57 | 13·3. |

**17.** Use Moseley's law with $b = 1$ to find the frequency of the $K_\alpha$ X-ray of La($Z = 57$) if the frequency of the $K_\alpha$ X-ray of Cu($Z = 29$) is known to be $1{\cdot}88 \times 10^{18}$ Hz.

**18.** The $K_\alpha$ and $K_\beta$ X-rays of molybdenum have wavelengths 0·71 Å and 0·63 Å respectively. Find the wavelength of $L_\alpha$ X-ray of molybdenum.

**19.** The wavelengths of $K_\alpha$ and $L_\alpha$ X-rays of a material are 21·3 pm and 141 pm respectively. Find the wavelength of $K_\beta$ X-ray of the material.

**20.** The energy of a silver atom with a vacancy in K shell is 25·31 keV, in L shell is 3·56 keV and in M shell is 0·530 keV higher than the energy of the atom with no vacancy. Find the frequency of $K_\alpha$, $K_\beta$ and $L_\alpha$ X-rays of silver.

**21.** Find the maximum potential difference which may be applied across an X-ray tube with tungsten target without emitting any characteristic K or L X-ray. The energy levels of the tungsten atom with an electron knocked out are as follows.

| Cell containing vacancy | K | L | M |
|---|---|---|---|
| Energy in keV | 69.5 | 11.3 | 2.3 |

**22.** The electric current in an X-ray tube (from the target to the filament) operating at 40 kV is 10 mA. Assume that on an average, 1% of the total kinetic energy of the electrons hitting the target are converted into X-rays. (a) What is the total power emitted as X-rays and (b) how much heat is produced in the target every second ?

**23.** Heat at the rate of 200 W is produced in an X-ray tube operating at 20 kV. Find the current in the circuit. Assume that only a small fraction of the kinetic energy of electrons is converted into X-rays.

**24.** Continuous X-rays are made to strike a tissue paper soaked with polluted water. The incoming X-rays excite the atoms of the sample by knocking out the electrons from the inner shells. Characteristic X-rays are subsequently emitted. The emitted X-rays are analysed and the intensity is plotted against the wavelength (figure 44-E1). Assuming that only $K_\alpha$ intensities are detected, list the elements present in the sample from the plot. Use Moseley's equation

$$\nu = (25 \times 10^{14}\ \text{Hz})\,(Z-1)^2.$$

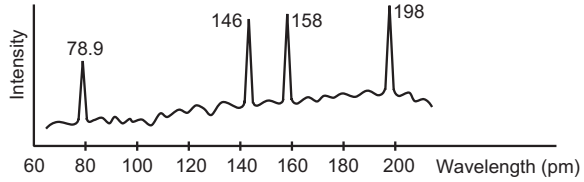



Figure 44-E1

**25.** A free atom of iron emits $K_\alpha$ X-rays of energy 6·4 keV. Calculate the recoil kinetic energy of the atom. Mass of an iron atom = $9{\cdot}3 \times 10^{-26}$ kg.

**26.** The stopping potential in a photoelectric experiment is linearly related to the inverse of the wavelength ($1/\lambda$) of the light falling on the cathode. The potential difference applied across an X-ray tube is linearly related to the inverse of the cutoff wavelength ($1/\lambda$) of the X-ray emitted. Show that the slopes of the lines in the two cases are equal and find its value.

**27.** Suppose a monochromatic X-ray beam of wavelength 100 pm is sent through a Young's double slit and the interference pattern is observed on a photographic plate placed 40 cm away from the slit. What should be the separation between the slits so that the successive maxima on the screen are separated by a distance of 0·1 mm ?

□

## ANSWERS

### OBJECTIVE I

1. (d) 2. (a) 3. (d) 4. (c) 5. (c) 6. (a)
7. (d) 8. (c) 9. (a) 10. (b) 11. (d) 12. (d)
13. (d) 14. (b)

### OBJECTIVE II

1. (c), (d) 2. (b) 3. (b), (c)
4. (c), (d) 5. (c), (d) 6. (b), (d)
7. all 8. (d)

### EXERCISES

1. 12·4 keV, $3 \times 10^{18}$ Hz, $6{\cdot}62 \times 10^{-24}$ kg m s$^{-1}$
2. 10 μs by both
3. 41·4 pm
4. 12·4 kV, $2{\cdot}0 \times 10^{-15}$ J
5. 15·5 keV
6. approximately 1%
7. 27·7 kV m$^{-1}$
8. 15·9 kV
9. 38·8 pm
10. $4{\cdot}12 \times 10^{-15}$ eVs
11. 44·3 pm, 148 pm, 493 pm
12. 69·5 kV
13. 3·47 keV
14. 198 pm
15. calcium
17. $7{\cdot}52 \times 10^{18}$ Hz
18. 5·64 Å
19. 18·5 pm
20. $5{\cdot}25 \times 10^{18}$ Hz, $5{\cdot}98 \times 10^{18}$ Hz, $7{\cdot}32 \times 10^{17}$ Hz
21. less than 11·3 kV
22. (a) 4 W (b) 396 J
23. 10 mA
24. Zr, Zn, Cu, Fe
25. $3{\cdot}9 \times 10^{-4}$ eV
26. $\frac{hc}{e} = 1{\cdot}242 \times 10^{-6}$ Vm
27. $4 \times 10^{-7}$ m

□

## CHAPTER 45

# SEMICONDUCTORS AND SEMICONDUCTOR DEVICES

### 45.1 INTRODUCTION

We have discussed some of the properties of conductors and insulators in earlier chapters. We assumed that there is a large number of almost free electrons in a conductor which wander randomly in the whole of the body, whereas, all the electrons in an insulator are tightly bound to some nucleus or the other. If an electric field $\vec{E}$ is established inside a conductor, the free electrons experience force due to the field and acquire a drift speed. This results in an electric current. The conductivity $\sigma$ is defined in terms of the electric field $\vec{E}$ existing in the conductor and the resulting current density $\vec{j}$. The relation between these quantities is

$$\vec{j} = \sigma \vec{E}.$$

Larger the conductivity $\sigma$, better is the material as a conductor.

The conductivity $\sigma$ of a conductor, such as copper, is fairly independent of the electric field applied and decreases as the temperature is increased. This is because as the temperature is increased, the random collisions of the free electrons with the particles in the conductor become more frequent. The electrons get less time to gain energy from the applied electric field. This results in a decrease in the drift speed and hence the conductivity decreases. The resistivity $\rho = 1/\sigma$ of a conductor increases as the temperature increases.

Almost zero electric current is obtained in insulators unless a very high electric field is applied.

We now introduce another kind of solid known as *semiconductor*. These solids do conduct electricity when an electric field is applied, but the conductivity is very small as compared to the usual metallic conductors. Silicon is an example of a semiconductor, its conductivity is about $10^{11}$ times smaller than that of copper and is about $10^{13}$ times larger than that of fused quartz. Another distinguishing feature about a semiconductor is that its conductivity increases as the temperature is increased. To understand the mechanism of conduction in solids, let us discuss qualitatively, formation of energy bands in solids.

### 45.2 ENERGY BANDS IN SOLIDS

The electrons of an isolated atom can have certain definite energies labelled as 1s, 2s, 2p, 3s, etc. Pauli exclusion principle determines the maximum number of electrons which can be accommodated in each energy level. An energy level consists of several quantum states and no quantum state can contain more than one electron. Consider a sodium atom in its lowest energy state. It has 11 electrons. The electronic configuration is $(1s)^2\ (2s)^2\ (2p)^6\ (3s)^1$. The levels 1s, 2s and 2p are completely filled and the level 3s contains only one electron although it has a capacity to accommodate 2. The next allowed energy level is 3p which can contain 6 electrons but is empty. All the energy levels above 3s are empty.

Now consider a group of $N$ sodium atoms separated from each other by large distances such as in sodium vapour. There are altogether $11N$ electrons. Assuming that each atom is in its ground state, what are the energies of these $11N$ electrons ? For each atom, there are two states in energy level 1s. There are $2N$ such states which have identical energy and are filled by $2N$ electrons. Similarly, there are $2N$ states having identical energy labelled 2s, $6N$ states having identical energy labelled 2p and $2N$ states having identical energy labelled 3s. The $2N$ states of 1s, $2N$ states of 2s and $6N$ states of 2p are completely filled whereas only $N$ of the $2N$ states of 3s are filled by the electrons and the remaining $N$ states are empty. These ideas are shown in table (45.1) and figure (45.1).

**Table 45.1 : *Quantum states in sodium vapour***

| Energy level | Total states available | Total states occupied |
|---|---|---|
| 1s | 2*N* | 2*N* |
| 2s | 2*N* | 2*N* |
| 2p | 6*N* | 6*N* |
| 3s | 2*N* | *N* |
| 3p | 6*N* | 0 |

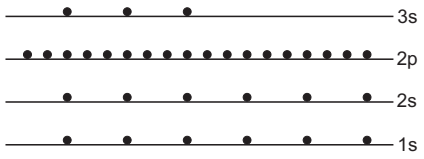



Figure 45.1

The value of energy in a particular state of an isolated sodium atom is determined by the mutual interactions among the nucleus and the 11 electrons. In the collection of the $N$ sodium atoms that we have considered, it is assumed that the atoms are widely separated from each other and hence the electrons of one atom do not interact with those of the others to any appreciable extent. As a result, the energy of 1s states of each atom is the same as that for an isolated atom. All the 1s states, therefore, have identical energy. Similarly, all the 2s, 2p, 3s states have identical energies, respectively. Now suppose, the atoms are drawn closer to one another to the extent that the outer 3s electron of one atom starts interacting with the 3s electrons of the neighbouring atoms. Because of these interactions, the energy of the 3s states will change. It turns out that the changes in energy in all the 2$N$ states of 3s level are not identical. Some of the states are shifted up in energy and some are pushed down. The magnitude of change is also different for different states of the 3s level. As a result, what was a sharply defined 3s energy, now becomes a combination of several closely spaced energies. We say that these 2$N$ states have formed an *energy band*. We label this band as 3s band. The inner electrons interact weakly with each other so that this splitting of sharp energy levels into bands is less for inner electrons.

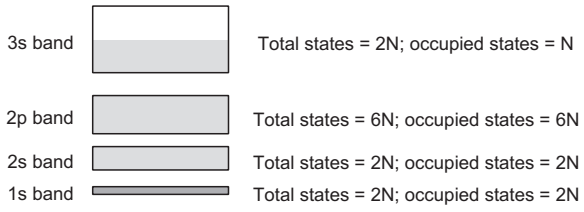



Figure 45.2

Figure (45.2) shows schematically this splitting of energy levels in bands. We have a 3s band which contains 2$N$ states with slightly different energies, $N$ of them are occupied by the $N$ electrons of sodium atoms and the remaining $N$ states are empty. Similarly, we talk of the 2p band which contains 6$N$ states with slightly different energies and all these states are filled. Similar is the case for other inner bands. The difference between the highest energy in a band and the lowest energy in the next higher band is called the *band gap* between the two bands.

Sodium was taken only as an example. We have energy bands separated by band gaps in all solids. At 0 K, the energy is the lowest and the electrons fill the bands from the bottom according to the Pauli exclusion principle till all the electrons are accommodated. As the temperature is raised, the electrons may collide with each other and with ions to exchange energy. At an absolute temperature $T$, the order of energy exchanged is $kT$ where $k$ is the Boltzmann constant. This is known as thermal energy. At room temperature (300 K), $kT$ is about 0·026 eV. Suppose the band gaps are much larger than $kT$. An electron in a completely filled band does not find an empty state with a slightly higher or a slightly lower energy. It, therefore, cannot accept or donate energy of the order of $kT$ and hence does not take part in processes involving energy exchange. This is the case with inner bands which are, in general, completely filled. The outermost electrons which are in the highest occupied energy band, may take up this energy $\approx kT$ if empty states are available in the same band. For example, a 3s electron in sodium can take up thermal energy and go to an empty state at a slightly more energy.

Similar is the scenario when an electric field is applied by connecting the sodium metal to a battery. The electric field, in general, can supply only a small amount of energy to the electrons. Only the electrons in the highest occupied band can accept this energy. These electrons can acquire kinetic energy and move according to the electric field. This results in electric current. The electrons in the inner bands cannot accept small amounts of energy from the electric field and hence do not take part in conduction.

The energy band structure in solids may be classified in four broad types as shown in figure (45.3).

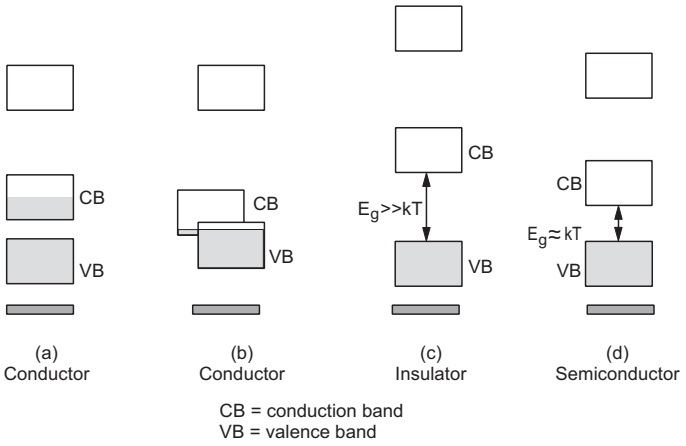



Figure 45.3

(a) The highest occupied energy band is only partially filled at 0 K (figure 45.3a). Sodium is an example of this kind. If the electronic configuration is such that the outermost subshell contains odd number of electrons, we get this type of band structure. They are good conductors of electricity because as an electric field is applied, the electrons in the partially filled band can receive energy from the field and drift accordingly.

(b) The highest occupied energy level is completely filled at 0 K and the next higher level is completely empty when the atoms are well-separated. But as the atoms come closer and these levels split into bands, the bands overlap with each other (figure 45.3b). Zinc is an example of this kind. The electronic configuration of zinc is $(1s)^2\ (2s)^2\ (2p)^6\ (3s)^2\ (3p)^6\ (3d)^{10}\ (4s)^2$. The highest energy level—that contains any electron when the zinc atoms are well-separated—is the 4s level and all the $2N$ states in it are occupied by electrons. However, in solid zinc, the 4p band overlaps with the 4*s* band. In this case also, there are empty states at energies close to the occupied states and hence these solids are also good conductors.

(c) The highest occupied energy band is completely filled and the next higher band, which is empty, is well above it (figure 45.3c). The band gap between these two bands is large. The electrons do not have empty states at an energy slightly above or below their existing energies. If an electric field is applied by connecting the two ends of such a solid to a battery, the electrons will refuse to receive energy from the field. This is because they do not find an empty state at a slightly higher energy. Diamond is an example of this kind. The gap between the lowest empty band and the highest filled band is about 6 eV. The electric field needed to supply 6 eV energy to an electron is of the order of $10^7\ \text{V m}^{-1}$ in copper (see example 45.1). Assuming the same value for diamond; if we take a 10 cm slab of diamond, we will have to use a battery of $10^6$ volts to get response from an electron. These solids are, therefore, insulators.

(d) The highest occupied band is completely filled at 0 K but the next higher band, which is empty, is only slightly above the filled band (figure 45.3d). The band structure is very similar to that of an insulator but the band gap between these two bands is small. An example is silicon in which the band gap is 1·1 eV. It is still difficult for an ordinary battery to supply an energy of the order of 1·1 eV to an electron. However, at temperatures well above 0 K, thermal collisions may push some of the electrons from the highest occupied band to the next empty band. These few electrons, in the otherwise empty band, can respond to even a weak battery because they have a large number of empty states just above their existing energy. As electrons from a filled band are pushed up in energy to land into a higher energy band, empty states are created in this filled band. These empty states allow some movement of electrons in that band and thus promote conduction. As the total number of electrons that can receive energy from the electric field is small, the conductivity is quite small as compared to common conductors. Such solids are called *semiconductors.*

The energy bands which are completely filled at 0 K are called *valence bands.* The bands with higher energies are called *conduction bands*. We are generally concerned with only the highest valence band and the lowest conduction band. So when we say valence band, it means the highest valence band. Similarly, when we say conduction band, it means the lowest conduction band. Study the labels 'conduction band (CB)' and 'valence band (VB)' in figure (45.3).

***Example 45.1***

*The mean free path of conduction electrons in copper is about* $4 \times 10^{-8}$ m. *For a copper block, find the electric field which can give, on an average,* 1 eV *energy to a conduction electron.*

***Solution*** **:** Let the electric field be $E$. The force on an electron is $eE$. As the electron moves through a distance $d$, the work done on it is $eEd$. This is equal to the energy transferred to the electron. As the electron travels an average distance of $4 \times 10^{-8}$ m before a collision, the energy transferred is $eE(4 \times 10^{-8}\ \text{m})$. To get 1 eV energy from the electric field,

$$eE(4 \times 10^{-8}\ \text{m}) = 1\ \text{eV}$$

or,
$$E = 2{\cdot}5 \times 10^7\ \text{V m}^{-1}.$$

Let us now consider the physical picture of conductors, insulators and semiconductors in a Bohr-type model. The inner electrons are tightly bound to the nuclei and move in their well-defined orbits. In a conductor, the outermost subshell is not completely filled and the electrons moving around one nucleus can jump to a similar orbit around some other nucleus (figure 45.4a). This is possible because there is an empty state there to accommodate the electron. For example, a 3s electron of one sodium atom can jump to the 3s orbit of some other sodium atom because out of the two 3s states only one, in general, is filled. Similar is the case with zinc-type metals. The 4p orbits in zinc atoms are, in general, empty. As the 4p band overlaps with the 4*s* band, the energy of a 4p state is not too different from a 4s state. Thus, without demanding for excessive energy, a 4s electron can jump

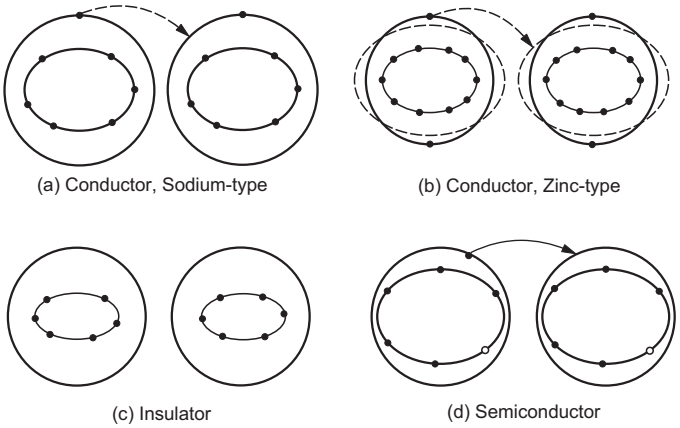



Figure 45.4

into the 4p orbit of its own atom or of some other atom (figure 45.4b).

In insulators, even the outermost electrons are tightly bound to their respective nuclei. The subshell in which they lie is completely filled and the next higher orbit is at a much higher energy. No drift is then possible (figure 45.4c).

In semiconductors at room temperature or above, some electrons move around the nuclei with a much larger radius. These large orbits are nearly empty and so the electrons in these orbits may jump from one atom to the other easily. These are the electrons in conduction band and are called *conduction electrons* (figure 45.4d).

### 45.3 THE SEMICONDUCTOR

As discussed above, in semiconductors the conduction band and the valence band are separated by a relatively small energy gap. For silicon, this gap is 1·1 eV. For another common semiconductor germanium, the gap is 0·68 eV.

Silicon has an atomic number of 14 and electronic configuration $(1s)^2(2s)^2(2p)^6(3s)^2(3p)^2$. The chemistry of silicon tells us that each silicon atom makes covalent bonds with the four neighbouring silicon atoms. To form a covalent bond, two silicon atoms contribute one electron each and the two electrons are shared by the two atoms. Both of these electrons are in the valence band. Due to collision, one of these valence electrons may acquire additional energy and it may start orbiting the silicon nucleus at a larger radius. Thus, the bond is broken. One electron has gone into the conduction band, it is moving in an orbit of large radius and is frequently jumping from one nucleus to another.

Figure (45.5a) represents a model of solid silicon. The four outer electrons of each atom form bonds with the four neighbouring atoms. Each bond consists of two electrons. Figure (45.5b) shows a broken bond

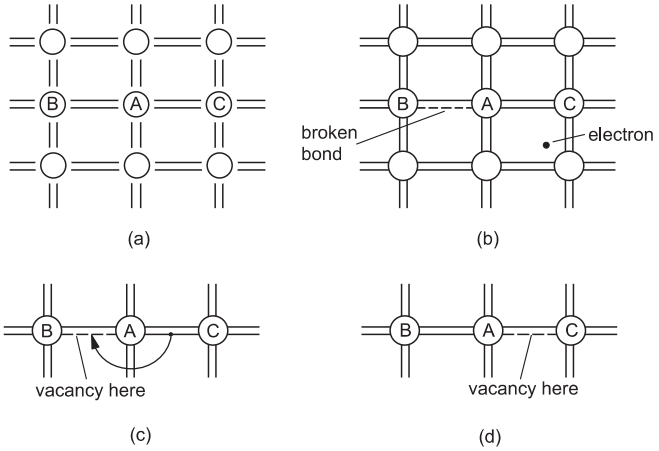



Figure 45.5

represented by the dashed line. The electron corresponding to this bond has acquired sufficient energy and has jumped into the conduction band. So there is a vacancy for an electron at the site of the broken bond. In a semiconductor, a number of such broken bonds and conduction electrons exist. Now consider the situation shown in figure (45.5c). The bond between the atoms $A$ and $B$ is broken. A bonding electron between $A$ and $C$ can make a jump towards the left and fill the broken bond between $A$ and $B$. Not much energy is needed to induce such a transfer. This is because, the electron makes transition from one bond to other only and all bond electrons have roughly the same energy.

As the broken bond $AB$ is filled, the bond $AC$ is broken (figure 45.5d). Thus, the vacancy has shifted towards the right. Any movement of a valence electron from one bond to a nearby broken bond may be described as the movement of vacancy in the opposite direction. It is customary in semiconductor physics to treat a vacancy in valence band as a particle having positive charge $+e$. Movement of electrons in valence band is then described in terms of movement of vacancies in the opposite directions. Such vacancies are also called *holes*.

Whenever a valence electron is shifted to conduction band, a hole is created. Thus, in a pure semiconductor, the number of conduction electrons equals the number of holes. When an electric field is applied, conduction electrons drift opposite to the field and holes drift along the field. That is why, holes are assumed to have positive charge. Conduction takes place due to the drift of conduction electrons as well as of holes. Such pure semiconductors are also called *intrinsic semiconductors.* The chemical structure of germanium is the same as that of silicon and hence the above discussion is equally applicable to it.

### 45.4 *p*-TYPE AND *n*-TYPE SEMICONDUCTORS

In an intrinsic semiconductor, like pure silicon, only a small fraction of the valence electrons are able

to reach the conduction band. The conduction properties of a semiconductor can be drastically changed by diffusing a small amount of impurity in it. The process of diffusing an impurity is also known as *doping*. Suppose a small amount of phosphorus ($Z = 15$) is diffused into a silicon crystal. Each phosphorus atom has five outer electrons in the valence band. Some of the phosphorus atoms displace the silicon atoms and occupy their place. A silicon atom has four valence electrons locked in covalent bonds with neighbouring four silicon atoms. Phosphorus comes in with five valence electrons. Four electrons are shared with the neighbouring four silicon atoms. The fifth one moves with a large radius ($\approx 30$ Å) round the phosphorus ion. The energy of this extra electron is much higher than the valence electrons locked in covalent bonds. In fact, the energy levels for these extra electrons—known as impurity levels—are only slightly below the conduction band (0·045 eV for phosphorus in silicon). This small gap is easily covered by the electrons during thermal collisions and hence a large fraction of them are found in the conduction band. Figure (45.6) shows qualitatively the situation in such a doped semiconductor.

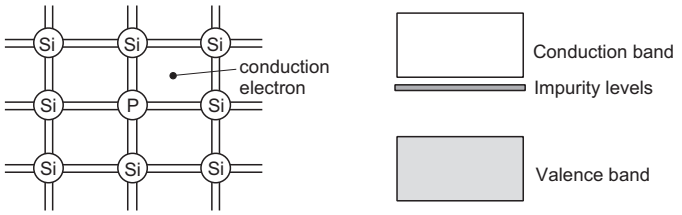



Figure 45.6

When a phosphorus atom with five outer electrons is substituted for a silicon atom, an extra electron is made available for conduction. Thus, the number of conduction electrons increases due to the introduction of a pentavalent impurity in silicon. The conduction properties are, therefore, very sensitive to the amount of the impurity. The introduction of phosphorus in the proportion of 1 in $10^6$ increases the conductivity by a factor of about $10^6$. Interesting desired results may be obtained by controlling the amount and distribution of impurity in a semiconductor.

Such impurities, which donate electrons for conduction, are called *donor* impurities. As the number of negative charge carriers is much larger than the number of positive charge carriers, these semiconductors are called *n-type semiconductors.* What happens if a trivalent impurity— such as aluminium— is doped into silicon ? Silicon atom with four valence electrons is substituted by an aluminium atom with three valence electrons. These three electrons are used to form covalent bonds with the neighbouring three silicon atoms but the bond with the fourth neighbour is not complete. The broken bond between the aluminium atom and its fourth neighbour can be filled by another valence electron if this electron obtains an extra energy of about 0·057 eV. A hole is then created in the valence band. We say that impurity levels are created a little above the valence band (0·057 eV for aluminium in silicon). The valence electrons can cross over to these levels leaving behind the holes, which are responsible for conduction (figure 45.7). As the energy gap $\Delta E$ between the valence band and the impurity levels is comparable to $kT$, large number of holes are created. The number of holes in such a doped semiconductor is much larger than the number of conduction electrons. As the majority charge carriers are holes, i.e., positive charges, these semiconductors are called *p-type semiconductors.* The impurity of this kind creates new levels which can accept valence electrons, hence these impurities are called *acceptor impurities.*

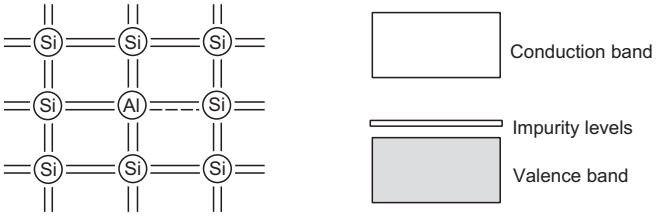



Figure 45.7

Semiconductors with an impurity doped into it are called *extrinsic semiconductor.* Figure (45.8) shows a schematic representation of intrinsic and extrinsic semiconductors.

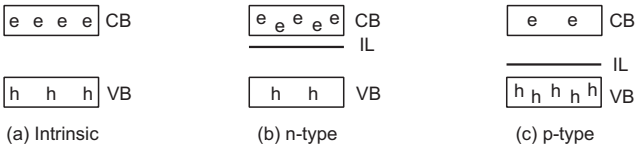



Figure 45.8

## 45.5 DENSITY OF CHARGE CARRIERS AND CONDUCTIVITY

Due to thermal collisions, an electron can take up or release energy. Thus, occasionally a valence electron takes up energy and the bond is broken. The electron goes to the conduction band and a hole is created. And occasionally, an electron from the conduction band loses some energy, comes to the valence band and fills up a hole. Thus, new electron–hole pairs are formed as well as old electron–hole pairs disappear. A steady-state situation is reached and the number of electron–hole pairs takes a nearly constant value. For silicon at room temperature (300 K), the number of

these pairs is about $7 \times 10^{15}\ m^{-3}$. For germanium, this number is about $6 \times 10^{1}\ m^{-3}$.

Table (45.2) gives a rough estimate of the densities of charge carriers in a typical conductor and in some of the semiconductors. Note that the product of density of conduction electrons and density of holes is constant for a semiconductor when it is doped with impurities.

**Table 45.2 : *Densities of charge carriers***

| Material | Type | Density of conduction electrons ($m^{-3}$) | Density of holes ($m^{-3}$) |
|---|---|---|---|
| Copper | Conductor | $9 \times 10^{28}$ | 0 |
| Silicon | Intrinsic semiconductor | $7 \times 10^{15}$ | $7 \times 10^{15}$ |
| Silicon doped with phosphorus (1 part in $10^6$) | *n*-type semiconductor | $5 \times 10^{22}$ | $1 \times 10^{9}$ |
| Silicon doped with aluminium (1 part in $10^6$) | *p*-type semiconductor | $1 \times 10^{9}$ | $5 \times 10^{22}$ |

The conductivity of a metal is given as

$$\sigma = \frac{j}{E} = ne\left(\frac{v}{E}\right)$$

where $n$ is the density of conduction electrons and $v$ is the drift speed when an electric field $E$ is applied. The quantity $v/E$ is known as the *mobility* of the electrons. Writing the mobility as $\mu$,

$$\sigma = ne\mu.$$

This equation is slightly modified for semiconductors. Here conduction is due to the conduction electrons as well as due to the holes. Electron mobility and hole mobility are also, in general, different. The conductivity of a semiconductor is, therefore, written as

$$\sigma = n_e e\,\mu_e + n_h e\,\mu_h$$

where $n_e$, $n_h$ are the densities of conduction electrons and the holes respectively and $\mu_e$, $\mu_h$ are their mobilities.

***Example 45.2***

*Calculate the resistivity of an n-type semiconductor from the following data: density of conduction electrons* $= 8 \times 10^{13}\ cm^{-3}$, *density of holes* $= 5 \times 10^{12}\ cm^{-3}$, *mobility of conduction electron* $= 2{\cdot}3 \times 10^{4}\ cm^{2}\ V^{-1}s^{-1}$ *and mobility of holes* $= 100\ cm^{2}\ V^{-1}s^{-1}$.

***Solution*** : The conductivity of the semiconductor is

$$\sigma = e(n_e\,\mu_e + n_h\mu_h)$$

$$= (1{\cdot}6 \times 10^{-19}\ C)\,[(8 \times 10^{19}\ m^{-3}) \times (2{\cdot}3\ m^{2}\ V^{-1}s^{-1}) + (5 \times 10^{18}\ m^{-3}) \times (10^{-2}\ m^{-2}\ V^{-1}s^{-1})]$$

$$\approx 2{\cdot}94\ Cm^{-1}\ V^{-1}\ s^{-1}.$$

The resistivity is $\rho = \frac{1}{\sigma} = \frac{1}{2{\cdot}94}\ m\ Vs\ C^{-1} \approx 0{\cdot}34\ \Omega m.$

**Temperature Dependence of Conductivity of a Semiconductor**

If temperature is increased, the average energy exchanged in a collision increases. More valence electrons cross the gap and the number of electron–hole pairs increases. It can be shown that the number of such pairs is proportional to the factor $T^{3/2} e^{-\Delta E/2kT}$, where $\Delta E$ is the band gap. The increase in the number of electron–hole pairs results in an increase in the conductivity, i.e., a decrease in the resistivity of the material.

There is a small opposing behaviour due to the increase in thermal collisions. The drift speed and hence the mobility decreases and this contributes towards increasing the resistivity just like a conductor. However, the effect of increasing the number of charge carriers is much more prominent than the effect of the decrease in drift speed. The resultant effect is that the resistivity decreases as the temperature increases. The temperature coefficient of resistivity is, therefore, negative. Its average value for silicon is $-0{\cdot}07\ K^{-1}$. This behaviour is opposite to that of a conductor where resistivity increases with increasing temperature.

## 45.6 *p-n* JUNCTION

When a semiconducting material such as silicon or germanium is doped with impurity in such a way that one side has a large number of acceptor impurities and the other side has a large number of donor impurities, we obtain a *p-n* junction. To construct a *p-n* junction, one may diffuse a donor impurity to a pure semiconductor so that the entire sample becomes *n*-type. The acceptor impurity may then be diffused in higher concentration from one side to make that side *p*-type. Figure (45.9) shows the physical structure of a typical *p-n* junction.

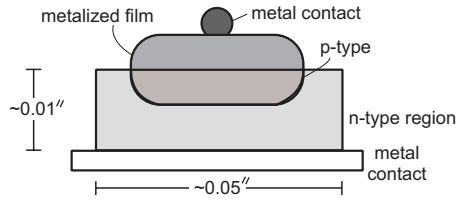



Figure 45.9

Consider the idealized situation of a *p-n* junction at the time of its formation shown in figure (45.10).

The symbol *e* represents a conduction electron and *h* represents a hole. Suppose that at the time of formation, the left-half is made *p*-type and the right-half *n*-type semiconductor. These two portions may be called *p*-side and *n*-side respectively.

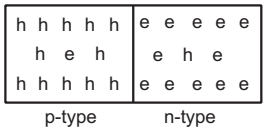



Figure 45.10

This cannot be the equilibrium situation. As there is a large concentration of holes in the left-half and only a small concentration of holes in the right-half, there will be diffusion of holes towards the right. Physically, this means that some of the valence electrons just to the right of the junction may fill up the vacancies just to the left of the junction. Similarly, because of the concentration difference, conduction electrons diffuse from the right to the left. The ions as such do not move because of their heavy masses. As the two halves (left-half and right-half) were electrically neutral in the beginning, diffusion of holes towards the right and diffusion of electrons towards the left make the right-half positively charged and the left-half negatively charged. This creates an electric field near the junction from the right to the left. Any hole near the junction is pushed by the electric field into the left-half. Similarly, any conduction electron near the junction is pushed by the electric field into the right-half. Thus, no charge carrier can remain in a small region near the junction. This region is called the *depletion layer* (figure 45.11).

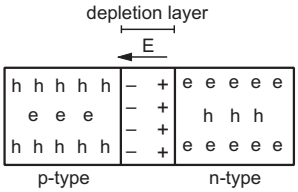



Figure 45.11

**Diffusion Current**

Because of the concentration difference, holes try to diffuse from the *p*-side to the *n*-side. In figure (45.11), this is from the left to the right. However, the electric field at the junction exerts a force on the holes towards the left as they come to the depletion layer. Only those holes which start moving towards the right with a high kinetic energy are able to cross the junction. Similarly, diffusion of electrons from the right to the left is opposed by the field and only those electrons which start towards the left with high kinetic energy are able to cross the junction. The electric potential of the *n*-side is higher than that of the *p*-side, the variation in potential is sketched in figure (45.12). We say that there is a *potential barrier* at the junction which allows only a small amount of diffusion. Nevertheless, there are some energetic holes and electrons which surmount the barrier and some diffusion does take place. This diffusion results in an electric current from the *p*-side to the *n*-side known as *diffusion current.*

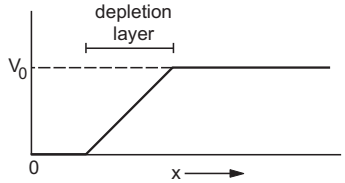



Figure 45.12

**Drift Current**

Because of thermal collisions, occasionally a covalent bond is broken and the electron jumps to the conduction band. An electron–hole pair is thus created. Also, occasionally a conduction electron fills up a vacant bond so that an electron–hole pair is destroyed. These processes continue in every part of the material. However, if an electron–hole pair is created in the depletion region, the electron is quickly pushed by the electric field towards the *n*-side and the hole towards the *p*-side. There is almost no chance of recombination of a hole with an electron in the depletion region. As electron–hole pairs are continuously created in the depletion region, there is a regular flow of electrons towards the *n*-side and of holes towards the *p*-side. This makes a current from the *n*-side to the *p*-side. This current is called the *drift current.*

The drift current and the diffusion current are in opposite directions. In steady state, the diffusion current equals the drift current in magnitude and there is no net transfer of charge at any cross-section. This is the case with a *p-n* junction kept in a cupboard.

## 45.7 *p-n* JUNCTION DIODE

Let us now discuss what happens if a battery is connected to the ends of a *p-n* junction. Figure (45.13) shows situations when (a) no battery is connected to the junction, (b) a battery is connected with its positive terminal connected to the *p*-side and the negative terminal connected to the *n*-side and (c) a battery is connected with its positive terminal connected to the *n*-side and the negative terminal connected to the *p*-side. If the positive terminal of the battery is connected to the *p*-side and the negative terminal to the *n*-side, we say that the junction is *forward-biased* (figure 45.13b). The potential of the *n*-side is higher than that of the *p*-side when no battery is connected to the junction. Due to the forward-bias connection, the potential of the *p*-side is raised and hence the height

of the potential barrier decreases. The width of the depletion region is also reduced in forward bias (figure 45.13b).

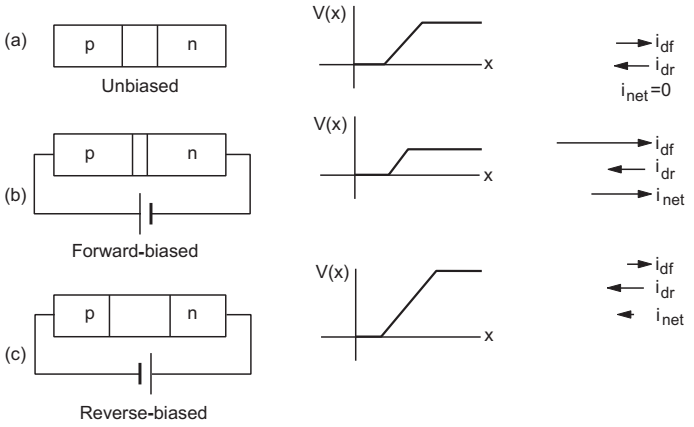



Figure 45.13

This allows more diffusion to take place. The diffusion current thus increases by connecting a battery in forward bias. The drift current remains almost unchanged because the rate of formation of new electron–hole pairs is fairly independent of the electric field unless the field is too large. Thus, the diffusion current exceeds the drift current and there is a net current from the *p*-side to the *n*-side. The diffusion increases as the applied potential difference is increased and the barrier height is decreased. When the applied potential difference is so high that the potential barrier is reduced to zero or is reversed, the diffusion increases very rapidly. The current *i* in the circuit thus changes nonlinearly with the applied potential difference. A *p-n* junction does not obey Ohm's law.

If the *p*-side of the junction is connected to the negative terminal and the *n*-side to the positive terminal of a battery, the junction is said to be *reverse-biased*. In this case, the potential barrier becomes higher as the battery further raises the potential of the *n*-side (figure 41.13c). The width of the depletion region is increased. Diffusion becomes more difficult and hence the diffusion current decreases. The drift current is not appreciably affected and hence it exceeds the diffusion current. So, there is a net current from the *n*-side to the *p*-side. However, this current is small as the drift current itself is small (typically in microamperes) and the net current is even smaller. Thus, during reverse bias, only a small current is allowed by the junction. We say that the junction offers a large resistance when reverse-biased.

Figure (45.14) shows a qualitative plot of current versus potential difference for a *p-n* junction. This is known as an *i–V characteristic* of the *p-n* junction. Note that the scales for the current are different for positive and negative current.

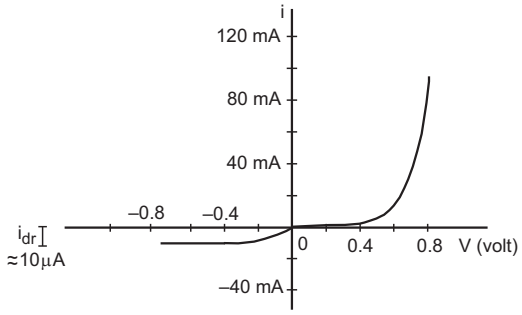



Figure 45.14

We see that the junction offers a little resistance if we try to pass an electric current from the *p*-side to the *n*-side and offers a large resistance if the current is passed from the *n*-side to the *p*-side. Any device which freely allows electric current in one direction but does not allow it in the opposite direction is called a diode. Thus, a *p-n* junction acts as a diode. An ideal diode should not allow any current in the reverse direction. A *p-n* junction diode is close to an ideal diode because the current in reverse bias is very small (few microamperes). The diode is symbolised as —▷|— , the arrow pointing in the direction in which the current can pass freely. For a *p-n* junction diode, the arrow points from the *p*-side to the *n*-side. We have already studied a vacuum-tube diode based on thermionic emission in an earlier chapter.

**Dynamic Resistance**

The dynamic resistance of a *p-n* junction diode is defined as

$$R = \frac{\Delta V}{\Delta i}$$

where $\Delta V$ denotes a small change in the applied potential difference and $\Delta i$ denotes the corresponding small change in the current. The dynamic resistance is a function of the operating potential difference. It is equal to the reciprocal of the slope of the *i–V* characteristic shown in figure (45.14).

***Example 45.3***

*The i–V characteristic of a p-n junction diode is shown in figure (45.15). Find the approximate dynamic resistance of the p-n junction when (a) a forward bias of* 1 *volt is applied, (b) a forward bias of* 2 *volt is applied.*

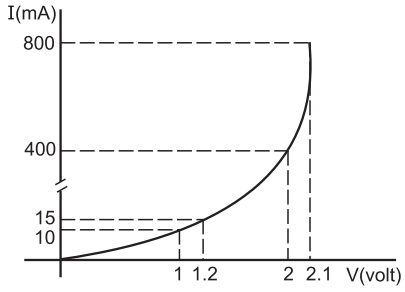



Figure 45.15

***Solution*** **:** (a) The current at 1 volt is 10 mA and at 1·2 volt it is 15 mA. The dynamic resistance in this region is

$$R = \frac{\Delta V}{\Delta i} = \frac{0{\cdot}2 \text{ volt}}{5 \text{ mA}} = 40\ \Omega.$$

(b) The current at 2 volt is 400 mA and at 2·1 volt it is 800 mA. The dynamic resistance in this region is

$$R = \frac{\Delta V}{\Delta i} = \frac{0{\cdot}1 \text{ volt}}{400 \text{ mA}} = 0{\cdot}25\ \Omega.$$

**Photodiode**

Photodiode is a *p-n* junction whose function is controlled by the light allowed to fall on it. Suppose, the wavelength is such that the energy of a photon, $hc/\lambda$, is sufficient to break a valence bond. When such light falls on the junction, new hole–electron pairs are created. The number of charge carriers increases and hence the conductivity of the junction increases. If the junction is connected in some circuit, the current in the circuit is controlled by the intensity of the incident light.

**Light-emitting Diode (LED)**

When a conduction electron makes a transition to the valence band to fill up a hole in a *p-n* junction, the extra energy may be emitted as a photon. If the wavelength of this photon is in the visible range (380 nm–780 nm), one can see the emitted light. Such a *p-n* junction is known as *light-emitting diode* abbreviated as LED. For silicon or germanium, the wavelength falls in the infrared region. LEDs may be made from semiconducting compounds like gallium such as, arsenide or indium phosphide. LEDs are very commonly used in electronic gadgets as indicator lights.

**Zener Diode**

If the reverse-bias voltage across a *p-n* junction diode is increased, at a particular voltage the reverse current suddenly increases to a large value. This phenomenon is called *breakdown* of the diode and the voltage at which it occurs is called the *breakdown voltage*. At this voltage, the rate of creation of hole–electron pairs is increased leading to the increased current.

There are two main processes by which breakdown may occur. The holes in the *n*-side and the conduction electrons in the *p*-side are accelerated due to the reverse-bias voltage. If these minority carriers acquire sufficient kinetic energy from the electric field and collide with a valence electron, the bond will be broken and the valence electron will be taken to the conduction band. Thus a hole–electron pair will be created. Breakdown occurring in this manner is called *avalanche breakdown*. Breakdown may also be produced by direct breaking of valence bonds due to high electric field. When breakdown occurs in this manner it is called *zener breakdown*.

A diode meant to operate in the breakdown region is called an *avalanche diode* or a *zener diode* depending on the mechanism of breakdown. Once the breakdown occurs, the potential difference across the diode does not increase even if the applied battery potential is increased. Such diodes are used to obtain constant voltage output. Figure (45.16) shows the *i–V* characteristic of a zener diode including the breakdown region and a typical circuit which gives constant voltage $V_0$ across the load resistance $R_L$. Even if there is a small change in the input voltage $V_i$, the current through $R_L$ remains almost the same. The current through the diode changes but the voltage across it remains essentially the same. Note the symbol used for the zener diode.

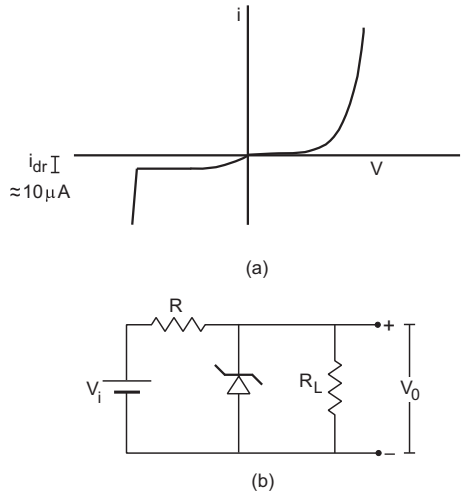



Figure 45.16

## 45.8 *p-n* JUNCTION AS A RECTIFIER

A rectifier is a device which converts an alternating voltage into a direct voltage. A *p-n*

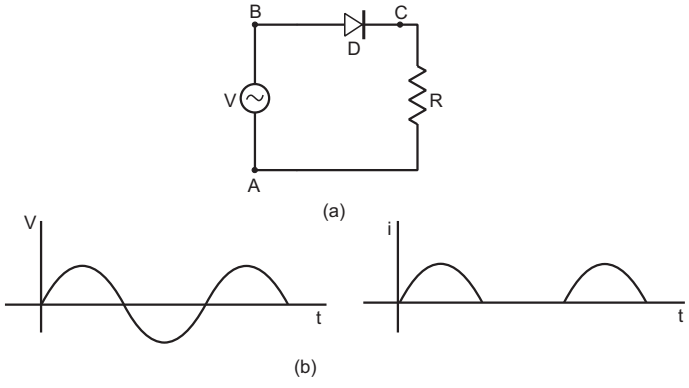



Figure 45.17

junction can be used as a rectifier because it permits current in one direction only. Figure (45.17a) shows an AC source connected to a load resistance through a *p-n* junction. The potential at the point $A$ is taken to be zero. The potential at $B$ varies with time as $V = V_0 \sin(\omega t + \varphi)$. During the positive half-cycle, $V > 0$ and $B$ is at a higher potential than $A$. In this case, the junction is forward-biased and a current $i$ is established in the resistance in the direction $C$ to $A$.

The current through the resistance during this half-cycle is given by

$$i = \frac{V_0 \sin(\omega t + \varphi)}{R + R_{junction}}$$

and the potential difference across it is

$$\frac{RV_0 \sin(\omega t + \varphi)}{R + R_{junction}}.$$

Here $R_{junction}$ is the resistance offered by the *p-n* junction.

In forward bias, $R_{junction} << R$ so that

$$i \approx \frac{V_0}{R} \sin(\omega t + \varphi).$$

During the next half-cycle, $V < 0$ and the potential at the point $B$ becomes smaller than that at $A$. The junction is thus reverse-biased and offers a large resistance during this half-cycle and there is only a negligible current in the circuit. The current in the resistance is thus unidirectional. The variations in voltage and current with time are sketched in figure (45.17b).

This is called half-wave rectification because there is practically no current during alternate half-cycles. A full-wave rectification can be achieved by using two diodes as shown in figure (45.18). The AC potential difference is obtained across the secondary of a transfomer and is connected in the circuit. In one half-cycle, $V_A > V_C > V_B$ so that the junction $D_1$ conducts but $D_2$ does not. The current is from $A$ to $D_1$ to $E$ to $C$. In the next half-cycle, $V_B > V_C > V_A$ so that $D_2$ conducts whereas $D_1$ does not. The current is from $B$ to $D_2$ to $E$ to $C$. In both the half-cycles, the current in the load resistance is from $E$ to $C$.

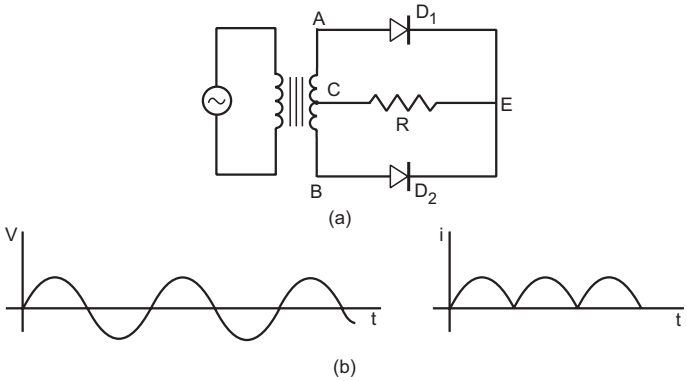



Figure 45.18

## 45.9 JUNCTION TRANSISTORS

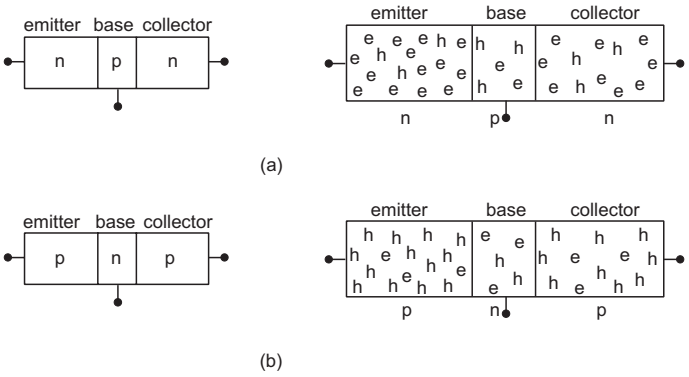



Figure 45.19

A junction transistor is formed by sandwiching a thin layer of a *p*-type semiconductor between two layers of *n*-type semiconductors or by sandwiching a thin layer of an *n*-type semiconductor between two layers of *p*-type semiconductors. In figure (45-19a), we show a transistor in which a thin layer of a *p*-type semiconductor is sandwiched between two *n*-type semiconductors. The resulting structure is called an *n-p-n* transistor. In figure (45.19b), we show a *p-n-p* transistor, where an *n*-type thin layer is sandwiched between two *p*-type layers. In actual design, the middle layer is very thin ($\approx 1\ \mu m$) as compared to the widths of the two layers at the sides. The middle layer is called the *base* and is very lightly doped with impurity. One of the outer layers is heavily doped and is called *emitter*. The other outer layer is moderately doped and is called *collector*. Usually, the emitter–base contact area is smaller than the collector–base contact area. Terminals come out from the emitter, the base and the collector for external connections. Thus, a transistor is a three-terminal device.

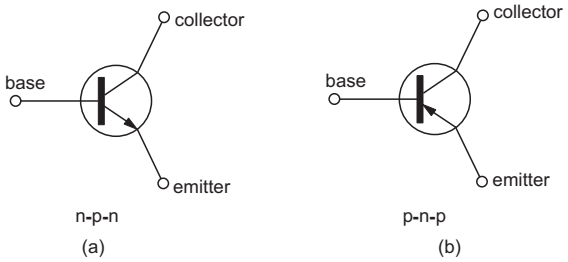



Figure 45.20

Figure (45.20) shows the symbols used for a junction transistor. In normal operation of a transistor, *the emitter–base junction is always forward-biased* whereas *the collector–base junction is reverse-biased.* The arrow on the emitter line shows the direction of the current through the emitter–base junction. In an *n-p-n* transistor, there are a large number of conduction electrons in the emitter and a large number of holes in the base. If the junction is forward-biased, the electrons will diffuse from the emitter to the base

and holes will diffuse from the base to the emitter. The direction of electric current at this junction is, therefore, from the base to the emitter. This is indicated by the outward arrow on the emitter line in figure (45.20a). Similarly, for a *p-n-p* transistor the current is from the emitter to the base when this junction is forward-biased which is indicated by the inward arrow in figure (45.20b).

**Biasing**

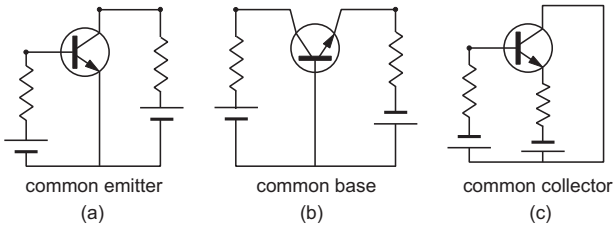



Figure 45.21

Suitable potential differences should be applied across the two junctions to operate the transitor. This is called *biasing* the transistor. A transistor can be operated in three different modes: common-emitter (or grounded-emitter), common-collector (or grounded-collector) and common-base (or grounded-base). In common-emitter mode, the emitter is kept at zero potential and the other two terminals are given appropriate potentials (figure 45.21a). Similarly, in common-base mode, the base is kept at zero potential (figure 45-21b) whereas in common-collector mode, the collector is kept at zero potential (figure 45-21c).

**Working of a Transistor**

Let us consider an *n-p-n* transistor connected to the proper biasing batteries as shown in figure (45.22). In part (a) of the figure, a physical picture of the transistor is used whereas in part (b), its symbol is used. Let us look at the current due to electrons. The emitter–base junction is forward-biased, so electrons are injected by the emitter into the base. The thickness of the base region is so small that most of the electrons diffusing into the base region cross over into the collector region. The reverse bias at the base–collector junction helps this process, because, as the electrons appear near this junction they are attracted by the collector. These electrons go through the batteries $V_{CC}$ and $V_{EE}$ and are then back to the emitter.

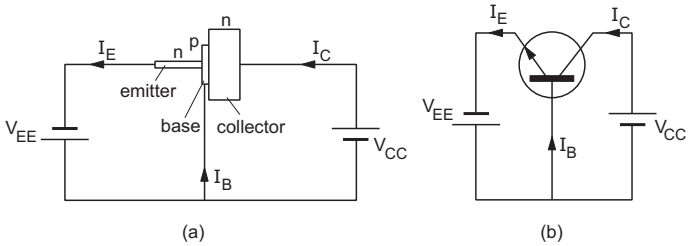



Figure 45.22

The electrons going from the battery $V_{EE}$ to the emitter constitute the electric current $I_E$ in the opposite direction. This is known as *emitter current*. Similarly, the electrons going from the collector to the battery $V_{CC}$ constitute the *collector current* $I_C$.

We have considered only the current due to the electrons. Similar is the story of the holes which move in the opposite direction but result in current in the same direction. Currents $I_E$ and $I_C$ refer to the net currents. However, since the base is only lightly doped, the hole concentration is very low and the current in an *n-p-n* transistor is mostly due to the electrons. As almost all the electrons injected into the emitter go through the collector, the collector current $I_C$ is almost equal to the emitter current. In fact, $I_C$ is slightly smaller than $I_E$ because some of the electrons coming to the base from the emitter may find a path directly from the base to the battery $V_{EE}$. This constitutes a *base current* $I_B$. The physical design of the transistor ensures that such events are small and hence $I_B$ is small. Typically, $I_B$ may be 1% to 5% of $I_E$.

Using Kirchhoff's law, we can write

$$I_E = I_B + I_C. \qquad \ldots (45.1)$$

**$\alpha$ and $\beta$ Parameters**

$\alpha$ and $\beta$ parameters of a transistor are defined as

$$\alpha = \frac{I_C}{I_E} \text{ and } \beta = \frac{I_C}{I_B}. \qquad \ldots (45.2)$$

Using equation (45.1),

$$\frac{I_E}{I_C} = \frac{I_B}{I_C} + 1 \qquad \ldots (i)$$

or, $$\frac{1}{\alpha} = \frac{1}{\beta} + 1$$

or, $$\beta = \frac{\alpha}{1-\alpha}. \qquad \ldots (45.3)$$

As $I_B$ is about 1–5% of $I_E$, $\alpha$ is about 0·95 to 0·99 and $\beta$ is about 20 to 100.

**Transistor Used in an Amplifier Circuit**

Figure (45.23) shows an amplifier circuit using an *n-p-n* transistor in common-emitter mode. The battery $E_B$ provides the biasing voltage $V_{BE}$ for the

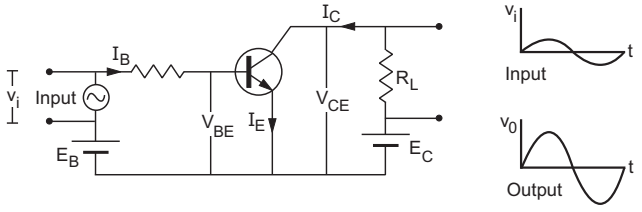



Figure 45.23

base–emitter junction. A potential difference $V_{CE}$ is maintained between the collector and the emitter by the battery $E_C$. The base–emitter junction is forward-biased and so the electrons of the emitter flow towards the base. As the base region is very thin—of the order of a micrometre—and the collector is also maintained at a positive potential, most of the electrons cross the base region and move into the collector. As discussed earlier, the current $I_C$ is about 0·95 $I_E$ to 0·99 $I_E$.

The holes in the base region may diffuse into the emitter due to the forward biasing of the base–emitter junction. Also, the electrons coming from the emitter may recombine with some of the holes in the base. If the holes are lost in this way, the base will become negatively charged and will obstruct the incoming electrons from the emitter. If the base current $I_B$ is increased by a small amount, the effect of hole-diffusion and hole–electron recombination may be neutralised and the collector current will be increased. Thus, a small change in the current $I_B$ in the base circuit controls the larger current $I_C$ in the collector circuit. This is the basis of amplification with the help of a transistor.

The input signal, to be amplified, is connected in series with the biasing battery $E_B$ in the base circuit. A load resistor having a large resistance $R_L$ is connected in the collector circuit and the output voltage is taken across this resistor. As the potential difference $V_{BE}$ changes with time due to the input signal, the base current $I_B$ changes. This results in a change $\Delta I_C$ in the collector current. The *current gain,* defined as $\Delta I_C / \Delta I_B$, is typically of the order of 50. The change in the voltage across $R_L$ is, accordingly,

$$\Delta V = R_L \Delta I_C.$$

Thus, an amplified output is obtained across $R_L$.

#### Voltage gain, current gain and power gain

When a signal voltage $v_i$ is added in the base circuit, the voltage across the load resistance changes by $v_o$. The ratio $\frac{v_o}{v_i}$ is called the *voltage gain* of the amplifier.

Suppose the input signal has a voltage $v_i$ at an instant. This produces a change in the base current $I_B$. As the base–emitter junction is forward-biased, it offers a small dynamic resistance $R_{BE}$. The change in the current in the base circuit is

$$\Delta I_B = \frac{v_i}{R_{BE}} .$$

The resistance $R_{BE}$ is also called the *input-resistance* of the circuit.

The collector current $I_c$ is related to $I_B$ as

$$I_c = \beta I_B.$$

Thus, the change in current $I_C$ due to the signal voltage is,

$$\Delta I_C = \beta \Delta I_B = \beta \frac{v_i}{R_{BE}} .$$

The output voltage, i.e., the change in the voltage across the load resistance is

$$v_o = \Delta V = R_L \Delta I_C = \frac{\beta v_i R_L}{R_{BE}} .$$

The voltage gain is

$$\frac{v_o}{v_i} = \beta \frac{R_L}{R_{BE}} .$$

As $\beta$ is of the order of 50 and $R_L$ may be much larger than $R_{BE}$, the voltage gain is high.

As mentioned earlier, the current gain is defined as the change in the collector current divided by the change in the base current when the signal is added in the base circuit.

Thus, the current gain is

$$\frac{\Delta I_C}{\Delta I_B} = \beta.$$

Power gain = voltage gain × current gain

$$= \frac{\beta^2 R_L}{R_{BE}} .$$

#### Transfer conductance

To have a large amplification, a small change in $V_{BE}$ should result in a large change in the collector current $I_C$. This property is measured by a quantity *transfer conductance* $g_m$ defined as

$$g_m = \frac{\Delta I_C}{\Delta V_{BE}} .$$

It is also known as *transconductance.*

### Transistor Used in an Oscillator Circuit

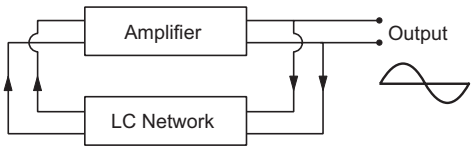



Figure 45.24

The function of an oscillator circuit is to produce an alternating voltage of desired frequency when only DC batteries are available. Figure (45.24) shows a schematic representation of an oscillator circuit. The basic parts in this circuit are (a) an amplifier and (b) an *LC* network.

The amplifier section is just a transistor used in common-emitter mode. The *LC* network consists of an inductor and a capacitance. This network resonates at a frequency

$$\nu_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}\ .$$

Batteries are used to bias the transistor and no external input signal is fed to the amplifying section. A part of the output signal is fed back to the input section after going through the $LC$ network. This signal is amplified by the transistor and a part is again fed back to its input section. Thus, it is a self-sustaining device. The component with the proper frequency $\nu_0$ gets resonantly amplified and the output acts as a source of alternating voltage of that frequency. The frequency can be varied by varying $L$ or $C$.

**Transistor Characteristics**

Let us consider an $n$-$p$-$n$ transistor in common-emitter configuration as in figure (45.23). We can view this circuit as made of an input section and an output section. The input section contains the base–emitter junction and the voltage source there whereas the output section contains the base–colletcor junction and the voltage source there. The current $I_B$ may then be called the *input current* and the current $I_C$ the *output current*. The voltage applied to the base–emitter junction, i.e., in the input section is $V_{BE}$ and that applied to the base–collector junction, i.e., in the output section, is $V_{CE}$. When the input current $I_B$ is plotted against the voltage $V_{BE}$ between the base and the emitter, we get the input characteristics. Similarly, when the output current $I_C$ is plotted against the voltage $V_{CE}$, we get the output characteristics.

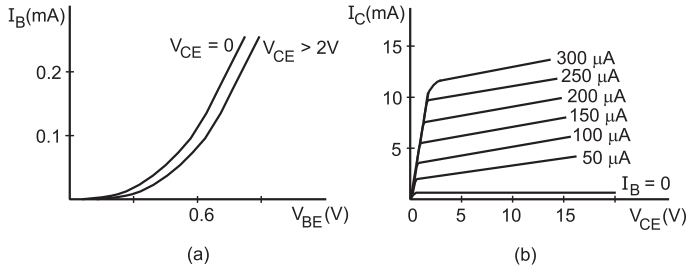



Figure 45.25

These characteristics are shown in figure (45.25). The input characteristics shown in figure (45.25a) are like those of a forward-biased $p$-$n$ junction. If the biasing voltage is small as compared to the height of the potential barrier at the junction, the current $I_B$ is very small. Once the voltage is more than the barrier height, the current rapidly increases. However, since most of the electrons diffused across the junction go to the collector, the net base current is very small (in microamperes) even at large values of $V_{BE}$.

The output characteristics are shown in figure (45.25b). For small values of the collector voltage, the collector–base junction is reverse–biased because the base is at a more positive potential. The current $I_C$ is then small. As the electrons are forced from the emitter side, the current $I_C$ is still quite large as compared to a single reverse-biased $p$-$n$ junction. As the voltage $V_C$ is increased, the current rapidly increases and becomes roughly constant once the junction is forward-biased. For higher base currents, the collector current is also high and increases more rapidly, even in forward bias.

## 45.10 LOGIC GATES

**Logical Variables and Logical Operations**

There are a number of questions which have only two answers, either YES or NO. There are a number of objects which can remain in either of two states only. An electric bulb can either be ON or OFF. A diode can either be conducting or nonconducting. A person can either be alive or dead. A statement may either be true or false. It is interesting to imagine a world in which each variable is allowed to take only two values. These values may be represented by two symbols, 0 and 1. The living state of a person is 0 if he/she is dead and is 1 if he/she is alive. The electric state of a diode is 0 if it is nonconducting and is 1 if it is conducting. Such a world may be very small because in the real world we do have quantities which assume more than two values. But let us concentrate on this small world where everything can either be answered in YES or in NO and so only two symbols 0 and 1 are needed to represent any variable.

Here is an example. A bulb, two switches and a power source are connected as shown in figure (45.26). We have three variables,

$A$ = state of switch $S_1$
$B$ = state of switch $S_2$
$C$ = state of the bulb.

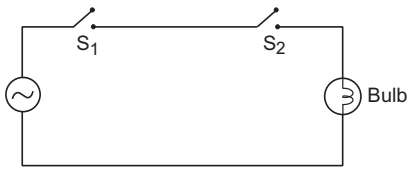



Figure 45.26

Let us assume that the variable $A$ is 0 if the switch $S_1$ is open and is 1 if the switch $S_1$ is closed. Similar is the case for $B$. The variable $C$ is 0 if the bulb is off and is 1 if it is on. Will the bulb be off or on will depend on the states of $S_1$ and $S_2$. Table (45.3) shows the dependence.

**Table 45.3**

| Switch $S_1$ | Switch $S_2$ | Bulb | $A$ | $B$ | $C$ |
|---|---|---|---|---|---|
| open | open | off | 0 | 0 | 0 |
| open | closed | off | 0 | 1 | 0 |
| closed | open | off | 1 | 0 | 0 |
| closed | closed | on | 1 | 1 | 1 |

Thus, $C$ is a *function* of $A$ and $B$. If we give the value of $C$ for all possible combinations of $A$ and $B$, the function is completely specified. Thus, a function may be specified by writing a table in which the value of the function is given for all possible combinations of the values of the independent variables. Such a table is known as the *truth table* for that function.

The particular function defined by table (45.3) is written as $A$ AND $B$. We say that $A$ and $B$ are ANDed to get $C$. It is also denoted by the symbol of dot. Thus $C = A \cdot B$ is the same as $C = A$ AND $B$. Quite often, the dot is omitted and we write just $AB$ to mean $A$ AND $B$. The function $C = A$ AND $B$ is 1 if each of $A$ and $B$ is 1. If any of $A$ and $B$ is zero or both are zero, $C$ is 0.

Let us take another example of a function of two variables. Figure (45.27) shows another circuit containing a power source, two switches $S_1$, $S_2$ and a bulb. Table (45.4) gives the state of the bulb for all possible combinations of the switches and the truth table for such a function.

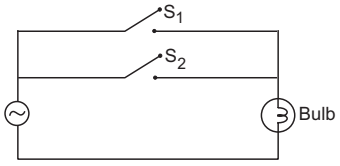



Figure 45.27

**Table 45.4**

| Switch $S_1$ | Switch $S_2$ | Bulb | $A$ | $B$ | $C$ |
|---|---|---|---|---|---|
| open | open | off | 0 | 0 | 0 |
| open | closed | on | 0 | 1 | 1 |
| closed | open | on | 1 | 0 | 1 |
| closed | closed | on | 1 | 1 | 1 |

The function of $A$ and $B$ defined by the truth table given in table (45.4) is written as $A$ OR $B$. We say that $A$ and $B$ are ORed to get $C$. It is also represented by the symbol of plus. Thus $C = A + B$ is the same as $C = A$ OR $B$. The function $A + B$ is 0 if each of $A$ and $B$ is 0. If one of the two is 1 or both of them are 1, the function is 1.

Let us now take an example of a function of a single variable. A bulb is short-circuited by a switch (figure 45.28). If the switch is open, the current goes through the bulb and it is on. If the switch is closed,

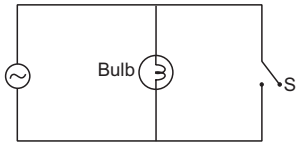



Figure 45.28

the current goes through the switch and the bulb is off (we assume ideally zero resistance in the switch). Let $A$ be the variable showing the state of the switch and $B$ be the variable showing the state of the bulb. Then $B$ is a function of $A$. Table (45.5) describes the function and its truth table.

**Table 45.5**

| Switch | Bulb | $A$ | $B$ |
|---|---|---|---|
| open | on | 0 | 1 |
| closed | off | 1 | 0 |

The function of $A$ described by this truth table is written as NOT $A$. The function NOT $A$ is also written as $\overline{A}$. Thus, $B$ = NOT $A$ is the same as $B = \overline{A}$.

A variable which can assume only two values is called a *logical variable.* A function of logical variables is called a *logical function.* AND, OR and NOT represent three basic operations on logical variables. The first two are operations between two logical variables. The third one is an operation on a single variable. A number of functions may be generated by using these operations.

***Example 45.4***

*Write the truth table for the logical function $Z$ = ($X$ AND $Y$) OR $X$.*

***Solution*** : $Z$ is a function of two variables $X$ and $Y$. The truth table is constructed in table (45.6). The third column gives the value of $W = X$ AND $Y$. It is 1 when $X = Y = 1$ and is 0 otherwise. The fourth column of this table gives the value of $Z = W$ OR $X$.

**Table 45.6**

| $X$ | $Y$ | $W$ = $X$ AND $Y$ | $Z$ = $W$ OR $X$ | $X$ | $Y$ | $Z$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 | 1 | 0 |
| 1 | 0 | 0 | 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |

In the first two rows, $W = 0$ and $X = 0$. Thus $W$ OR $X = 0$. In the third row, $W = 0$ and $X = 1$. Thus $W$ OR $X = 1$. In the fourth row, $W = X = 1$. Thus $W$ OR $X = 1$. The last three columns of the table collect the values of $X$, $Y$ and $Z$ which is the required truth table.

This function may also be written as

$$Z = (X \cdot Y) + X = XY + X.$$

**Logic Gates**

A logic gate is an electronic circuit which evaluates a particular logical function. The circuit has one or more *input* terminals and an *output* terminal. A potential of zero (equal to the earth's potential in general) denotes the logical value 0 and a fixed positive potential $V$ (say, + 5 V) denotes the logical value 1. Each input terminal denotes an independent variable. If zero potential is applied to an input terminal, the corresponding independent variable takes the value 0. If the positive potential $V$ is applied to the terminal, the corresponding variable takes the value 1. The potential appearing at the output terminal denotes the value of the function. If the potential is zero, the value of the function is 0. If it is $V$, the value of the function is 1.

A gate may have more than one output terminals. Each output terminal then represents a separate function and the same circuit may be used to evaluate more than one functions.

Figure (45.29) shows the symbols for the logic gates to evaluate the functions AND, OR and NOT. They are known as AND gate, OR gate and NOT gate respectively. The terminals shown on the left are the input terminals and the terminal on the right is the output terminal in each case.

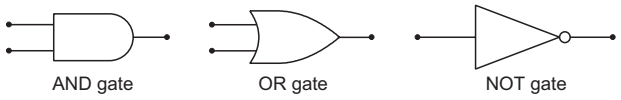



Figure 45.29

**Realisation of AND and OR gates with diodes**

An AND gate and an OR gate may be constructed with two *p-n* junction diodes. Figure (45.30) shows the construction for an AND gate. The circuit evaluates the function $X = A$ AND $B$, i.e., $X = AB$. A potential of 5 V at $A$ denotes the logical value $A = 1$ and a potential of zero at $A$ denotes $A = 0$, similarly for $B$ and $X$.

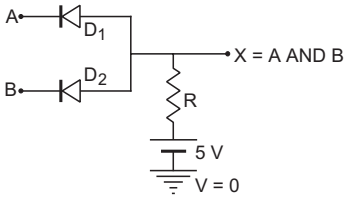



Figure 45.30

Suppose $A = 0$ and $B = 0$. The potentials at $A$ and $B$ are both zero so that both the diodes are forward-biased and offer no resistance. The potential at $X$ is equal to the potential at $A$ or $B$. Thus, $X = 0$. Now suppose, $A = 0$ and $B = 1$. The potential is zero at $A$ and 5 V at $B$ so that the diode $D_1$ is forward-biased. The potential at $X$ is equal to the potential at $A$ which is zero. Thus, if $A = 0$ and $B = 1$ then $X = 0$. Similarly, when $A = 1$ and $B = 0$ then $X = 0$. Finally, suppose $A = B = 1$. The potentials at both $A$ and $B$ are 5 V so that neither of the diodes is conducting. This is because if either of the diodes conducts, a current will go through the resistance $R$ and the potential at $X$ will become less than 5 V making the diode reverse-biased. As the diodes are not conducting, there will be no current through $R$ and the potential at $X$ will be equal to 5 V, i.e., $X = 1$.

Thus, the output is $X = 1$ if both the inputs $A$ and $B$ are 1. If any of the inputs is 0, the output is $X = 0$. Hence $X = AB$ and the circuit evaluates the AND function.

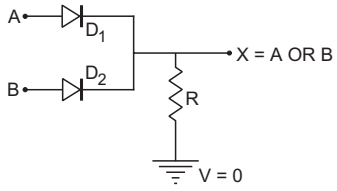



Figure 45.31

Figure (45.31) shows the construction of an OR gate using two diodes. The circuit evaluates $X$ OR $A$, i.e., $X = A + B$. If $A = B = 0$, there is no potential difference anywhere in the circuit so that $X = 0$. If $A = 1$ and $B = 0$, the potential is 5 V at $A$ and zero at $B$. The diode $D_1$ is forward-biased and offers no resistance. Thus, the potential at $X$ is equal to the potential at $A$, i.e., 5 V. Thus $X = 1$. Similarly, if $A = 0$ and $B = 1$, $X = 1$. Also, if both $A$ and $B$ are 1, both the diodes are forward-biased and the potential at $X$ is the same as the common potential at $A$ and $B$ which is 5 V. This also gives $X = 1$. Hence the circuit evaluates $X = A$ OR $B = A + B$.

**Realisation of NOT gate with a transistor**

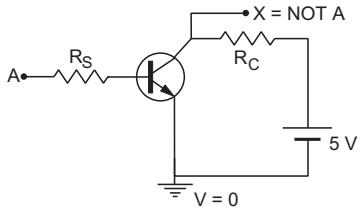



Figure 45.32

A NOT gate cannot be constructed with diodes. Figure (45.32) shows a circuit using an *n-p-n* transistor to evaluate the NOT function.

If $A = 0$, the emitter–base junction is unbiased and there is no current through it. Correspondingly, there is no current through the resistance $R_C$. The potential at $X$ is equal to the potential at the positive terminal of the battery which is 5 V. Thus, if $A = 0$, $X = 1$. Note that the collector–base junction is also reverse-biased

which is consistent with the fact that there is no current in the circuit.

On the other hand if the potential at $A$ is 5 V, the base–emitter junction is forward-biased and there is a large current in the circuit. The direction of the current in the resistance $R_C$ is from right to left in figure (45.33). The potential drops across $R$ and its value at $X$ becomes zero. Thus, if $A = 1, X = 0$.

**NAND and NOR gates**

The function $X = \text{NOT } (A \text{ AND } B)$ of two logical variables $A$ and $B$ is called NAND function. It is written as $X = A$ NAND $B$. It is also written as $X = \overline{A \cdot B}$ or simply $X = \overline{AB}$.

Tables (45.7) shows the evaluation of $\overline{AB}$ and its truth table. A NAND gate can be made by an AND gate followed by a NOT gate. Figure (45.33) shows the combination and the symbol used for a NAND gate.

**Table 45.7**

| $A$ | $B$ | $AB$ = $A$ AND $B$ | $\overline{AB}$ = NOT($A$ AND $B$) | $A$ | $B$ | $\overline{AB}$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| 0 | 1 | 0 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 0 | 1 | 1 | 0 |

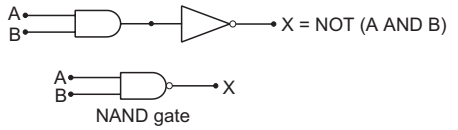



Figure 45.33

The function $X = \text{NOT } (A \text{ OR } B)$ is called a NOR function and is written as $X = A$ NOR $B$. It is also written as $X = \overline{A + B}$.

Table (45.8) shows the evaluation of $\overline{A + B}$ and its truth table. A NOR gate can be made by an OR gate followed by a NOT gate. Figure (45.34) shows the combination and the symbol used for a NOR gate.

**Table 45.8**

| $A$ | $B$ | $A + B$ | $\overline{A + B}$ | $A$ | $B$ | $\overline{A + B}$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| 0 | 1 | 1 | 0 | 0 | 1 | 0 |
| 1 | 0 | 1 | 0 | 1 | 0 | 0 |
| 1 | 1 | 1 | 0 | 1 | 1 | 0 |

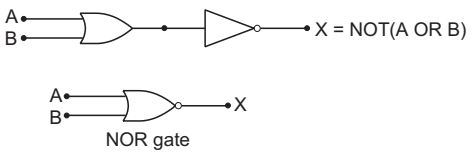



Figure 45.34

**XOR gate**

XOR is a function of two logical variables $A$ and $B$ which evaluates to 1 if one of the two variables is 0 and the other is 1. If both of the variables are 0 or both are 1, the function is zero. It is also called the *exclusive* OR function. The truth table for XOR function is given in table (45.9). Verify that

$$A \text{ XOR } B = A\overline{B} + \overline{A}B. \quad \ldots \text{ (i)}$$

**Table 45.9**

| $A$ | $B$ | $A$ XOR $B$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

An XOR gate can be constructed with AND, OR and NOT gates as shown in figure (45.35a). The symbol for an XOR gate is shown in figure (45.35b). We have made use of equation (i) in constructing this circuit.

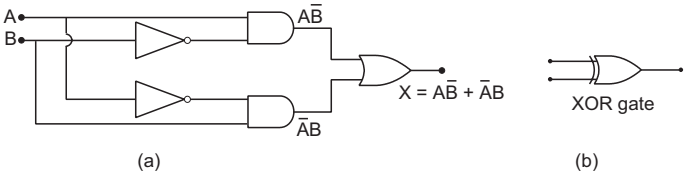



Figure 45.35

**NAND and NOR as the basic building blocks**

Any logical gate can be constructed by using only NAND gates or only NOR gates. In this sense, a NAND gate or a NOR gate is called a basic building block of logic circuits.

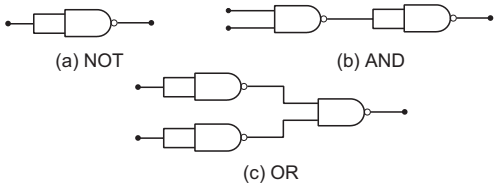



Figure 45.36

Figure (45.36) shows the construction of NOT, AND and OR gates using NAND gates. A NAND gate can be used as a NOT gate by simply connecting the two input terminals (figure 45.36a). In other words, the value of the independent variable $A$ is fed to both the input terminals. As $A$ AND $A$ is $A$,

$$X = \overline{AA} = \overline{A}.$$

To construct an AND gate, the output of a NAND gate should be fed to a NOT gate. This is because, the output of the NAND gate is $\overline{AB}$, i.e., NOT of $AB$. If we pass it through a NOT gate, it is again inverted and becomes $AB$. Figure (45.36b) does exactly the same.

Figure (45.36c) shows the construction of an OR gate. The two input variables are first inverted by passing them through two NOT gates. The inverted

signals are NANDed. Let us construct the truth table of this combination and verify that it represents an OR gate. This is done in table (45.10).

**Table 45.10**

| $A$ | $B$ | $\overline{A}$ | $\overline{B}$ | $\overline{A}\,\overline{B}$ | $\overline{\overline{A}\,\overline{B}}$ |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 | 0 |
| 0 | 1 | 1 | 0 | 0 | 1 |
| 1 | 0 | 0 | 1 | 0 | 1 |
| 1 | 1 | 0 | 0 | 0 | 1 |

Thus, $\overline{\overline{A}\,\overline{B}}$ is 1 if either of $A$ and $B$ is 1 or both are 1. It is zero only if both $A$ and $B$ are zero. Thus, $\overline{\overline{A}\,\overline{B}} = A + B$.

**Binary Mathematics : Half Adder and Full Adder**

Logic gates are the basic elements in the electronic circuits used to perform mathematical calculations such as that in a computer or in a calculator. The numbers are converted into *binary system* where only two digits 0 and 1 are used. In this system, the natural numbers (1, 2, 3, 4, … in the decimal system) are represented as 1, 10, 11, 100, 101, 110, 111, …, etc. The mathematical operation of addition is governed by the following basic rules:

$$0 + 0 = 0$$
$$0 + 1 = 1$$
$$1 + 0 = 1$$
$$1 + 1 = 10.$$

The plus sign here represents addition of arithmetic and not the logical operation OR. When 1 is added to 1 in binary system, we get a two digit number 10. It is read as *one zero* and not as ten. It is equivalent to 'two' in decimal system. When two one-digit numbers are added and the result is a two digit number, the more significant digit (that on left) is called the *carry digit* and the less significant digit (that on right) is called the *sum digit.* The word *sum* is often used for the sum digit and *carry* for the carry digit. One should be careful about the word 'sum' because it is also used to mean the net result of addition. Table (45.11) shows the carry digit and the sum digit for the addition of one-digit binary numbers. If we treat *A, B, C* and *S* as logical variables (they can take only two values 0 and 1), each of $C$ and $S$ is a logical function of $A$ and $B$.

**Table 45.11**

| First number $A$ | Second number $B$ | Carry digit $C$ | Sum digit $S$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 1 |
| 1 | 0 | 0 | 1 |
| 1 | 1 | 1 | 0 |

It can be easily verified that the carry digit $C$ is $A$ AND $B$ and the sum digit $S$ is $A$ XOR $B$, i.e.,

$$C = AB$$

and

$$S = A\overline{B} + \overline{A}B.$$

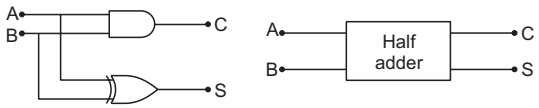



Figure 45.37

Figure (45.37) shows the circuit which takes $A$ and $B$ as inputs and gives $C$ and $S$ as outputs. Note that $A$ and $B$ are passed through an XOR gate to get $S$. The circuit described above is called a *half adder.* The symbol for a half adder is also shown in the figure.

If two numbers of more than one digit are to be added, one needs to know the addition rules for three one-digit numbers. Here are two examples, one with decimal numbers and one with binary numbers.

$$\begin{array}{r} 11 \\ 768 \\ +\ 353 \\ \hline 1121 \end{array} \qquad \begin{array}{r} 111 \\ 101 \\ +\ 111 \\ \hline 1100 \end{array}$$

In the first example, one has to evaluate 1 + 6 + 5 to get the sum digit as 2 and the carry digit as 1 for the second place from right. Similarly, one needs 1 + 7 + 3. In the second example, one needs the sum 1 + 0 + 1 to get the sum digit as 0 and the carry digit as 1 for the second place from right. Similarly, one needs 1 + 1 + 1 to get the sum digit as 1 and the carry digit as 1. Thus, we need a circuit which can take three inputs $A_1$, $A_2$ and $A_3$ and produce two outputs $C$ and $S$. There are eight possible combinations of $A_1$, $A_2$ and $A_3$ and table (45.12) gives the values of $C$ and $S$ for all these combinations.

**Table 45.12**

| $A_1$ | $A_2$ | $A_3$ | $C$ | $S$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 |
| 0 | 1 | 1 | 1 | 0 |
| 1 | 0 | 0 | 0 | 1 |
| 1 | 0 | 1 | 1 | 0 |
| 1 | 1 | 0 | 1 | 0 |
| 1 | 1 | 1 | 1 | 1 |

One can add the digits $A_1$, $A_2$ and $A_3$ as follows. First add $A_1$ and $A_2$ to get the sum digit $S_1$ and the carry digit $C_1$. Now add $A_3$ and $S_1$ to get the sum digit $S$ and the carry digit $C_2$. The sum digit $S$ is the final sum digit. Now consider the two carry digits $C_1$ and $C_2$. Verify that both of $C_1$ and $C_2$ cannot be 1. If both are 0, the final carry digit is 0. If one of them is 1 and

the other is 0, the final carry digit is 1. Thus, the final carry digit may be obtained as $C = C_1$ OR $C_2$. The circuit shown in figure (45.38) is constructed on these lines to perform the three-digit addition. Such a circuit is called a *full adder*. The symbol of full adder is also shown in figure (45.38).

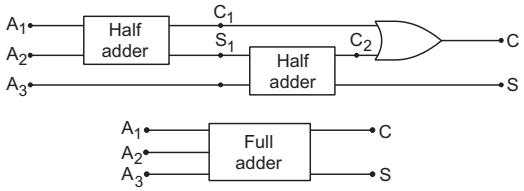



Figure 45.38

## *Worked Out Examples*

**1.** *A doped semiconductor has impurity levels* 30 meV *below the conduction band. (a) Is the material n-type or p-type ? (b) In a thermal collision, an amount kT of energy is given to the extra electron loosely bound to the impurity ion and this electron is just able to jump into the conduction band. Calculate the temperature T.*

***Solution*** : (a) The impurity provides impurity levels close to the conduction band and a number of electrons from the impurity level will populate the conduction band. Thus, the majority carriers are electrons and the material is *n*-type.

(b) According to the question, $kT = 30$ meV

or, $$T = \frac{30 \text{ meV}}{k}$$

$$= \frac{0{\cdot}03 \text{ eV}}{8{\cdot}62 \times 10^{-5} \text{ eV K}^{-1}} = 348 \text{ K}.$$

**2.** *The energy of a photon of sodium light* ($\lambda = 589$ nm) *equals the band gap of a semiconducting material. (a) Find the minimum energy E required to create a hole–electron pair. (b) Find the value of E/kT at a temperature of* 300 K.

***Solution*** : (a) The energy of the photon is $E = \frac{hc}{\lambda}$

$$= \frac{1242 \text{ eV nm}}{589 \text{ nm}} = 2{\cdot}1 \text{ eV}.$$

Thus the band gap is 2·1 eV. This is also the minimum energy $E$ required to push an electron from the valence band into the conduction band. Hence, the minimum energy required to create a hole–electron pair is 2·1 eV.

(b) At $T = 300$ K,

$$kT = (8{\cdot}62 \times 10^{-5} \text{ eV K}^{-1})\ (300 \text{ K})$$

$$= 25{\cdot}86 \times 10^{-3} \text{ eV}.$$

Thus, $$\frac{E}{kT} = \frac{2{\cdot}1 \text{ eV}}{25{\cdot}86 \times 10^{-3} \text{ eV}} = 81.$$

So it is difficult for the thermal energy to create the hole–electron pair but a photon of light can do it easily.

**3.** *A p-type semiconductor has acceptor levels* 57 meV *above the valence band. Find the maximum wavelength of light which can create a hole.*

***Solution*** : To create a hole, an electron from the valence band should be given sufficient energy to go into one of the acceptor levels. Since the acceptor levels are 57 meV above the valence band, at least 57 meV is needed to create a hole.

If $\lambda$ be the wavelength of light, its photon will have an energy $hc/\lambda$. To create a hole,

$$\frac{hc}{\lambda} \geq 57 \text{ meV}$$

or, $$\lambda \leq \frac{hc}{57 \text{ meV}}$$

$$= \frac{1242 \text{ eV nm}}{57 \times 10^{-3} \text{ eV}} = 2{\cdot}18 \times 10^{-5} \text{ m}.$$

**4.** *The band gap in germanium is* $\Delta E = 0{\cdot}68$ eV. *Assuming that the number of hole–electron pairs is proportional to* $e^{-\Delta E/2kT}$, *find the percentage increase in the number of charge carriers in pure germanium as the temperature is increased from* 300 K *to* 320 K.

***Solution*** : The number of charge carriers in an intrinsic semiconductor is double the number of hole–electron pairs. If $N_1$ be the number of charge carriers at temperature $T_1$ and $N_2$ at $T_2$, we have

$$N_1 = N_0 e^{-\Delta E/2kT_1}$$

and $$N_2 = N_0 e^{-\Delta E/2kT_2}.$$

The percentage increase as the temperature is raised from $T_1$ to $T_2$ is

$$f = \frac{N_2 - N_1}{N_1} \times 100 = \left(\frac{N_2}{N_1} - 1\right) \times 100$$

$$= 100 \left[ e^{\frac{\Delta E}{2k}\left(\frac{1}{T_1} - \frac{1}{T_2}\right)} - 1 \right].$$

Now $$\frac{\Delta E}{2k}\left(\frac{1}{T_1} - \frac{1}{T_2}\right)$$

$$= \frac{0{\cdot}68 \text{ eV}}{2 \times 8{\cdot}62 \times 10^{-5} \text{ eV K}^{-1}} \left(\frac{1}{300 \text{ K}} - \frac{1}{320 \text{ K}}\right)$$

$$= 0{\cdot}82.$$

Thus, $$f = 100 \times [e^{0{\cdot}82} - 1] \approx 127.$$

Thus, the number of charge carriers increases by about 127%.

**5.** *The concentration of hole–electron pairs in pure silicon at* $T = 300$ K *is* $7 \times 10^{15}$ *per cubic metre. Antimony is doped into silicon in a proportion of* 1 *atom in* $10^7$ *atoms. Assuming that half of the impurity atoms contribute electrons in the conduction band, calculate the factor by which the number of charge carriers increases due to doping. The number of silicon atoms per cubic metre is* $5 \times 10^{28}$.

***Solution*** **:** The number of charge carriers before doping is equal to the number of holes plus the number of conduction electrons. Thus, the number of charge carriers per cubic metre before doping

$$= 2 \times 7 \times 10^{15} = 14 \times 10^{15}.$$

Since antimony is doped in a proportion of 1 in $10^7$, the number of antimony atoms per cubic metre is $10^{-7} \times 5 \times 10^{28} = 5 \times 10^{21}$. As half of these atoms contribute electrons to the conduction band, the number of extra conduction electrons produced is $2{\cdot}5 \times 10^{21}$ per cubic metre. Thus, the number of charge carriers per cubic metre after the doping is

$$2{\cdot}5 \times 10^{21} + 14 \times 10^{15}$$

$$\approx 2{\cdot}5 \times 10^{21}.$$

The factor by which the number of charge carriers is increased

$$= \frac{2{\cdot}5 \times 10^{21}}{14 \times 10^{15}} = 1{\cdot}8 \times 10^5.$$

In fact, as the *n*-type impurity is doped, the number of holes will decrease. This is because the product of the concentrations of holes and conduction electrons remains almost the same. However, this does not affect our result as the number of holes is anyway too small as compared to the number of conduction electrons.

**6.** *A potential barrier of* 0·50 V *exists across a p-n junction. (a) If the depletion region is* $5{\cdot}0 \times 10^{-7}$ m *wide, what is the intensity of the electric field in this region ? (b) An electron with speed* $5{\cdot}0 \times 10^5$ m s$^{-1}$ *approaches the p-n junction from the n-side. With what speed will it enter the p-side ?*

***Solution*** **:** (a) The electric field is $E = V/d$

$$= \frac{0{\cdot}50 \text{ V}}{5{\cdot}0 \times 10^{-7} \text{ m}} = 1{\cdot}0 \times 10^6 \text{ V m}^{-1}.$$

(b)

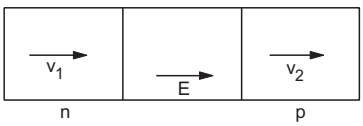



Figure 45-W1

Suppose the electron has a speed $v_1$ when it enters the depletion layer and $v_2$ when it comes out of it (figure 45-W1). As the potential energy increases by $e \times 0{\cdot}50$ V, from the principle of conservation of energy,

$$\frac{1}{2} m v_1^2 = e \times 0{\cdot}50 \text{ V} + \frac{1}{2} m v_2^2$$

or, $$\frac{1}{2} \times (9{\cdot}1 \times 10^{-31} \text{ kg}) \times (5{\cdot}0 \times 10^5 \text{ m s}^{-2})^2$$

$$= 1{\cdot}6 \times 10^{-19} \times 0{\cdot}5 \text{ J} + \frac{1}{2} (9{\cdot}1 \times 10^{-31} \text{ kg}) v_2^2$$

or, $$1{\cdot}13 \times 10^{-19} \text{ J} = 0{\cdot}8 \times 10^{-19} \text{ J} + (4{\cdot}55 \times 10^{-31} \text{ kg}) v_2^2.$$

Solving this, $v_2 = 2{\cdot}7 \times 10^5$ m s$^{-1}$.

**7.** *The reverse-biased current of a particular p-n junction diode increases when it is exposed to light of wavelength less than or equal to* 600 nm. *Assume that the increase in carrier concentration takes place due to the creation of new hole–electron pairs by the light. Find the band gap.*

***Solution*** **:** The reverse-biased current is caused mainly due to the drift current. The drift current in a *p-n* junction is caused by the formation of new hole–electron pairs and their subsequent motions in the depletion layer. When the junction is exposed to light, it may absorb energy from the light photons. If this energy supplied by a photon is greater than (or equal to) the band gap, a hole–electron pair may be formed. Thus, the reverse-biased current will increase if the light photons have energy greater than (or equal to) the band gap. Hence the band gap is equal to the energy of a photon of 600 nm light which is

$$\frac{hc}{\lambda} = \frac{1242 \text{ eV nm}}{600 \text{ nm}} = 2{\cdot}07 \text{ eV}.$$

**8.** *A* 2 V *battery may be connected across the points A and B as shown in figure (45-W2). Assume that the resistance of each diode is zero in forward bias and infinity in reverse bias. Find the current supplied by the battery if the positive terminal of the battery is connected to (a) the point A (b) the point B.*

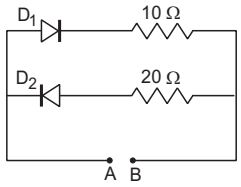



Figure 45-W2

***Solution*** **:** (a) When the positive terminal of the battery is connected to the point *A*, the diode $D_1$ is forward-biased and $D_2$ is reverse-biased. The resistance of the diode $D_1$ is zero, and it can be replaced by a resistanceless wire. Similarly, the resistance of the diode $D_2$ is infinity, and it can be replaced by a broken wire. The equivalent circuit is shown in figure (45-W3a). The current supplied by the battery is 2 V/10 Ω = 0·2 A.

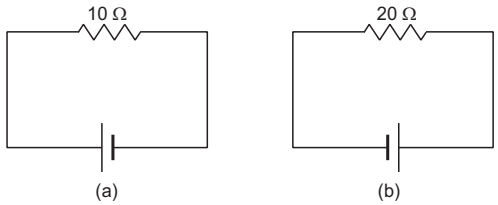



Figure 45-W3

(b) When the positive terminal of the battery is connected to the point $B$, the diode $D_2$ is forward-biased and $D_1$ is reverse biased. The equivalent circuit is shown in figure (45-W3b). The current through the battery is 2 V/20 Ω = 0·1 A.

**9.** *A change of* 8·0 mA *in the emitter current brings a change of* 7·9 mA *in the collector current. How much change in the base current is required to have the same change* 7·9 mA *in the collector current ? Find the values of* α *and* β.

***Solution*** : We have,

$$I_E = I_B + I_C$$

or, $$\Delta I_E = \Delta I_B + \Delta I_C.$$

From the question, when $\Delta I_E = 8{\cdot}0$ mA, $\Delta I_C = 7{\cdot}9$ mA.

Thus,

$$\Delta I_B = 8{\cdot}0 \text{ mA} - 7{\cdot}9 \text{ mA} = 0{\cdot}1 \text{ mA}.$$

So a change of 0·1 mA in the base current is required to have a change of 7·9 mA in the collector current.

$$\alpha = \frac{I_C}{I_E} = \frac{\Delta I_C}{\Delta I_E}$$

$$= \frac{7{\cdot}9 \text{ mA}}{8{\cdot}0 \text{ mA}} \approx 0{\cdot}99.$$

$$\beta = \frac{I_C}{I_B} = \frac{\Delta I_C}{\Delta I_B}$$

$$= \frac{7{\cdot}9 \text{ mA}}{0{\cdot}1 \text{ mA}} = 79.$$

Check if these values of α and β satisfy the equation

$$\beta = \frac{\alpha}{1-\alpha}.$$

**10.** *A transistor is used in common-emitter mode in an amplifier circuit. When a signal of* 20 mV *is added to the base–emitter voltage, the base current changes by* 20 μA *and the collector current changes by* 2 mA. *The load resistance is* 5 kΩ. *Calculate (a) the factor* β, *(b) the input resistance* $R_{BE}$, *(c) the transconductance and (d) the voltage gain.*

***Solution*** : (a) $\beta = \dfrac{\Delta I_C}{\Delta I_B} = \dfrac{2 \text{ mA}}{20 \text{ μA}} = 100.$

(b) The input resistance $R_{BE} = \dfrac{\Delta V_{BE}}{\Delta I_B}$

$$= \frac{20 \text{ mV}}{20 \text{ μA}} = 1 \text{ kΩ}.$$

(c) Transconductance $= \dfrac{\Delta I_C}{\Delta V_{BE}}$

$$= \frac{2 \text{ mA}}{20 \text{ mV}} = 0{\cdot}1 \text{ mho}.$$

(d) The change in output voltage is $R_L \, \Delta I_c$

$$= (5 \text{ kΩ}) \, (2 \text{ mA}) = 10 \text{ V}.$$

The applied signal voltage = 20 mV.

Thus, the voltage gain is,

$$\frac{10 \text{ V}}{20 \text{ mV}} = 500.$$

**11.** *Construct the truth table for the function X of A and B represented by figure (45-W4).*

Figure 45-W4

***Solution*** : Here an AND gate and an OR gate are used. Let the output of the OR gate be $Y$. Clearly, $Y = A + B$. The AND gate receives $A$ and $A + B$ as input. The output of this gate is $X$. So $X = A(A + B)$. The following table evaluates $X$ for all combinations of $A$ and $B$. The last three columns give the truth table.

| $A$ | $B$ | $Y = A + B$ | $X = A(A + B)$ | $A$ | $B$ | $X$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 | 0 | 1 | 0 |
| 1 | 0 | 1 | 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |

□

## QUESTIONS FOR SHORT ANSWER

**1.** How many 1s energy states are present in one mole of sodium vapour ? Are they all filled in normal conditions ? How many 3s energy states are present in one mole of sodium vapour ? Are they all filled in normal conditions ?

**2.** There are energy bands in a solid. Do we have really continuous energy variation in a band or do we have very closely spaced but still discrete energy levels ?

**3.** The conduction band of a solid is partially filled at 0 K. Will it be a conductor, a semiconductor or an insulator ?

**4.** In semiconductors, thermal collisions are responsible for taking a valence electron to the conduction band. Why does the number of conduction electrons not go on increasing with time as thermal collisions continuously take place ?

**5.** When an electron goes from the valence band to the conduction band in silicon, its energy is increased by 1·1 eV. The average energy exchanged in a thermal collision is of the order of $kT$ which is only 0·026 eV at room temperature. How is a thermal collision able to take some of the electrons from the valence band to the conduction band ?

**6.** What is the resistance of an intrinsic semiconductor at 0 K ?

**7.** We have valence electrons and conduction electrons in a semiconductor. Do we also have 'valence holes' and 'conduction holes' ?

**8.** When a $p$-type impurity is doped in a semiconductor, a large number of holes are created. This does not make the semiconductor charged. But when holes diffuse from the $p$-side to the $n$-side in a $p$-$n$ junction, the $n$-side gets positively charged. Explain.

**9.** The drift current in a reverse-biased $p$-$n$ junction increases in magnitude if the temperature of the junction is increased. Explain this on the basis of creation of hole–electron pairs.

**10.** An ideal diode should pass a current freely in one direction and should stop it completely in the opposite direction. Which is closer to ideal—vacuum diode or a $p$-$n$ junction diode ?

**11.** Consider an amplifier circuit using a transistor. The output power is several times greater than the input power. Where does the extra power come from ?

## OBJECTIVE I

**1.** Electric conduction in a semiconductor takes place due to
(a) electrons only (b) holes only
(c) both electrons and holes
(d) neither electrons nor holes.

**2.** An electric field is applied to a semiconductor. Let the number of charge carriers be $n$ and the average drift speed be $v$. If the temperature is increased,
(a) both $n$ and $v$ will increase
(b) $n$ will increase but $v$ will decrease
(c) $v$ will increase but $n$ will decrease
(d) both $n$ and $v$ will decrease.

**3.** Let $n_p$ and $n_e$ be the numbers of holes and conduction electrons in an intrinsic semiconductor.
(a) $n_p > n_e$ (b) $n_p = n_e$ (c) $n_p < n_e$ (d) $n_p \neq n_e$

**4.** Let $n_p$ and $n_e$ be the numbers of holes and conduction electrons in an extrinsic semiconductor.
(a) $n_p > n_e$ (b) $n_p = n_e$ (c) $n_p < n_e$ (d) $n_p \neq n_e$

**5.** A $p$-type semiconductor is
(a) positively charged (b) negatively charged
(c) uncharged
(d) uncharged at 0 K but charged at higher temperatures.

**6.** When an impurity is doped into an intrinsic semiconductor, the conductivity of the semiconductor
(a) increases (b) decreases
(c) remains the same (d) becomes zero.

**7.** If the two ends of a $p$-$n$ junction are joined by a wire,
(a) there will not be a steady current in the circuit
(b) there will be a steady current from the $n$-side to the $p$-side
(c) there will a steady current from the $p$-side to the $n$-side
(d) there may or may not be a current depending upon the resistance of the connecting wire.

**8.** The drift current in a $p$-$n$ junction is
(a) from the $n$-side to the $p$-side
(b) from the $p$-side to the $n$-side
(c) from the $n$-side to the $p$-side if the junction is forward-biased and in the opposite direction if it is reverse-biased
(d) from the $p$-side to the $n$-side if the junction is forward-biased and in the opposite direction if it is reverse-biased.

**9.** The diffusion current in a $p$-$n$ junction is
(a) from the $n$-side to the $p$-side
(b) from the $p$-side to the $n$-side
(c) from the $n$-side to the $p$-side if the junction is forward-biased and in the opposite direction if it is reverse-biased
(d) from the $p$-side to the $n$-side if the junction is forward-biased and in the opposite direction if it is reverse-biased.

**10.** Diffusion current in a $p$-$n$ junction is greater than the drift current in magnitude
(a) if the junction is forward-biased
(b) if the junction is reverse-biased
(c) if the junction is unbiased
(d) in no case.

**11.** Two identical $p$-$n$ junctions may be connected in series with a battery in three ways (figure 45-Q1). The potential difference across the two $p$-$n$ junctions are equal in
(a) circuit 1 and circuit 2 (b) circuit 2 and circuit 3
(c) circuit 3 and circuit 1 (d) circuit 1 only.

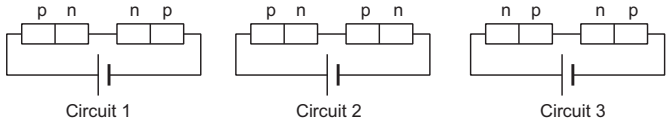



Figure 45-Q1

**12.** Two identical capacitors $A$ and $B$ are charged to the same potential $V$ and are connected in two circuits at $t = 0$ as shown in figure (45-Q2). The charges on the

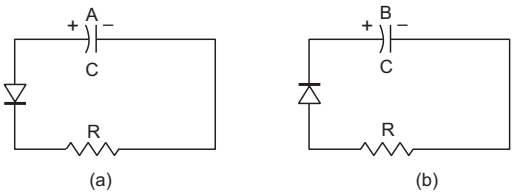



Figure 45-Q2

capacitors at a time $t = CR$ are, respectively,
(a) $VC$, $VC$ (b) $VC/e$, $VC$ (c) $VC$, $VC/e$ (d) $VC/e$, $VC/e$.

**13.** A hole diffuses from the $p$-side to the $n$-side in a $p$-$n$ junction. This means that
(a) a bond is broken on the $n$-side and the electron freed from the bond jumps to the conduction band
(b) a conduction electron on the $p$-side jumps to a broken bond to complete it
(c) a bond is broken on the $n$-side and the electron freed from the bond jumps to a broken bond on the $p$-side to complete it
(d) a bond is broken on the $p$-side and the electron freed from the bond jumps to a broken bond on the $n$-side to complete it.

**14.** In a transistor,
(a) the emitter has the least concentration of impurity
(b) the collector has the least concentration of impurity
(c) the base has the least concentration of impurity
(d) all the three regions have equal concentrations of impurity.

**15.** An incomplete sentence about transistors is given below: The emitter–..... junction is ___ and the collector– ..... junction is ___. The appropriate words for the dotted empty positions are, respectively,
(a) 'collector' and 'base' (b) 'base' and 'emitter'
(c) 'collector' and 'emitter' (d) 'base' and 'base'.

## OBJECTIVE II

**1.** In a semiconductor,
(a) there are no free electrons at 0 K
(b) there are no free electrons at any temperature
(c) the number of free electrons increases with temperature
(d) the number of free electrons is less than that in a conductor.

**2.** In a $p$-$n$ junction with open ends,
(a) there is no systematic motion of charge carriers
(b) holes and conduction electrons systematically go from the $p$-side to the $n$-side and from the $n$-side to the $p$-side respectively
(c) there is no net charge transfer between the two sides
(d) there is a constant electric field near the junction.

**3.** In a $p$-$n$ junction,
(a) new holes and conduction electrons are produced continuously throughout the material
(b) new holes and conduction electrons are produced continuously throughout the material except in the depletion region
(c) holes and conduction electrons recombine continuously throughout the material
(d) holes and conduction electrons recombine continuously throughout the material except in the depletion region.

**4**. The impurity atoms with which pure silicon may be doped to make it a $p$-type semiconductor are those of
(a) phosphorus (b) boron (c) antimony (d) aluminium.

**5.** The electrical conductivity of pure germanium can be increased by
(a) increasing the temperature
(b) doping acceptor impurities
(c) doping donor impurities
(d) irradiating ultraviolet light on it.

**6.** A semiconducting device is connected in a series circuit with a battery and a resistance. A current is found to pass through the circuit. If the polarity of the battery is reversed, the current drops to almost zero. The device may be
(a) an intrinsic semiconductor
(b) a $p$-type semiconductor
(c) an $n$-type semiconductor
(d) a $p$-$n$ junction.

**7.** A semiconductor is doped with a donor impurity.
(a) The hole concentration increases.
(b) The hole concentration decreases.
(c) The electron concentration increases.
(d) The electron concentration decreases.

**8.** Let $i_E$, $i_C$ and $i_B$ represent the emitter current, the collector current and the base current respectively in a transistor. Then
(a) $i_C$ is slightly smaller than $i_E$
(b) $i_C$ is slightly greater than $i_E$
(c) $i_B$ is much smaller than $i_E$
(d) $i_B$ is much greater than $i_E$.

**9.** In a normal operation of a transistor,
(a) the base–emitter junction is forward-biased
(b) the base–collector junction is forward-biased
(c) the base–emitter junction is reverse-biased
(d) the base–collector junction is reverse-biased.

**10.** An AND gate can be prepared by repetitive use of
(a) NOT gate (b) OR gate
(c) NAND gate (d) NOR gate.

## EXERCISES

Planck constant = $4{\cdot}14 \times 10^{-15}$ eV s,
Boltzmann constant = $8{\cdot}62 \times 10^{-5}$ eV $K^{-1}$.

1. Calculate the number of states per cubic metre of sodium in 3s band. The density of sodium is 1013 kgm$^{-3}$. How many of them are empty ?
2. In a pure semiconductor, the number of conduction electrons is $6 \times 10^{19}$ per cubic metre. How many holes are there in a sample of size 1 cm × 1 cm × 1 mm ?
3. Indium antimonide has a band gap of 0·23 eV between the valence and the conduction band. Find the temperature at which $kT$ equals the band gap.
4. The band gap for silicon is 1·1 eV. (a) Find the ratio of the band gap to $kT$ for silicon at room temperature 300 K. (b) At what temperature does this ratio become one tenth of the value at 300 K ? (Silicon will not retain its structure at these high temperatures.)
5. When a semiconducting material is doped with an impurity, new acceptor levels are created. In a particular thermal collision, a valence electron receives an energy equal to $2kT$ and just reaches one of the acceptor levels. Assuming that the energy of the electron was at the top edge of the valence band and that the temperature $T$ is equal to 300 K, find the energy of the acceptor levels above the valence band.
6. The band gap between the valence and the conduction bands in zinc oxide (ZnO) is 3·2 eV. Suppose an electron in the conduction band combines with a hole in the valence band and the excess energy is released in the form of electromagnetic radiation. Find the maximum wavelength that can be emitted in this process.
7. Suppose the energy liberated in the recombination of a hole–electron pair is converted into electromagnetic radiation. If the maximum wavelength emitted is 820 nm, what is the band gap ?
8. Find the maximum wavelength of electromagnetic radiation which can create a hole–electron pair in germanium. The band gap in germanium is 0·65 eV.
9. In a photodiode, the conductivity increases when the material is exposed to light. It is found that the conductivity changes only if the wavelength is less than 620 nm. What is the band gap ?
10. Let $\Delta E$ denote the energy gap between the valence band and the conduction band. The population of conduction electrons (and of the holes) is roughly proportional to $e^{-\Delta E/2kT}$. Find the ratio of the concentration of conduction electrons in diamond to that in silicon at room temperature 300 K. $\Delta E$ for silicon is 1·1 eV and for diamond is 6·0 eV. How many conduction electrons are likely to be in one cubic metre of diamond ?
11. The conductivity of a pure semiconductor is roughly proportional to $T^{3/2} e^{-\Delta E/2kT}$ where $\Delta E$ is the band gap. The band gap for germanium is 0·74 eV at 4 K and 0·67 eV at 300 K. By what factor does the conductivity of pure germanium increase as the temperature is raised from 4 K to 300 K ?
12. Estimate the proportion of boron impurity which will increase the conductivity of a pure silicon sample by a factor of 100. Assume that each boron atom creates a hole and the concentration of holes in pure silicon at the same temperature is $7 \times 10^{15}$ holes per cubic metre. Density of silicon is $5 \times 10^{28}$ atoms per cubic metre.
13. The product of the hole concentration and the conduction electron concentration turns out to be independent of the amount of any impurity doped. The concentration of conduction electrons in germanium is $6 \times 10^{19}$ per cubic metre. When some phosphorus impurity is doped into a germanium sample, the concentration of conduction electrons increases to $2 \times 10^{23}$ per cubic metre. Find the concentration of the holes in the doped germanium.
14. The conductivity of an intrinsic semiconductor depends on temperature as $\sigma = \sigma_0 e^{-\Delta E/2kT}$, where $\sigma_0$ is a constant. Find the temperature at which the conductivity of an intrinsic germanium semiconductor will be double of its value at $T = 300$ K. Assume that the gap for germanium is 0·650 eV and remains constant as the temperature is increased.
15. A semiconducting material has a band gap of 1 eV. Acceptor impurities are doped into it which create acceptor levels 1 meV above the valence band. Assume that the transition from one energy level to the other is almost forbidden if $kT$ is less than 1/50 of the energy gap. Also, if $kT$ is more than twice the gap, the upper levels have maximum population. The temperature of the semiconductor is increased from 0 K. The concentration of the holes increases with temperature and after a certain temperature it becomes approximately constant. As the temperature is further increased, the hole concentration again starts increasing at a certain temperature. Find the order of the temperature range in which the hole concentration remains approximately constant.
16. In a $p$-$n$ junction, the depletion region is 400 nm wide and an electric field of $5 \times 10^5$ V m$^{-1}$ exists in it. (a) Find the height of the potential barrier. (b) What should be the minimum kinetic energy of a conduction electron which can diffuse from the $n$-side to the $p$-side ?
17. The potential barrier existing across an unbiased $p$-$n$ junction is 0·2 volt. What minimum kinetic energy a hole should have to diffuse from the $p$-side to the $n$-side if (a) the junction is unbiased, (b) the junction is forward-biased at 0·1 volt and (c) the junction is reverse-biased at 0·1 volt ?
18. In a $p$-$n$ junction, a potential barrier of 250 meV exists across the junction. A hole with a kinetic energy of 300 meV approaches the junction. Find the kinetic energy of the hole when it crosses the junction if the hole approached the junction (a) from the $p$-side and (b) from the $n$-side.
19. When a $p$-$n$ junction is reverse-biased, the current becomes almost constant at 25 μA. When it is forward-biased at 200 mV, a current of 75 μA is obtained. Find the magnitude of diffusion current when the diode is

(a) unbiased, (b) reverse-biased at 200 mV and (c) forward-biased at 200 mV.

**20.** The drift current in a *p-n* junction is 20·0 μA. Estimate the number of electrons crossing a cross section per second in the depletion region.

**21.** The current–voltage characteristic of an ideal *p-n* junction diode is given by

$$i = i_0 (e^{eV/kT} - 1)$$

where the drift current $i_0$ equals 10 μA. Take the temperature $T$ to be 300 K. (a) Find the voltage $V_0$ for which $e^{eV/kT} = 100$. One can neglect the term 1 for voltages greater than this value. (b) Find an expression for the dynamic resistance of the diode as a function of $V$ for $V > V_0$. (c) Find the voltage for which the dynamic resistance is 0·2 Ω.

**22.** Consider a *p-n* junction diode having the characteristic $i = i_0(e^{eV/kT} - 1)$ where $i_0 = 20$ μA. The diode is operated at $T = 300$ K. (a) Find the current through the diode when a voltage of 300 mV is applied across it in forward bias. (b) At what voltage does the current double ?

**23.** Calculate the current through the circuit and the potential difference across the diode shown in figure (45-E1). The drift current for the diode is 20 μA.

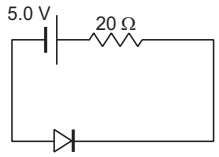



Figure 45-E1

**24.** Each of the resistances shown in figure (45-E2) has a value of 20 Ω. Find the equivalent resistance between *A* and *B*. Does it depend on whether the point *A* or *B* is at higher potential ?

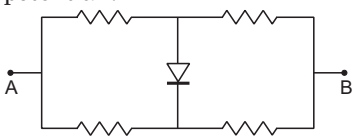



Figure 45-E2

*In problems* 25 *to* 30, *assume that the resistance of each diode is zero in forward bias and is infinity in reverse bias.*

**25.** Find the currents through the resistances in the circuits shown in figure (45-E3).

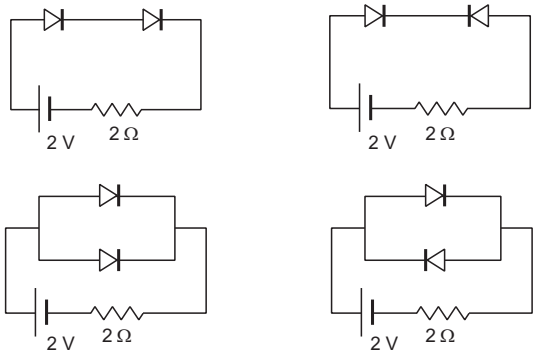



Figure 45-E3

**26.** What are the readings of the ammeters $A_1$ and $A_2$ shown in figure (45-E4). Neglect the resistances of the meters.

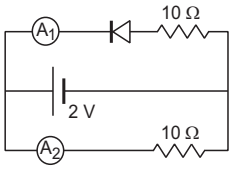



Figure 45-E4

**27.** Find the current through the battery in each of the circuits shown in figure (45-E5).

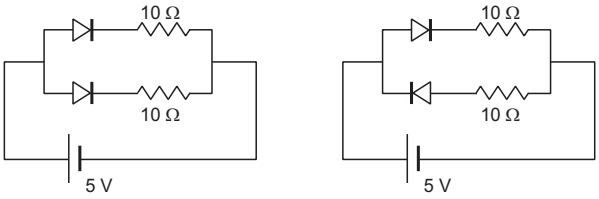



Figure 45-E5

**28.** Find the current through the resistance $R$ in figure (45-E6) if (a) $R = 12$ Ω (b) $R = 48$ Ω.

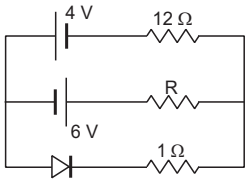



Figure 45-E6

**29.** Draw the current–voltage characteristics for the device shown in figure (45-E7) between the terminals *A* and *B*.

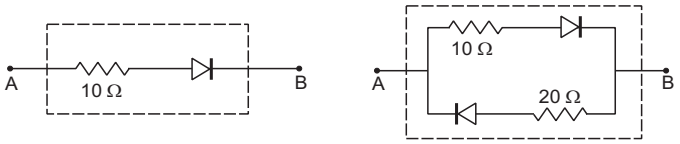



Figure 45-E7

**30.** Find the equivalent resistance of the network shown in figure (45-E8) between the points *A* and *B*.

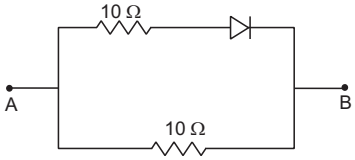



Figure 45-E8

**31.** When the base current in a transistor is changed from 30 μA to 80 μA, the collector current is changed from 1·0 mA to 3·5 mA. Find the current gain β.

**32.** A load resistor of 2 kΩ is connected in the collector branch of an amplifier circuit using a transistor in common-emitter mode. The current gain $\beta = 50$. The input resistance of the transistor is 0·50 kΩ. If the input current is changed by 50 μA, (a) by what amount does the output voltage change, (b) by what amount does the input voltage change and (c) what is the power gain ?

**33.** Let $X = A\overline{BC} + B\overline{CA} + C\overline{AB}$. Evaluate $X$ for
(a) $A = 1, B = 0, C = 1$, (b) $A = B = C = 1$, and
(c) $A = B = C = 0$.

**34.** Design a logical circuit using AND, OR and NOT gates to evaluate $A\overline{BC} + B\overline{CA}$.

**35.** Show that $AB + \overline{AB}$ is always 1.

□

## ANSWERS

### OBJECTIVE I

1. (c) 2. (b) 3. (b) 4. (d) 5. (c) 6. (a)
7. (a) 8. (a) 9. (b) 10. (a) 11. (b) 12. (b)
13. (c) 14. (c) 15. (d)

### OBJECTIVE II

1. (a), (c), (d) 2. (b), (c), (d) 3. (a), (d)
4. (b), (d) 5. all 6. (d)
7. (b), (c) 8. (a), (c) 9. (a), (d)
10. (c), (d)

### EXERCISES

1. $5{\cdot}3 \times 10^{28}$, $2{\cdot}65 \times 10^{28}$
2. $6 \times 10^{12}$
3. 2670 K
4. (a) 43 (b) 3000 K
5. 50 meV
6. 390 nm
7. 1·5 eV
8. $1{\cdot}9 \times 10^{-6}$ m
9. 2·0 eV
10. $2{\cdot}3 \times 10^{-33}$, almost zero
11. approximately $10^{463}$
12. 1 in about $3{\cdot}5 \times 10^{10}$
13. $1{\cdot}8 \times 10^{16}$ per cubic metre
14. 318 K
15. 20 to 230 K
16. (a) 0·2 V (b) 0·2 eV
17. (a) 0·2 eV (b) 0·1 eV (c) 0·3 eV
18. (a) 50 meV (b) 550 meV
19. (a) 25 μA (b) zero (c) 100 μA
20. $3{\cdot}1 \times 10^{13}$
21. (a) 0·12 V (b) $\frac{kT}{ei_0} e^{-eV/kT}$ (c) 0·25 V
22. (a) 2 A (b) 318 mV
23. 20 μA, 4·996 V ≅ 5 V
24. 20 Ω
25. (a) 1 A (b) zero (c) 1 A (d) 1 A
26. zero, 0·2 A
27. (a) 1A (b) 0·5 A
28. (a) 0·42 A, 0·13 A
30. 5 Ω if $V_A > V_B$ and 10 Ω if $V_A < V_B$
31. 50
32. (a) 5·0 V (b) 25 mV (c) $10^4$
33. (a) 1 (b) 0 (c) 0

□

## CHAPTER 46

# THE NUCLEUS

At the centre of an atom exists the *nucleus* which contains protons and neutrons. The electrons surround this nucleus to form the atom. As discussed earlier, this structure of atom was revealed by the experiments of Rutherford in which a beam of alpha particles was made to strike a thin gold foil. Most of the alpha particles crossed the foil without being appreciably deviated, but there were some alpha particles which suffered large deviation from their original lines of motion. The data suggested that positive charges in an atom are concentrated in a small volume which we call nucleus and this nucleus is responsible for the large deviation of alpha particles. Later on, the existence of protons and neutrons in the nucleus was established. In this chapter, we shall discuss the physics of the nucleus.

### 46.1 PROPERTIES OF A NUCLEUS

#### Nuclear Constituents

A nucleus is made of protons and neutrons. A proton has a positive charge of magnitude equal to that of an electron and has a mass of about 1840 times the mass of an electron. A neutron has a mass slightly greater than that of a proton. The masses of a proton and a neutron are

$$m_p = 1{\cdot}6726231 \times 10^{-27} \text{ kg}$$

and $m_n = 1{\cdot}6749286 \times 10^{-27}$ kg.

It is customary in nuclear physics and high energy physics to represent mass in energy units according to the conversion formula $E = mc^2$. (Matter can be viewed as a condensed form of energy. Theory of relativity reveals that a mass $m$ is equivalent to an energy $E$ where $E = mc^2$.) For example, the mass of an electron is $m_e = 9{\cdot}1093897 \times 10^{-31}$ kg and the equivalent energy is

$$m_e c^2 = 510{\cdot}99 \text{ keV}.$$

Thus, the mass of an electron is 510·99 keV $c^{-2}$. Similarly, the mass of a proton is 938·27231 MeV $c^{-2}$ and the mass of a neutron is 939·56563 MeV $c^{-2}$. The energy corresponding to the mass of a particle when it is at rest is called its *rest mass energy*.

Another unit which is widely used in describing mass in nuclear physics as well as in atomic physics is *unified atomic mass unit* denoted by the symbol u. It is 1/12 of the mass of a neutral carbon atom in its lowest energy state which contains six protons, six neutrons and six electrons. We have

$$1 \text{ u} = 1{\cdot}6605402 \times 10^{-27} \text{ kg} = 931{\cdot}478 \text{ MeV } c^{-2}.$$

Protons and neutrons are fermions and obey the Pauli exclusion principle like electrons. No two protons or two neutrons can have the same quantum state. But one proton and one neutron can exist in the same quantum state. Protons and neutrons are collectively called *nucleons*.

The number of protons in a nucleus is denoted by $Z$, the number of neutrons by $N$ and the total number of nucleons by $A$. Thus, $A = Z + N$. The total number of nucleons $A$ is also called the *mass number* of the nucleus. The number of protons $Z$ is called the *atomic number*. A nucleus is symbolically expressed as ${}^{A}_{Z}\text{X}$ in which X is the chemical symbol of the element. Thus, ${}^{4}_{2}\text{He}$ represents helium nucleus which contains 2 protons and a total of 4 nucleons. So it contains 2 neutrons. Similarly, ${}^{238}_{92}\text{U}$ represents a uranium nucleus which contains 92 protons and 146 neutrons.

The distribution of electrons around the nucleus is determined by the number of protons $Z$ and hence the chemical properties of an element are also determined by $Z$. The nuclei having the same number of protons but different number of neutrons are called *isotope*s. Nuclei with the same neutron number $N$ but different atomic number $Z$ are called *isotones* and the nuclei with the same mass number $A$ are called *isobar*s. All nuclei with a given $Z$ and $N$ are collectively called a *nuclide*. Thus, all the ${}^{56}_{26}\text{Fe}$ nuclei taken together is one nuclide and all the ${}^{32}_{16}\text{S}$ nuclei taken together is another nuclide.

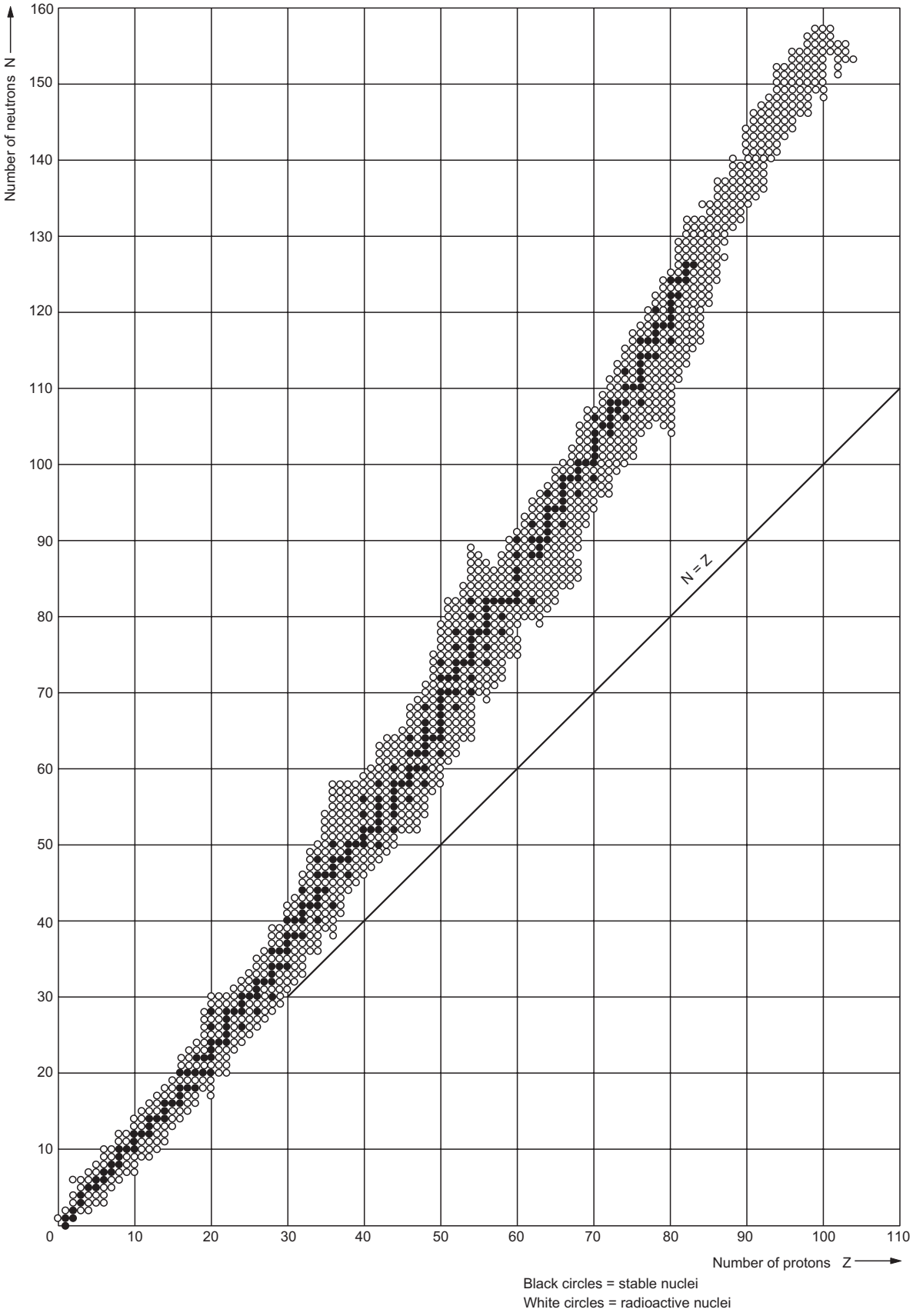

Number of neutrons N
160
150
140
130
120
110
100
90
80
70
60
50
40
30
20
10
0
10
20
30
40
50
60
70
80
90
100
110
Number of protons Z
N = Z
Black circles = stable nuclei
White circles = radioactive nuclei

Figure 46.1

**Nuclear Stability**

More than 1000 nuclides have been identified but not all are stable. An unstable nucleus emits some kind of particle and changes its constitution. A stable nucleus maintains its constitution all the time. Figure (46.1) shows a plot of neutron number $N$ versus proton number $Z$ for the nuclides observed. The black circles represent the stable nuclides. For light stable nuclides, the neutron number is equal to the proton number so that the ratio $N/Z$ is equal to 1. The ratio $N/Z$ increases for the heavier nuclides and becomes about 1·6 for heaviest stable nuclides.

The points ($Z$, $N$) for stable nuclides fall in a rather well-defined narrow region. There are nuclides to the left of the stability region as well as to the right of it. They are represented in figure (46.1) by white circles. The nuclides to the left of the stability region have excess neutrons, whereas, those to the right of the stability region have excess protons. These nuclides are unstable and decay with time according to the laws of radioactive disintegration discussed later in this chapter. They are called *radioactive nuclide.*

**Nuclear Radius**

The nucleus is so small a particle that we cannot define a sharp boundary for it. For such small particles, the description must be given in terms of the wave functions only. The magnitude of the wave function becomes very small as one moves some distance away from the centre of the nucleus. A rough estimate of nuclear size may be made by finding the region where the wave function has appreciable magnitude. Experiments show that the average radius $R$ of a nucleus may be written as

$$R = R_0 A^{1/3} \quad \ldots \quad (46.1)$$

where $R_0 = 1{\cdot}1 \times 10^{-15}$ m = 1·1 fm and $A$ is the mass number.

The volume of a nucleus is

$$V = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi R_0^3 A. \quad \ldots \quad \text{(i)}$$

As the masses of a proton and a neutron are roughly equal, say $m$, the mass of a nucleus $M$ is also roughly proportional to the mass number $A$.

We have, $M = mA.$ … (ii)

From (i) and (ii), the density within a nucleus (mass per unit volume) $\rho = M/V$ is independent of $A$.

***Example 46.1***

*Calculate the radius of* $^{70}$Ge.

***Solution*** : We have,

$$R = R_0 A^{1/3} = (1{\cdot}1 \text{ fm})\,(70)^{1/3}$$

$$= (1{\cdot}1 \text{ fm})\,(4{\cdot}12) = 4{\cdot}53 \text{ fm}.$$

**Nuclear Spin**

The protons and neutrons inside a nucleus move in well-defined quantum states and thus have orbital angular momenta. Apart from this, protons and neutrons also have internal spin angular momentum. The spin quantum number $m_s$ is $+\frac{1}{2}$ or $-\frac{1}{2}$. The total angular momentum of the nucleus is the resultant of all the spin and orbital angular momenta of the individual nucleons. This total angular momentum of a nucleus is called the *nuclear spin* of that nucleus.

## 46.2 NUCLEAR FORCES

When nucleons are kept at a separation of the order of a femtometre ($10^{-15}$ m), a new kind of force, called *nuclear force* starts acting. Nuclear force is much stronger than gravitational and electromagnetic forces if the separation between the interacting nucleons is of the order of 1 fm. Nuclear forces are basically attractive and are responsible for keeping the nucleons bound in a nucleus. The protons exert repulsive Coulomb forces on each other. The neutrons take no part in electric interaction as they are chargeless particles. The nuclear forces operate between proton and proton, neutron and neutron as well as between proton and neutron. The overall effect of this attractive nuclear force is much stronger than that of the repulsive Coulomb forces between the protons and thus the nucleus stays bound.

Unlike gravitational or electromagnetic force, nuclear force is not represented by a simple formula. In fact, the nuclear force is not yet completely understood and physicists are still working out the details. Some of the qualitative properties of nuclear forces are as follows:

(a) Nuclear forces are short-ranged. They are most effective only up to a distance of the order of a femtometre or less. The nuclear force between two nucleons decreases rapidly as the separation between them increases and becomes negligible at separations more than, say, 10 fm. The range up to which the nuclear force is effective is called *nuclear range.*

(b) Nuclear forces are, on an average, much stronger than electromagnetic forces (≈ 50–60 times stronger) in the nuclear range.

(c) Nuclear forces are independent of charge. The nuclear force between two protons is the same as that between two neutrons or between a proton and a neutron. Remember, the Coulomb force between two protons acts according to the well-defined Coulomb's

law, over and above the nuclear force. The nuclear force itself is independent of charge.

(d) An important property of nuclear force is that it is not a central force. The force between a pair of nucleons is not solely determined by the distance between the necleons. For example, the nuclear force depends on the directions of the spins of the nucleons. The force is stronger if the spins of the nucleons are parallel (i.e., both nucleons have $m_s = +1/2$ or $-1/2$) and is weaker if the spins are antiparallel (i.e., one nucleon has $m_s = +1/2$ and the other has $m_s = -1/2$).

Many of the nuclear properties may be understood on the basis of these qualitative properties of nuclear forces.

Because of the short-range nature of nuclear force, each nucleon in a nucleus interacts only with a small number of neighbouring nucleons through the nuclear force. This explains why the density of the nucleons is roughly the same in all the nuclei.

Because of the Pauli exclusion principle, each quantum state can contain at the most two protons (with opposite spins) and two neutrons (again with opposite spins). Thus, nuclear forces favour pairing of two protons and two neutrons together. In light nuclei, the nuclear forces between the nucleons are much dominant over the Coulomb repulsion and hence the neutron number $N$ and the proton number $Z$ tend to be equal in light nuclei. In a heavier nucleus, the radius is large and for many nucleon-pairs, the interaction through nuclear force is not effective. On the other hand, Coulomb force is a long-range force and even the diametrically opposite protons repel each other. Thus, Coulomb repulsion becomes more effective for nuclei of larger mass number $A$. Stability is achieved by having more neutrons than protons because neutrons do not take part in Coulomb interaction. That is why $N/Z$ increases with $A$ for stable nuclides. However, one should not expect greater stability with too many neutrons because then many of these neutrons will not have pairing with protons. This will increase the energy and hence, decrease the stability. A very large nucleus cannot be stable for any value of $N/Z$. The heaviest stable nuclide is $^{209}_{83}$Bi.

### 46.3 BINDING ENERGY

We already know about the concept of binding energy of a hydrogen atom. If the constituents of a hydrogen atom (a proton and an electron) are brought from infinity to form the atom, 13·6 eV of energy is released. Thus, the binding energy of a hydrogen atom in ground state is 13·6 eV. Also, 13·6 eV energy must be supplied to the hydrogen atom in ground state to separate the constituents to large distances. Similarly, the nucleons are bound together in a nucleus and energy must be supplied to the nucleus to separate the constituent nucleons to large distances (figure 46.2a). The amount of energy needed to do this is called the *binding energy of the nucleus*. If the nucleons are initially well-separated and are brought to form the nucleus, this much energy is released (figure 46.2b).

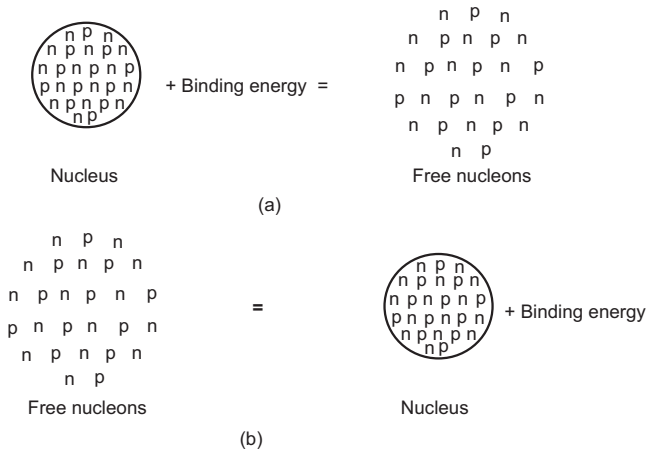



Figure 46.2

Thus, the rest mass energy of a nucleus is smaller than the rest mass energy of its constituent nucleons in free state. The difference of the two energies is the binding energy. The rest mass energy of a free proton is $m_p c^2$ and that of a free neutron is $m_n c^2$. If the nucleus has a mass $M$, its rest mass energy is $Mc^2$. If it contains $Z$ protons and $N$ neutrons, the rest mass energy of its nucleons in free state is $Zm_p c^2 + Nm_n c^2$. If the binding energy of the nucleus is $B$, we have,

$$B = (Zm_p + Nm_n - M)c^2. \quad \ldots \text{(i)}$$

We can also use the atomic masses in place of nuclear masses. The above equation then becomes

$$B = \left[ Zm\left(^1_1\mathrm{H}\right) + N\,m_n - m\left(^{Z+N}_{Z}\mathrm{X}\right)\right] c^2. \quad \ldots \text{(46.2)}$$

Here, $m\left(^1_1\mathrm{H}\right)$ is the mass of a hydrogen atom and $m\left(^{Z+N}_{Z}\mathrm{X}\right)$ is the mass of an atom with $Z$ protons and $N$ neutrons. Verify that the masses of electrons cancel out in this equation. There are $Z$ electrons in $Z$ hydrogen atoms as well as in the atom $^{Z+N}_{Z}$X. Hence, equation (46.2) is equivalent to (i) above. Such cancellations often occur and atomic masses are rather frequently used in place of nuclear masses. The small difference due to binding energy of electrons with the nucleus is neglected. As we shall see later, such cancellations do not work in the case of β-decays. Unless stated otherwise, mass of $^{A}_{Z}$X in this chapter will refer to the atomic mass which includes the mass of $Z$ electrons.

***Example 46.2***

*Calculate the binding energy of an alpha particle from the following data*:

*mass of* $^1_1$H *atom* = 1·007825 u

*mass of neutron* = 1·008665 u

*mass of* $^4_2$He *atom* = 4·00260 u.

*Take* 1 u = 931 MeV c$^{-2}$.

***Solution*** : The alpha particle contains 2 protons and 2 neutrons. The binding energy is

$$B = (2 \times 1{\cdot}007825 \text{ u} + 2 \times 1{\cdot}008665 \text{ u} - 4{\cdot}00260 \text{ u})c^2$$

$$= (0{\cdot}03038 \text{ u})c^2$$

$$= 0{\cdot}03038 \times 931 \text{ MeV} = 28{\cdot}3 \text{ MeV}.$$

If two protons and two neutrons combine to form an alpha particle, 28·3 MeV of energy will be released.

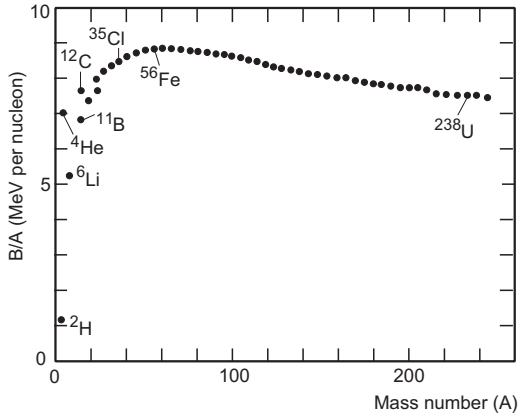



Figure 46.3

A very useful quantity in nuclear physics is binding energy per nucleon, i.e., binding energy divided by the mass number. Figure (46.3) shows a plot of binding energy per nucleon against the mass number. The binding energy of deuteron (the nucleus of heavy hydrogen containing a proton and a neutron) is 2·22 MeV so that the binding energy per nucleon is $\frac{2{\cdot}22 \text{ MeV}}{2} = 1{\cdot}11$ MeV. As we consider nuclei with increasing mass number, the binding energy per nucleon increases on an average and reaches a maximum of about 8·7 MeV for $A \approx 50$–80. For still heavier nuclei, the binding energy per nucleon slowly decreases as $A$ increases. For uranium, one of the heaviest natural elements ($A \approx 238$), the value drops to about 7·5 MeV. It follows that the nuclei in the intermediate region, $A \approx 50$–80, are most stable as maximum energy is needed to bring out any individual nucleon from such a nucleus.

The behaviour of binding energy per nucleon versus $A$ can be roughly understood in terms of the short-range nature of nuclear force. Because of the short range, a nucleon inside the nucleus can interact with a fixed number of nucleons surrounding it. If each nucleon is visualised as a sphere and these spheres are assumed to be closely packed, each nucelon has 12 neighbouring nucleons touching it. Thus, 12 pairs are formed with each nucleon for nuclear interaction. If all the $A$ nucleons could be in the interior of the nucleus, there would have been 6$A$ pairs. The binding energy resulting from these pairs will, therefore, be proportional to $A$. This energy is called *volume energy* and is written as $b_v = a_1 A$. However, all the nucleons are not in the interior. A nucleon near the surface does not interact with 12 nucleons. Thus, there is a decrease in binding energy and this decrease will be proportional to the surface area or to $R^2$. As $R = R_0 A^{1/3}$, this surface energy is $b_s = -a_2 A^{2/3}$. The negative sign is used because the surface effect decreases the binding energy whereas the volume effect increases it. Another factor contributing to the binding energy is the Coulomb repulsion between the protons. As Coulomb force is a long-range force, all proton-pairs participate in it. The Coulomb potential energy is, therefore, proportional to $Z(Z-1)/2$ and is inversely proportional to the nuclear radius $R$ as the average separation between the protons will be proportional to the nuclear radius. The Coulomb contribution to the binding energy is, therefore, written as $b_c = -a_3 Z(Z-1)/A^{1/3}$. The binding energy per nucleon is, therefore,

$$\frac{B}{A} = a_1 - \frac{a_2}{A^{1/3}} - a_3 \frac{Z(Z-1)}{A^{4/3}} \cdot \qquad \ldots \quad (46.3)$$

With suitable choices of $a_1$, $a_2$ and $a_3$, this equation agrees well with the general nature of the plot shown in figure (46.3). There are other effects such as that due to Pauli exclusion principle which should be included in (46.3) to make it more accurate.

**Mass Excess**

Consider a nucleus of mass number $A$. Let the mass of the neutral atom containing this nucleus be $m$ atomic mass units. Also, let $A'$ represent the mass $A$ atomic mass units. Thus,

mass of the atom = $m$ u

and $A' = A$ u.

We define *mass excess* as

$$(\text{mass of the atom} - A')\, c^2 \qquad \ldots \quad (46.4)$$

$$= (m \text{ u} - A \text{ u})c^2$$

$$= (m - A)\left(\frac{931 \text{ MeV}}{\text{c}^2}\right)c^2$$

$$= 931(m - A)\text{MeV}. \qquad \ldots \quad (46.5)$$

***Example 46.3***

*The atomic mass of* $^{1}_{1}$H *is* 1·00783 u. *Calculate the mass excess of hydrogen.*

***Solution*** **:** The mass excess of hydrogen is $931(m - A)$MeV
$= 931(1{\cdot}00783 - 1)\text{MeV} = 7{\cdot}29$ MeV.

## 46.4 RADIOACTIVE DECAY

Stable nuclides have definite $Z$, $N$ combinations as shown by the black circles in figure (46.1). Nuclides with other $Z$, $N$ combinations are also found in nature or may be prepared in laboratory. However, these nuclides are unstable and they decay into other nuclides by various processes. Two main processes by which an unstable nucleus decays are *alpha decay* and *beta decay*.

### Alpha Decay

In alpha decay, the unstable nucleus emits an alpha particle reducing its proton number $Z$ as well as its neutron number $N$ by 2. The alpha decay process may be represented as

$$^{A}_{Z}\text{X} \rightarrow {}^{A-4}_{Z-2}\text{Y} + {}^{4}_{2}\text{He}. \qquad \ldots \ (46.6)$$

As the proton number $Z$ is changed, the element itself is changed and hence the chemical symbol of the residual nucleus is different from that of the original nucleus. The nucleus before the decay is called the *parent nucleus* and that resulting after the decay is called the *daughter nucleus*. An example of alpha decay is

$$^{212}_{83}\text{Bi} \rightarrow {}^{208}_{81}\text{Tl} + {}^{4}_{2}\text{He}.$$

The parent nucleus bismuth had 83 protons and 129 neutrons, the daughter nucleus is thallium with 81 protons and 127 neutrons.

Alpha decay occurs in all nuclei with mass number $A > 210$. We have seen that too heavy a nucleus will be unstable because of the Coulomb repulsive force. By emitting alpha particle the nucleus decreases its mass number to move towards stability.

The rest mass energy of $^{212}_{83}$Bi is larger than the sum of the rest mass energies of the products $^{208}_{81}$Tl and $^{4}_{2}$He. The difference between the rest mass energy of the initial constituents and that of the final products is called the *Q-value* of the process. Thus, if $U_i$ is the rest mass energy of the initial constituents and $U_f$ is that of the final products,

$$Q = U_i - U_f .$$

This definition is valid not only for alpha decay but for any nuclear process. This much energy is made available as the kinetic energy of the products. In an α-decay given by equation (46.6), the $Q$-value is

$$Q = \left[ m\left({}^{A}_{Z}\text{X}\right) - m\left({}^{A-4}_{Z-2}\text{Y}\right) - m\left({}^{4}_{2}\text{He}\right)\right] c^{2}. \qquad \ldots \ (46.7)$$

A stream of alpha particles coming from a bulk material is called *alpha ray*.

### Beta Decay

Beta decay is a process in which either a neutron is converted into a proton or a proton is converted into a neutron. Thus, the ratio $N/Z$ is altered in beta decay. If a nucleus is formed with more number of neutrons than needed for stability, a neutron will convert itself into a proton to move towards stability. Similarly, if a nucleus is formed with more number of protons than needed for stability, a proton will convert itself into a neutron. Such transformations take place because of *weak forces* operating within a neutron or a proton. When a neutron is converted into a proton, an electron and a new particle named *antineutrino* are created and emitted from the nucleus,

$$n \rightarrow p + e + \bar{\nu}. \qquad \ldots \ (46.8)$$

The antineutrino is denoted by the symbol $\bar{\nu}$. It is supposed to have zero rest mass like photon, is chargeless and has spin quantum number ± 1/2. The electron emitted from the nucleus is called a *beta particle* and is denoted by the symbol $\beta^{-}$. A stream of such beta particles coming from a bulk of unstable nuclei is called *beta ray*. The beta decay process may be represented by

$$^{A}_{Z}\text{X} \rightarrow {}^{A}_{Z+1}\text{Y} + e + \bar{\nu}$$

or,
$$^{A}_{Z}\text{X} \rightarrow {}^{A}_{Z+1}\text{Y} + \beta^{-} + \bar{\nu}. \qquad \ldots \ (46.9)$$

It is also called *beta minus decay* as negatively charged beta particles are emitted. The rest mass energy of the initial constituents is

$$U_i = \left[ m\left({}^{A}_{Z}\text{X}\right) - Zm_e \right] c^{2}$$

and that of the final constituents is

$$U_f = \left[ m\left({}^{A}_{Z+1}\text{Y}\right) - (Z+1)m_e + m_e \right] c^{2}.$$

The kinetic energy available to the product particles is,

$$Q = U_i - U_f = \left[ m\left({}^{A}_{Z}\text{X}\right) - m\left({}^{A}_{Z+1}\text{Y}\right)\right] c^{2} \qquad \ldots \ (46.10)$$

where atomic masses are used. It may be noted that the rest mass energy of the electron created is not explicitly subtracted in this equation. Because of the large mass, the residual nucleus $^{A}_{Z+1}$Y does not share appreciable kinetic energy. Thus, the energy $Q$ is shared by the antineutrino and the beta particle. Depending on the fraction taken away by the antineutrino, the kinetic energy of the beta particle can be anything between zero and a maximum value $Q$.

If the unstable nucleus has excess protons than needed for stability, a proton converts itself into a neutron. In the process, a *positron* and a *neutrino* are created and emitted from the nucleus,

$$p \rightarrow n + e^{+} + \nu. \qquad \ldots (46.11)$$

The positron $e^{+}$ has a positive electric charge equal in magnitude to the charge on an electron and has a mass equal to the mass of an electron. Positron is called the *antiparticle* of electron. When an electron and a positron collide, both the particles are destroyed and energy is made available. Similarly, neutrino and antineutrino are antiparticles of each other. When a proton in a nucleus converts itself into a neutron, the decay process is represented as

$$ {}^{A}_{Z}\mathrm{X} \rightarrow {}^{A}_{Z-1}\mathrm{Y} + e^{+} + \nu$$

or,
$$ {}^{A}_{Z}\mathrm{X} \rightarrow {}^{A}_{Z-1}\mathrm{Y} + \beta^{+} + \nu. \qquad \ldots (46.12)$$

This process is called *beta plus decay*. The positron so emitted is called a *beta plus* particle.

Verify that the $Q$-value of this decay is given by

$$Q = \left[ m\left({}^{A}_{Z}\mathrm{X}\right) - m\left({}^{A}_{Z-1}\mathrm{Y}\right) - 2m_e \right] c^{2}. \qquad \ldots (46.13)$$

Can an isolated proton decay to a neutron emitting a positron and a neutrino as suggested by equation (46.11) ? The mass of a neutron is larger than the mass of a proton and hence the $Q$-value of such a process would be negative. So, an isolated proton does not beta decay to a neutron. On the other hand, an isolated neutron decays to a proton as suggested by equation (46.8).

A similar process, known as *electron capture*, takes place in certain nuclides. In this process, the nucleus captures one of the atomic electrons (most likely an electron from the K shell). A proton in the nucleus combines with this electron and converts itself into a neutron. A neutrino is created in the process and emitted from the nucleus,

$$p + e \rightarrow n + \nu.$$

The process may be represented as

$$ {}^{A}_{Z}\mathrm{X} + e \rightarrow {}^{A}_{Z-1}\mathrm{Y} + \nu. \qquad \ldots (46.14)$$

The $Q$-value of the process is

$$Q = \left[ m\left({}^{A}_{Z}\mathrm{X}\right) - m\left({}^{A}_{Z-1}\mathrm{Y}\right) \right] c^{2}. \qquad \ldots (46.15)$$

When an atomic electron is captured, a vacancy is created in the atomic shell and X-rays are emitted following the capture.

The daughter nucleus formed as a result of an alpha decay or a beta decay may not be stable and undergo another alpha or beta decay. Thus, a series of decays proceed till a stable nucleus is formed. An example of such a series of decays starting from $^{238}$U and ending at $^{206}$Pb is shown in figure (46.4).

**Gamma Decay**

The protons and neutrons inside a nucleus move in discrete quantum states with definite energies. In the ground state, the nucleons occupy such quantum states which minimise the total energy of the nucleus. However, higher energy states are also available to the nucleons and if appropriate energy is supplied, the nucleus may be excited to higher energies. The energy differences in the allowed energy levels of a nucleus are generally large, of the order of MeVs. It is, therefore, difficult to excite the nucleus to higher energy levels by usual methods of supplying energy as heating. However, when an alpha or a beta decay takes place, the daughter nucleus is generally formed in one of its excited states. Such a nucleus in an excited state eventually comes to its ground state by emitting a photon or photons of electromagnetic radiation. The process is similar to that in a hydrogen atom when an electron jumps from a higher energy orbit to a lower energy orbit emitting a photon. A typical situation is schematically shown in figure (46.5). The parent nucleus $^{57}$Co in its ground state decays to the daughter nucleus $^{57}$Fe by $\beta^{+}$-decay. The nucleus $^{57}$Fe is formed in its second excited state with energy 136 keV above the ground state. This nucleus in excited state may emit a photon of 136 keV and reach its ground state. Or, it may emit a photon of 122 keV and drop to its first excited state and then drop to the ground state by emitting another photon of energy 14 keV. If bulk $^{57}$Co is taken, many $^{57}$Fe nuclei will be formed; some will drop directly to their ground states and the rest will go via the first excited states. Thus, one will observe a stream of $\beta^{+}$-particles, 136 keV photons, 122 keV photons and 14 keV photons coming from the $^{57}$Co source.

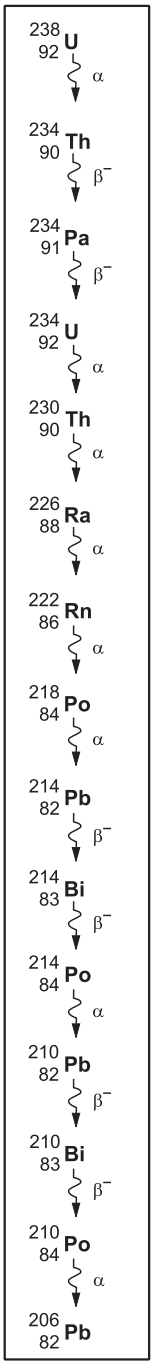


Figure 46.4

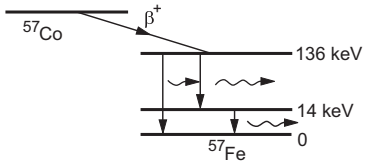


Figure 46.5

The electromagnetic radiation emitted in nuclear transitions is called *gamma ray*. The wavelength of this radiation is given by the usual relation

$$\lambda = \frac{hc}{E}$$

where $E$ is the energy of the photon. The process of a nucleus coming down to a lower energy level by emitting a photon is called *gamma decay*.

Note that in gamma decay neither the proton number nor the neutron number changes. Only the quantum states of the nucleons change.

Alpha, beta and gamma decays are collectively called *radioactive decay* and the materials capable of undergoing radioactive decay are called *radioactive materials*. The α-, β- and γ-rays are collectively called *nuclear radiation*.

## 46.5 LAW OF RADIOACTIVE DECAY

A radioactive nucleus decays by emitting some nuclear radiation (α, β or γ). Suppose an alpha-active nucleus is prepared at $t = 0$. When will this nucleus emit alpha particle ? The answer to this question is that there is no fixed time at which it must decay. No rule in physics predicts the time at which it will decay. Only when it decays we know that it has done so. Putting in another way, suppose several identical active nuclei are prepared at the same instant and are kept in identical environment. They will, in general, not decay simultaneously. Some will decay quite early and some will live longer, some might live for very long periods. In classical physics of Newton, if all the conditions at present are known, the equations of motion completely determine the future course of all particles. This is called a *deterministic world* in which everything is predetermined. Radioactive decay cannot be described by classical physics. One has to use quantum mechanics to understand it. In quantum mechanics, the equations of physics do not predict the exact future of a system in terms of the present conditions. The equations only give the probability of a particular particle behaving in a particular fashion. In fact, Einstein never wholeheartedly accepted this indeterministic nature of the world.

Suppose there are $N$ active nuclei at an instant $t$. How many of these nuclei will decay in the next small time interval $dt$ ? The number will be proportional to $N$ and to $dt$. Each active nucleus has a chance to decay in time interval $dt$. So, more the number of active nuclei at time $t$, more will decay in the next $dt$. Similarly, if you take $dt$ slightly longer, more nuclei will decay in $dt$ because each nucleus will have an increased chance of decaying. Thus,

$$dN = -\lambda N dt. \qquad \ldots (46.16)$$

The minus sign is used because the number of active nuclei is decreasing. The constant of proportionality λ is called *decay constant* and is a constant for a given decay scheme. From equation (46.16),

$$\frac{dN}{N} = -\lambda\, dt$$

or, $$\int_{N_0}^{N} \frac{dN}{N} = -\lambda \int_0^t dt$$

or, $$\ln \frac{N}{N_0} = -\lambda t$$

or, $$N = N_0 e^{-\lambda t} \qquad \ldots (46.17)$$

where $N_0$ is the number of active nuclei at $t = 0$. Also from (46.16), the rate of decay is

$$-\frac{dN}{dt} = \lambda N. \qquad \ldots (46.18)$$

The quantity $\left(-\frac{dN}{dt}\right)$ gives the number of decays per unit time and is called the *activity* of the sample. Thus, the activity of a radioactive sample is $A = \lambda N$.

From equation (46.17), we have

$$A = A_0 e^{-\lambda t}. \qquad \ldots (46.19)$$

### Unit of Activity

The activity of a radioactive material is measured in terms of the disintegrations per unit time. Its SI unit is becquerel which is the same as 1 disintegration per second. It is denoted by the symbol Bq. However, the popular unit of activity is curie defined as

$$1 \text{ curie} = 3{\cdot}7 \times 10^{10} \text{ disintegrations s}^{-1}.$$

The unit 'curie' is represented by the symbol Ci.

The activity per unit mass is called *specific activity*.

***Example 46.4***

*The decay constant for the radioactive nuclide* $^{64}$Cu *is* $1{\cdot}516 \times 10^{-5}\ s^{-1}$. *Find the activity of a sample containing* 1 μg *of* $^{64}$Cu. *Atomic weight of copper* = 63·5 g mole$^{-1}$. *Neglect the mass difference between the given radioisotope and normal copper.*

***Solution*** : 63·5 g of copper has $6 \times 10^{23}$ atoms. Thus, the number of atoms in 1 μg of Cu is

$$N = \frac{6 \times 10^{23} \times 1\ \mu g}{63{\cdot}5\ g} = 9{\cdot}45 \times 10^{15}.$$

The activity = $\lambda N$

$$= (1{\cdot}516 \times 10^{-5}\ s^{-1}) \times (9{\cdot}45 \times 10^{15})$$

$$= 1{\cdot}43 \times 10^{11} \text{ disintegrations s}^{-1}$$

$$= \frac{1{\cdot}43 \times 10^{11}}{3{\cdot}7 \times 10^{10}} \text{ Ci} = 3{\cdot}86 \text{ Ci}.$$

**Half-life**

The time elapsed before half the active nuclei decay is called *half-life* and is denoted by $t_{1/2}$. Suppose there are $N_0$ active nuclei at $t = 0$. The half-life $t_{1/2}$ is the time elapsed before $N_0/2$ nuclei have decayed and $N_0/2$ remain active. From equation (46.17),

$$\frac{N_0}{2} = N_0 e^{-\lambda t_{1/2}}$$

or, $$e^{\lambda t_{1/2}} = 2$$

or, $$\lambda t_{1/2} = \ln 2$$

or, $$t_{1/2} = \frac{\ln 2}{\lambda} = \frac{0{\cdot}693}{\lambda} . \quad \ldots (46.20)$$

Equation (46.20) relates the decay constant λ and the half-life $t_{1/2}$. As the activity $\left(-\frac{dN}{dt}\right)$ is proportional to $N$, the activity also reduces to half its value in one half-life.

Using equation (46.20), equation (46.17) may be rewritten as

$$N = N_0 e^{-(\ln 2)\frac{t}{t_{1/2}}} = \frac{N_0}{\left(e^{\ln 2}\right)^{t/t_{1/2}}}$$

$$= \frac{N_0}{2^{t/t_{1/2}}} . \quad \ldots (46.21)$$

Similarly, the activity at time $t$ is

$$A = \frac{A_0}{2^{t/t_{1/2}}} . \quad \ldots (46.22)$$

***Example 46.5***

*The half-life of a radioactive nuclide is* 20 *hours. What fraction of original activity will remain after* 40 *hours* ?

***Solution*** : We have

$$\frac{t}{t_{1/2}} = \frac{40 \text{ hours}}{20 \text{ hours}} = 2.$$

Thus,

$$A = \frac{A_0}{2^{t/t_{1/2}}} = \frac{A_0}{2^2} = \frac{A_0}{4}$$

or, $$\frac{A}{A_0} = \frac{1}{4} .$$

So one fourth of the original activity will remain after 40 hours.

**Average-life**

Consider a sample containing $N_0$ radioactive nuclei at time $t = 0$. The number of nuclei which decay between the time $t$ and $t + dt$ is $\lambda N\, dt$. The life of these nuclei is approximately $t$ each. The sum of the lives of these $dN$ nuclei is $t\,\lambda N\, dt$. The sum of all the lives of all the $N$ nuclei that were active at $t = 0$ will be

$$S = \int_0^\infty t\, \lambda N\, dt$$

$$= \lambda N_0 \int_0^\infty t\, e^{-\lambda t}\, dt$$

$$= \lambda N_0 \left[ \left( t \frac{e^{-\lambda t}}{-\lambda} \right)_0^\infty - \int_0^\infty \frac{e^{-\lambda t}}{-\lambda}\, dt \right]$$

$$= -\lambda N_0 \left( \frac{e^{-\lambda t}}{\lambda^2} \right)_0^\infty = \frac{N_0}{\lambda} .$$

Thus, the average-life of the nuclei is

$$t_{av} = \frac{S}{N_0} = \frac{1}{\lambda} . \quad \ldots (46.23)$$

Using equation (46.20),

$$t_{av} = \frac{t_{1/2}}{0{\cdot}693} . \quad \ldots (46.24)$$

All the equations derived above are statistical in nature. They do not predict the exact behaviour of each radioactive nucleus, they only predict the total numbers. In one half-life, half of the active particles will decay. But which of these particles will decay in one half-life is never predicted. Suppose a traffic controller is stationed at the junction of three roads *A*, *B* and *C*. He counts the total number of vehicles coming towards the junction from the side *A* and the number turning towards *B* and towards *C*. From his observations over a long period, he can formulate a statistical law that out of the vehicles coming from the side *A*, 60% turn towards *B* and 40% turn towards *C*. This rule is statistical and will work well on a normal day (it will not work, say, on the Independence Day when a procession of hundreds of vehicles on road *A* may turn towards *B*). But the traffic controller will not be able to predict whether a particular vehicle coming from the side *A* will turn towards *B* or towards *C*. Also, the rule works only when a large number of vehicles are considered. If he considers just 5 vehicles, 4 may turn towards *B*, and only 1 towards *C* and the rule may be a total failure. Similarly, the equations developed for radioactive decay work well only when $N$ is large.

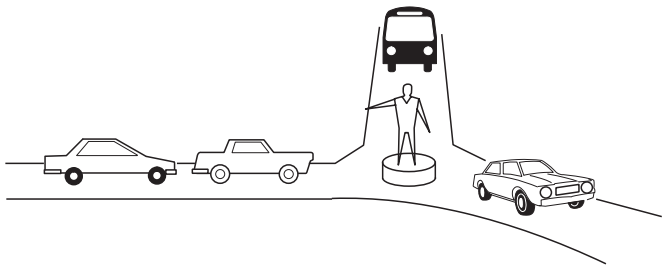


Figure 46.6

No description of radioactivity can be complete without mentioning the Curie couple. Marie Curie and

her teacher-turned-husband Pierre Curie worked hard to separate the radioactive material radium chloride ($RaCl_2$) from uranium ore. They succeeded in 1902 when about 0·19 g of $RaCl_2$ was separated and its radioactivity was studied. They shared the Nobel Prize in physics for 1903 with Henri Becquerel for this work. The unit Ci is in their honour.

## 46.6 PROPERTIES AND USES OF NUCLEAR RADIATION

### Alpha ray

(a) It is a stream of alpha particles, each particle containing two protons and two neutrons. An alpha particle is nothing but a helium nucleus.

(b) Being made of positively charged particles, alpha ray can be deflected by an electric field as well as by a magnetic field.

(c) Its penetrating power is low. Even in air, its intensity falls down to very small values within a few centimetres.

(d) Alpha rays coming from radioactive materials travel at large speeds of the order of $10^6$ m $s^{-1}$.

(e) All the alpha particles coming from a particular decay scheme have the same energy.

(f) Alpha ray produces scintillation (flashes of light) when it strikes certain fluorescent materials, such as barium platinocyanide.

(g) It causes ionization in gases.

### Beta ray

(a) It is a stream of electrons coming from the nuclei. Thus, the properties of beta ray, cathode ray, thermions, photoelectrons, etc., are all identical except for their origin. Beta particles are created at the time of nuclear transformation, whereas, in cathode ray, thermions, etc., the electrons are already present and get ejected.

(b) Being made of negatively charged particles, beta ray can be deflected by an electric field as well as by a magnetic field.

(c) Its penetrating power is greater than that of alpha ray. Typically, it can travel several metres in air before its intensity drops to small values.

(d) The ionizing power is less than that of alpha rays.

(e) Beta ray also produces scintillation in fluorescent materials, but the scintillation is weak.

(f) The energy of the beta particles coming from the same decay scheme are not equal. This is because the available energy is shared by antineutrinos. The energy of beta particles thus varies between zero and a maximum.

### Beta-plus ray

Beta-plus ray has all the properties of beta ray, except that it is made of positively charged particles.

### Gamma ray

(a) Gamma ray is an electromagnetic radiation of short wavelength. Its wavelength is, in general, smaller than X-rays. Many of its properties are the same as those of X-rays.

(b) Being chargeless, it is not deflected by electric or magnetic field.

(c) It has the least ionizing power and the largest penetrating power among different types of nuclear radiation.

(d) All the photons coming from a particular gamma decay scheme have the same energy.

(e) Being an electromagnetic wave, gamma ray travels in vacuum with the velocity $c$.

Nuclear radiation, specially gamma ray, is used in medicine for cancer therapy and other treatments. The ionizing power of nuclear radiation is used in factories to avoid accumulation of charge on moving parts due to friction. Presence of radioactive material ionizes the air and any charge accumulated leaks away. Carbon dating, which is based on radioactive decay of $^{14}C$, is a reliable technique to estimate the 'age' of archeological samples which have carbon contents. Excess exposure to nuclear radiation is harmful for human body.

## 46.7 ENERGY FROM THE NUCLEUS

Energy in various forms is available around us. Matter itself is a concentrate of energy. All atoms, molecules, nuclei, etc., are in continuous motion and have large amount of kinetic energy. But the energy that we need for our daily tasks is required in specific forms. We require energy in the form of heat to cook our food. We require energy in the form of electric current for our fans and electric lamps. Cooking gas and oxygen in air contain energy but this internal energy is not in the form required to cook our food. When the two are engaged in a chemical reaction, heat is produced which is in usable form. Sources of usable energy is something not in plenty and man is now concerned about energy conservation. Traditional sources of usable energy are wood, coal, petroleum, etc., and they are only in limited amounts and might be exhausted in a few hundred years. We are getting a large amount of energy from the sun but to use an appreciable fraction of it has been a challenging task. Several centres throughout the world are working hard to develop efficient solar cells which can convert the energy from the sun to usable forms. Satellites and

spaceships receive energy from the sun through such solar cells.

A solution to the energy crisis has been presented by nuclear energy. Nuclear energy may be obtained either by breaking a heavy nucleus into two nuclei of middle weight or by combining two light nuclei to form a middle-weight nucleus. The former process is called *nuclear fission* and the latter *nuclear fusion.*

The physics of fission or fusion lies in the relation between the binding energy per nucleon versus $A$ (figure 46.3). The middle-weight nuclei are more tightly bound than heavy-weight nuclei. When the nucleons of a heavy nucleus regroup in two middle-weight nuclei, called fragments, the total binding energy increases and hence the rest mass energy decreases. The difference in energy appears as the kinetic energy of the fragments or in some other form. This is the basic principle of fission.

Similar arguments hold for fusion. Again, the light-weight nuclei are less-tightly bound than the middle-weight nuclei. Thus, if two light nuclei combine to form a middle-weight nucleus, the binding energy increases and the rest mass decreases. Energy is released in the form of kinetic energy or in some other external form.

***Example 46.6***

*The binding energy per nucleon is* 8·5 MeV *for* $A \approx 120$ *and is* 7·6 MeV *for* $A = 240$ *(see figure 46.3). Suppose a nucleus with* $A = 240$ *breaks into two nuclei of nearly equal mass numbers. Calculate the energy released in the process.*

***Solution*** :

Suppose the heavy nucleus had $Z$ protons and $N$ neutrons. The rest mass energy of this nucleus would be

$$E = Mc^2 = (Zm_p + Nm_n)c^2 - B_1$$
$$= (Zm_p + Nm_n)c^2 - 7{\cdot}6 \times 240 \text{ MeV}.$$

If there are $Z_1$ protons and $N_1$ neutrons in the first fragment, its rest mass energy will be

$$E_1 = M_1c^2 = (Z_1m_p + N_1m_n)c^2 - B_2$$
$$= (Z_1m_p + N_1m_n)c^2 - (8{\cdot}5 \text{ MeV})\,(Z_1 + N_1).$$

Similarly, if there are $Z_2$ protons and $N_2$ neutrons in the second fragment, its rest mass energy will be

$$E_2 = (Z_2m_p + N_2m_n)c^2 - (8{\cdot}5 \text{ MeV})\,(Z_2 + N_2).$$

The energy released due to the breaking is

$$E - (E_1 + E_2)$$
$$= \left[(Z - Z_1 - Z_2)m_p c^2 + (N - N_1 - N_2)m_n c^2\right]$$
$$+ [(Z_1 + Z_2 + N_1 + N_2) \times 8{\cdot}5 - 240 \times 7{\cdot}6] \text{ MeV}$$
$$= 240 \times (8{\cdot}5 - 7{\cdot}6) \text{ MeV} = 216 \text{ MeV}.$$

We have used the fact that $Z_1 + Z_2 = Z$, $N_1 + N_2 = N$ and $Z_1 + Z_2 + N_1 + N_2 = Z + N = 240$. Thus, 216 MeV of energy will be released when this nucleus breaks.

## 46.8 NUCLEAR FISSION

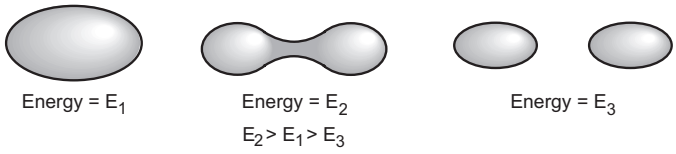



Figure 46.7

We have seen that a heavy nucleus has larger rest mass energy than that of its two middle-weight fragments. It is thus energetically favourable for the heavy nucleus to break into two middle-weight nuclei. However, before finally breaking into two parts, the heavy nucleus has to undergo a distortion which gradually increases to break the nucleus. The situation is shown in figure (46.7). The rest mass energy $E_1$ of the heavy nucleus is larger than the combined rest mass energy $E_3$ of its fragments but the energy $E_2$ in the intermediate state happens to be larger than $E_1$. Thus, it is not simple for the heavy nucleus to break spontaneously. In fact, according to classical physics, the process is impossible unless energy is supplied to the heavy nucleus to reach the intermediate state. Once it reaches the intermediate state, it can break into two parts and release energy. But the amount $E_2 - E_1$ has to be supplied to the heavy nucleus so that it may reach the intermediate state. Left to itself, the heavy nucleus will not break according to classical physics.

The world of subatomic particles is much different from that of our common day experience. According to quantum mechanics, if the final state has lesser energy than the energy in the initial state, there is a chance that the process will take place even if the intermediate state has energy greater than the initial one and no energy is supplied externally (figure 46.8).

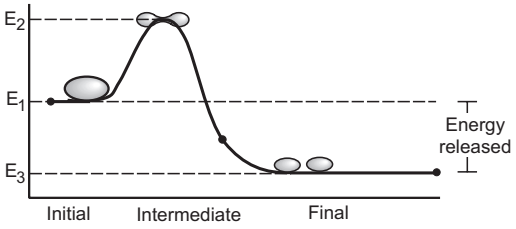



Figure 46.8

Such processes are called *barrier penetration*. The energy seems to be created out of nothing, a violation of energy conservation ! But this is a fact of the physics of small particles. The energy conservation in the usual sense may be violated for 'short times'. The amount of energy seems to be created and the time for which it is created are related through Heisenberg uncertainty

relation,

$$\Delta E.\Delta t \approx h/2\pi$$

where $h$ is the Planck constant.

Barrier penetration, though possible, is not easy. Greater the energy difference $\Delta E = E_2 - E_1$, smaller is the probability of a successful barrier penetration. This extra energy $\Delta E$ is called height of the barrier. Similarly, larger the duration of intermediate state, smaller is the probability of barrier penetration. These parameters are different for different nuclides and hence the probability that a heavy nucleus will break in a given time is different for different nuclides. Generally, this probability is extremely small except in a few nuclides. For example, the half-life of $^{238}_{92}U$ for fission reaction is about $10^{16}$ years. If you start with $N$ nuclei of $^{238}_{92}U$ today, only $N/2$ will disintegrate in the next $10^{16}$ years. Table (46.1) shows some of the better cases where the probability of fission is appreciable.

**Table 46.1 : *Fission probability***

| Nuclide | Fission probability relative to $^{236}U$ |
|---|---|
| $^{236}U$ | 1 (arbitrarily assumed) |
| $^{239}U$ | $< 1 \times 10^{-3}$ |
| $^{240}Pu$ | 1·5 |
| $^{244}Am$ | $< 2 \times 10^{-4}$ |

## 46.9 URANIUM FISSION REACTOR

The most attractive bid, from a practical point of view, to achieve energy from nuclear fission is to use $^{236}_{92}U$ as the fission material. This nuclide is highly fissionable and hence is not found in nature. Natural uranium contains about 99·3% of $^{238}_{92}U$ and 0·7% of $^{235}_{92}U$. The technique is to hit a uranium sample by slow-moving neutrons (kinetic energy ≈ 0·04 eV, also called thermal neutrons). A $^{235}_{92}U$ nucleus has large probability of absorbing a slow neutron and forming $^{236}_{92}U$ nucleus. This nucleus then fissions into two parts. A variety of combinations of the middle-weight nuclei may be formed due to the fission. For example, one may have

$$^{236}_{92}U \rightarrow {}^{137}_{53}I + {}^{97}_{39}Y + 2n,$$

$$^{236}_{92}U \rightarrow {}^{140}_{56}Ba + {}^{94}_{36}Kr + 2n$$

and a number of other combinations.

During a fission event, in general, two or three neutrons are emitted. If the total number of neutrons emitted in a large number of events is divided by the number of events, the average comes out to be around 2·47. We say that on an average 2·47 neutrons (or 2·5 as a round figure) are emitted in each fission event. The two fragments generally have unequal mass numbers as is the case in the above examples. If the relative yield of different nuclei are plotted against their mass number, a plot of the type shown in figure (46.9) results. The most probable mass numbers of the fragments are around $A = 95$ and 140. The probability of having nearly equal fragments is small.

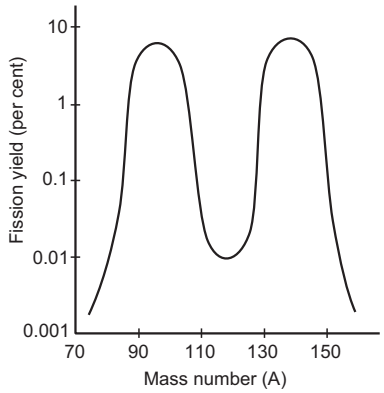



Figure 46.9

The ratio $N/Z$ is larger in heavier nuclei than in the middle-weight nuclei (figure 46.1). Thus, the fragments will have $N/Z$ ratio larger than that needed for stability. As a result, 2 or 3 neutrons are emitted together with the fission fragments. The fragments reduce their $N/Z$ ratio further via beta decay in which a neutron is converted into a proton. These daughter nuclei are generally formed in excited states and consequently emit gamma rays. At some stage, a daughter nucleus can also emit another neutron. Thus, neutrons, beta particles (electrons), antineutrinos and gamma photons accompany nuclear fission. For example, in one of the reactions, $^{236}U$ breaks into $^{97}_{39}Y$ and $^{137}_{53}I$. These nuclei undergo the following changes:

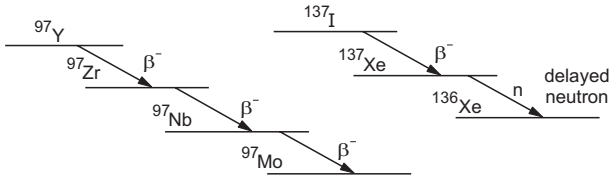



Figure 46.10

The neutron emitted by $^{137}Xe$ comes out an appreciable time interval after the fission as against the two neutrons emitted almost simultaneously with the fission. This is because $^{137}I$ decays to $^{137}Xe$ with its own half-life and only then $^{137}Xe$ emits the neutron. Such neutrons are called *delayed neutrons* and play an important role in controlling the fission rate.

In each fission event, about 200 MeV of energy is released a large part of which appears in the form of kinetic energies of the two fragments. Neutrons take away about 5 MeV. As the fragments decay, an

additional energy of about 15–20 MeV is released in the form of kinetic energy of beta particles, antineutrino and photons. The fragments, formed with so much kinetic energy, are immediately stopped in the bulk uranium solid in which they are formed. This produces large amount of heat which is taken away by passing a cold liquid in pipes through the reaction area.

**Chain Reaction**

A very important and interesting feature of neutron-induced fission is the chain reaction. Once a neutron starts the fission by being absorbed in $^{235}U$, the fission itself produces 2 or 3 new neutrons which may be absorbed by another nearby $^{235}U$ causing another fission. Such a process is known as a chain reaction. The number of neutrons may thus go on increasing in each generation of fission and the rate of fission may likewise increase in geometrical progression. If that happens, the whole of the material will fission out in a small time. The large amount of heat produced in such a short time will be uncontrollable and will only lead to disaster. There are ways to control the rate of chain reaction. If the fission event takes place near the surface of the bulk uranium material taken, there is a good chance that the neutrons produced will escape from the material without coming in contact with another $^{235}U$. The fraction of the neutrons lost in this way will be larger if smaller pieces of uranium are taken. Controlling the size will thus control the rate of fission. Another important point is that the neutrons produced in the fission have kinetic energy $\approx 2$ MeV. They are called *fast neutrons*. The $^{235}U$ nucleus has a good probability of absorbing slow neutrons ($\approx 0{\cdot}04$ eV), but has a poor chance of absorbing fast neutrons. The neutron may not get absorbed in $^{235}U$ even if the two meet and collide. If the material is large enough, the neutron will suffer a number of collisions with other nuclei. In each collision, it will lose its kinetic energy and after some time may slow down to thermal energies $\approx 0{\cdot}04$ eV. It may then be absorbed by $^{235}U$. But $^{238}U$ nuclei, which are present in large numbers (99·3%), are extremely good neutron absorbers if they get neutrons of energy 1–100 eV. These neutrons get absorbed in $^{238}U$ to form $^{239}U$ which decays generally by means other than fission. A fast neutron has to go through this 1–100 eV range before slowing down to thermal energies $\approx 0{\cdot}04$ eV and has a fair chance of being absorbed by $^{238}U$. This neutron is, therefore, lost as far as fission is concerned.

**Nuclear Reactor**

Let us now consider the design and working of a typical uranium nuclear reactor (figure 46.11). Uranium is taken in the form of cylindrical rods arranged in a regular pattern in the active reactor core. The volume in the core is filled with a low-$Z$ material such as heavy water ($D_2O$), graphite, beryllium, etc. This material is called *moderator*.

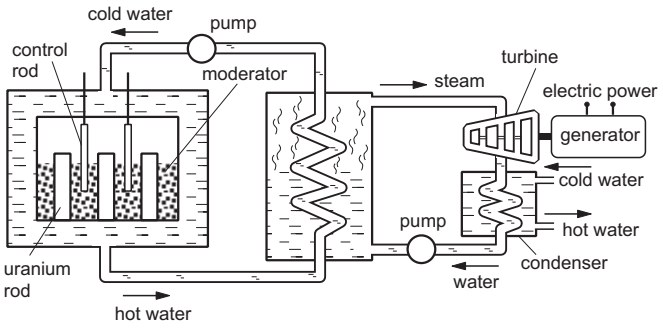



Figure 46.11

When fission takes place in a uranium rod, most of the fast neutrons produced escape from the rod and enter into the moderator. These neutrons make collisions with the particles of the moderator and thus slow down. About 25 collisions with deuteron (present in heavy water) or 100 collisions with carbon or beryllium are sufficient to slow down a neutron from 2 MeV to thermal energies. The distances between the rods are adjusted in such a way that a neutron coming from one rod is generally slowed down to thermal energies before entering the other rod. This eliminates the possibility of a neutron being absorbed by $^{238}U$ in 1–100 eV region. The geometry of the core is such that out of the average 2·5 neutrons produced per fission, 1 neutron is used to trigger the next fission and the remaining are lost without triggering any fission. The reaction is then sustained at a constant rate. If the rate of the loss of neutrons is decreased further, the fission rate will keep on increasing which may lead to explosion. If the rate of loss of neutrons is increased, the rate of fission will keep on decreasing and ultimately the chain reaction will stop. The finer control of fission rate is made by the *control rods* which are made of cadmium and are inserted up to a certain depth in the moderator. Cadmium is a very good neutron absorber. If the stage is set for stable chain reaction and the cadmium rods are pushed into the moderator, the reactor will be shut off. Pulling the cadmium rods out will start the reactor.

Some coolant liquid such as water at high pressure or molten sodium is passed through the reactor-core area which withdraws the heat produced in the core. The heat is used to prepare steam from water. The steam so prepared is used to run steam turbines and

produce electric power. The amount of $^{235}U$ goes on decreasing as the uranium rod is used for fission. When $^{235}U$ is finished and only $^{238}U$ is left, the rods have to be changed.

**Breeder Reactors**

Although fission generates large amount of energy and the world is heavily depending on fission for its energy requirement, uranium resources are also limited. Table (46.1) shows that fission can easily take place with $^{240}Pu$ besides $^{236}U$. But $^{239}Pu$ is not a naturally occurring isotope. However, $^{238}U$ can capture a neutron to produce $^{239}Pu$ which can be used as a nuclear fuel.

$$^{238}_{92}U + n \rightarrow {}^{239}_{92}U \xrightarrow{\beta^-} {}^{239}_{93}Np \xrightarrow{\beta^-} {}^{239}_{94}Pu.$$

Suppose, used uraninum rods, which contain only $^{238}U$, are kept in or around a uranium-reactor core. Also suppose, the geometry is such that out of the average 2·5 neutrons produced in a fission, one neutron is absorbed by a $^{238}U$ nucleus in these rods resulting in $^{239}Pu$. Then we produce as much nuclear fuel in the form of $^{239}Pu$ as we consume in the form of $^{235}U$. If more than one neutron can be absorbed by these $^{238}U$ rods per fission then we produce more fuel than what we consume. Thus, apart from nuclear energy, these reactors give us fresh nuclear fuel which often exceeds the nuclear fuel used. Such a reactor is called a *breeder reactor.*

## 46.10 NUCLEAR FUSION

When two light nuclei come close to one another, within the range of attractive nuclear force (≈ 1 fm), they may combine to form a bigger nucleus. The process is possible from an energy point of view because the binding energy per nucleon is small for light nuclei and increases with $A$ until $A$ is about 50. To bring the light nuclei within the separation of about a femtometre is, however, a difficult task. Any bulk material is composed of atoms and an atom typically has a radius of a few angstroms (1 angstrom $= 10^{-10}$ m $= 10^{5}$ fm). When atoms are pushed closer, their electrons cause them to repel each other. Even if all the electrons are stripped off, the nuclei themselves are positively charged and strongly repel each other. The technique thus is to heat a gas to an extremely high temperature so that the electrons are completely detached and the nuclei move within the gas with large random speeds. Two nuclei moving towards each other may come close enough to fuse into one nucleus. What must be the order of temperature which will ensure enough fusion ?

***Example 46.7***

*Consider two deuterons moving towards each other with equal speeds in a deuteron gas. What should be their kinetic energies (when they are widely separated) so that the closest separation between them becomes* 2 fm ? *Assume that the nuclear force is not effective for separations greater than* 2 fm. *At what temperature will the deuterons have this kinetic energy on an average* ?

***Solution*** :

As the deuterons move, the Coulomb repulsion will slow them down. The loss in kinetic energy will be equal to the gain in Coulomb potential energy. At the closest separation, the kinetic energy is zero and the potential energy is $\frac{e^2}{4\pi\varepsilon_0 r}$. If the initial kinetic energy of each deuteron is $K$ and the closest separation is 2 fm, we shall have

$$2K = \frac{e^2}{4\pi\varepsilon_0 (2\text{ fm})}$$

$$= \frac{(1{\cdot}6 \times 10^{-19}\text{ C})^2 \times (9 \times 10^9\text{ N m}^2\text{C}^{-2})}{2 \times 10^{-15}\text{ m}}$$

or, $$K = 5{\cdot}7 \times 10^{-14}\text{ J}.$$

If the temperature of the gas is $T$, the average kinetic energy of random motion of each nucleus will be 1·5 $kT$. The temperature needed for the deuterons to have the average kinetic energy of $5{\cdot}7 \times 10^{-14}$ J will be given by

$$1{\cdot}5\ kT = 5{\cdot}7 \times 10^{-14}\text{ J}$$

or, $$T = \frac{5{\cdot}7 \times 10^{-14}\text{ J}}{1{\cdot}5 \times 1{\cdot}38 \times 10^{-23}\text{ J K}}$$

$$= 2{\cdot}8 \times 10^9\text{ K}.$$

One needs a temperature of the order of $10^9$ K if deuterons are to be fused. The temperature inside the sun is estimated to be around $1{\cdot}5 \times 10^7$ K. Yet fusion is the main source of energy in the sun which it ultimately radiates to the universe including the earth. There are two main reasons why fusion can take place at a temperature even hundred times smaller than that calculated in example (46.7). One is that the energy of all the particles is not equal to the average energy. Although the average kinetic energy is 1·5 $kT$, there are particles which have kinetic energy much larger than 1·5 $kT$. Secondly, even if the kinetic energy of the two interacting nuclei is less than that needed to bring them within the nuclear range, there is a small chance of fusion through the process of *barrier penetration.*

As these reactions take place at high temperatures, they are also called *thermonuclear fusion* or *thermonuclear reactions.*

**Fusion in Sun**

Among the celestial bodies in which energy is produced, the sun is relatively cooler. There are stars with temperatures around $10^8$ K inside. In sun and other stars, where the temperature is less than or around $10^7$ K, fusion takes place dominantly by *proton–proton cycle* as follows:

$$^1\text{H} + {}^1\text{H} \rightarrow {}^2\text{H} + e^+ + \nu$$

$$^2\text{H} + {}^1\text{H} \rightarrow {}^3\text{He} + \gamma$$

$$^3\text{He} + {}^3\text{He} \rightarrow {}^4\text{He} + 2\,{}^1\text{H}$$

$$4\,{}^1\text{H} \rightarrow {}^4\text{He} + 2e^+ + 2\nu + 2\gamma$$

Note that the first two reactions should occur twice to produce two $^3$He nuclei and initiate the third reaction. As a result of this cycle, effectively, four hydrogen nuclei combine to form a helium nucleus. About 26·7 MeV energy is released in the cycle. Thus, hydrogen is the fuel which 'burns' into helium to release energy. The sun is estimated to have been radiating energy for the last $4{\cdot}5 \times 10^9$ years and will continue to do so till all the hydrogen in it is used up. It is estimated that the present store of hydrogen in the sun is sufficient for the next $5 \times 10^9$ years.

In hotter stars where the temperature is $\approx 10^8$ K, another cycle known as proton–carbon cycle takes place.

$$^1\text{H} + {}^{12}\text{C} \rightarrow {}^{13}\text{N} + \gamma$$

$$^{13}\text{N} \rightarrow {}^{13}\text{C} + e^+ + \nu$$

$$^1\text{H} + {}^{13}\text{C} \rightarrow {}^{14}\text{N} + \gamma$$

$$^1\text{H} + {}^{14}\text{N} \rightarrow {}^{15}\text{O} + \gamma$$

$$^{15}\text{O} \rightarrow {}^{15}\text{N} + e^+ + \nu$$

$$^1\text{H} + {}^{15}\text{N} \rightarrow {}^{12}\text{C} + {}^4\text{He}$$

The end result of this cycle is again the fusion of four hydrogen nuclei into a helium nucleus. Carbon nucleus acts only as a catalyst.

It is clear that Coulomb repulsion becomes more and more obstructive to fusion as $Z$ increases. Thus, it needs still higher temperatures for heavier elements to fuse. When the temperature inside a star rises, such fusions do take place to produce heavier nuclei such as

$$^4\text{He} + {}^4\text{He} + {}^4\text{He} \rightarrow {}^{12}\text{C} + \gamma.$$

The process can continue, finally producing elements in iron region ($A = 56$) where the binding energy per nucleon is maximum. Elements heavier than iron can be produced by neutron absorption and subsequent beta decay.

## 46.11 FUSION IN LABORATORY

In stellar objects, the material remains confined at high temperature due to gravitational pull. If we wish to make fusion as energy-producing device in laboratory, the major problem is to confine the hot plasma (when all the electrons are detached from the atoms, we get a plasma) in a small volume for extended time intervals. Producing a high temperature is obviously a major task but confinement at such high temperatures is more challenging. Solid walls can't be used as containers because no solid can sustain the high temperatures needed for fusion. The easiest thermonuclear reaction that can be handled on earth is the fusion of two deuterons (*D–D* reaction) or fusion of a deuteron with a triton (*D–T* reaction).

$$^2_1\text{H} + {}^2_1\text{H} \rightarrow {}^3_2\text{He} + n + 3{\cdot}3 \text{ MeV } (D\text{–}D)$$

$$^2_1\text{H} + {}^2_1\text{H} \rightarrow {}^3_1\text{H} + {}^1_1\text{H} + 4{\cdot}0 \text{ MeV } (D\text{–}D)$$

$$^2_1\text{H} + {}^3_1\text{H} \rightarrow {}^4_2\text{He} + n + 17{\cdot}6 \text{ MeV } (D\text{–}T)$$

One starts with deuterium gas (heavy hydrogen) or a mixture of deuterium with tritium, heat the gas to high temperatures, ensuring its confinement for reasonable period, and looks for the fusion.

**Lawson criterion**

J.D. Lawson showed that in order to get an energy output greater than the energy input, a fusion reactor should achieve

$$n\tau > 10^{14} \text{ s cm}^{-3}$$

where $n$ is the density of the interacting particles and $\tau$ is the confinement time. The quantity $n\tau$ in s cm$^{-3}$ is called *Lawson number*.

The ratio of the energy output to the energy input is known as $Q$ of the fusion machine. For a viable fusion machine, $Q$ should be greater than 1.

**Tokamak Design**

In one of the methods receiving serious attention, one uses the so-called *Tokamak* design. The deuterium plasma is contained in a toroidal region by specially designed magnetic field. The directions and magnitudes of the magnetic field are so managed in the toroidal space that whenever a charged plasma particle attempts to go out, the $q\vec{v} \times \vec{B}$ force tends to push it back into the toroidal volume. It is a difficult task to design a magnetic field which will push the particles moving in random directions with random speeds into a specified volume, but it is possible and has been done. The plasma is, therfore, confined by the magnetic field. Such confinement has been achieved for short durations (≈ few microseconds) in which some fusion occurs. Fusion thus proceeds in

bursts or pulses. The heating is accomplished by passing high frequency oscillating current through the plasma gas. A schematic design is shown in figure (46.12).

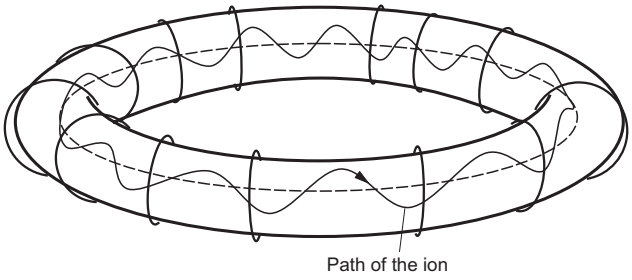



Figure 46.12

A large fusion machine known as *Joint European Torus* (JET) is designed to achieve fusion energy on this principle. A value of $Q \approx 1$ is already achieved with JET. Scientists working on this machine expect to get $Q \approx 40$ in the next 40 to 50 years.

At the Institute for Plasma Research (IPR), Ahmedabad, a small machine named *Aditya* is functioning on the Tokamak design. This machine is being used to study the properties of a plasma and a value of $n\tau \approx 10^{11}$ s cm$^{-3}$ has been achieved for Lawson number.

**Inertial Confinement**

In another method known as inertial confinement, laser beams are used to confine the plasma. A small solid pellet is made which contains deuterium and tritium. Intense laser beams are directed on the pellet from many directions distributed over all sides. The lasers first vaporize the pellet converting it into plasma and then compress it from all directions because of the large pressure exerted. The density increases by $10^3$ to $10^4$ times the initial density and the temperature rises to high values. The fusion occurs in this period. The α-particles (He nuclei) generated by the fusion are also forced to remain inside the plasma. Their kinetic energy is lost into the plasma itself contributing further rise in temperature. Again the lasers are operated in pulses of short duration.

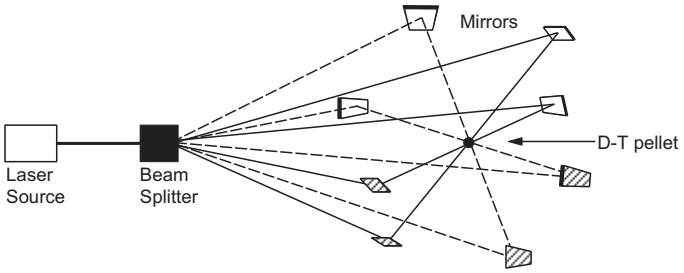



Figure 46.13

The research in fusion energy is going on. Fusion is the definite and ultimate answer to our energy problems. The 'fuel' used for fusion on earth is deuterium which is available in natural water (0·03%). And with oceans as the almost unlimited source of water, we can be sure of fuel supply for thousands of years. Secondly, fusion reactions are neat and clean. Radioactive radiation accompanying fission reactors will not be there with fusion reactors.

## *Worked Out Examples*

**1.** *Calculate the electric potential energy due to the electric repulsion between two nuclei of* $^{12}$C *when they 'touch' each other at the surface.*

***Solution*** : The radius of a $^{12}$C nucleus is

$$R = R_0 A^{1/3}$$

$$= (1{\cdot}1 \text{ fm}) (12)^{1/3} = 2{\cdot}52 \text{ fm}.$$

The separation between the centres of the nuclei is $2R = 5{\cdot}04$ fm. The potential energy of the pair is

$$U = \frac{q_1 q_2}{4\pi\varepsilon_0 r}$$

$$= (9 \times 10^9 \text{ N m}^2\text{C}^{-2}) \frac{(6 \times 1{\cdot}6 \times 10^{-19} \text{ C})^2}{5{\cdot}04 \times 10^{-15} \text{ m}}$$

$$= 1{\cdot}64 \times 10^{-12} \text{ J} = 10{\cdot}2 \text{ MeV}.$$

**2.** *Find the binding energy of* $^{56}_{26}$Fe. *Atomic mass of* $^{56}$Fe *is* 55·9349 u *and that of* $^1$H *is* 1·00783 u. *Mass of neutron* = 1·00867 u.

***Solution*** : The number of protons in $^{56}_{26}$Fe = 26 and the number of neutrons = 56 – 26 = 30. The binding energy of $^{56}_{26}$Fe is

$$= [26 \times 1{\cdot}00783 \text{ u} + 30 \times 1{\cdot}00867 \text{ u} - 55{\cdot}9349 \text{ u}]c^2$$

$$= (0{\cdot}52878 \text{ u})c^2$$

$$= (0{\cdot}52878 \text{ u}) (931 \text{ MeV u}^{-1}) = 492 \text{ MeV}.$$

**3.** *Find the kinetic energy of the* α*-particle emitted in the decay* $^{238}$Pu $\rightarrow$ $^{234}$U + α. *The atomic masses needed are as follows:*

| $^{238}$Pu | $^{234}$U | $^4$He |
|---|---|---|
| 238·04955 u | 234·04095 u | 4·002603 u |

*Neglect any recoil of the residual nucleus.*

***Solution*** : Using energy conservation,

$$m(^{238}\text{Pu})\, c^2 = m(^{234}\text{U})c^2 + m(^4\text{He})c^2 + K$$

or, $$K = [m(^{238}\text{Pu}) - m(^{234}\text{U}) - m(^4\text{He})]c^2$$

$$= [238{\cdot}04955 \text{ u} - 234{\cdot}04095 \text{ u} - 4{\cdot}002603 \text{ u}] (931 \text{ MeV u}^{-1})$$

$$= 5{\cdot}58 \text{ MeV}.$$

**4.** *Calculate the Q-value in the following decays:*

*(a)* $^{19}O \rightarrow {}^{19}F + e + \bar{\nu}$

*(b)* $^{25}Al \rightarrow {}^{25}Mg + e^{+} + \nu$.

*The atomic masses needed are as follows*:

| $^{19}O$ | $^{19}F$ | $^{25}Al$ | $^{25}Mg$ |
|---|---|---|---|
| 19·003576 u | 18·998403 u | 24·990432 u | 24·985839 u |

***Solution*** :

(a) The $Q$-value of $\beta^{-}$-decay is

$$Q = [m(^{19}O) - m(^{19}F)]c^{2}$$

$$= [19{\cdot}003576 \text{ u} - 18{\cdot}998403 \text{ u}] (931 \text{ MeV u}^{-1})$$

$$= 4{\cdot}816 \text{ MeV}.$$

(b) The $Q$-value of $\beta^{+}$-decay is

$$Q = [m(^{25}Al) - m(^{25}Mg) - 2m_e]c^{2}$$

$$= [24{\cdot}990432 \text{ u} - 24{\cdot}985839 \text{ u} - 2 \times 0{\cdot}511 \text{ MeV c}^{-2}]c^{2}$$

$$= (0{\cdot}004593 \text{ u}) (931 \text{ MeV u}^{-1}) - 1{\cdot}022 \text{ MeV}$$

$$= 4{\cdot}276 \text{ MeV} - 1{\cdot}022 \text{ MeV} = 3{\cdot}254 \text{ MeV}.$$

**5.** *Find the maximum energy that a beta particle can have in the following decay*

$$^{176}Lu \rightarrow {}^{176}Hf + e + \bar{\nu}.$$

*Atomic mass of* $^{176}Lu$ *is* 175·942694 u *and that of* $^{176}Hf$ is 175·941420 u.

***Solution*** : The kinetic energy available for the beta particle and the antineutrino is

$$Q = [m(^{176}Lu) - m(^{176}Hf)]c^{2}$$

$$= (175{\cdot}942694 \text{ u} - 175{\cdot}941420 \text{ u}) (931 \text{ MeVu}^{-1})$$

$$= 1{\cdot}182 \text{ MeV}.$$

This energy is shared by the beta particle and the antineutrino. The maximum kinetic energy of a beta particle in this decay is, therefore, 1·182 MeV when the antineutrino practically does not get any share.

**6.** *Consider the beta decay*

$$^{198}Au \rightarrow {}^{198}Hg^{*} + \beta^{-} + \bar{\nu}$$

*where* $^{198}Hg^{*}$ *represents a mercury nucleus in an excited state at energy* 1·088 MeV *above the ground state. What can be the maximum kinetic energy of the electron emitted* ? *The atomic mass of* $^{198}Au$ *is* 197·968233 u *and that of* $^{198}Hg$ *is* 197·966760 u.

***Solution*** : If the product nucleus $^{198}Hg$ is formed in its ground state, the kinetic energy available to the electron and the antineutrino is

$$Q = [m(^{198}Au) - m(^{198}Hg)]c^{2}.$$

As $^{198}Hg^{*}$ has energy 1·088 MeV more than $^{198}Hg$ in ground state, the kinetic energy actually available is

$$Q = [m(^{198}Au) - m(^{198}Hg)]c^{2} - 1{\cdot}088 \text{ MeV}$$

$$= (197{\cdot}968233 \text{ u} - 197{\cdot}966760 \text{ u}) (931 \text{ MeV u}^{-1}) - 1{\cdot}088 \text{ MeV}$$

$$= 1{\cdot}3686 \text{ MeV} - 1{\cdot}088 \text{ MeV} = 0{\cdot}2806 \text{ MeV}$$

This is also the maximum possible kinetic energy of the electron emitted.

**7.** *The half-life of* $^{198}Au$ *is* 2·7 days. *Calculate (a) the decay constant, (b) the average-life and (c) the activity of* 1·00 mg *of* $^{198}Au$. *Take atomic weight of* $^{198}Au$ *to be* 198 g mol$^{-1}$.

***Solution*** : (a) The half-life and the decay constant are related as

$$t_{1/2} = \frac{\ln 2}{\lambda} = \frac{0{\cdot}693}{\lambda}$$

or,
$$\lambda = \frac{0{\cdot}693}{t_{1/2}} = \frac{0{\cdot}693}{2{\cdot}7 \text{ days}}$$

$$= \frac{0{\cdot}693}{2{\cdot}7 \times 24 \times 3600 \text{ s}} = 2{\cdot}9 \times 10^{-6} \text{ s}^{-1}.$$

(b) The average-life is $t_{av} = \frac{1}{\lambda} = 3{\cdot}9$ days.

(c) The activity is $A = \lambda N$. Now, 198 g of $^{198}Au$ has $6 \times 10^{23}$ atoms. The number of atoms in 1·00 mg of $^{198}Au$ is

$$N = 6 \times 10^{23} \times \frac{1{\cdot}0 \text{ mg}}{198 \text{ g}} = 3{\cdot}03 \times 10^{18}.$$

Thus,
$$A = \lambda N$$

$$= (2{\cdot}9 \times 10^{-6} \text{ s}^{-1}) (3{\cdot}03 \times 10^{18})$$

$$= 8{\cdot}8 \times 10^{12} \text{ disintegrations s}^{-1}$$

$$= \frac{8{\cdot}8 \times 10^{12}}{3{\cdot}7 \times 10^{10}} \text{ Ci} = 240 \text{ Ci}.$$

**8.** *A radioactive sample has* $6{\cdot}0 \times 10^{18}$ *active nuclei at a certain instant. How many of these nuclei will still be in the same active state after two half-lives* ?

***Solution*** : In one half-life the number of active nuclei reduces to half the original number. Thus, in two half-lives the number is reduced to $\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$ of the original number. The number of remaining active nuclei is, therefore,

$$6{\cdot}0 \times 10^{18} \times \left(\frac{1}{2}\right) \times \left(\frac{1}{2}\right)$$

$$= 1{\cdot}5 \times 10^{18}.$$

**9.** *The activity of a radioactive sample falls from* 600 s$^{-1}$ *to* 500 s$^{-1}$ *in* 40 minutes. *Calculate its half-life.*

***Solution*** : We have,

$$A = A_0 e^{-\lambda t}$$

or, $500 \text{ s}^{-1} = (600 \text{ s}^{-1}) e^{-\lambda t}$

or, $e^{-\lambda t} = \frac{5}{6}$

or, $\lambda t = \ln(6/5)$

or, $\lambda = \frac{\ln(6/5)}{t} = \frac{\ln(6/5)}{40 \text{ min}}$.

The half-life is $t_{1/2} = \frac{\ln 2}{\lambda}$

$= \frac{\ln 2}{\ln(6/5)} \times 40 \text{ min}$

$= 152 \text{ min}.$

**10.** *The number of* $^{238}$U *atoms in an ancient rock equals the number of* $^{206}$Pb *atoms. The half-life of decay of* $^{238}$U *is* $4{\cdot}5 \times 10^9$ y. *Estimate the age of the rock assuming that all the* $^{206}$Pb *atoms are formed from the decay of* $^{238}$U.

***Solution*** : Since the number of $^{206}$Pb atoms equals the number of $^{238}$U atoms, half of the original $^{238}$U atoms have decayed. It takes one half-life to decay half of the active nuclei. Thus, the sample is $4{\cdot}5 \times 10^9$ y old.

**11.** *Equal masses of two samples of charcoal A and B are burnt separately and the resulting carbon dioxide are collected in two vessels. The radioactivity of* $^{14}$C *is measured for both the gas samples. The gas from the charcoal A gives* 2100 *counts per week and the gas from the charcoal B gives* 1400 *counts per week. Find the age difference between the two samples. Half-life of* $^{14}C$ = 5730 y.

***Solution*** : The activity of sample $A$ is 2100 counts per week. After a certain time $t$, its activity will be reduced to 1400 counts per week. This is because a fraction of the active $^{14}$C nuclei will decay in time $t$. The sample $B$ must be a time $t$ older than the sample $A$.

We have,

$A = A_0 e^{-\lambda t}$

or, $1400 \text{ s}^{-1} = 2100 \text{ s}^{-1} e^{-\lambda t}$

or, $e^{-\lambda t} = \frac{2}{3}$

$t = \frac{\ln(3/2)}{\lambda}$

$= \frac{\ln(3/2)}{0{\cdot}693} t_{1/2}$

$= \frac{0{\cdot}4055}{0{\cdot}693} \times 5730 \text{ y} = 3352 \text{ y}.$

**12.** *Suppose, the daughter nucleus in a nuclear decay is itself radioactive. Let* $\lambda_p$ *and* $\lambda_d$ *be the decay constants of the parent and the daughter nuclei. Also, let* $N_p$ *and* $N_d$ *be the number of parent and daughter nuclei at time t. Find the condition for which the number of daughter nuclei becomes constant.*

***Solution*** : The number of parent nuclei decaying in a short time interval $t$ to $t + dt$ is $\lambda_p N_p dt$. This is also the number of daughter nuclei produced in this interval. The number of daughter nuclei decaying during the same time interval is $\lambda_d N_d dt$. The number of the daughter nuclei will be constant if

$\lambda_p N_p dt = \lambda_d N_d dt$

or, $\lambda_p N_p = \lambda_d N_d.$

**13.** *A radioactive sample decays with an average-life of* 20 ms. *A capacitor of capacitance* 100 μF *is charged to some potential and then the plates are connected through a resistance R. What should be the value of R so that the ratio of the charge on the capacitor to the activity of the radioactive sample remains constant in time* ?

***Solution*** : The activity of the sample at time $t$ is given by

$A = A_0 e^{-\lambda t}$

where $\lambda$ is the decay constant and $A_0$ is the activity at time $t = 0$ when the capacitor plates are connected. The charge on the capacitor at time $t$ is given by

$Q = Q_0 e^{-t/CR}$

where $Q_0$ is the charge at $t = 0$ and $C = 100$ μF is the capacitance. Thus,

$\frac{Q}{A} = \frac{Q_0}{A_0} \frac{e^{-t/CR}}{e^{-\lambda t}}$.

It is independent of $t$ if $\lambda = \frac{1}{CR}$

or, $R = \frac{1}{\lambda C} = \frac{t_{av}}{C} = \frac{20 \times 10^{-3} \text{ s}}{100 \times 10^{-6} \text{ F}} = 200 \ \Omega.$

**14.** *A radioactive nucleus can decay by two different processes. The half-life for the first process is* $t_1$ *and that for the second process is* $t_2$. *Show that the effective half-life t of the nucleus is given by*

$\frac{1}{t} = \frac{1}{t_1} + \frac{1}{t_2}$.

***Solution*** : The decay constant for the first process is $\lambda_1 = \frac{\ln 2}{t_1}$ and for the second process it is $\lambda_2 = \frac{\ln 2}{t_1}$. The probability that an active nucleus decays by the first process in a time interval $dt$ is $\lambda_1 dt$. Similarly, the probability that it decays by the second process is $\lambda_2 dt$. The probability that it either decays by the first process or by the second process is $\lambda_1 dt + \lambda_2 dt$. If the effective decay constant is $\lambda$, this probability is also equal to $\lambda dt$. Thus,

$\lambda dt = \lambda_1 dt + \lambda_2 dt$

or, $\lambda = \lambda_1 + \lambda_2$

or, $$\frac{1}{t}=\frac{1}{t_1}+\frac{1}{t_2}.$$

**15.** *Calculate the energy released when three alpha particles combine to form a* $^{12}$C *nucleus. The atomic mass of* $^4_2$He *is* 4·002603 u.

***Solution*** : The mass of a $^{12}$C atom is exactly 12 u. The energy released in the reaction $3\left(^4_2\text{He}\right) \rightarrow {}^{12}_6\text{C}$ is

$$[3\, m(^4_2\text{He}) - m(^{12}_6\text{C})]c^2$$

$$= [3 \times 4{\cdot}002603 \text{ u} - 12 \text{ u}] (931 \text{ MeV u}^{-1}) = 7{\cdot}27 \text{ MeV}.$$

□

## QUESTIONS FOR SHORT ANSWER

**1.** If neutrons exert only attractive force, why don't we have a nucleus containing neutrons alone ?

**2.** Consider two pairs of neutrons. In each pair, the separation between the neutrons is the same. Can the force between the neutrons have different magnitudes for the two pairs ?

**3.** A molecule of hydrogen contains two protons and two electrons. The nuclear force between these two protons is always neglected while discussing the behaviour of a hydrogen molecule. Why ?

**4.** Is it easier to take out a nucleon (a) from carbon or from iron (b) from iron or from lead ?

**5.** Suppose we have 12 protons and 12 neutrons. We can assemble them to form either a $^{24}$Mg nucleus or two $^{12}$C nuclei. In which of the two cases more energy will be liberated ?

**6.** What is the difference between cathode rays and beta rays ? When the two are travelling in space, can you make out which is the cathode ray and which is the beta ray ?

**7.** If the nucleons of a nucleus are separated from each other, the total mass is increased. Where does this mass come from ?

**8.** In beta decay, an electron (or a positron) is emitted by a nucleus. Does the remaining atom get oppositely charged ?

**9.** When a boron nucleus ($^{10}_5$B) is bombarded by a neutron, an $\alpha$-particle is emitted. Which nucleus will be formed as a result ?

**10.** Does a nucleus lose mass when it suffers gamma decay ?

**11.** In a typical fission reaction, the nucleus is split into two middle-weight nuclei of unequal masses. Which of the two (heavier or lighter) has greater kinetic energy ? Which one has greater linear momentum ?

**12.** If three helium nuclei combine to form a carbon nucleus, energy is liberated. Why can't helium nuclei combine on their own and minimise the energy ?

## OBJECTIVE I

**1.** The mass of a neutral carbon atom in ground state is
(a) exact 12 u (b) less than 12 u
(c) more than 12 u
(d) depends on the form of carbon such as graphite or charcoal.

**2.** The mass number of a nucleus is equal to
(a) the number of neutrons in the nucleus
(b) the number of protons in the nucleus
(c) the number of nucleons in the nucleus
(d) none of them.

**3.** As compared to $^{12}$C atom, $^{14}$C atom has
(a) two extra protons and two extra electrons
(b) two extra protons but no extra electron
(c) two extra neutrons and no extra electron
(d) two extra neutrons and two extra electrons.

**4.** The mass number of a nucleus is
(a) always less than its atomic number
(b) always more than its atomic number
(c) equal to its atomic number
(d) sometimes more than and sometimes equal to its atomic number.

**5.** The graph of $\ln(R/R_0)$ versus $\ln A$ ($R$ = radius of a nucleus and $A$ = its mass number) is
(a) a straight line (b) a parabola
(c) an ellipse (d) none of them.

**6.** Let $F_{pp}$, $F_{pn}$ and $F_{nn}$ denote the magnitudes of the nuclear force by a proton on a proton, by a proton on a neutron and by a neutron on a neutron respectively. When the separation is 1 fm,
(a) $F_{pp} > F_{pn} = F_{nn}$ (b) $F_{pp} = F_{pn} = F_{nn}$
(c) $F_{pp} > F_{pn} > F_{nn}$ (d) $F_{pp} < F_{pn} = F_{nn}$.

**7.** Let $F_{pp}$, $F_{pn}$ and $F_{nn}$ denote the magnitudes of the net force by a proton on a proton, by a proton on a neutron and by a neutron on a neutron respectively. Neglect gravitational force. When the separation is 1 fm,
(a) $F_{pp} > F_{pn} = F_{nn}$ (b) $F_{pp} = F_{pn} = F_{nn}$
(c) $F_{pp} > F_{pn} > F_{nn}$ (d) $F_{pp} < F_{pn} = F_{nn}$.

**8.** Two protons are kept at a separation of 10 nm. Let $F_n$ and $F_e$ be the nuclear force and the electromagnetic force between them.
(a) $F_e = F_n$ (b) $F_e >> F_n$ (c) $F_e << F_n$
(d) $F_e$ and $F_n$ differ only slightly.

**9.** As the mass number $A$ increases, the binding energy per nucleon in a nucleus
(a) increases (b) decreases (c) remains the same
(d) varies in a way that depends on the actual value of $A$.

**10.** Which of the following is a wrong description of binding energy of a nucleus ?
(a) It is the energy required to break a nucleus into its constituent nucleons.
(b) It is the energy made available when free nucleons combine to form a nucleus.
(c) It is the sum of the rest mass energies of its nucleons minus the rest mass energy of the nucleus.
(d) It is the sum of the kinetic energy of all the nucleons in the nucleus.

**11.** In one average-life,
(a) half the active nuclei decay
(b) less than half the active nuclei decay
(c) more than half the active nuclei decay
(d) all the nuclei decay.

**12.** In a radioactive decay, neither the atomic number nor the mass number changes. Which of the following particles is emitted in the decay ?
(a) Proton (b) Neutron (c) Electron (d) Photon

**13.** During a negative beta decay,
(a) an atomic electron is ejected
(b) an electron which is already present within the nucleus is ejected
(c) a neutron in the nucleus decays emitting an electron
(d) a proton in the nucleus decays emitting an electron.

**14.** A freshly prepared radioactive source of half-life 2 h emits radiation of intensity which is 64 times the permissible safe level. The minimum time after which it would be possible to work safely with this source is
(a) 6 h (b) 12 h (c) 24 h (d) 128 h.

**15.** The decay constant of a radioactive sample is $\lambda$. The half-life and the average-life of the sample are respectively
(a) $1/\lambda$ and $(\ln 2/\lambda)$ (b) $(\ln 2/\lambda)$ and $1/\lambda$
(c) $\lambda(\ln 2)$ and $1/\lambda$ (d) $\lambda/(\ln 2)$ and $1/\lambda$.

**16.** An $\alpha$-particle is bombarded on $^{14}N$. As a result, a $^{17}O$ nucleus is formed and a particle is emitted. This particle is a
(a) neutron (b) proton (c) electron (d) positron.

**17.** Ten grams of $^{57}Co$ kept in an open container beta-decays with a half-life of 270 days. The weight of the material inside the container after 540 days will be very nearly
(a) 10 g (b) 5 g (c) 2·5 g (d) 1·25 g.

**18.** Free $^{238}U$ nuclei kept in a train emit alpha particles. When the train is stationary and a uranium nucleus decays, a passenger measures that the separation between the alpha particle and the recoiling nucleus becomes $x$ in time $t$ after the decay. If a decay takes place when the train is moving at a uniform speed $v$, the distance between the alpha particle and the recoiling nucleus at a time $t$ after the decay, as measured by the passenger will be
(a) $x + vt$ (b) $x - vt$ (c) $x$
(d) depends on the direction of the train.

**19.** During a nuclear fission reaction,
(a) a heavy nucleus breaks into two fragments by itself
(b) a light nucleus bombarded by thermal neutrons breaks up
(c) a heavy nucleus bombarded by thermal neutrons breaks up
(d) two light nuclei combine to give a heavier nucleus and possibly other products.

## OBJECTIVE II

**1.** As the mass number $A$ increases, which of the following quantities related to a nucleus do not change ?
(a) Mass (b) Volume (c) Density (d) Binding energy

**2.** The heavier nuclei tend to have larger $N/Z$ ratio because
(a) a neutron is heavier than a proton
(b) a neutron is an unstable particle
(c) a neutron does not exert electric repulsion
(d) Coulomb forces have longer range compared to the nuclear forces.

**3.** A free neutron decays to a proton but a free proton does not decay to a neutron. This is because
(a) neutron is a composite particle made of a proton and an electron whereas proton is a fundamental particle
(b) neutron is an uncharged particle whereas proton is a charged particle
(c) neutron has larger rest mass than the proton
(d) weak forces can operate in a neutron but not in a proton.

**4.** Consider a sample of a pure beta-active material.
(a) All the beta particles emitted have the same energy.
(b) The beta particles originally exist inside the nucleus and are ejected at the time of beta decay.
(c) The antineutrino emitted in a beta decay has zero mass and hence zero momentum.
(d) The active nucleus changes to one of its isobars after the beta decay.

**5.** In which of the following decays the element does not change ?
(a) $\alpha$-decay (b) $\beta^+$-decay (c) $\beta^-$-decay (d) $\gamma$-decay

**6.** In which of the following decays the atomic number decreases ?
(a) $\alpha$-decay (b) $\beta^+$-decay (c) $\beta^-$-decay (d) $\gamma$-decay

**7.** Magnetic field does not cause deflection in
(a) $\alpha$-rays (b) beta-plus rays
(c) beta-minus rays (d) gamma rays.

**8.** Which of the following are electromagnetic waves ?
(a) $\alpha$-rays (b) Beta-plus rays
(c) Beta-minus rays (d) Gamma rays

**9.** Two lithium nuclei in a lithium vapour at room temperature do not combine to form a carbon nucleus because
(a) a lithium nucleus is more tightly bound than a

carbon nucleus
(b) carbon nucleus is an unstable particle
(c) it is not energetically favourable
(d) Coulomb repulsion does not allow the nuclei to come very close.

**10.** For nuclei with $A > 100$,
(a) the binding energy of the nucleus decreases on an average as $A$ increases
(b) the binding energy per nucleon decreases on an average as $A$ increases
(c) if the nucleus breaks into two roughly equal parts, energy is released
(d) if two nuclei fuse to form a bigger nucleus, energy is released.

## EXERCISES

Mass of proton $m_p = 1{\cdot}007276$ u, Mass of ${}^{1}_{1}\mathrm{H}$ atom = 1·007825 u, Mass of neutron $m_n = 1{\cdot}008665$ u, Mass of electron = 0·0005486 u ≈ 511 keV c$^{-2}$, 1 u = 931 MeV c$^{-2}$.

**1.** Assume that the mass of a nucleus is approximately given by $M = Am_p$ where $A$ is the mass number. Estimate the density of matter in kgm$^{-3}$ inside a nucleus. What is the specific gravity of nuclear matter ?

**2.** A neutron star has a density equal to that of the nuclear matter. Assuming the star to be spherical, find the radius of a neutron star whose mass is $4{\cdot}0 \times 10^{30}$ kg (twice the mass of the sun).

**3.** Calculate the mass of an α-particle. Its binding energy is 28·2 MeV.

**4.** How much energy is released in the following reaction:

$$^{7}\mathrm{Li} + p \rightarrow \alpha + \alpha.$$

Atomic mass of ${}^{7}\mathrm{Li} = 7{\cdot}0160$ u and that of ${}^{4}\mathrm{He} = 4{\cdot}0026$ u.

**5.** Find the binding energy per nucleon of ${}^{197}_{79}\mathrm{Au}$ if its atomic mass is 196·96 u.

**6.** (a) Calculate the energy released if ${}^{238}\mathrm{U}$ emits an α-particle. (b) Calculate the energy to be supplied to ${}^{238}\mathrm{U}$ if two protons and two neutrons are to be emitted one by one. The atomic masses of ${}^{238}\mathrm{U}$, ${}^{234}\mathrm{Th}$ and ${}^{4}\mathrm{He}$ are 238·0508 u, 234·04363 u and 4·00260 u respectively.

**7.** Find the energy liberated in the reaction

$$^{223}\mathrm{Ra} \rightarrow {}^{209}\mathrm{Pb} + {}^{14}\mathrm{C}.$$

The atomic masses needed are as follows.

| ${}^{223}\mathrm{Ra}$ | ${}^{209}\mathrm{Pb}$ | ${}^{14}\mathrm{C}$ |
|---|---|---|
| 223·018 u | 208·981 u | 14·003 u |

**8.** Show that the minimum energy needed to separate a proton from a nucleus with $Z$ protons and $N$ neutrons is

$$\Delta E = (M_{Z-1,\,N} + M_H - M_{Z,\,N})c^2$$

where $M_{Z,\,N}$ = mass of an atom with $Z$ protons and $N$ neutrons in the nucleus and $M_H$ = mass of a hydrogen atom. This energy is known as *proton-separation energy*.

**9.** Calculate the minimum energy needed to separate a neutron from a nucleus with $Z$ protons and $N$ neutrons in terms of the masses $M_{Z,\,N}$, $M_{Z,\,N-1}$ and the mass of the neutron.

**10.** ${}^{32}\mathrm{P}$ beta-decays to ${}^{32}\mathrm{S}$. Find the sum of the energy of the antineutrino and the kinetic energy of the β-particle. Neglect the recoil of the daughter nucleus. Atomic mass of ${}^{32}\mathrm{P} = 31{\cdot}974$ u and that of ${}^{32}\mathrm{S} = 31{\cdot}972$ u.

**11.** A free neutron beta-decays to a proton with a half-life of 14 minutes. (a) What is the decay constant ? (b) Find the energy liberated in the process.

**12.** Complete the following decay schemes.
(a) ${}^{226}_{88}\mathrm{Ra} \rightarrow \alpha +$
(b) ${}^{19}_{8}\mathrm{O} \rightarrow {}^{19}_{9}\mathrm{F} +$
(c) ${}^{25}_{13}\mathrm{Al} \rightarrow {}^{25}_{12}\mathrm{Mg} +$

**13.** In the decay ${}^{64}\mathrm{Cu} \rightarrow {}^{64}\mathrm{Ni} + e^+ + \nu$, the maximum kinetic energy carried by the positron is found to be 0·650 MeV. (a) What is the energy of the neutrino which was emitted together with a positron of kinetic energy 0·150 MeV ? (b) What is the momentum of this neutrino in kg m s$^{-1}$ ? Use the formula applicable to a photon.

**14.** Potassium-40 can decay in three modes. It can decay by β$^-$-emission, β$^+$-emission or electron capture. (a) Write the equations showing the end products. (b) Find the $Q$-values in each of the three cases. Atomic masses of ${}^{40}_{18}\mathrm{Ar}$, ${}^{40}_{19}\mathrm{K}$ and ${}^{40}_{20}\mathrm{Ca}$ are 39·9624 u, 39·9640 u and 39·9626 u respectively.

**15.** Lithium ($Z = 3$) has two stable isotopes ${}^{6}\mathrm{Li}$ and ${}^{7}\mathrm{Li}$. When neutrons are bombarded on lithium sample, electrons and α-particles are ejected. Write down the nuclear processes taking place.

**16.** The masses of ${}^{11}\mathrm{C}$ and ${}^{11}\mathrm{B}$ are respectively 11·0114 u and 11·0093 u. Find the maximum energy a positron can have in the β$^+$-decay of ${}^{11}\mathrm{C}$ to ${}^{11}\mathrm{B}$.

**17.** ${}^{228}\mathrm{Th}$ emits an alpha particle to reduce to ${}^{224}\mathrm{Ra}$. Calculate the kinetic energy of the alpha particle emitted in the following decay:

$$^{228}\mathrm{Th} \rightarrow {}^{224}\mathrm{Ra}^{*} + \alpha$$

$$^{224}\mathrm{Ra}^{*} \rightarrow {}^{224}\mathrm{Ra} + \gamma\,(217\ \mathrm{keV}).$$

Atomic mass of ${}^{228}\mathrm{Th}$ is 228·028726 u, that of ${}^{224}\mathrm{Ra}$ is 224·020196 u and that of ${}^{4}_{2}\mathrm{He}$ is 4·00260 u.

**18.** Calculate the maximum kinetic energy of the beta particle emitted in the following decay scheme:

$$^{12}\mathrm{N} \rightarrow {}^{12}\mathrm{C}^{*} + e^+ + \nu$$

$$^{12}\mathrm{C}^{*} \rightarrow {}^{12}\mathrm{C} + \gamma\,(4{\cdot}43\ \mathrm{MeV}).$$

The atomic mass of ${}^{12}\mathrm{N}$ is 12·018613 u.

**19.** The decay constant of ${}^{197}_{80}\mathrm{Hg}$ (electron capture to ${}^{197}_{79}\mathrm{Au}$) is $1{\cdot}8 \times 10^{-4}$ s$^{-1}$. (a) What is the half-life ? (b) What is the average-life ? (c) How much time will it take to convert 25% of this isotope of mercury into gold ?

**20.** The half-life of $^{198}$Au is 2·7 days. (a) Find the activity of a sample containing 1·00 μg of $^{198}$Au. (b) What will be the activity after 7 days ? Take the atomic weight of $^{198}$Au to be 198 g mol$^{-1}$.

**21.** Radioactive $^{131}$I has a half-life of 8·0 days. A sample containing $^{131}$I has activity 20 μCi at $t = 0$. (a) What is its activity at $t = 4·0$ days ? (b) What is its decay constant at $t = 4·0$ days ?

**22.** The decay constant of $^{238}$U is $4·9 \times 10^{-18}$ s$^{-1}$. (a) What is the average-life of $^{238}$U ? (b) What is the half-life of $^{238}$U ? (c) By what factor does the activity of a $^{238}$U sample decrease in $9 \times 10^{9}$ years ?

**23.** A certain sample of a radioactive material decays at the rate of 500 per second at a certain time. The count rate falls to 200 per second after 50 minutes. (a) What is the decay constant of the sample ? (b) What is its half-life ?

**24.** The count rate from a radioactive sample falls from $4·0 \times 10^{6}$ per second to $1·0 \times 10^{6}$ per second in 20 hours. What will be the count rate 100 hours after the beginning ?

**25.** The half-life of $^{226}$Ra is 1602 y. Calculate the activity of 0·1 g of $RaCl_2$ in which all the radium is in the form of $^{226}$Ra. Taken atomic weight of Ra to be 226 g mol$^{-1}$ and that of Cl to be 35·5 g mol$^{-1}$.

**26.** The half-life of a radioisotope is 10 h. Find the total number of disintegrations in the tenth hour measured from a time when the activity was 1 Ci.

**27.** The selling rate of a radioactive isotope is decided by its activity. What will be the second-hand rate of a one month old $^{32}$P($t_{1/2} = 14·3$ days) source if it was originally purchased for 800 rupees ?

**28.** $^{57}$Co decays to $^{57}$Fe by $\beta^+$-emission. The resulting $^{57}$Fe is in its excited state and comes to the ground state by emitting γ-rays. The half-life of $\beta^+$-decay is 270 days and that of the γ-emission is $10^{-8}$ s. A sample of $^{57}$Co gives $5·0 \times 10^{9}$ gamma rays per second. How much time will elapse before the emission rate of gamma rays drops to $2·5 \times 10^{9}$ per second ?

**29.** Carbon ($Z = 6$) with mass number 11 decays to boron ($Z = 5$). (a) Is it a $\beta^+$-decay or a $\beta^-$-decay ? (b) The half-life of the decay scheme is 20·3 minutes. How much time will elapse before a mixture of 90% carbon-11 and 10% boron-11 (by the number of atoms) converts itself into a mixture of 10% carbon-11 and 90% boron-11 ?

**30.** $4 \times 10^{23}$ tritium atoms are contained in a vessel. The half-life of decay of tritium nuclei is 12·3 y. Find (a) the activity of the sample, (b) the number of decays in the next 10 hours (c) the number of decays in the next 6·15 y.

**31.** A point source emitting alpha particles is placed at a distance of 1 m from a counter which records any alpha particle falling on its 1 cm$^2$ window. If the source contains $6·0 \times 10^{16}$ active nuclei and the counter records a rate of 50000 counts/second, find the decay constant. Assume that the source emits alpha particles uniformly in all directions and the alpha particles fall nearly normally on the window.

**32.** $^{238}$U decays to $^{206}$Pb with a half-life of $4·47 \times 10^{9}$ y. This happens in a number of steps. Can you justify a single half-life for this chain of processes ? A sample of rock is found to contain 2·00 mg of $^{238}$U and 0·600 mg of $^{206}$Pb. Assuming that all the lead has come from uranium, find the life of the rock.

**33.** When charcoal is prepared from a living tree, it shows a disintegration rate of 15·3 disintegrations of $^{14}$C per gram per minute. A sample from an ancient piece of charcoal shows $^{14}$C activity to be 12·3 disintegrations per gram per minute. How old is this sample ? Half-life of $^{14}$C is 5730 y.

**34.** Natural water contains a small amount of tritium ($^3_1$H). This isotope beta-decays with a half-life of 12·5 years. A mountaineer while climbing towards a difficult peak finds debris of some earlier unsuccessful attempt. Among other things he finds a sealed bottle of whisky. On return he analyses the whisky and finds that it contains only 1·5 per cent of the $^3_1$H radioactivity as compared to a recently purchased bottle marked '8 years old'. Estimate the time of that unsuccessful attempt.

**35.** The count rate of nuclear radiation coming from a radioactive sample containing $^{128}$I varies with time as follows.

| Time $t$ (minute): | 0 | 25 | 50 | 75 | 100 |
|---|---|---|---|---|---|
| Count rate $R$($10^9$ s$^{-1}$): | 30 | 16 | 8·0 | 3·8 | 2·0 |

(a) Plot $\ln(R_0/R)$ against $t$. (b) From the slope of the best straight line through the points, find the decay constant λ. (c) Calculate the half-life $t_{1/2}$.

**36.** The half-life of $^{40}$K is $1·30 \times 10^{9}$ y. A sample of 1·00 g of pure KCl gives 160 counts s$^{-1}$. Calculate the relative abundance of $^{40}$K (fraction of $^{40}$K present) in natural potassium.

**37.** $^{197}_{80}$Hg decays to $^{197}_{79}$Au through electron capture with a decay constant of 0·257 per day. (a) What other particle or particles are emitted in the decay ? (b) Assume that the electron is captured from the K shell. Use Moseley's law $\sqrt{\nu} = a(Z - b)$ with $a = 4·95 \times 10^{7}$ s$^{-1/2}$ and $b = 1$ to find the wavelength of the $K_\alpha$ X-ray emitted following the electron capture.

**38.** A radioactive isotope is being produced at a constant rate $dN/dt = R$ in an experiment. The isotope has a half-life $t_{1/2}$. Show that after a time $t >> t_{1/2}$, the number of active nuclei will become constant. Find the value of this constant.

**39.** Consider the situation of the previous problem. Suppose the production of the radioactive isotope starts at $t = 0$. Find the number of active nuclei at time $t$.

**40.** In an agricultural experiment, a solution containing 1 mole of a radioactive material ($t_{1/2} = 14·3$ days) was injected into the roots of a plant. The plant was allowed 70 hours to settle down and then activity was measured in its fruit. If the activity measured was 1 μCi, what per cent of activity is transmitted from the root to the fruit in steady state ?

**41.** A vessel of volume 125 cm$^3$ contains tritium ($^3$H, $t_{1/2} = 12·3$ y) at 500 kPa and 300 K. Calculate the activity of the gas.

**42.** $^{212}_{83}$Bi can disintegrate either by emitting an α-particle or by emitting a $\beta^-$-particle. (a) Write the two equations showing the products of the decays. (b) The probabilities of disintegration by α- and β-decays are in the ratio 7/13. The overall half-life of $^{212}$Bi is one hour. If 1 g of pure $^{212}$Bi is taken at 12·00 noon, what will be the composition of this sample at 1 p.m. the same day ?

**43.** A sample contains a mixture of $^{110}$Ag and $^{108}$Ag isotopes each having an activity of $8{\cdot}0 \times 10^8$ disintegrations per second. $^{108}$Ag is known to have larger half-life than $^{110}$Ag. The activity $A$ is measured as a function of time and the following data are obtained.

| Time (s) | Activity ($A$) ($10^8$ disintegrations $s^{-1}$) | Time (s) | Activity ($A$) ($10^8$ disintegrations $s^{-1}$) |
|---|---|---|---|
| 20 | 11·799 | 200 | 3·0828 |
| 40 | 9·1680 | 300 | 1·8899 |
| 60 | 7·4492 | 400 | 1·1671 |
| 80 | 6·2684 | 500 | 0·7212 |
| 100 | 5·4115 | | |

(a) Plot $\ln(A/A_0)$ versus time. (b) See that for large values of time, the plot is nearly linear. Deduce the half-life of $^{108}$Ag from this portion of the plot. (c) Use the half-life of $^{108}$Ag to calculate the activity corresponding to $^{110}$Ag in the first 50 s. (d) Plot $\ln(A/A_0)$ versus time for $^{110}$Ag for the first 50 s. (e) Find the half-life of $^{110}$Ag.

**44.** A human body excretes (removes by waste discharge, sweating, etc.) certain materials by a law similar to radioactivity. If technitium is injected in some form in a human body, the body excretes half the amount in 24 hours. A patient is given an injection containing $^{99}$Tc. This isotope is radioactive with a half-life of 6 hours. The activity from the body just after the injection is 6 μCi. How much time will elapse before the activity falls to 3 μCi ?

**45.** A charged capacitor of capacitance $C$ is discharged through a resistance $R$. A radioactive sample decays with an average-life τ. Find the value of $R$ for which the ratio of the electrostatic field energy stored in the capacitor to the activity of the radioactive sample remains constant in time.

**46.** Radioactive isotopes are produced in a nuclear physics experiment at a constant rate $dN/dt = R$. An inductor of inductance 100 mH, a resistor of resistance 100 Ω and a battery are connected to form a series circuit. The circuit is switched on at the instant the production of radioactive isotope starts. It is found that $i/N$ remains constant in time where $i$ is the current in the circuit at time $t$ and $N$ is the number of active nuclei at time $t$. Find the half-life of the isotope.

**47.** Calculate the energy released by 1 g of natural uranium assuming 200 MeV is released in each fission event and that the fissionable isotope $^{235}$U has an abundance of 0·7% by weight in natural uranium.

**48.** A uranium reactor develops thermal energy at a rate of 300 MW. Calculate the amount of $^{235}$U being consumed every second. Average energy released per fission is 200 MeV.

**49.** A town has a population of 1 million. The average electric power needed per person is 300 W. A reactor is to be designed to supply power to this town. The efficiency with which thermal power is converted into electric power is aimed at 25%. (a) Assuming 200 MeV of thermal energy to come from each fission event on an average, find the number of events that should take place every day. (b) Assuming the fission to take place largely through $^{235}$U, at what rate will the amount of $^{235}$U decrease ? Express your answer in kg per day. (c) Assuming that uranium enriched to 3% in $^{235}$U will be used, how much uranium is needed per month (30 days) ?

**50.** Calculate the $Q$-values of the following fusion reactions:
(a) $^2_1\text{H} + {}^2_1\text{H} \rightarrow {}^3_1\text{H} + {}^1_1\text{H}$
(b) $^2_1\text{H} + {}^2_1\text{H} \rightarrow {}^3_2\text{He} + n$
(c) $^2_1\text{H} + {}^3_1\text{H} \rightarrow {}^4_2\text{He} + n$.

Atomic masses are $m(^2_1\text{H}) = 2{\cdot}014102$ u, $m(^3_1\text{H}) = 3{\cdot}016049$ u, $m(^3_2\text{He}) = 3{\cdot}016029$ u, $m(^4_2\text{He}) = 4{\cdot}002603$ u.

**51.** Consider the fusion in helium plasma. Find the temperature at which the average thermal energy $1{\cdot}5\ kT$ equals the Coulomb potential energy at 2 fm.

**52.** Calculate the $Q$-value of the fusion reaction

$$^4\text{He} + {}^4\text{He} = {}^8\text{Be}.$$

Is such a fusion energetically favourable ? Atomic mass of $^8$Be is 8·0053 u and that of $^4$He is 4·0026 u.

**53.** Calculate the energy that can be obtained from 1 kg of water through the fusion reaction

$$^2\text{H} + {}^2\text{H} \rightarrow {}^3\text{H} + p.$$

Assume that $1{\cdot}5 \times 10^{-2}$ % of natural water is heavy water $D_2O$ (by number of molecules) and all the deuterium is used for fusion.

□

## ANSWERS

### OBJECTIVE I

| | | | | | |
|---|---|---|---|---|---|
| 1. (a) | 2. (c) | 3. (c) | 4. (d) | 5. (a) | 6. (b) |
| 7. (d) | 8. (b) | 9. (d) | 10. (d) | 11. (c) | 12. (d) |
| 13. (c) | 14. (b) | 15. (b) | 16. (b) | 17. (a) | 18. (c) |
| 19. (c) | | | | | |

### OBJECTIVE II

| | | |
|---|---|---|
| 1. (c) | 2. (c), (d) | 3. (c) |
| 4. (d) | 5. (d) | 6. (a), (b) |
| 7. (d) | 8. (d) | 9. (d) |
| 10. (b), (c) | | |

## EXERCISES

1. $3 \times 10^{17}$ kg m$^{-3}$, $3 \times 10^{14}$
2. 15 km
3. 4·0016 u
4. 17·34 MeV
5. 7·94 MeV
6. (a) 4·255 MeV (b) 24·03 MeV
7. 31·65 MeV
9. $(M_{Z,N-1} + m_n - M_{Z,N})c^2$
10. 1·86 MeV
11. (a) $8{\cdot}25 \times 10^{-4}$ s$^{-1}$ (b) 782 keV
12. (a) $^{222}_{86}$Rn (b) $\bar{e} + \bar{\nu}$ (c) $e^+ + \nu$
13. (a) 500 keV (b) $2{\cdot}67 \times 10^{-22}$ kg m s$^{-1}$
14. (a) $^{40}_{19}\mathrm{K} \rightarrow {}^{40}_{20}\mathrm{Ca} + e^- + \bar{\nu}$, $^{40}_{19}\mathrm{K} \rightarrow {}^{40}_{18}\mathrm{Ar} + e^+ + \nu$, $^{40}_{19}\mathrm{K} + e^- \rightarrow {}^{40}_{18}\mathrm{Ar} + \nu$
    (b) 1·3034 MeV, 0·4676 MeV, 1·490 MeV
15. $^{6}_{3}\mathrm{Li} + n \rightarrow {}^{7}_{3}\mathrm{Li}$, $^{7}_{3}\mathrm{Li} + n \rightarrow {}^{8}_{3}\mathrm{Li} \rightarrow {}^{8}_{4}\mathrm{Be} + e^- + \bar{\nu}$, $^{8}_{4}\mathrm{Be} \rightarrow {}^{4}_{2}\mathrm{He} + {}^{4}_{2}\mathrm{He}$
16. 933·6 keV
17. 5·304 MeV
18. 11·88 MeV
19. (a) 64 min (b) 92 min (c) 1600 s
20. (a) 0·244 Ci (b) 0·040 Ci
21. (a) 14 μCi (b) $1{\cdot}4 \times 10^{-6}$ s$^{-1}$
22. $6{\cdot}49 \times 10^{9}$ y (b) $4{\cdot}5 \times 10^{9}$ y (c) 4
23. $3{\cdot}05 \times 10^{-4}$ s (b) 38 min
24. $3{\cdot}9 \times 10^{3}$ per second
25. $2{\cdot}8 \times 10^{9}$ disintegrations s$^{-1}$
26. $6{\cdot}91 \times 10^{13}$
27. 187 rupees
28. 270 days
29. (a) $\beta^+$ (b) 64 min
30. (a) $7{\cdot}146 \times 10^{14}$ disintegrations s$^{-1}$
    (b) $2{\cdot}57 \times 10^{19}$ (c) $1{\cdot}17 \times 10^{23}$
31. $1{\cdot}05 \times 10^{-7}$ s$^{-1}$
32. $1{\cdot}92 \times 10^{9}$ y
33. 1800 y
34. about 83 years ago
35. (b) 0·028 min$^{-1}$ approx. (c) 25 min approx.
36. 0·12%
37. (a) neutrino (b) 20 pm
38. $\dfrac{Rt_{1/2}}{0{\cdot}693}$
39. $\dfrac{R}{\lambda}(1 - e^{-\lambda t})$
40. $1{\cdot}26 \times 10^{-11}$ %
41. 724 Ci
42. (a) $^{212}_{83}\mathrm{Bi} \rightarrow {}^{208}_{81}\mathrm{Tl} + \alpha$, $^{212}_{83}\mathrm{Bi} \rightarrow {}^{212}_{84}\mathrm{Bi} \rightarrow {}^{212}_{84}\mathrm{Po} + e^- + \bar{\nu}$
    (b) 0·50 g Bi, 0·175 g Tl, 0·325 g Po
43. the half-life of $^{110}$Ag = 24·4 s and of $^{108}$Ag = 144 s
44. 4·8 hours
45. $2\tau/C$
46. $6{\cdot}93 \times 10^{-4}$ s
47. $5{\cdot}7 \times 10^{8}$ J
48. 3·7 mg
49. (a) $3{\cdot}24 \times 10^{24}$ (b) 1·264 kg per day (c) 1263 kg
50. (a) 4·05 MeV (b) 3·25 MeV (c) 17·57 MeV
51. $2{\cdot}23 \times 10^{10}$ K
52. –93·1 keV, no
53. 3200 MJ

□

CHAPTER 47

# THE SPECIAL THEORY OF RELATIVITY

Special theory of relativity stems from two bold postulates put forth by Albert Einstein—regarded by many as the greatest scientific mind of all times. The ideas put forward by special theory of relativity are fascinating and to the first-time learners they seem to go against their everyday experiences—what they think is 'common sense'. However, in our brief discussion here, we shall see that the whole of special theory of relativity is *logically deduced* from Einstein's postulates. In this chapter, we shall use only the inertial frames of reference and we shall frequently use the word *frame* for *inertial frame*.

## 47.1 THE PRINCIPLE OF RELATIVITY

The principle of relativity is not a new concept for us. We have seen that all frames of reference that move with uniform velocities with respect to an inertial frame are themselves inertial. Newton's laws of motion are valid in the same form in all such frames. Standing on a railway platform, you can drop a stone in such a way that it hits your left foot. If you repeat the same experiment in a train moving smoothly with a uniform velocity, the stone will again strike your left foot. One cannot distinguish between two inertial frames by repeating the same experiment in the two frames. No experiment done inside the train can tell whether the train is at rest at a platform or is moving at 120 km $h^{-1}$ with respect to the platform provided there are no jerks, the train does not speed up or speed down and it does not bend.

This is the principle of relativity. There is no preferred inertial frame. All frames are equivalent. The motion between two frames is relative—you can choose any of the frames and call it at rest and the other in motion.

We can understand this on the basis of Newton's laws of motion. These laws have the same form in all inertial frames. Whether you measure acceleration, force and mass on the platform or on the train, force always equals mass times acceleration. As the results of experiments are governed by Newton's law, identical experiments will give identical results irrespective of the frame involved. But Newton's laws govern only the experiments of mechanics ! Can we do an experiment related to electricity inside a train and tell if the train is moving or is at rest ?

## 47.2 ARE MAXWELL'S LAWS INDEPENDENT OF FRAME ?

Maxwell's laws tell us that electromagnetic waves propagate in vacuum with a speed $c = \sqrt{\frac{1}{\mu_0 \varepsilon_0}}$ $\approx 3 \times 10^8$ m $s^{-1}$. Light is an electromagnetic wave. If Maxwell's laws are valid in the same form in all inertial frames, light must travel with the same speed $c = 3 \times 10^8$ m $s^{-1}$ in all such frames. Experiments show that this is true. Figure (47.1) shows a representative situation when two observers $A$ and $B$ look at a light pulse $W$.

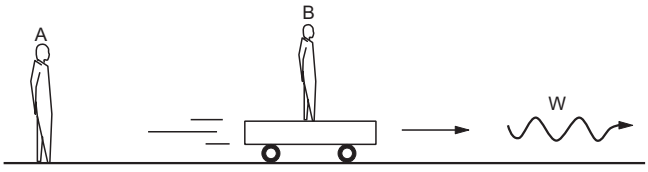



Figure 47.1

Suppose $B$ moves away from $A$ at a speed $u = c/2$ towards the right. The light pulse $W$ moves away from $A$ at a speed $c$ towards the right, and $W$ also moves away from $B$ at the same speed $c$ towards the right. Suppose at $t = 0$; $A$, $B$ and $W$ were at the same place. $A$ notes down the distances of $B$ and $W$ from himself as a function of time and $B$ notes down the distances of $A$ and $W$ from himself as a function of time. If you collect their diaries the next day, you will find something as given below.

Diary of $A$:

| $t$ | $AB$ | $AW$ |
|---|---|---|
| 0 | 0 | 0 |
| 1 s | $1{\cdot}5 \times 10^8$ m | $3 \times 10^8$ m |
| 2 s | $3 \times 10^8$ m | $6 \times 10^8$ m |
| 3 s | $4{\cdot}5 \times 10^8$ m | $9 \times 10^8$ m |

Diary of $B$:

| $t$ | $BA$ | $BW$ |
|---|---|---|
| 0 | 0 | 0 |
| 1 s | $1{\cdot}5 \times 10^8$ m | $3 \times 10^8$ m |
| 2 s | $3 \times 10^8$ m | $6 \times 10^8$ m |
| 3 s | $4{\cdot}5 \times 10^8$ m | $9 \times 10^8$ m |

It seems quite reasonable that $AB = BA$ at each $t$. But we also find that $AW = BW$ at each $t$ even though $AB \neq 0$. Do we not have $AW = AB + BW$? The two diaries suggest that something is wrong somewhere. May be, $t = 1$ s of $A$ is not the same as $t = 1$ s of $B$. Then $AW$, $AB$ and $BW$ are measured at different instants. Or may be, 1 m of $A$ is not the same as 1 m of $B$ so that $AW$ and $BW$ are measured in different units.

If Maxwell's equations are valid in the same form for both $A$ and $B$, something is fishy here. Either the clocks of $A$ and $B$ do not run at the same rate or the metre sticks used by $A$ and $B$ are not of equal length, or both. There are two possibilities.

1. Maxwell's equations have the same form in all inertial frames. Our understanding of clocks and metre sticks, i.e., of time and length has to be revised.

2. Maxwell's equations have different forms in different inertial frames.

If the second option is correct, the experiments of electricity, magnetism and optics should behave differently in different inertial frames. The experiments of mechanics could not distinguish between a train resting at a platform and a train moving uniformly with respect to the platform. That was because Newton's laws had the same form in the two frames. But now the experiments of electricity, magnetism and optics done inside a train should be able to tell whether the train is at rest or it is moving and if it is moving, with what velocity.

Experiments show that this is not true. Even experiments of electricity, magnetism and optics done inside a train cannot tell whether it is moving or not with respect to a platform. A very sophisticated experiment of this kind was designed by Michelson and Morley which is by any standard one of the greatest experiments in physics till date. The experiment attempted to measure the earth's velocity with respect to the imagined ether frame. However, it failed and created history.

We have to accept the first option as we cannot accept the second. The speed of electromagnetic wave must be the same in all inertial frames. Einstein put these ideas in the form of two postulates known as the *postulates of special relativity*.

Postulate 1: *The laws of nature have identical form in all inertial frames.*

Postulate 2: *The speed of light in vacuum has the same value c in all inertial frames.*

## 47.3 KINEMATICAL CONSEQUENCES

We have seen that the postulates of special relativity require that we must revise our concepts of length and time. If we construct two identical metre sticks, put one on a railway platform and another on a moving train, they behave differently. Similarly, if we construct two identical clocks, use one in the platform frame and the other in the train frame and measure the time interval between the occurrences of two events, the results may be different. Let us investigate these phenomena in detail.

### (A) A Rod Moving Perpendicular to its Length

Let us imagine a hypothetical experiment as follows. Construct two identical rods $L_1$ and $L_2$. Place them together and verify that the ends match against each other, i.e., they are of equal length. Put some red paint at the ends of $L_1$ and blue paint at the ends of $L_2$. Separate $L_1$ from $L_2$ by moving it perpendicular to its length. Now move $L_1$ towards $L_2$ with a uniform velocity $v$ and look from the frame of reference of $L_2$, that is, from the $L_2$-frame (figure 47.2a).

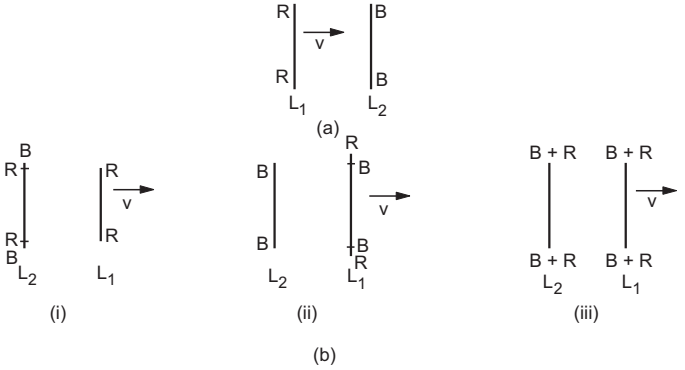



Figure 47.2

$L_1$ is moving, $L_2$ is at rest. Are the two rods still equal in length or is the moving rod shorter or is the moving rod longer? This can be checked by our experiment. As the rod $L_1$ passes over $L_2$, there are three possibilities (figure 47.2b).

(i) There are red marks on $L_2$ near the ends. This means that *the moving rod shrinks in its length.*

(ii) There are blue marks on $L_1$ near the ends. This means that *the moving rod extends in its length.*

(iii) Red and blue intermix at all the four ends. This means that *the moving rod has the same length as the stationary rod.*

Suppose option (i) is correct, i.e., $L_1$ leaves red marks on $L_2$ as it passes over $L_2$. This means that *the*

*moving rod shrinks*. Now the same experiment may be observed by a person fixed with $L_1$, that is, from the $L_1$-frame. For this observer, $L_1$ is at rest, $L_2$ comes from the right and passes over $L_1$. Since $L_2$ gets red paint on it during the crossing, the observer concludes that the *moving rod extended in its length*. But by the principle of relativity you can choose any of the two rods to be at rest and the other moving. Laws of nature should have the same form in the $L_1$-frame and the $L_2$-frame. Thus, option (i) is wrong. Similarly, option (ii) is wrong. We conclude that *a rod's length remains unchanged when it is moved perpendicular to its length*.

**(B) Moving Clocks (Time Dilation)**

Consider another hypothetical experiment. Take a rod of length $L$ as measured by an observer fixed with the rod and suppose that there are two mirrors fixed at the ends. Suppose a light pulse is reflected back and forth by the mirrors. Let us find the time interval between successive reflections from the mirror $M_1$. Let us call the first reflection 'event $E_1$' and the next reflection 'event $E_2$'. (The word 'event' also has a specialised meaning in the mathematical theory of relativity but we are using the literal meaning only.)

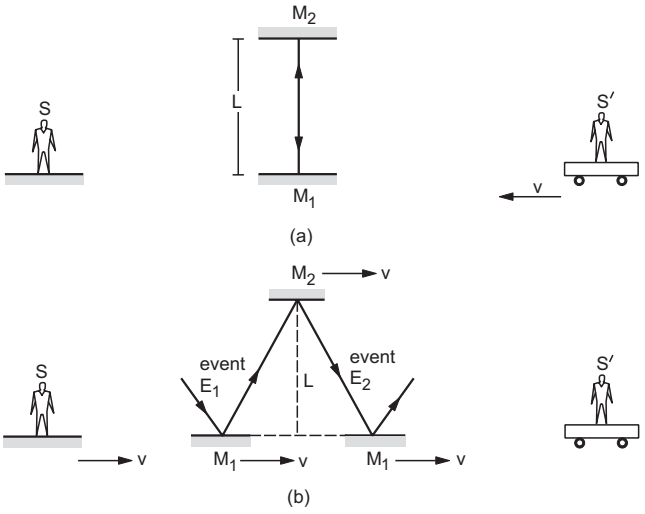



Figure 47.3

Figure (47.3a) shows the situation from a frame $S$ in which the mirrors are at rest. The rod connecting the mirrors is not shown for clarity. The light pulse travels a distance $2L$ between successive reflections from $M_1$. As the speed of light is $c$, the time elapsed between these reflections is $\Delta t = \frac{2L}{c}$.

Now consider another frame $S'$ moving with respect to the frame $S$ with a speed $v$ towards the left. From this frame, the rod and the mirrors are moving towards the right with a velocity $v$. The mirror $M_1$ at the time of the second reflection is at a place different from where it was at the time of the first reflection (figure 47.3b). If the time interval between these reflections is $\Delta t'$, the simple geometry of figure (47.3b) shows that the light pulse travels a distance

$$2\sqrt{L^2 + \left(\frac{v\Delta t'}{2}\right)^2}$$

between these reflections. Note that the length of the rod is unaltered as it moves in a direction perpendicular to its length. As the speed of light is $c$, the time interval between the successive reflections from $M_1$, is

$$\Delta t' = \frac{2}{c}\sqrt{L^2 + \left(\frac{v\Delta t'}{2}\right)^2}$$

or, $$\left(\frac{c\Delta t'}{2}\right)^2 = L^2 + \left(\frac{v\Delta t'}{2}\right)^2$$

or, $$(c^2 - v^2)\left(\frac{\Delta t'}{2}\right)^2 = L^2$$

or, $$\Delta t' = \frac{2L/c}{\sqrt{1 - v^2/c^2}}$$

or, $$\Delta t' = \frac{\Delta t}{\sqrt{1 - v^2/c^2}} = \gamma\,\Delta t \qquad \ldots (47.1)$$

where $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$.

We shall use this factor again and again and hence a symbol $\gamma$ is assigned to it. Note that $\gamma$ is greater than 1. The time interval between the occurrences of the same two events is different as measured from different frames. In frame $S$, both $E_1$ and $E_2$ occur at the same place. The time interval measured from such a frame where the two events occur at the same place is called *proper time interval*. The time interval measured by a frame where the events occur at different places is called *improper time interval*. Here $\Delta t$ is proper and $\Delta t'$ is improper time interval. According to equation (47.1),

*The proper time interval between the occurrences of two events is smaller than the improper time interval by the factor $\gamma$.*

This phenomenon is called *time dilation*.

The apparatus described above may be treated as a clock. Each reflection from the mirror $M_1$ can be thought of as a *tick* of the clock. We shall call it a *light-beam clock*. We see that when the clock is stationary with respect to the observer, it ticks at an interval $2L/c$ and when it moves with respect to the observer, it ticks at an interval $\gamma\,(2L/c)$. Thus,

*A moving clock runs slower than a stationary clock by a factor of $\gamma$.*

Remember, any of the two clocks may be taken to be stationary. Suppose $A$ and $B$ are two clocks moving with respect to each other. As seen from the frame of $A$, $B$ runs slower and as seen from the frame of $B$, $A$ runs slower.

**Proper time interval**

The concept of proper and improper time interval is valid for any two events and not only for two ticks of a clock.

Consider two events $E_1$ and $E_2$. Suppose, $E_1$ occurs at $x = 0$ at a time $t = 0$ and $E_2$ occurs at $x = L$ at a time $t$ as seen from a frame $S$ (figure 47.4).

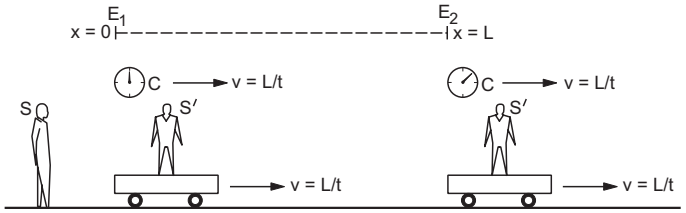



Figure 47.4

Suppose a clock $C$ is at $x = 0$ at $t = 0$ and moves along the $x$-axis with a speed $L/t$. At time $t$, this clock will be at $x = L$. So this same clock is present at both the events $E_1$ and $E_2$.

Consider a frame $S'$ moving along the $x$-axis at a speed $L/t$ with respect to $S$. In this frame, the clock is at rest and both the events are measured on the same clock. In other words, both the events take place at the *same place in S′*.

As seen from $S$, the time interval between the events is $t$. The clock $C$ is moving with respect to $S$ and hence runs slower by a factor $\gamma$. The time interval between the events as measured by this clock is $t' = t/\gamma$. This is also the time interval between $E_1$ and $E_2$ in frame $S'$ and hence is the proper time interval.

In the frame $S'$, the events can be recorded by a single clock. All other frames where two clocks are needed to record the events, give improper time intervals. The proper time interval is smaller than an improper time interval by a factor of $\gamma$.

***Example 47.1***

*A person in a train moving at a speed* $3 \times 10^7 \text{ m s}^{-1}$ *sleeps at* 10·00 p.m. *by his watch and gets up at* 4·00 a.m. *How long did he sleep according to the clocks at the stations* ?

***Solution*** : The time interval measured by the watch is the proper time interval because the events, 'sleeping' and 'getting up', are recorded by the single clock (the watch). The clocks at the stations represent the ground frame and in this frame he sleeps at one place and gets up at another place. Thus, the time interval measured by the station clocks is improper time interval and is more than the proper time interval.

The duration of his sleep in the ground frame is

$$\Delta t' = \gamma \Delta t = \frac{\Delta t}{\sqrt{1 - v^2/c^2}} = \frac{6 \text{ h}}{\sqrt{1 - \left(\dfrac{3 \times 10^7 \text{ m s}^{-1}}{3 \times 10^8 \text{ m s}^{-1}}\right)^2}}$$

$$= 6 \text{ h} \sqrt{\frac{100}{99}} = 6 \text{ hours } 1{\cdot}8 \text{ minutes.}$$

The speed of the train in this example is hypothetical. A typical fast train today runs at about $300 \text{ km h}^{-1}$. Repeat the exercise with such a train.

**(C) A Rod Moving Parallel to its Length (Length Contraction)**

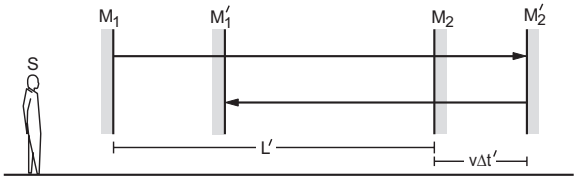



Figure 47.5

Consider the light-beam clock that we discussed. Suppose it is moved at a velocity $v$ along its length (figure 47.5) with respect to an observer. As the rod in the light-beam clock is now moving parallel to its length, we do not know whether the rod retains its length or not. Suppose the length of the rod is $L'$ in the frame $S$. Consider a light pulse reflected from $M_1$ and moving towards $M_2$. Now, $M_2$ is itself moving with velocity $v$ in the same direction. Suppose that the pulse strikes $M_2$ at the position $M_2'$ and that it has taken a time $\Delta t'$ to go from the position $M_1$ to $M_2'$. The mirror $M_2$ has moved ahead a distance $v\Delta t_1'$ so that the pulse has moved a distance $L' + v\Delta t_1'$ before striking $M_2$. But the speed of light pulse is $c$ so that it must travel a distance $c\Delta t_1'$ in time $\Delta t_1'$. Thus,

$$c\Delta t_1' = L' + v\Delta t_1'$$

or,
$$\Delta t_1' = \frac{L'}{c - v} .$$

Similarly, the time taken by the pulse in its return journey from $M_2$ to $M_1$ (it strikes $M_1$ at the position $M_1'$) is

$$\Delta t_2' = \frac{L'}{c + v} .$$

The total time elapsed between successive reflections from $M_1$ is, therefore,

$$\Delta t' = \Delta t_1' + \Delta t_2' = \frac{L'}{c - v} + \frac{L'}{c + v}$$

$$= \frac{2L'c}{c^2 - v^2} . \qquad \ldots \text{ (i)}$$

But $\Delta t'$ is the improper time interval between the two reflections as they occur at different places. An observer, stationary with respect to the rod of the light-beam block, measures this interval to be $\Delta t = 2L/c$ which is the proper time interval between the same two events. Thus, from the equation (47.1),

$$\Delta t' = (\Delta t)\gamma = \left(\frac{2L}{c}\right)\gamma. \quad \ldots \text{(ii)}$$

Using (i) and (ii),

$$\frac{2L'c}{c^2 - v^2} = \frac{2L}{c\sqrt{1 - v^2/c^2}}$$

or, $$\frac{L'}{(1 - v^2/c^2)} = \frac{L}{\sqrt{1 - v^2/c^2}}$$

or, $$L' = L\sqrt{1 - v^2/c^2} = L/\gamma. \quad \ldots \text{(47.2)}$$

The length of a rod is contracted by a factor of $\gamma$ if it moves parallel to its length. The length measured by an observer at rest with respect to the rod is called its *rest length* or *proper length*. Thus,

*The length of a rod moving parallel to itself is shorter than its rest length by the factor* $\gamma$. This phenomenon is called *length contraction.*

***Example 47.2***

*The passenger of example 47.1 slept with his head towards the engine and feet towards the guard's coach. If he measured* 6 ft *in the train frame, how tall is he in the ground frame* ?

***Solution*** : In the ground frame, the passenger is moving with a velocity $c/10$. His length is thus contracted. The length measured in the train frame is the rest length of the passenger as the passenger is at rest in the train. Thus, his length in the ground frame is

$$L' = 6 \text{ ft}\sqrt{1 - \left(\frac{1}{10}\right)^2} = 6 \text{ ft}\sqrt{\frac{99}{100}} = 5 \text{ feet } 11{\cdot}6 \text{ inches.}$$

**(D) Which Event Occurred Earlier ?**

A very important result of special relativity that often surprises beginners is that the concept of simultaneity and ordering of events depends on frame. It is possible that an event $E_1$ occurs before another event $E_2$ in one frame but after $E_2$ in some other frame.

Suppose, a long box of rest length $L$, having two doors $D_1$ and $D_2$ at the ends, lies on the ground (figure 47.6a). At the middle point of the box, there is a light source which can be switched on or off rapidly. Suppose the mechanism is such that when a light pulse strikes a door, the door opens. There are two trains $T_1$ and $T_2$, the first moving towards the left and the other towards the right with respect to the ground, both at speed $v$. Figure (47.6a) shows the situation from the ground frame.

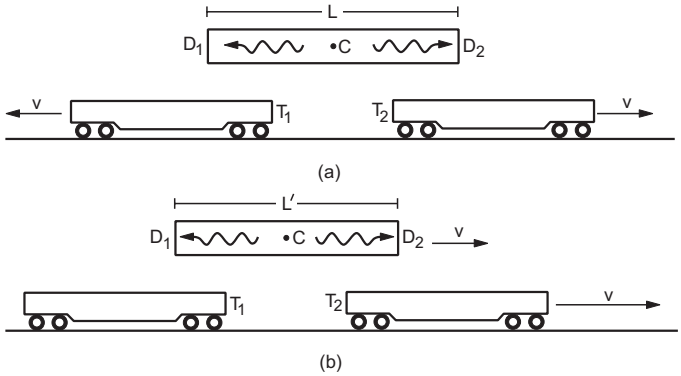



Figure 47.6

Suppose the light source at $C$ is switched on at $t = 0$. Light pulses travel towards $D_1$ and $D_2$, finally striking and opening them. Which of the two doors opened first ? In the ground frame, $D_1$ and $D_2$ do not move and $C$ is at the middle point. Both the light pulses travel with the same speed $c$ and cover equal distances $L/2$ before striking the doors. Both the doors were opened at $t = \frac{L}{2c}$, i.e., the two events are simultaneous in the ground frame.

Now analyse the situation from the train $T_1$. The scene from $T_1$ is shown in figure (47.6b). The box is moving towards the right at a speed $v$. The length of the box is $L' = L\sqrt{1 - v^2/c^2}$. Consider the light pulse moving towards $D_1$. The door $D_1$ is coming towards the pulse with a velocity $v$. If the time taken by the pulse to reach $D_1$ is $\Delta t_1'$, the door has moved a distance $v\Delta t_1'$ and the pulse had to travel a distance $L'/2 - v\Delta t_1'$. Thus,

$$c\Delta t_1' = \frac{L'}{2} - v\Delta t_1'$$

or $$\Delta t_1' = \frac{L'}{2(c + v)}. \quad \ldots \text{(47.3)}$$

Similarly, the time taken by the pulse to reach $D_2$ is

$$\Delta t_2' = \frac{L'}{2(c - v)}$$

or, $$\Delta t_2' - \Delta t_1' = \frac{L'}{2}\left(\frac{1}{c - v} - \frac{1}{c + v}\right) = \frac{L'v}{c^2(1 - v^2/c^2)} = \frac{Lv}{c^2\sqrt{(1 - v^2/c^2)}}$$

or, $$\Delta t_2' - \Delta t_1' = \frac{Lv}{c^2}\gamma. \quad \ldots \text{(47.4)}$$

As $\Delta t_2' > \Delta t_1'$, the door $D_2$ opens after $D_1$ in the frame of $T_1$. Similar analysis from $T_2$ shows that $D_2$ opens before $D_1$ in the frame of $T_2$.

Which of the door opened first ? The answer depends on the frame. In the ground frame both opened together, in the frame of the train $T_1$ the door $D_1$ opened before $D_2$ and in the frame of the train $T_2$ the door $D_2$ opened before $D_1$.

***Example 47.3***

*Suppose the rest length of the box in figure (47.6) is* 30 *light seconds. The train* $T_1$ *travels at a speed of* 0·8*c. Find the time elapsed between opening of* $D_1$ *and* $D_2$ *in the frame of* $T_1$.

***Solution*** : The box moves in the frame of $T_1$ with a speed of 0·8$c$ so that

$$\gamma = \frac{1}{\sqrt{1-(0{\cdot}8)^2}} = \frac{1}{0{\cdot}6}.$$

In this frame, $D_2$ opens after $D_1$. The time elapsed between the openings of the doors is

$$\frac{Lv}{c^2}\gamma = \frac{(30 \text{ light seconds}) \times (0{\cdot}8c)}{c^2 \times 0{\cdot}6}$$

$$= \frac{(30 \text{ s})c \times (0{\cdot}8c)}{c^2 \times 0{\cdot}6} = 40 \text{ s}.$$

**(E) Are the Clocks Synchronized ?**

How do we synchronize two clocks separated from each other ? The readings of the two clocks must be the same at the same instant. One way of doing this is to place a light source at the middle point between the clocks and send light pulses simultaneously towards the clocks. The time $t = 0$ on a clock may be set at the arrival of the pulse at the clock. As the light pulses travel at identical speeds and the source is placed exactly midway between the clocks, this process ensures that the clocks simultaneously read $t = 0$.

Once again consider the situation of figure (47.6a) redrawn in figure (47.7). Suppose the train $T_1$ is very long and a series of clocks are kept on it along its length. Suppose there are clocks fixed in the box at the doors $D_1$ and $D_2$ and the hands are set at zero as the light pulses reach the doors. These clocks are synchronized in the ground frame. The doors open simultaneously in the ground frame and hence the clocks simultaneously read $t = 0$.

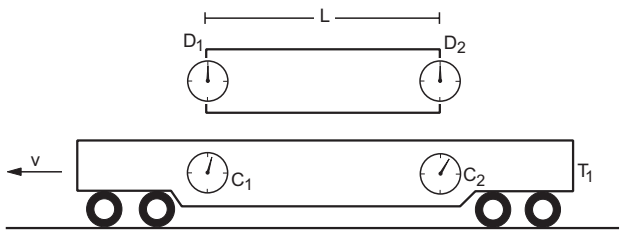



Figure 47.7

The clocks in $T_1$ are synchronized by the same procedure performed in the train. As the door $D_1$ opens, some clock $C_1$ on the train will be opposite to $D_1$ and it will have some reading, say $t'_1$. Similarly, as the door $D_2$ opens, there will be some clock $C_2$ on the train opposite to $D_2$ and it will have a reading $t'_2$.

We have seen that (equation 47.4) according to these clocks on $T_1$, door $D_2$ opens after $D_1$ and the time lag is

$$t'_2 - t'_1 = \frac{Lv}{c^2}\gamma. \qquad \ldots \text{ (i)}$$

In ground frame, the doors open at the same instant $t = 0$. So the clock $C_1$ reads $t'_1$ at the same instant when $C_2$ reads $t'_2$. So in the ground frame, the train clocks are out of synchronization. The clock $C_2$, that is at the rear, is ahead of the clock $C_1$, that is at the front, by an amount

$$\Delta t = \frac{Lv}{c^2}\gamma. \qquad \ldots \text{ (47.5)}$$

Here $L$ is the separation between the doors $D_1$ and $D_2$ as measured from the ground frame. It is also the separation between $C_1$ and $C_2$ as measured from the ground frame. But the length $C_1C_2$ measured from the ground frame is the moving length as $C_1$ and $C_2$ are moving with respect to the ground. The distance between $C_1$ and $C_2$ in the train frame, i.e., the rest length $C_1C_2$, will be larger than the moving length. Thus, the rest separation of the clock $C_1C_2$ is $L_0 = L\gamma$ and equation (47.5) can be reframed as

$$\Delta t = \frac{L_0 v}{c^2}. \qquad \ldots \text{ (47.6)}$$

*The clocks of a moving frame are out of synchronization. The clock at the rear leads the one at the front by* $L_0v/c^2$, *where* $L_0$ *is the rest separation between the clocks, and* $v$ *is the speed of the moving frame.*

Remember that the first postulate asserts that you can call any inertial frame at rest and the conclusions above are valid for all frames. In the previous example, we can very well take the train $T_1$ as the rest frame and then the clocks of the ground frame will be out of synchronization.

## 47.4 DYNAMICS AT LARGE VELOCITY

When velocities comparable to the velocity of light are involved, Newton's second law of motion, $\vec{F} = m\vec{a}$, does not adequately govern the dynamics. We shall not deduce the correct laws but state them.

The linear momentum $\vec{p}$ of a particle is defined as

$$\vec{p} = m_0 \gamma \vec{v} \qquad \ldots \text{ (47.7)}$$

where $m_0$ is the mass of the particle as we know it in Newtonian mechanics. The quantity

$$m = m_0\gamma = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

is called the *moving mass* of the particle when it moves at a speed $v$ with respect to the observer. Thus, the mass of the particle is different for different observers. The mass $m_0$, measured by an observer at rest with respect to the particle, is called its *rest mass*.

With equation (47.7) as the definition of momentum, the law of dynamics is

$$\frac{d\vec{p}}{dt} = \vec{F}. \qquad \ldots (47.8)$$

Equation (47.8) leads to the law of conservation of relativistic momentum. If no force acts on a particle, its momentum remains constant.

***Example 47.4***

*A particle is kept at rest at the origin. A constant force $\vec{F}$ starts acting on it at $t = 0$. Find the speed of the particle at time t.*

***Solution :***

The equation of motion is,

$$\frac{d\vec{p}}{dt} = \vec{F}.$$

As the particle starts from rest and the force is always in the same direction, the motion will be along this direction only. Thus, we can write

$$\frac{dp}{dt} = F$$

or, $$\int_0^p dp = \int_0^t F\,dt$$

or, $$p = Ft$$

or, $$\frac{m_0 v}{\sqrt{1 - v^2/c^2}} = Ft$$

or, $$m_0^2 v^2 = F^2 t^2 - \frac{F^2 t^2}{c^2} v^2$$

or, $$v^2\left(m_0^2 + \frac{F^2 t^2}{c^2}\right) = F^2 t^2$$

or, $$v = \frac{Ftc}{\sqrt{m_0^2 c^2 + F^2 t^2}}.$$

Note from example (47.4) that however large $t$ may be, $v$ can never exceed $c$. No matter how long you apply a force, the speed of a particle will be less than the speed $c$.

## 47.5 ENERGY AND MOMENTUM

According to relativistic dynamics, matter is a condensed form of energy. The energy $E$ equivalent to a mass $m$ is given by the equation

$$E = mc^2. \qquad \ldots (47.9)$$

Thus, matter can be converted into energy and energy into matter. If work is done on a particle, energy is supplied to it. Its energy increases and hence the mass increases. Energy and mass are names for one and the same physical quantity in this viewpoint. When a particle is at rest, its mass is $m_0$ which is called its rest mass. The energy concentrated in it is, therefore, $E_0 = m_0c^2$. This is called the *rest mass energy* of the particle. If the particle moves at a speed $v$, its mass changes to

$$m = m_0\gamma = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

and the total energy in it becomes $E = mc^2$

$$= m_0c^2\left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

$$= m_0c^2\left(1 + \frac{v^2}{2c^2} + \frac{1}{2}\cdot\frac{3}{4}\frac{v^4}{c^4} + \cdots\right)$$

$$= m_0c^2 + \frac{1}{2}m_0v^2 + \ldots \qquad \ldots (i)$$

The extra energy ($mc^2 - m_0c^2$) is called the *kinetic energy*. If $v << c$, the higher order terms in (i) are negligible and hence the kinetic energy is $K = \frac{1}{2}m_0v^2$ as usual. Combining the equations

$$p = mv = \frac{m_0 v}{\sqrt{1 - v^2/c^2}}$$

and $$E = mc^2 = \frac{m_0c^2}{\sqrt{1 - v^2/c^2}},$$

one can deduce that

$$E^2 = m_0^2c^4 + p^2c^2. \qquad \ldots (47.10)$$

For particles having zero rest mass like photons, $m_0 = 0$ and hence from equation (47.10),

$$E = pc$$

or, $$p = E/c. \qquad \ldots (47.11)$$

This result has already been used in previous chapters for photons.

***Example 47.5***

*If a mass of 3·6 g is fully converted into energy, how many kilowatt hour of electrical energy will be obtained ?*

***Solution :***

The energy obtained is

$$E = mc^2 = (3{\cdot}6 \times 10^{-3}\ \text{kg})\,(3 \times 10^8\ \text{m s}^{-1})^2 = 32{\cdot}4 \times 10^{13}\ \text{J}.$$

Now 1 kilowatt hour $= 10^3\ \text{J s}^{-1} \times 3600\ \text{s} = 3{\cdot}6 \times 10^6\ \text{J}$.

Thus, $$E = \frac{32{\cdot}4 \times 10^{13}}{3{\cdot}6 \times 10^6}\ \text{kWh} = 9 \times 10^7\ \text{kWh}.$$

### 47.6 THE ULTIMATE SPEED

We saw in example (47.4) that even if we continue to apply a force on a particle for a long time, its speed cannot exceed *c*. This is a very general result in special relativity. In fact, no information can be sent with a speed greater than *c*. If we assume that information can be sent with a speed greater than *c*, it turns out that we shall have frames in which a bullet will hit the bird before it is actually fired, a dog can die before it is born and so on. If the *effect* cannot precede its *cause* in any frame, then *c* is the ultimate speed for any material particle or information.

Scientists have worked out the mathematics for a world in which all the particles are moving with respect to each other with speeds greater than *c*. Such a world can exist without violating the postulates of relativity but these particles can never be slowed down to a speed less than *c*. The particles of this hypothetical world cannot interact with ours and in that world, effect will always precede its cause. Such particles are named *tachyons* and a group of physicists is working to explore the possibility of the actual existence of such particles. These large speeds and the unworldly results remind us of several stories from Indian Scriptures and no wonder the idea of tachyons was mooted by an Indian scientist E.C.G. Sudarshan.

### 47.7 TWIN PARADOX

As the postulates of special relativity lead to results which contradict 'common sense', a number of interesting paradoxes have been floated. We shall describe one of the most famous paradoxes of relativity—the *twin paradox*. Consider the twins Ram and Balram living happily on the earth. Ram decides to make a trip to a distant planet *P*, which is at rest with respect to the earth, and come back. He boards a spaceship $S_1$, going towards the planet with a uniform velocity. When he reaches the planet, he jumps from the spaceship $S_1$ to another spaceship $S_2$ which is going towards the earth. When he reaches the earth, he jumps out and meets his brother Balram.

As Ram returns from his trip and stands next to Balram, do they have equal age ? Or is Ram younger than Balram or is he older than Balram ?

To keep the calculations simple, let us assume the following data:

Distance between the earth and the planet = 8 light-years,

speed of $S_1$ with respect to earth = 0·8c, and

speed of $S_2$ with respect to earth = 0·8c.

When we said that the distance between the earth and the planet *P* is 8 light-years, was it clear to you that this length is the length as measured from the earth frame ?

First, let us analyse the events from the point of view of Balram who is on the earth. For him, both the spaceships move at a speed 0·8c. So,

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{0{\cdot}6}.$$

When Ram is on $S_1$, he is moving and all his clocks run slower because of time dilation. His heartbeat, pulse beat, etc., represent clocks in themselves and they all run slower. Balram calculates that Ram will take 8 light-year/0·8c = 10 years to reach the planet *P*. But during all these 10 years, time is passing slowly on $S_1$ and the clocks will read only 10 years × 0·6 = 6 years in this period. The number of breaths taken by Ram corresponds to 6 years only.

Ram jumped into $S_2$ for the return journey. This spaceship is also moving at 0·8*c* and for Balram, time passes slowly on $S_2$ as well. Although 10 years passed on the earth during Ram's return journey, on the spaceship the journey was clocked at 6 years. Thus, Ram has aged only 12 years whereas Balram has aged 20 years during this expedition. Ram has become younger than Balram by 8 years. This difference in aging is real in the sense that Ram shows lesser signs of aging like he has lesser white hairs than his brother.

The observation of Balram is quite consistent with the special theory of relativity. Such experiments are indeed performed in laboratories with radioactive particles. Particles are accelerated to large speeds and are kept at these speeds for quite some time by magnetic fields. These particles with large speeds have longer lives than their counterparts kept at rest in the laboratory.

The paradox arises when we analyse the events from the point of veiw of Ram. When he is in the spaceship $S_1$, to him the distance between the earth and the planet is not 8 light-years. The earth and the planet *P* are moving with respect to Ram and hence he is measuring contracted length. The separation is, therefore, 8 light-years × 0·6 = 4·8 light-years. As the planet is approaching Ram at 0·8*c*, the time taken by the planet to reach Ram is 4·8 light-year/0·8*c* = 6 years. So according to Ram's clock, he jumped from $S_1$ to $S_2$ 6 years after getting into $S_1$. Once he is on $S_2$, the earth and the planet are again moving with the same speed 0·8*c*. Again, the earth is 4·8 light-years from the planet and is approaching at 0·8*c*. It takes 6 years for the earth to reach Ram. Thus, according to Ram's clock, he was out for 12 years from the earth, the same result as Balram had expected.

But how about Ram's calculation of Balram's age ? When Ram is on $S_1$, the earth is going away from him

with a speed 0·8$c$. Ram will find that the time on the earth is passing slower by a factor of 0·6 so that Balram is aging slower than he is. The same is true when he is on $S_2$. During this period also, Balram is moving (towards Ram) with a speed 0·8$c$ and hence time is passing slowly for Balram. As 12 years passes on Ram's clock, he calculates that Balram's clocks have advanced only by 12 years $\times$ 0·6 = 7·2 years in this period. According to this analysis, Ram should find that Balram is 12 – 7·2 = 4·8 years younger than him.

This is the paradox. According to Ram, Balram's clocks are running slow and according to Balram, Ram's clocks are running slow. Each thinks the other is younger. Where lies the fallacy ?

The fallacy lies in the fact that Ram has changed frames whereas Balram has stayed in an inertial frame. Thus, the roles of the twins are not symmetrical. The ordering of events are different in different frames and Ram must take that into account when he changes frames. Suppose Ram gets into the spaceship $S_1$ when his clock reads zero. So does Balram's clock. What is the reading of the planet's clock at this instant ? According to Balram, it is zero because both the earth and the planet are at rest and the clocks are synchronized in his frame. But that is not so in $S_1$. As Ram gets into $S_1$, he may have the following conversation with the captain of the ship.

*Captain*: Welcome aboard $S_1$. I saw you on the earth, coming towards us. Your jump to board this ship was perfect. Where are you going ?

*Ram*: Thank you. I am going to the planet $P$. How far is it from here and how long will it take for the planet to come to us ?

*Captain*: Planet $P$ is 4·8 light-years from us at the moment. It is coming towards us at a speed of 0·8$c$ so it will take 4·8 light-years/0·8$c$ = 6 years for the planet $P$ to reach us.

*Ram*: Well, the clocks on the earth and the planet are running a bit slower than ours. I have been taught that moving clocks run slow by a factor of $\gamma$. This factor is 1/0·6 for these clocks. So they will advance by 6 years $\times$ 0·6 = 3·6 years by the time the planet reaches us.

*Captain*: Yes, both the clocks will advance by 3·6 years by the time you jump on the planet $P$.

*Ram*: The earth-clock was reading $t = 0$ as we passed the earth. This means when I jump on the planet $P$ the clocks on the earth and the planet will be reading 3·6 years.

*Captain*: Here you are mistaken. Don't you remember that the planet's clock is not synchronized with the earth's clock ? The planet's clock is at the rear end, and hence is running 6·4 years ahead of the earth's clock. At the instant the earth's clock was reading zero, the planet's clock was reading 6·4 years. As the planet reaches us, both the clocks will advance by 3·6 years. So when you jump out of $S_1$, the earth's clock will be reading 3·6 years but the planet's clock will be reading 10 years.

Ram understands the logic. In the earth's frame, the two clocks read zero simultaneously. But in $S_1$-frame, the event "planet's clock reading zero" occurred several years before "earth's clock reading zero". Six years pass in $S_1$ and Ram finds that the planet $P$ has reached him. He finds another spaceship $S_2$ which is heading towards the earth. Ram jumps onto $S_2$. In the process he looks at the planet's clock and finds that it is reading 10 years as calculated by him on $S_1$. On $S_2$, he starts talking to the commander of the ship.

*Commander*: Welcome to $S_2$. How long will you be with us ?

*Ram*: Thank you. I am going to Earth. Earth is at present 4·8 light-years from here and is coming towards us with a speed of 0·8$c$. So I will be with you for 6 years. The captain of $S_1$ told me that the earth's clock is reading 3·6 years at this moment whereas the planet's clock reads 10 years. There is a difference of 6·4 years in the reading because the two clocks are not synchronized. Also ….

*Commander*: Sorry for interrupting you, but you are mistaken. It is true that the earth's clock and the planet's clock are not synchronized as they are moving past us. Also the difference in the readings of the two clocks is 6·4 years. But the planet's clock is at the front and the earth's clock is at the rear. It is the earth's clock that is leading by 6·4 years. At the moment the planet's clock reads 10 years and hence the earth's clock must be reading 16·4 years.

*Ram*: Hmm… you are right. In $S_1$, the earth was at the front and its clock lagged behind the planet's clock. But in $S_2$ it is the other way round. Indeed the earth's clock reads 16·4 years whereas the planet's clock reads 10 years.

*Commander*: That's right. The earth's clock is reading 16·4 years at present. It will advance by another 3·6 years during the 6 years you will be with us. So it will be reading 20 years when the earth reaches you.

We see that the paradox is resolved.

## *Worked Out Examples*

**1.** *A hypothetical train moving with a speed of* 0·6*c passes by the platform of a small station without being slowed down. The observers on the platform note that the length of the train is just equal to the length of the platform which is* 200 m. *(a) Find the rest length of the train. (b) Find the length of the platform as measured by the observers in the train.*

***Solution*** : (a) The length $L'$ of the train at a speed 0·6$c$ is 200 m. If the rest length is $L$,

$$L' = L\sqrt{1 - v^2/c^2}$$

or,
$$L = \frac{L'}{\sqrt{1 - v^2/c^2}} = \frac{200 \text{ m}}{\sqrt{1 - (0{\cdot}6)^2}}$$
$$= 250 \text{ m}.$$

(b) The rest length of the platform is 200 m. For the observers in the train, the platform is moving at a speed of 0·6 $c$. The length as measured by the observers in the train is, therefore,

$$L' = 200 \text{ m}\sqrt{1 - (0{\cdot}6)^2} = 160 \text{ m}.$$

**2.** *Unstable pions are produced as a beam in a nuclear reaction experiment. The pions leave the target at a speed of* 0·995$c$. *The intensity of the beam reduces to half its original value as the beam travels a distance of* 39 m. *Find the half-life of pions (a) in the laboratory frame, (b) in their rest frame.*

***Solution*** : (a) The intensity of the pion beam reduces to half its original value in one half-life. The half-life of the pions as measured in the laboratory is

$$t_{1/2} = \frac{39 \text{ m}}{0{\cdot}995\ c} = \frac{39 \text{ m}}{0{\cdot}995 \times 3 \times 10^8 \text{ m s}^{-1}}$$
$$= 1{\cdot}3 \times 10^{-7} \text{ s}.$$

(b) The events—a pion leaving the target and its decaying—occur at the same place in the pion-frame. Thus, the time measured in the pion-frame is the proper time and is the smallest. It is equal to

$$t'_{1/2} = t_{1/2}\sqrt{1 - v^2/c^2} = (1{\cdot}3 \times 10^{-7} \text{ s})\sqrt{1 - (0{\cdot}995)^2}$$
$$= 1{\cdot}3 \times 10^{-8} \text{ s}.$$

**3.** *Two events A and B occur at places separated by* $10^6$ km, *B occurring* 5 s *after A. (a) Find the velocity of a frame in which these events occur at the same place. (b) What is the time interval between the events in this frame* ?

***Solution*** :

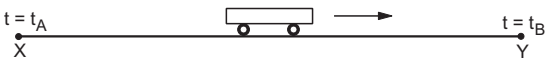



Figure 47-W1

(a) Suppose the events $A$ and $B$ occur at points $X$ and $Y$ at times $t_A$ and $t_B$ where $t_B = t_A + 5$ s. Consider a small train which is at the point $X$ when the event $A$ occurs. Suppose, this same train moves towards $Y$ and reaches the point $Y$ when the event $B$ occurs. Thus, the events $A$ and $B$ occur at the same place in the train frame. This frame moves $10^6$ km in 5 s as seen from the original frame. Thus, the velocity of the train frame is

$$v = \frac{10^6 \text{ km}}{5 \text{ s}} = 2 \times 10^8 \text{ m s}^{-1}.$$

(b) As the events $A$ and $B$ occur at the same place in the train frame, the time interval between the events measured in this frame is the proper interval. Thus, this time interval is

$$= (5 \text{ s})\sqrt{1 - v^2/c^2} = (5 \text{ s})\sqrt{1 - \left(\frac{2}{3}\right)^2}$$
$$= 3{\cdot}7 \text{ s}.$$

**4.** *A satellite orbits the earth near its surface. By what amount does the satellite's clock fall behind the earth's clock in one revolution* ? *Assume that nonrelativistic analysis can be made to compute the speed of the satellite and only the time dilation is to be taken into account for calculation of clock speeds.*

***Solution*** : The speed of the satellite may be obtained from the equation,

$$\frac{GMm}{R^2} = \frac{mv^2}{R}$$

or,
$$v = \sqrt{\frac{GM}{R}}$$
$$= \left[\frac{(6{\cdot}67 \times 10^{-11} \text{ N m}^2\text{kg}^{-2})\ (6 \times 10^{24} \text{ kg})}{6400 \times 10^3 \text{ m}}\right]^{1/2}$$
$$= 7910 \text{ m s}^{-1}. \quad \ldots \text{ (i)}$$

Thus, $v/c = \frac{7910}{3 \times 10^8} = 2{\cdot}637 \times 10^{-5}$

or,
$$\sqrt{1 - \left(\frac{v}{c}\right)^2} = [1 - 6{\cdot}95 \times 10^{-10}]^{1/2}$$
$$\approx 1 - 3{\cdot}48 \times 10^{-10}.$$

The time taken by the satellite to complete one revolution is

$$T = \frac{2\pi R}{v} = \frac{6{\cdot}28 \times 6400 \times 10^3 \text{ m}}{7910 \text{ m s}^{-1}} = 5080 \text{ s}.$$

The clock on the satellite will slow down as observed from the earth. If the time elapsed on the satellite's clock is $t$ as the satellite completes one revolution (this is proper time and 5080 s is improper time),

$$t = (1 - 3{\cdot}48 \times 10^{-10}) \times (5080 \text{ s})$$

or,
$$\frac{t}{5080 \text{ s}} = 1 - 3{\cdot}48 \times 10^{-10}$$

or, $$\frac{(t - 5080\text{ s})}{5080\text{ s}} = -3{\cdot}48 \times 10^{-10}$$

or, $$(t - 5080\text{ s}) = -1{\cdot}77 \times 10^{-6}\text{ s}.$$

The satellite's clock falls behind by $1{\cdot}77 \times 10^{-6}$ s in one revolution.

**5.** *The radius of our galaxy is about* $3 \times 10^{20}$ m. *With what speed should a person travel so that he can reach from the centre of the galaxy to its edge in* 20 years *of his lifetime* ?

***Solution*** : Let the speed of the person be $v$. As seen by the person, the edge of the galaxy is coming towards him at a speed $v$. In 20 years (as measured by the person), the edge moves (20 y) $v$ and reaches the person. The radius of the galaxy as measured by the person is, therefore, (20 y)$v$. The rest length of the radius of the galaxy is $3 \times 10^{20}$ m. Thus,

$$(20\text{ y})v = (3 \times 10^{20}\text{ m})\sqrt{1 - v^2/c^2}$$

or, $$(6{\cdot}312 \times 10^8\text{ s})^2 v^2 = (9 \times 10^{40}\text{ m}^2)(1 - v^2/c^2).$$

Solving this,

$$v = 0{\cdot}9999996\ c.$$

**6.** *Find the speed at which the mass of an electron is double of its rest mass.*

***Solution*** : The mass of an electron at speed $v$ is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where $m_0$ is its rest mass. If $m = 2\ m_0$,

$$2 = \frac{1}{\sqrt{1 - v^2/c^2}}$$

or, $$1 - \frac{v^2}{c^2} = \frac{1}{4}$$

or, $$v = \frac{\sqrt{3}}{2}\ c = 2{\cdot}598 \times 10^8\text{ m s}^{-1}.$$

**7.** *Calculate the increase in mass when a body of rest mass* 1 kg *is lifted up through* 1 m *near the earth's surface.*

***Solution*** : The increase in energy = $mgh$

$$= (1\text{ kg})(9{\cdot}8\text{ m s}^{-2})(1\text{ m}) = 9{\cdot}8\text{ J}.$$

The increase in mass $= \frac{9{\cdot}8\text{ J}}{c^2}$

$$= 1{\cdot}11 \times 10^{-16}\text{ kg}.$$

**8.** *A body of rest mass* $m_0$ *collides perfectly inelastically at a speed of* 0·8$c$ *with another body of equal rest mass kept at rest. Calculate the common speed of the bodies after the collision and the rest mass of the combined body.*

***Solution*** : The linear momentum of the first body

$$= \frac{m_0 v}{\sqrt{1 - v^2/c^2}} = \frac{m_0 \times 0{\cdot}8c}{0{\cdot}6}$$

$$= \frac{4}{3}\ m_0 c.$$

This should be the total linear momentum after the collision. If the rest mass of the combined body is $M_0$ and it moves at speed $v'$,

$$\frac{M_0 v'}{\sqrt{1 - v'^2/c^2}} = \frac{4}{3}\ m_0 c. \qquad \ldots \text{(i)}$$

The energy before the collison is

$$\frac{m_0}{\sqrt{1 - v^2/c^2}}\ c^2 + m_0 c^2 = m_0 c^2\left(\frac{1}{0{\cdot}6} + 1\right)$$

$$= \frac{8}{3}\ m_0 c^2.$$

The energy after the collision is

$$\frac{M_0 c^2}{\sqrt{1 - v'^2/c^2}}.$$

Thus, $$\frac{M_0 c^2}{\sqrt{1 - v'^2/c^2}} = \frac{8}{3}\ m_0 c^2. \qquad \ldots \text{(ii)}$$

Dividing (i) by (ii),

$$\frac{v'}{c^2} = \frac{1}{2c} \quad \text{or,} \quad v' = \frac{c}{2}.$$

Putting this value of $v'$ in (ii),

$$M_0 = \frac{8}{3}\ m_0 \sqrt{1 - \frac{1}{4}}$$

or, $$M_0 = 2{\cdot}309\ m_0.$$

The rest mass of the combined body is greater than the sum of the rest masses of the individual bodies.

□

## QUESTIONS FOR SHORT ANSWER

**1.** The speed of light in glass is $2{\cdot}0 \times 10^8$ m s$^{-1}$. Does it violate the second postulate of special relativity ?

**2.** A uniformly moving train passes by a long platform. Consider the events 'engine crossing the beginning of the platform' and 'engine crossing the end of the platform'. Which frame (train frame or the platform frame) is the proper frame for the pair of events ?

**3.** An object may be regarded to be at rest or in motion depending on the frame of reference chosen to view the object. Because of length contraction it would mean that the same rod may have two different lengths depending on the state of the observer. Is this true ?

**4.** Mass of a particle depends on its speed. Does the attraction of the earth on the particle also depend on the particle's speed ?

**5.** A person travelling in a fast spaceship measures the distance between the earth and the moon. Is it the same, smaller or larger than the value quoted in this book ?

## OBJECTIVE I

**1.** The magnitude of linear momentum of a particle moving at a relativistic speed $v$ is proportional to
(a) $v$ (b) $1 - v^2/c^2$
(c) $\sqrt{1 - v^2/c^2}$ (d) none of these.

**2.** As the speed of a particle increases, its rest mass
(a) increases (b) decreases
(c) remains the same (d) changes.

**3.** An experimenter measures the length of a rod. Initially the experimenter and the rod are at rest with respect to the lab. Consider the following statements.
(A) If the rod starts moving parallel to its length but the observer stays at rest, the measured length will be reduced.
(B) If the rod stays at rest but the observer starts moving parallel to the measured length of the rod, the length will be reduced.
(a) A is true but B is false. (b) B is true but A is false.
(c) Both A and B are true. (d) Both A and B are false.

**4.** An experimenter measures the length of a rod. In the cases listed, all motions are with respect to the lab and parallel to the length of the rod. In which of the cases the measured length will be minimum ?
(a) The rod and the experimenter move with the same speed $v$ in the same direction.
(b) The rod and the experimenter move with the same speed $v$ in opposite directions.
(c) The rod moves at speed $v$ but the experimenter stays at rest.
(d) The rod stays at rest but the experimenter moves with the speed $v$.

**5.** If the speed of a particle moving at a relativistic speed is doubled, its linear momentum will
(a) become double (b) become more than double
(c) remain equal (d) become less than double.

**6.** If a constant force acts on a particle, its acceleration will
(a) remain constant (b) gradually decrease
(c) gradually increase (d) be undefined.

**7.** A charged particle is projected at a very high speed perpendicular to a uniform magnetic field. The particle will
(a) move along a circle
(b) move along a curve with increasing radius of curvature
(c) move along a curve with decreasing radius of curvature
(d) move along a straight line.

## OBJECTIVE II

**1.** Mark the correct statements:
(a) Equations of special relativity are not applicable for small speeds.
(b) Equations of special relativity are applicable for all speeds.
(c) Nonrelativistic equations give exact result for small speeds.
(d) Nonrelativistic equations never give exact result.

**2.** If the speed of a rod moving at a relativistic speed parallel to its length is doubled,
(a) the length will become half of the original value
(b) the mass will become double of the original value
(c) the length will decrease
(d) the mass will increase.

**3.** Two events take place simultaneously at points $A$ and $B$ as seen in the lab frame. They also occur simultaneously in a frame moving with respect to the lab in a direction
(a) parallel to $AB$ (b) perpendicular to $AB$
(c) making an angle of 45° with $AB$
(d) making an angle of 135° with $AB$.

**4.** Which of the following quantities related to an electron has a finite upper limit ?
(a) Mass (b) Momentum (c) Speed (d) Kinetic energy

**5.** A rod of rest length $L$ moves at a relativistic speed. Let $L' = L/\gamma$. Its length
(a) must be equal to $L'$ (b) may be equal to $L$
(c) may be more than $L'$ but less than $L$
(d) may be more than $L$.

**6.** When a rod moves at a relativistic speed $v$, its mass
(a) must incrase by a factor of $\gamma$
(b) may remain unchanged
(c) may increase by a factor other than $\gamma$
(d) may decrease.

## EXERCISES

**1.** The *guru* of a *yogi* lives in a Himalyan cave, 1000 km away from the house of the yogi. The yogi claims that whenever he thinks about his guru, the guru immediately knows about it. Calculate the minimum possible time interval between the yogi thinking about the guru and the guru knowing about it.

**2.** A suitcase kept on a shop's rack is measured 50 cm × 25 cm × 10 cm by the shop's owner. A traveller takes this suitcase in a train moving with velocity 0·6*c*. If the suitcase is placed with its length along the train's velocity, find the dimensions measured by (a) the traveller and (b) a ground observer.

**3.** The length of a rod is exactly 1 m when measured at rest. What will be its length when it moves at a speed of (a) $3 \times 10^5$ m s$^{-1}$, (b) $3 \times 10^6$ m s$^{-1}$ and (c) $3 \times 10^7$ m s$^{-1}$ ?

**4.** A person standing on a platform finds that a train moving with velocity 0·6*c* takes one second to pass by him. Find (a) the length of the train as seen by the person and (b) the rest length of the train.

**5.** An aeroplane travels over a rectangular field 100 m × 50 m, parallel to its length. What should be the speed of the plane so that the field becomes square in the plane frame ?

**6.** The rest distance between Patna and Delhi is 1000 km. A nonstop train travels at 360 km h$^{-1}$. (a) What is the distance between Patna and Delhi in the train frame ? (b) How much time elapses in the train frame between Patna and Delhi ?

**7.** A person travels by a car at a speed of 180 km h$^{-1}$. It takes exactly 10 hours by his wristwatch to go from the station *A* to the station *B*. (a) What is the rest distance between the two stations ? (b) How much time is taken in the road frame by the car to go from the station *A* to the station *B* ?

**8.** A person travels on a spaceship moving at a speed of 5*c*/13. (a) Find the time interval calculated by him between the consecutive birthday celebrations of his friend on the earth. (b) Find the time interval calculated by the friend on the earth between the consecutive birthday celebrations of the traveller.

**9.** According to the station clocks, two babies are born at the same instant, one in Howrah and other in Delhi. (a) Who is elder in the frame of 2301 Up Rajdhani Express going from Howrah to Delhi ? (b) Who is elder in the frame of 2302 Dn Rajdhani Express going from Delhi to Howrah.

**10.** Two babies are born in a moving train, one in the compartment adjacent to the engine and other in the compartment adjacent to the guard. According to the train frame, the babies are born at the same instant of time. Who is elder according to the ground frame ?

**11.** Suppose Swarglok (heaven) is in constant motion at a speed of 0·9999*c* with respect to the earth. According to the earth's frame, how much time passes on the earth before one day passes on Swarglok ?

**12.** If a person lives on the average 100 years in his rest frame, how long does he live in the earth frame if he spends all his life on a spaceship going at 60% of the speed of light.

**13.** An electric bulb, connected to a make and break power supply, switches off and on every second in its rest frame. What is the frequency of its switching off and on as seen from a spaceship travelling at a speed 0·8*c* ?

**14.** A person travelling by a car moving at 100 km h$^{-1}$ finds that his wristwatch agrees with the clock on a tower *A*. By what amount will his wristwatch lag or lead the clock on another tower *B*, 1000 km (in the earth's frame) from the tower *A* when the car reaches there ?

**15.** At what speed the volume of an object shrinks to half its rest value ?

**16.** A particular particle created in a nuclear reactor leaves a 1 cm track before decaying. Assuming that the particle moved at 0·995*c*, calculate the life of the particle (a) in the lab frame and (b) in the frame of the particle.

**17.** By what fraction does the mass of a spring change when it is compressed by 1 cm ? The mass of the spring is 200 g at its natural length and the spring constant is 500 N m$^{-1}$.

**18.** Find the increase in mass when 1 kg of water is heated from 0°C to 100°C. Specific heat capacity of water = 4200 J kg$^{-1}$K$^{-1}$.

**19.** Find the loss in the mass of 1 mole of an ideal monatomic gas kept in a rigid container as it cools down by 10°C. The gas constant $R = 8{\cdot}3$ J K$^{-1}$mol$^{-1}$.

**20.** By what fraction does the mass of a boy increase when he starts running at a speed of 12 km h$^{-1}$ ?

**21.** A 100 W bulb together with its power supply is suspended from a sensitive balance. Find the change in the mass recorded after the bulb remains on for 1 year.

**22.** The energy from the sun reaches just outside the earth's atmosphere at a rate of 1400 W m$^{-2}$. The distance between the sun and the earth is $1{\cdot}5 \times 10^{11}$ m. (a) Calculate the rate at which the sun is losing its mass. (b) How long will the sun last assuming a constant decay at this rate ? The present mass of the sun is $2 \times 10^{30}$ kg.

**23.** An electron and a positron moving at small speeds collide and annihilate each other. Find the energy of the resulting gamma photon.

**24.** Find the mass, the kinetic energy and the momentum of an electron moving at 0·8*c*.

**25.** Through what potential difference should an electron be accelerated to give it a speed of (a) 0·6*c*, (b) 0·9*c* and (c) 0·99*c* ?

**26.** Find the speed of an electron with kinetic energy (a) 1 eV, (b) 10 keV and (c) 10 MeV.

**27.** What is the kinetic energy of an electron in electronvolts with mass equal to double its rest mass ?

**28.** Find the speed at which the kinetic energy of a particle will differ by 1% from its nonrelativistic value $\frac{1}{2} m_0 v^2$.

□

## ANSWERS

### OBJECTIVE I

1. (d) 2. (c) 3. (c) 4. (b) 5. (b) 6. (b)
7. (b)

### OBJECTIVE II

1. (b), (d) 2. (c), (d) 3. (b)
4. (c) 5. (b), (c) 6. (a)

### EXERCISES

1. 1/300 s
2. (a) 50 cm × 25 cm × 10 cm (b) 40 cm × 25 cm × 10 cm
3. (a) 0·9999995 m (b) 0·99995 m (c) 0·995 m
4. (a) $1{\cdot}8 \times 10^{8}$ m (b) $2{\cdot}25 \times 10^{8}$ m
5. 0·866$c$
6. (a) 56 nm less than 1000 km
   (b) 0·56 ns less than $\frac{500}{3}$ min
7. (a) 25 nm more than 1800 km
   (b) 0·5 ns more than 10 hours
8. $\frac{13}{12}$ y in both cases
9. (a) Delhi baby is elder (b) Howrah baby is elder
10. the baby adjacent to the guard is elder
11. 70·7 days
12. 125 y
13. $0{\cdot}6\ s^{-1}$
14. will lag by 0·154 ns
15. $\frac{\sqrt{3}c}{2}$
16. (a) 33·5 ps (b) 3·35 ps
17. $1{\cdot}4 \times 10^{-18}$
18. $4{\cdot}7 \times 10^{-12}$ kg
19. $1{\cdot}38 \times 10^{-15}$ kg
20. $6{\cdot}17 \times 10^{-17}$
21. $3{\cdot}5 \times 10^{-8}$ kg
22. (a) $4{\cdot}4 \times 10^{9}$ kg $s^{-1}$ (b) $1{\cdot}44 \times 10^{13}$ y
23. 1·02 MeV
24. $15{\cdot}2 \times 10^{-31}$ kg, $5{\cdot}5 \times 10^{-14}$ J, $3{\cdot}65 \times 10^{-22}$ kg m $s^{-1}$
25. (a) 128 kV (b) 661 kV (c) 3·1 MV
26. (a) $5{\cdot}92 \times 10^{5}$ m $s^{-1}$ (b) $5{\cdot}85 \times 10^{7}$ m $s^{-1}$
    (c) $2{\cdot}996 \times 10^{8}$ m $s^{-1}$
27. 511 keV
28. $3{\cdot}46 \times 10^{7}$ m $s^{-1}$

□

# APPENDIX A

## Units and Dimensions of Physical Quantities

| Quantity | Common Symbol | SI Unit | Dimension |
|---|---|---|---|
| Displacement | $s$ | METRE (m) | L |
| Mass | $m, M$ | KILOGRAM (kg) | M |
| Time | $t$ | SECOND (s) | T |
| Area | $A$ | $m^2$ | $L^2$ |
| Volume | $V$ | $m^3$ | $L^3$ |
| Density | $\rho$ | $kg\ m^{-3}$ | $M\ L^{-3}$ |
| Velocity | $v, u$ | $m\ s^{-1}$ | $L\ T^{-1}$ |
| Acceleration | $a$ | $m\ s^{-2}$ | $L\ T^{-2}$ |
| Force | $F$ | newton (N) | $ML\ T^{-2}$ |
| Work | $W$ | joule (J)(= N–m) | $ML^2\ T^{-2}$ |
| Energy | $E, U, K$ | joule (J) | $ML^2\ T^{-2}$ |
| Power | $P$ | watt (W)(= $J\ s^{-1}$) | $ML^2\ T^{-3}$ |
| Momentum | $p$ | $kg–m\ s^{-1}$ | $ML\ T^{-1}$ |
| Gravitational constant | $G$ | $N–m^2\ kg^{-2}$ | $L^3\ M^{-1}T^{-2}$ |
| Angle | $\theta, \varphi$ | radian | |
| Angular velocity | $\omega$ | $rad\ s^{-1}$ | $T^{-1}$ |
| Angular acceleration | $\alpha$ | $rad\ s^{-2}$ | $T^{-2}$ |
| Angular momentum | $L$ | $kg–m^2\ s^{-1}$ | $ML^2\ T^{-1}$ |
| Moment of inertia | $I$ | $kg–m^2$ | $ML^2$ |
| Torque | $\tau$ | N–m | $ML^2\ T^{-2}$ |
| Angular frequency | $\omega$ | $rad\ s^{-1}$ | $T^{-1}$ |
| Frequency | $\nu$ | hertz (Hz) | $T^{-1}$ |
| Period | $T$ | s | T |
| Young's modulus | $Y$ | $N\ m^{-2}$ | $M\ L^{-1}T^{-2}$ |
| Bulk modulus | $B$ | $N\ m^{-2}$ | $M\ L^{-1}T^{-2}$ |
| Shear modulus | $\eta$ | $N\ m^{-2}$ | $M\ LT^{-2}$ |
| Surface tension | $S$ | $N\ m^{-1}$ | $M\ T^{-2}$ |
| Coefficient of viscosity | $\eta$ | $N–s\ m^{-2}$ | $M\ L^{-1}T^{-1}$ |
| Pressure | $P, p$ | $N\ m^{-2}$, Pa | $M\ L^{-1}T^{-2}$ |
| Wavelength | $\lambda$ | m | L |
| Intensity of wave | $I$ | $W\ m^{-2}$ | $M\ T^{-3}$ |
| Temperature | $T$ | KELVIN (K) | $\Theta$ |
| Specific heat capacity | $c$ | $J\ kg^{-1}–K^{-1}$ | $L^2\ T^{-2}\ K^{-1}$ |
| Stefan's constant | $\sigma$ | $W\ m^{-2}–K^{-4}$ | $M\ T^{-3}K^{-4}$ |
| Heat | $Q$ | J | $ML^2\ T^{-2}$ |
| Thermal conductivity | $K$ | $W\ m^{-1}–K^{-1}$ | $ML\ T^{-3}K^{-1}$ |
| Current | $I$ | AMPERE(A) | I |
| Charge | $q, Q$ | coulomb (C) | IT |
| Current density | $j$ | $A\ m^{-2}$ | $IL^{-2}$ |
| Electrical conductivity | $\sigma$ | $1/\Omega–m$ (= mho/m) | $I^2T^3\ M^{-1}L^{-3}$ |
| Dielectric constant | $k$ | | |
| Electric dipole moment | $p$ | C–m | LIT |
| Electric field | $E$ | $V\ m^{-1}$(= $N\ C^{-1}$) | $ML\ I^{-1}T^{-3}$ |
| Potential (voltage) | $V$ | volt (V)(= $J\ C^{-1}$) | $ML^2\ I^{-1}T^{-3}$ |
| Electric flux | $\Phi$ | V–m | $ML^3\ I^{-1}T^{-3}$ |
| Capacitance | $C$ | farad (F) | $I^2T^4\ M^{-1}L^{-2}$ |
| Electromotive force | $E$ | volt (V) | $ML^2\ I^{-1}T^{-3}$ |
| Resistance | $R$ | ohm ($\Omega$) | $ML^2\ I^{-2}T^{-3}$ |
| Permittivity of space | $\varepsilon_0$ | $C^2\ N^{-1}–m^{-2}$ (= $F\ m^{-1}$) | $I^2T^4\ M^{-1}L^{-3}$ |
| Permeability of space | $\mu_0$ | $N\ A^{-2}$ | $ML\ I^{-2}T^{-2}$ |
| Magnetic field | $B$ | tesla (T)(= $Wb\ m^{-2}$) | $M\ I^{-1}T^{-2}$ |
| Magnetic flux | $\Phi_B$ | weber (Wb) | $ML^2\ I^{-1}T^{-2}$ |
| Magnetic dipole moment | $\mu$ | $N–m\ T^{-1}$ | $IL^2$ |
| Inductance | $L$ | henry (H) | $ML^2\ I^{-2}T^{-2}$ |

# APPENDIX B

## Universal Constants (as revised in 1986)

| Quantity | Symbol | Value | Unit | Uncertainty in the last two digits |
|---|---|---|---|---|
| Constant of gravitation | $G$ | $6{\cdot}67259 \times 10^{-11}$ | $N–m^2 kg^{-2}$ | 85 |
| Speed of light in vacuum | $c$ | $2{\cdot}99792458 \times 10^{8}$ | $m\ s^{-1}$ | exact |
| Avogadro constant | $N_A$ | $6{\cdot}0221367 \times 10^{23}$ | $mol^{-1}$ | 36 |
| Gas constant | $R$ | 8·314510 | $J\ K^{-1}–mol^{-1}$ | 70 |
| Boltzmann constant | $k$ | $1{\cdot}380658 \times 10^{-23}$<br>$8{\cdot}617385 \times 10^{-5}$ | $J\ K^{-1}$<br>$eV\ K^{-1}$ | 12<br>73 |
| Stefan–Boltzmann constant | $\sigma$ | $5{\cdot}67051 \times 10^{-8}$ | $W\ m^{-2}–K^{-4}$ | 19 |
| Wien's displacement law constant | $b$ | $2{\cdot}897756 \times 10^{-3}$ | m–K | 24 |
| Charge of proton | $e$ | $1{\cdot}60217733 \times 10^{-19}$ | C | 49 |
| Mass of electron | $m_e$ | $9{\cdot}1093897 \times 10^{-31}$<br>$5{\cdot}48579903 \times 10^{-4}$ | kg<br>u | 54<br>13 |
| Mass of proton | $m_p$ | $1{\cdot}6726231 \times 10^{-27}$<br>1·007276470 | kg<br>u | 10<br>12 |
| Mass of neutron | $m_n$ | $1{\cdot}6749286 \times 10^{-27}$<br>1·008664904 | kg<br>u | 10<br>14 |
| Permeability of vacuum | $\mu_0$ | $4\pi \times 10^{-7}$<br>$= 12{\cdot}566370614\ldots \times 10^{-7}$ | $N\ A^{-2}$ | exact |
| Permittivity of vacuum | $\epsilon_0$ | $\frac{1}{\mu_0 c^2}$<br>$= 8{\cdot}854187817\ldots \times 10^{-12}$ | $C^2\ N^{-1}–m^{-2} = F\ m^{-1}$ | exact |
| Faraday constant | $F$ | 96485·3029 | $C\ mol^{-1}$ | 29 |
| Planck constant | $h$ | $6{\cdot}6260755 \times 10^{-34}$<br>$4{\cdot}1356692 \times 10^{-15}$ | J–s<br>eV–s | 40<br>12 |
| Rydberg constant | $R$ | $1{\cdot}0973731534 \times 10^{7}$ | $m^{-1}$ | 13 |
| Ground state energy of hydrogen atom | | 13·605698 | eV | 40 |
| Bohr radius | $a_0$ | $5{\cdot}29177249 \times 10^{-11}$ | m | 24 |

## Astronomical Constant

| Quantity | Value | Unit |
|---|---|---|
| Mass of the sun | $1.99 \times 10^{30}$ | kg |
| Radius of the sun | $6.95 \times 10^{8}$ | m |
| Mass of the earth | $5.98 \times 10^{24}$ | kg |
| Mean radius of the earth | $6.37 \times 10^{6}$ | m |
| Mass of the moon | $7.36 \times 10^{22}$ | kg |
| Radius of the moon | $1.74 \times 10^{6}$ | m |
| Mean earth–sun distance | $1.50 \times 10^{11}$ | m |
| Mean earth–moon distance | $3.84 \times 10^{8}$ | m |
| Escape speed from the earth | 11.2 | $km\ s^{-1}$ |
| Escape speed from the moon | 2.38 | $km\ s^{-1}$ |

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